Elliptic curves exercise sheet 10

David Holmes

Abstract

Questions 1 and 2 will be graded.

This is due in on 27/4/2015 before the start of the lecture (13:45). Please email your solutions to Giulio at ellipticcurvesleiden@gmail.com, or put them in his mailbox. Please include your student number on your answer sheet.

You may work together on the problems, but please write up your answers separately.

The grade for this work is out of 25. Of this, 20 points are for the content, and 5 points are for clarity and style. This is about mathematical style, not handwriting (though the latter must be legible - if you have terrible handwriting, it may help to use LATEX).

0. Read up to page ??? of the online lecture notes (this is approximately what was covered in the lecture, but contains some extra details).

1. We define the height of a rational number $a \in \mathbb{Q}$ to be the height of the point $(a : 1) \in \mathbb{P}^1(\mathbb{Q})$. Prove the following statements:

   (a) For all $x \in \mathbb{Q}$, we have $H(x) = H(-x)$.

   (b) For all $x \in \mathbb{Q}^*$ we have $H(x) = H(x^{-1})$.

   (c) For all $x_1, \dots x_n \in \mathbb{Q}$, we have $H(x_1 \cdot x_2 \cdots x_n) \leq H(x_1) \cdots H(x_n)$.

2. Let $E/\mathbb{Q}$ be an elliptic curve with $\#E(\mathbb{Q})[2] = 4$, and with discriminant Δ. Let b be the number of prime numbers dividing 2Δ. Show that

$$\operatorname{Rank} E(\mathbb{Q}) \leq 2b.$$