Algebraic Geometry–Midterm assignment

- Rules of the game: solutions are due in on 13 December, to be handed in on paper at the beginning of the lecture. Keep a copy of the solutions you hand in. Include your name, student number, and university on the top of each sheet. You can write your solutions in TEX and hand in a printed version. If you hand in something handwritten then please try to write very neatly - any parts we cannot easily read will be ignored.
- Your solutions should convince us that you have a good understanding of what has been discussed so far. You may use results from Lectures 1–8, but please give precise references. If you want to use the results of exercises from those lectures, you should include solutions to those exercises. What you hand in should be your own, individual work (you can work together to solve the problems, but please write up your solutions on your own).
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. Let $n \in \mathbb{Z}_{\geq 1}$ and assume that $\operatorname{char} k \nmid 3(3n-1)$. Let

$$X_1 \colon y^3 = x^{3n-1} - 1 \,, \quad X_2 \colon v^3 = u - u^{3n}$$

be affine varieties in $\mathbb{A}^2$ with coordinates (x, y) respectively (u, v). Write $X_{12} = D(x) \subset X_1$ and $X_{21} = D(u) \subset X_2$.

(i) Show that the map $\varphi \colon X_{12} \to X_{21}$ given by $(x, y) \mapsto (x^{-1}, yx^{-n})$ is an isomorphism of varieties.

Let X be the variety obtained from glueing X_1, X_2 along the isomorphism $\varphi \colon X_{12} \xrightarrow{\sim} X_{21}$.

(ii) Show that the variety X is

(a) irreducible;

(b) smooth of dimension one;

(c) separated.

The variety X is projective (you don't need to prove this).

(iii) Compute the genus of X.

2. Let X be a variety together with morphisms $m \colon X \times X \to X$, $i \colon X \to X$ and a distinguished element $e \in X$. If (X, m, i, e) is a group with multiplication m, inverse i and neutral element e we call (X, m, i, e) a group variety.

If X is any variety, recall that the *Yoneda functor* h_X of X is the functor $\mathrm{VaVar}_k^{op} \to \mathrm{Set}$ sending a variety T to the set $\mathrm{Hom}(T, X)$, and which acts on morphisms by suitable composition (see for example the notes from the last lecture of the intensive course on categories and modules).

(i) Let (X, m, i, e) be a group variety. Writing h_X for the Yoneda functor of X, construct a factorisation of h_X via the forgetful functor from the category of groups to the category of sets.

One can use the Yoneda Lemma to show that given a variety X together with a factorisation of its Yoneda functor over the category of groups (with its forgetful functor to Set) there is a natural structure of group variety (X, m, i, e) on X.

(ii) Let X be a variety together with a factorisation of its Yoneda functor over the category of groups (with its forgetful functor to Set). Explain how to construct the morphisms m and i, and the distinguished point $e \in X$, from the given factorisation. You do not need to write out the proofs of the group axioms for m, i, e, though you should check them for yourself. You do need to show that your m and i are morphisms, not just maps of sets.

(iii) True or false? Every group variety is

(a) smooth;

(b) separated;

(c) compact;

(d) irreducible.

Motivate each answer with a proof or counterexample.