Algebraic Geometry–Retake Exam 21 June 2016

- Time allowed: 3 hours.
- You should answer all 6 questions.
- You may quote results from the lectures without proof in the exam. If you wish to use results from the exercises then you are expected to re-prove them in the exam.
- Pen and paper only allowed - no books, notes, calculators etc.
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. (a) If one identifies $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

   (b) Let X be the variety $\mathbb{A}^1 \times \mathbb{P}^1$. Prove that X is not affine and not projective.

2. Let X be an affine variety. For f in $\mathcal{O}(X)$ define $D(f) = \{p \in X \,:\, f(p) \neq 0\}$ and $Z(f) = X \setminus D(f) = \{p \in X \,:\, f(p) = 0\}$. Prove:

   (a) $X = \cup_{i\in I} D(f_i) \Leftrightarrow \cap_{i\in I} Z(f_i) = \varnothing \Leftrightarrow$ the f_i generate $\mathcal{O}(X)$ as an ideal. You may use without proof that every proper ideal in a commutative ring is contained in at least one maximal ideal.

   (b) The $D(f)$ with f running through $\mathcal{O}(X)$ form a basis for the Zariski topology on X.

   (c) X is compact, i.e., every covering of X with open subsets has a finite subcovering.

3. Let X be a smooth projective irreducible curve and D a divisor on X.

   (a) State the definition of $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$.

   (b) Define the genus of X.

   (c) Compute the genus of $\mathbb{P}^1$.

4. (a) Let $U \subset \mathbb{A}^1$ be non-empty open and let $f\colon U \to \mathbb{A}^1$ be a morphism. Show that f extends uniquely as a morphism $\bar{f}\colon \mathbb{P}^1 \to \mathbb{P}^1$.

   (b) Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbb{A}^1$. Assume that $\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\}$ and $\mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}$ are isomorphic. Show that $r = s$.

   (c) Describe explicitly $\mathrm{Aut}(\mathbb{P}^1)$ (you don't need to prove your answer).

   (d) Does the converse to (b) hold? More precisely, is it true that for every positive integer r and every set $P_1, \ldots, P_r, Q_1, \ldots, Q_r$ of distinct points in $\mathbb{A}^1$ we have

$$\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\} \cong \mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}?$$

Questions 5 and 6 are on the next page.

5. Let X be a smooth irreducible projective curve.

   (a) State the Riemann-Roch theorem for X.

   (b) State Serre Duality for X.

   From now on assume X has genus 1.

   (c) Show that the degree of a canonical divisor K on X is zero.

   (d) Let $\varphi\colon \operatorname{Div} X \to \operatorname{Pic} X$ be the map sending a divisor to its class modulo principal divisors. Show that the class $\varphi(K)$ of K in $\operatorname{Pic} X$ is zero.

   (e) Let $O \in X$. Show that the map $\psi\colon X \to \operatorname{Pic}^0 X$ given by $P \mapsto \varphi(P - O)$ is bijective (here we write $\operatorname{Pic}^0 X$ for the image of the degree-zero divisors in $\operatorname{Pic} X$).

6. Assume that k is an algebraic closure of a finite field $\mathbb{F}_q$ with q elements. Let X be a smooth irreducible projective curve defined over $\mathbb{F}_q$ and of genus g. For a divisor D on X we write $h^0(D) = \dim H^0(X, \mathcal{O}_X(D))$. Let

$$G(t) = \sum_{\substack{[D]\in \operatorname{Pic} X_0 \\ 0\le \deg D \le 2g-2}} q^{h^0(D)} t^{\deg D}\,.$$

   (a) Assume that $X(\mathbb{F}_q) \neq \varnothing$. Show that $G(t)$ is a polynomial.

   (b) Show that $G(t)$ satisfies the functional equation

$$G(t) = q^{g-1} t^{2g-2} G(1/qt)\,.$$