Definition 6.6.1 (Local ring at a point) Let X be a variety, and let $P \in X$ a point. Let

$$\mathcal{O}_{X,P} := \{(U, f) : U \subset X \text{ is open, } P \in U, \text{ and } f \in \mathcal{O}_X(U)\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if there is an open and dense $W \subset U \cap V$ with $P \in W$ such that $f = g$ on W.

Notice the similarity of this definition with the definition of $K(X)$; in fact if X is irreducible we have a natural injective map $\mathcal{O}_{X,P} \to K(X)$. The difference is that we only take regular functions defined in a neighborhood of our fixed point P. In order to show that $\mathcal{O}_{X,P}$ is a local ring, consider the (well-defined) subset

$$\mathfrak{m}_{X,P} = \{(U, f) : U \subset X \text{ is open, } P \in U,\ f \in \mathcal{O}_X(U), \text{ and } f(P) = 0\}/\sim$$

of $\mathcal{O}_{X,P}$. Then $\mathfrak{m}_{X,P}$ is a maximal ideal, as it is the kernel of the evaluation map $\mathcal{O}_{X,P} \to k$ that sends $[(U, f)] \mapsto f(P)$. Moreover, if $[(U, f)] \notin \mathfrak{m}_{X,P}$, then $f(P) \neq 0$, and $[(U, f)] = [(U \setminus Z(f), f)]$ is invertible in $\mathcal{O}_{X,P}$.

If X is irreducible then $K(X)$ is naturally the fraction field of $\mathcal{O}_{X,P}$. If X is affine and irreducible, let $\mathfrak{m}_P \subset \mathcal{O}_X(X)$ be the maximal ideal at P, and let

$$\mathcal{O}_X(X)_{\mathfrak{m}_P} := \{g/h : g, h \in \mathcal{O}_X(X), h \notin \mathfrak{m}_P\} \subset Q(\mathcal{O}_X(X))$$

be the *localization* of $\mathcal{O}_X(X)$ at m_P. Then under the identification of $K(X)$ with the fraction field of $\mathcal{O}_X(X)$ (cf. Proposition 6.5.3(ii)) we have that $\mathcal{O}_{X,P} \subset K(X)$ is identified with $\mathcal{O}_X(X)_{\mathfrak{m}_P}$. Thus, on arbitrary irreducible varieties X, local rings can be computed by first choosing a suitable affine open neighborhood, and then localizing. The reader is encouraged to verify that the rings $\mathcal{O}_X(X)_{\mathfrak{m}_P}$ are Noetherian. It follows that the local rings of varieties are Noetherian.

6.7 Exercises

Exercise 6.7.1 Let $\Psi\colon \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{mn-1}$ be the Segre map (of sets):

$$((a_1 : \cdots : a_m), (b_1 : \cdots : b_n)) \mapsto (a_1 b_1 : \cdots : a_m b_n).$$

Let $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ be closed.

i. Show that Ψ is a morphism of varieties.

ii. Show that $\Psi(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$ is closed in $\mathbb{P}^{mn-1}$.

iii. Show that Ψ is an isomorphism from the product variety $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ to the projective variety $\Psi(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$.

iv. Show that Ψ restricts to an isomorphism from the product variety $X \times Y$ to the projective variety $\Psi(X \times Y)$.

v. Show that the diagonal $\Delta_{\mathbb{P}^{n-1}}$ is closed in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

vi. Show that projective varieties are separated.

Exercise 6.7.2 Let $X = Z(xy) \subset \mathbb{A}^2$. Show that $K(X)$ is not a field.

Exercise 6.7.3 Let X be the variety obtained from the following gluing data: $X_1 = X_2 = \mathbb{A}^1$ and $X_{12} = X_{21} = \mathbb{A}^1 - \{0\}$ with $\varphi_{12} = \text{id}$. Give the presentation of X corresponding to this glueing data. Describe the topology on X and the sheaf of regular functions on X. What is the diagonal $\Delta_X \subset X \times X$? What is the closure of the diagonal? Conclude that X is not separated.