1. Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V. (This is enough to show that V is *not* a vector space.)

2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is not in W. This is enough to show that W is *not* a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

7. All polynomials of degree at most 3, with integers as coefficients.

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

9. Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

10. Let H be the set of all vectors of the form $\begin{bmatrix} 3t \\ 0 \\ -7t \end{bmatrix}$, where t is any real number. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)

11. Let W be the set of all vectors of the form $\begin{bmatrix} 2b + 3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

12. Let W be the set of all vectors of the form $\begin{bmatrix} 2s + 4t \\ 2s \\ 2s - 3t \\ 5t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)

13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

Now read the list of axioms for a vector space in the book, and answer the following questions:

Exercises 25–29 show how the axioms for a vector space V can be used to prove the elementary properties described after the definition of a vector space. Fill in the blanks with the appropriate axiom numbers. Because of Axiom 2, Axioms 4 and 5 imply, respectively, that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $-\mathbf{u} + \mathbf{u} = \mathbf{0}$ for all $\mathbf{u}$.

25. Complete the following proof that the zero vector is unique. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u} + \mathbf{w} = \mathbf{w} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V. In particular, $\mathbf{0} + \mathbf{w} = \mathbf{0}$. But $\mathbf{0} + \mathbf{w} = \mathbf{w}$, by Axiom ______. Hence $\mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{0}$.

26. Complete the following proof that $-\mathbf{u}$ is the *unique vector* in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. Suppose that $\mathbf{w}$ satisfies $\mathbf{u} + \mathbf{w} = \mathbf{0}$. Adding $-\mathbf{u}$ to both sides, we have

$(-\mathbf{u}) + [\mathbf{u} + \mathbf{w}] = (-\mathbf{u}) + \mathbf{0}$

$[(-\mathbf{u}) + \mathbf{u}] + \mathbf{w} = (-\mathbf{u}) + \mathbf{0}$ by Axiom ______ (a)

$\mathbf{0} + \mathbf{w} = (-\mathbf{u}) + \mathbf{0}$ by Axiom ______ (b)

$\mathbf{w} = -\mathbf{u}$ by Axiom ______ (c)

27. Fill in the missing axiom numbers in the following proof that $0\mathbf{u} = \mathbf{0}$ for every $\mathbf{u}$ in V.

$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$ by Axiom ______ (a)

Add the negative of $0\mathbf{u}$ to both sides:

$0\mathbf{u} + (-0\mathbf{u}) = [0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u})$

$0\mathbf{u} + (-0\mathbf{u}) = 0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})]$ by Axiom ______ (b)

$\mathbf{0} = 0\mathbf{u} + \mathbf{0}$ by Axiom ______ (c)

$\mathbf{0} = 0\mathbf{u}$ by Axiom ______ (d)

1. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

In Exercises 3–6, find an explicit description of Nul A, by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

In Exercises 7–14, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

7. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$ **8.** $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t \right\}$

In Exercises 25 and 26, A denotes an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

25. a. The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

b. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

c. The column space of A is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

d. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then Col A is $\mathbb{R}^m$.

e. The kernel of a linear transformation is a vector space.

f. Col A is the set of all vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$.

28. Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

It can be shown that the first system has a solution. Use this fact and the theory from this section to explain why the second system must also have a solution. (Make no row operations.)

35. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\mathbf{x})$, where $\mathbf{x}$ is in U. Show that $T(U)$ is a subspace of W.