Elliptic curves exercise sheet 7

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Abstract

Giulio is away this week, so all solutions must be emailed to him at ellipticcurvesleiden@gmail.com (either latex or a scan).

Questions 3 and 4 will be graded, the others not. However, you are very strongly recommended to do the other questions.

This is due in on 23/3/2015 before the start of the lecture (13:45). **Giulio is away this week, so all solutions must be emailed to him at ellipticcurvesleiden@gmail.com (either latex or a scan)**. Please include your student number on your answer sheet.

You may work together on the problems, but please write up your answers separately.

The grade for this work is out of 25. Of this, 20 points are for the content, and 5 points are for clarity and style. This is about mathematical style, not handwriting (though the latter must be legible - if you have terrible handwriting, it may help to use LATEX).

0. Read up to page 48 of the online lecture notes (this is approximately what was covered in the lecture, but contains some extra details).

1. Consider the following three conditions on an abelian group G:

   (a) The torsion subgroup G_{tors} is finite;

   (b) There exists an integer $n \geq 2$ such that $G/nG = 0$.

   (c) There exists a function $q\colon G \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ such that $q(2g) = 4q(g)$ and the function $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is bi-additive (this is a slightly weakened version of being a quadratic form), *and* such that $q(x) = x \implies x = 0$ (non-degenerate).

   By giving three examples of groups, show that no two of these conditions is enough to imply that G is finitely generated. Later, we will see that all three conditions together *do* force G to be finitely generated.

2. Consider the following situation:

   - X a compact topological space, $z \in X$ a point;
   - $(Z_i)_i$ a decreasing sequence of subsets of X with $\cap_i Z_i = \{z\}$;
   - $(x_i)_i$ a sequence of points in X converging to a point $x \in X$;

   such that for every $i \in \mathbb{N}$, we have $x_i \in Z_i$.

(a) Show that, if the subsets Z_i are all closed, then it must hold that $x = z$;

(b) give an example to show that this is false without the assumption that the Z_i be closed.

3. For this question, please do not use any of the theorems stated without proof on page 45 of the notes. Let E be the elliptic curve over $\mathbb{Q}$ given by $y^2 = x^3 + 2x$. Compute $E(\mathbb{Q})_{tors}$.

4. For this question, you may use the following result from page 45 of the notes:

   Theorem 0.1. *Let $E/\mathbb{Q}$ be an elliptic curve, and p a prime of good reduction. Then the map*

   $$E(\mathbb{Q})_{tors} \to \bar{E}(\mathbb{F}_p)$$

   is injective.

   (a) Let E be the elliptic curve over $\mathbb{Q}$ given by $y^2 = x^3 + 2x + 3$. Compute $E(\mathbb{Q})_{tors}$.

   (b) Let E be the elliptic curve over $\mathbb{Q}$ given by $y^2 = x^3 + 1$. Compute $E(\mathbb{Q})_{tors}$.

5. (Very optional exercise) Last week, you may have used in your homework that a complete totally bounded metric space is compact. Recall that 'totally bounded' means that for every $\epsilon > 0$ there exists a finite cover of your space by sets of radius at most ϵ. In this exercise, we see why 'bounded' is not enough, we really need 'totally bounded'. Define a metric d on $\mathbb{R}$ by

   $$d(x, y) = \min(1, |x - y|),$$

   where $|-|$ denotes the usual euclidean absolute value. Show the metric space $(\mathbb{R}, d)$ is complete and bounded but not compact.