Elliptic curves exercise sheet 8

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Abstract

Questions 2 and 3 will be graded, the others not. However, you are very strongly recommended to do the other questions.

This is due in on 30/3/2015 before the start of the lecture (13:45). Please email your solutions to Giulio at ellipticcurvesleiden@gmail.com, or put them in his mailbox. Please include your student number on your answer sheet.

You may work together on the problems, but please write up your answers separately.

The grade for this work is out of 25. Of this, 20 points are for the content, and 5 points are for clarity and style. This is about mathematical style, not handwriting (though the latter must be legible - if you have terrible handwriting, it may help to use LATEX).

0. Read up to page ??? of the online lecture notes (this is approximately what was covered in the lecture, but contains some extra details).

1. On pages 52 and 53 of the lecture notes, we gave two definitions of the Krull topology on the absolute Galois group G of a perfect field K.

   (a) Show these definitions really are equivalent ;

   (b) Show that a subgroup $H \leq G$ is open if and only if $\bar{K}^H$ is a finite extension of K;

   (c) Show that G is compact;

   (d) Show that every closed subgroup of G is an intersection of open subgroups [hint: use the Galois correspondence].

2. Let k denote a perfect field, and $\bar{k}$ a fixed algebraic closure of k. Write G for the Galois group of $\bar{k}$ over k, and let M be a discrete G-module.

   (a) Show that
   $$H^1(G, M) = \lim_{H\to} H^1(G/H, M^H)$$
   as H runs over the open normal subgroups of G. Explicitly, this means that:
   (i) $H^1(G, M)$ is the union of the images of the inflation maps $\mathrm{Inf} : H^1(G/H, M^H) \to H^1(G, M)$, where H runs over the open normal subgroup of G;
   (ii) for any open normal subgroup H of G, an element $g \in H^1(G/H, M^H)$ maps to zero in $H^1(G, M)$ if and only if it maps to zero in $H^1(G/H', M^{H'})$ for some open normal subgroup H' of G contained in H.

(b) Show that $H^1(G, M)$ is torsion (that is, every element has finite order).

3. Let G be a finite group, H a normal subgroup of G, and M a G-module.

   a Show the inflation-restriction sequence

   $$0 \to H^1(G/H, M^H) \stackrel{Inf}{\to} H^1(G, M) \stackrel{Res}{\to} H^1(H, M)$$

   is exact.

   b Show that there is a natural action of G/H on $H^1(H, M)$.

   c Show that
   $$\operatorname{Res}(H^1(G, M)) \subset H^1(H, M)^{G/H}.$$

   Hence the inflation-restriction exact sequence can be refined to

   $$0 \to H^1(G/H, M^H) \stackrel{Inf}{\to} H^1(G, M) \stackrel{Res}{\to} H^1(H, M)^{G/H}.$$