The purpose of this note is to provide some background material for Exercise 11.3.4 and Exercise 12.4.4. We will use the following fact:

1. $X := \mathbb{P}^m \times \mathbb{P}^n$ is an irreducible closed subvariety of $\mathbb{P}^{(m+1)(n+1)-1}$ of dimension $m+n$ through Segre embedding. Here m, n are nonnegative integers.

2. Let $x_0 : \ldots : x_m$ be the coordinates of $\mathbb{P}^m$, and $y_0 : \ldots : y_n$ the coordinates of $\mathbb{P}^n$. Then X admits an open covering given by all $D_+(x_i) \times D_+(y_j)$, with $0 \le i \le m, 0 \le j \le n$.

We give the following statements to help the readers to have a better understanding of X. Key hints will be given below.

1. Each closed subset of X is given by a set of bihomogeneous polynomials in $S := k[x_0, \ldots, x_m, y_0, \ldots, y_n]$, i.e., each nonempty closed subset of X is of the form

   $$Z(f_i, i \in I) = \{((a_0 : \ldots, a_m), (b_0 : \ldots : b_n)) \in \mathbb{P}^m \times \mathbb{P}^n | f_i(a_0, \ldots, a_m, b_0, \ldots, b_n) = 0, \forall i \in I\}$$

   where $f_i (i \in I)$ is a set of bihomogeneous polynomials in S such that the ideal generated all f_i is strictly contained in $(x_0, \ldots, x_m, y_0, \ldots, y_n) \subset S$.

   Hint:

   (a) use Segre embedding

   $$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}, \ \big((x_0 : \ldots : x_m), (y_0, \ldots, y_n)\big) \mapsto (w_{ij} = x_i y_j).$$

   (b) If $f \in S$ is bihomogeneous polynomial of degree (d, d), then there exists a homogenous polynomial

   $$F \in k[w_{ij}, 0 \le i \le m, 0 \le j \le n],$$

   such that $F|_{w_{ij} = x_i y_j} = f(x_0, \ldots, x_m, y_0, \ldots, y_n)$.

   (c) If $f \in S$ is a bihomogeneous polynomial of bidgree (d, e) such that $d > e$. Then $f(a_0, \ldots, a_m, b_0, \ldots, b_n) = 0$ if and only if

   $$(y_j^{d-e} f)(a_0, \ldots, a_m, b_0, \ldots, b_n) = 0$$

   for all j.

2. For each nonzero irreducible bihomogenous polynomial f in S, the closed subset $Z(f)$ is a prime divisor. Hint:

   (a) $f(x_0, \ldots, x_m, y_0, \ldots, y_n)$ implies that

   $$f_{ij} := f(x_0, \ldots, x_{i-1}, 1, x_i, \ldots, x_m, y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n).$$

   is irreducible in $S_{ij} := k[x_0, \ldots, x_{i-1}, 1, x_i, \ldots, x_m, y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n]$

If $Z(f)_{ij} := Z(f) \cap (D_+(x_i) \times D_+(y_j)) \neq \emptyset$, then

$$Z(f)_{ij} \subset D_+(x_i) \times D_+(y_j) \cong \mathbb{A}^{m+n}$$

is equal to $Z(f_{ij}) \subset D_+(x_i) \times D_+(y_j)$, which is prime divisor (namely, an irreducible closed subset of dimension $m+n-1$) in $D_+(x_i) \times D_+(y_j)$. Where we use S_{ij} to denote the coordinate ring of $D_+(x_i) \times D_+(y_j)$. Since $D_+(x_i) \times D_+(y_j)$ is dense in X, the closure of $Z(f)_{ij}$ in X is $Z(f)$. Note that a subset W of a topological space Y is irreducible, then the closure of W in Y is irreducible. In particular, we have that $Z(f)$ is irreducible. On the other hand, by Exercise 1.6.11, we have $\dim Z(f)_{ij} = m+n$.

3. Each prime divisor Y of X is of the form $Z(f)$, where f is an irreducible bihomogenous polynomial.

   Hint: let $Y = Z(f_i, i \in I)$ with each f_i nonzero bihomogenous. Take any $i \in I$, then $Y \subset Z(f_i)$. Let g be an irreducible factor of f_i (this makes sense since S is a unique factorial domain), which is necessarily bihomogenous (this is easy to check). Then we have $Y \subset Z(g)$. But since $Y, Z(g)$ are both prime divisors, we must have $Y = Z(g)$ by dimension consideration.

4. Any rational function of X has the form $\frac{F}{G}$ such that F, G are homogeneous polynomials such that

   (a) F, G are both bihomogeneous polynomials in S;

   (b) If F, G have bidegrees $(d_1, e_1), (d_2, e_2)$ respectively, then $d_1 + e_1 = d_2 + e_2$.

   Conversely, if $F, G \in S$ both satisfy requirement (b) above, then $\frac{F}{G}$ is a rational function of X.

   Hint: This is easy.

5. A divisor Z of X is a principle divisor if and only if it has bidegree $(0, 0)$. Here given a divisor $Y = \Sigma_i n_i Z(f_i)$, with f_i bihomogeneous of bidgree (d_i, e_i), the bidgree of Y is defined to

$$(\Sigma n_i d_i, \Sigma n_i e_i).$$

   Hint: this is also easy.