Localisation

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Abstract

Here we give the definition and some basic properties of localisation in commutative algebra. All suggested exercises are optional, and need not be handed in.

All rings are commutative and have units, and morphisms send units to units.

Definition 0.1. Let A be a ring. A *multiplicative subset* of A is a subset $S \subset A$ such that

1. for all a, $b \in S$, we have $ab \in S$;
2. $1 \in S$.

Note that the image of a multiplicative subset under a morphism of rings is again multiplicative.

For us the main example of a multiplicative subset is to take $p \triangleleft A$ a prime ideal, and define S to be the complement $A \setminus p$. You will easily check that, because the ideal is prime, the set $A \setminus p$ is multiplicative.

Definition 0.2. Let M be an A-module. The localisation $S^{-1}M$ is defined to be the set of pairs $(m, s) \in M \times S$ modulo the equivalence relation $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $s \in S$ such that

$$s(m_1 s_2 - m_2 s_1) = 0.$$

The set $S^{-1}M$ has a natural addition given by

$$(m_1, s_1) + (m_2, s_2) = (m_1 s_2 + m_2 s_1, s_1 s_2),$$

which has obvious inverse and identity (cf. remark below about fractions). It also has a multiplication by A, by $a(m, s) = (am, s)$, making it into an A-module.

If we consider the case $M = A$, we also see that $S^{-1}A$ has a multiplication given by $(a_1, s_1)(a_2, s_2) = (a_1 a_2, s_1 s_2)$. This makes $S^{-1}A$ into a ring, and even an A-algebra, by mapping A to $S^{-1}A$ by sending a to $(a, 1)$.

We actually find that $S^{-1}M$ is naturally an $S^{-1}A$-module.

Remark 0.3.

1. We usually write m/s for the pair (m, s), which makes the definition of addition etc seem more reasonable;
2. If $M = A$ is a domain, then you may always take the element s in the definition of $\sim$ to be 1. With a little more work, you will recover the definition given in class in the case when A is a domain;
3. if $S = A \setminus p$ for some prime ideal p, we often write $S^{-1}M = M_p$;

4. is S is a subset of the units of A, then $S^{-1}A = A$;

5. if $f\colon A \to B$ is a ring homomorphism and S a multiplicative subset of A, then there are two natural ways to localise B at S: we can view B as an A-module, and think about $S^{-1}B$, or we can use that the image of a multiplicative set is multiplicative, and think about the localisation $f(S)^{-1}B$, i.e. localise B at $f(S)$ as a B-module. In fact, these two A-modules $S^{-1}B$ and $f(S)^{-1}B$ are canonically isomorphic (not hard to check).

1 Localisation and quotients

Localisation is very well-behaved with respect to quotienting by ideals. This is a special case of the fact that localisation is flat, but we will not give more details here. In this course, you may use without proof the following proposition:

Proposition 1.1. *Let A be a ring, $I \lhd A$ an ideal, and $S \subset A$ a multiplicative subset. Write $f\colon A \to A/I$ for the quotient map. Then the sequence of A-module homomorphisms*

$$0 \to I \cdot (S^{-1}A) \to S^{-1}A \to f(S)^{-1}(A/I) \to 0 \tag{1}$$

is exact. Here the map $S^{-1}A \to f(S)^{-1}(A/I)$ sends (a, s) to $(a + I, s + I)$.

A proof of this can be obtained by combining corollary 1.2.11 and definition 1.2.1 of [Liu, algebraic geometry and arithmetic curves], or look in almost any commutative algebra book for 'flatness of localisation'. In fact the non-trivial part is the left-exactness, which is not strictly necessary for what we will do below.

Note that, by the above remark, the ring $f(S)^{-1}(A/I)$ (which we can think of as localisation of the ring A/I at the multiplicative subset $f(S)$) is actually canonically isomorphic as an A-module to the A-module $S^{-1}(A/I)$.

Why is this proposition useful? Let us use it to compute an intersection number.

Let k be an (algebraically closed) field, and $A = k[x, y]$. Consider the curves defined by the polynomials x and $x - y^3$. The intersection number of these curves at $x = y = 0$ is given by

$$\dim_k \frac{k[x, y]_{(x,y)}}{(x, x - y^3)},$$

where the dimension is as a k-module, also known as a k-vector space. Let $B = k[x, y]/(x, x - y^3)$, and $f\colon A \to B$ the quotient map. By the above proposition, we find that we have an isomorphism of $k[x, y]$-algebras (and in particular of k-algebras)

$$\frac{k[x, y]_{(x,y)}}{(x, x - y^3)} \to f(A \setminus (x, y))^{-1}B.$$

Finally, I claim the natural map

$$B \to f(A \setminus (x, y))^{-1}B$$

(sending b to $b/1$) is an isomorphism. To see this, note simply that every element of $f(A \setminus (x, y))$ is a unit in B, and (by a remark above) localising at a set of units does nothing. Finally, B is of dimension 3 as a k-vector space, for example with basis 1, y, y^2.

This example was rather simplified by the fact that the naive intersection consists only of the point $x = y = 0$. In general, one should not expect the localisation to do nothing; rather, it will 'cut out' parts of the ring that you do not care about. Maybe a good exercise to see this would be to compute carefully the intersection numbers of the curves given by y and $y - (x^2 - 1)$ at the two points $x = 1, y = 0$ and $x = -1, y = 0$. You should get intersection number 1 at both points.