Lecture 1.

§ Basic Q

Fix some integers:

$g \geq 0, n \geq 1, q \geq 0, \quad m_1, \ldots, m_n, \qquad \sum_i m_i = q(2g-2)$

Let C be a smooth (proper, connected) alg curve / compact connected R-surface of genus g.

Let $p_1, \ldots, p_n \in C$ distinct pts.

Q: When is $\sum_{i=1}^{n} m_i p_i \underset{lin}{\sim} q \cdot K_C$?

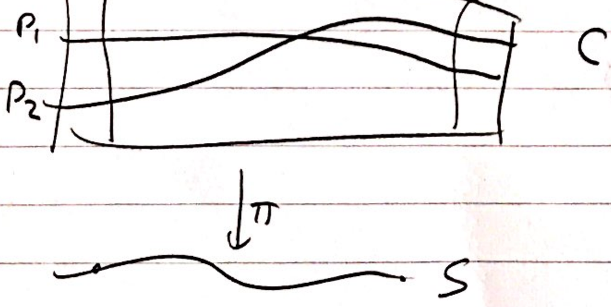
K_C canonical divisor class degree $2g-2$, divisor of any mero. differential.

i.e. $\exists$ a mero. fctn f on C s.t. $\text{zeros}(f) - \text{poles}(f) = \sum m_i p_i - q K_C$.

This is v. vague! We will explore various ways to make this more precise, & see some answers.

§ Families (vagueish) Say we have a family of R-surfaces C/S. Precise meaning to be discussed, but for every $s \in S$ the fibre C_s should be a R-surface, & should 'fit together nicely'.

Say $p_1, \ldots, p_n$ sections of family:



Given $s \in S$, fibre $C_s = \pi^{-1}(s)$ is a R-surface, & has pts $p_1(s), \ldots, p_n(s)$.

Def (to be refined):

$$\underset{\substack{\cap\mid \\ S}}{DRL} = \{ s \in S \mid \sum_i m_i p_i(s) \underset{lin}{\sim} q . K_{C_s} \text{ on } C_s \}.$$

Qs: What does this subset look like?
Is it algebraic / analytic?
~~(co) dim~~ Open / closed?
(Co)-dimension?
⋮

§ Maps to $\mathbb{P}^1$:

What is $\underbrace{DR}_{\text{double ramification}}\underbrace{L}_{\text{locus}}$

Say $q = 0$. Then $\sum_{i \geq 1} m_i p_i \sim q K_C = 0$ is $\equiv$ to

$\exists$ map

$$f : C \longrightarrow \mathbb{P}^1$$

s.t. $f^{-1}(0) = \sum_{i : m_i > 0} m_i p_i$ 'ramification profile over 0'

$f^{-1}(\infty) = \sum_{i : m_i < 0} m_i p_i$ 'ramification profile over ∞'.

Relation to Hurwitz numbers etc...

§ Jacobians

To get a better handle on DRL ⊆ S, want a more geometric characterisation of " $\sum m_i P_i \sim q k_i$ ".

Let C a (cpct conn) R-surface. Have Jacobian

$J = J_C$:
- compact $\mathbb{C}$-mfd, dim g.
- points in J_C = iso. classes of deg 0 holo. line bdles on C
  = $\underset{\text{lin}}{\sim}$ classes of deg-0 divisors on C.

- gp structure from ⊗ of line bdles / Σ of divisors

$$\underset{[O]}{e \in J}, \quad m: J \times J \to J, \quad i: J \to J$$
$$\mathcal{L}_1, \mathcal{L}_2 \mapsto \mathcal{L}_1 \otimes \mathcal{L}_2 \qquad \mathcal{L} \mapsto \mathcal{L}^\vee.$$

- Formal def & generalisation later...

In J_C, have two distinguished pts:

$e = [O]$ $\qquad \sigma = [q k_c - \sum_i m_i P_i]$.

Then $\sum_i m_i P_i \sim q k_c \iff e = \sigma$ in J_C.

To ~~use~~ this to understand $DRL \subseteq S$, need jacobians in families. Again, to be made precise in next hour, but idea is that, given a family C/S of (cpct conn) R-surfaces, the jacobians fit together into a family J/S.

Given also $p_1, \dots, p_n \in C(S)$, can define sections

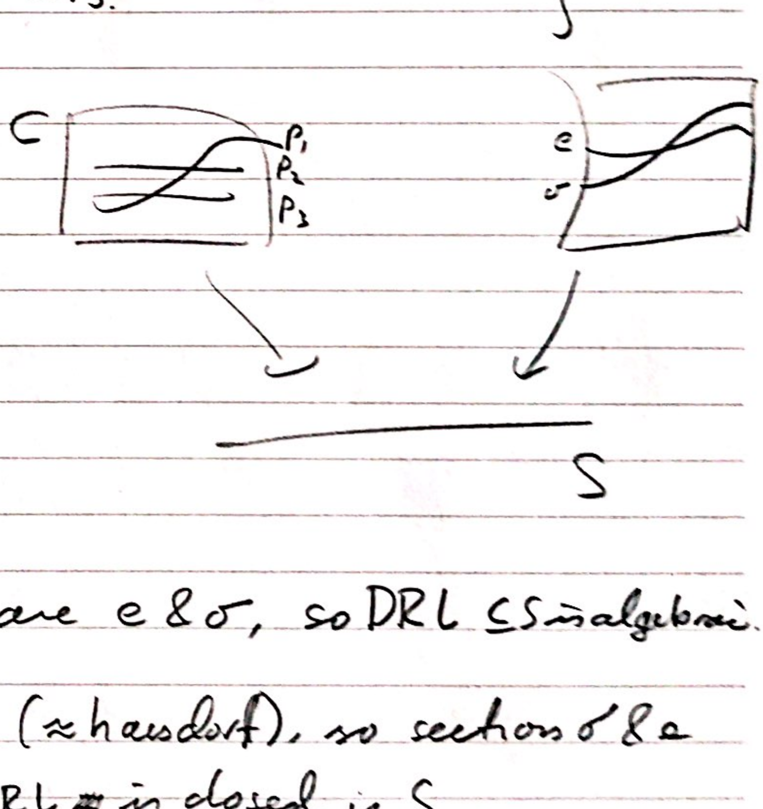
$$e = [0] \in J(S)$$

$$\sigma = [q k_C - \sum m_i p_i] \in J(S)$$

Then

$$DRL = \{ s \in S \mid e(s) = \sigma(s) \}.$$

$$= e^{-1}\sigma = \sigma^{-1} e$$



Consequences:

- J/S is algebraic, as are e & σ, so $DRL \subseteq S$ is algebraic.
- J/S is separated ($\approx$ hausdorff), so sections σ & e are closed, so DRL is closed in S.
- J/S has rel. dim g so 'often' DRL has codim g

  (eg. fails if $q = 0 = m_1, .., m_n$!).

§ Jacobians – more formal!

~~To get a better handle on DRL⊆S, want a more ~~intrinsic~~ geometric characterization of $\sum_i m_i p_i \sim q K_C$.~~

First for a single R-surface / alg. curve C (sm, proper, conn, w-a pt)

1st approximation of def: The jacobian $J = J_C$ is the moduli / parameter space for ~~the bdl~~ (hol) line bdles on C of degree 0.

(equivalently, lin ≡ classes of degree-zero divisors).

Q: What does this mean? As a set, J is the set of iso. classes of deg. 0 line bdles on C.
But we want more than a set! Want a top. space / $\mathbb{C}$-mfd / scheme.

Recall
lemma (Yoneda): $\mathcal{C}$ a cat, then

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \underline{\mathrm{Set}}) \quad \text{is fully faithful.}$$
$$c \longmapsto \mathrm{Hom}(-, c)$$

Thus, to tell you what J is, I just have to give a functor

$$\underline{\mathrm{Sch}}^{op} \longrightarrow \underline{\mathrm{Set}},$$

& hope you believe me when I say this J actually exists ('the functor is representable').

For families, $\underline{\mathrm{Sch}_S}^{op} \to \underline{\mathrm{Set}}$.

2^{nd} approximation of def

$$J_2 : \underline{Sch}^{op} \longrightarrow \underline{Set}$$

$$T \longmapsto \left\{ \begin{array}{l} \text{iso. classes of line bdles} \\ \text{on } C \times T \text{, fibrewise deg } 0 \end{array} \right\} / \sim$$

This is better; it's actually enough data to uniquely determine a J.
~~But~~ it still doesn't work; there's no representing object, i.e. there doesn't actually exist a schem / $\mathbb{C}$-mfd ~~whose Yoneda~~

$$J \text{ s.t. } h(J) = J_2 .$$

(See ex).
Problem is that isos don't need to fit together nicely, so an equivariant map from a cover of T to J_2 need not descend to a map $T \to J_2$.

3^{rd} (Correct) def: Fix $p \in C(S)$

$$J : \underline{Sch}_S^{op} \longrightarrow \underline{Set} \text{ or } \underline{Ab}.$$

$$T \longmapsto \left\{ (\mathcal{L}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bdle on } C \times_S T \\ \varphi : p^*\mathcal{L} \simeq \mathcal{O}_T \\ \text{deg } 0 \text{ on each fibre} \end{array} \right\} / \sim$$

where $(\mathcal{L}, \varphi) \sim (\mathcal{L}', \varphi')$ iff $\exists\ \psi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$

s.t.
$$\begin{tikzcd} p^*\mathcal{L} \arrow[r, "p^*\psi"] \arrow[dr, "\varphi"'] & p^*\mathcal{L}' \arrow[d, "\varphi'"] \\ & \mathcal{O}_T \end{tikzcd}$$
commutes.

Then ~~automor~~ objects have no non-trivial automorphisms (uses connectedness of fibres), so descent is easy.

For representability, see [BLR] or [FGA explained].

~~1.8~~

1.7

~~[illegible]~~

Thm ~~[Raynaud]: Let C/S proper, [illegible] fin pres, & connecte~~

[Raynaud]: Let C/S sm. proper family of curves w. conn. geom. fibres, $P \in C(S)$.
Then
$\mathcal{J}$ is representable by a smooth, proper separated gp scheme over S.

"nice family of R-surfaces ⇒ jacobians fit together nicely."

The formulae
$e = [(\mathcal{O}, id)]$ gives a section $S \to \mathcal{J}$.

For σ, take care bcs $p_i^* \Omega^{\otimes g}(-\Sigma m_i p_i)$ need not be trivial on S, but it is so locally on S, so can build sections locally on S & glue to a global one (ex: details...)

Since $\mathcal{J}/S$ is sep. it follows that e, σ are closed immersions, so $\sigma^* e = e^* \sigma$ is closed in S.

Can define elt DRC = ~~[illegible]~~ $\sigma^! [e]$ in $CH^g S$, (double ram. cycle), always codim g.

or: $\sigma := \left[\Omega^{\otimes g}(-\Sigma_i m_i p_i) \otimes \left(\pi^* p^* \Omega^{a}(-\Sigma_i m_i p_i) \right)^{\vee} \right]$,

canon, triv. along p,
fibrewise what we want.