Final Exam – Linear Algebra and Image Processing

22 May 2013

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question ?

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) Two row-equivalent linear systems can have different solution sets.

b) A linear map $\mathbb{R}^3 \to \mathbb{R}^3$ is onto (surjective) if and only if it is one-to-one (injective).

c) Every linear map $\mathbb{R}^3 \to \mathbb{R}^2$ is onto (surjective).

d) For any two $n \times n$ matrices A and B, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).

e) Given an $n \times n$ matrix A, let B denote the matrix obtained from A by interchanging two rows. Then $\det A = \det B$.

Question ?

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) If a finite-dimensional vector space V contains 5 linearly independent vectors, then $\dim V \geq 5$.

b) Every linear map $\mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (injective).

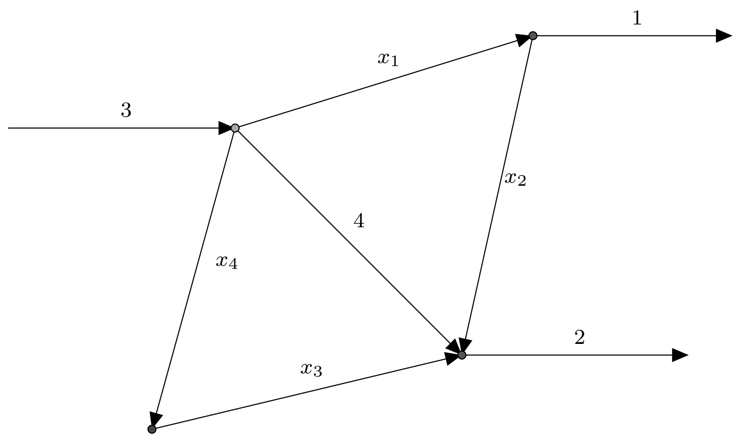
c) Let $V = \{\mathbf{0}\}$ be the zero vector space. Then any function $V \to \mathbb{R}^2$ is linear.

d) For any two $n \times n$ matrices A and B, we have that $\det(AB) = \det(BA)$.

e) If a linear system has at least one free variable, then it has infinitely many solutions.

Question ?

Consider the following network:



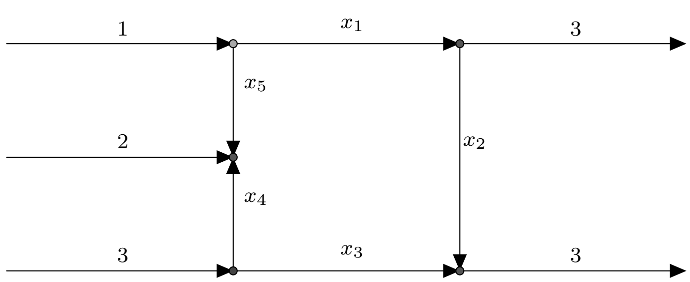
a) Write down a linear system describing the flow in the network.

b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

c) Does there exist a solution with all flows non-negative (≥ 0)?

Question ?

Consider the following network:



a) Write down a linear system describing the flow in the network.

b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

c) Write the general solution of the linear system in parametric vector form.

Question ?

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}, \qquad , B = \begin{bmatrix} k & 1 \\ 0 & 2 \end{bmatrix}.$$

a) For which values of k is $A + B = B + A$?

b) For which values of k is $AB = BA$?

c) Find a non-zero matrix C such that $AC = 0$.

Question (econ final)

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 3 & 2 & 1 \end{bmatrix}, \qquad , B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 6 & 5 & 4 & 0 & 0 \\ 7 & 5 & 3 & 1 & 1 \\ 5 & -2 & -3 & -4 & 5 \end{bmatrix}.$$

a) Compute the determinant $\det A$.

b) Compute the determinant $\det B$.

c) Compute the determinant $\det(A^3)$.

Question (Econ final)

Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 - x_3 \\ -3x_1 + x_3 \\ 2x_1 + x_2x_3 \end{bmatrix}.$$

Write $V = \mathrm{Im}T$ for the image of T, and $W = \ker T$ for the kernel of T.

a) Write down the standard matrix of T.

b) Compute the dimension of V.

c) Compute the dimension of W.

[Hint: Note that $\dim V + \dim W = \dim \mathbb{R}^3$].

Question ??

Write the general solution to the linear system $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & 1 \\ 2 & -2 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}.$$

Question ??

Exactly one of the following sets of vectors in $\mathbb{R}^4$ is linearly independent:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 17 \\ 2 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

a) Which of the sets S_1, S_2 is linearly independent?

b) For the set which is linearly independent, expand it to a basis of $\mathbb{R}^4$.

Question ??

Exactly one of the following sets of vectors in $\mathbb{R}^3$ spans the whole of $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 17 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

a) Which of the sets S_1, S_2 spans the whole of $\mathbb{R}^3$?

b) For the set which spans the whole of $\mathbb{R}^3$, find a subset of it which is a basis of $\mathbb{R}^3$.

Question ??

Recall that $\mathbb{P}_3$ denotes the set of polynomials in one variable t of degree at most 3, which form a vector space with the usual addition and scalar multiplication of polynomials. A basis of $\mathbb{P}_3$ is given by

$$B = \{1 + t, 1 - t, 2t^2, 3t^3\}.$$

a) What is the dimension of $\mathbb{P}_3$?

b) Consider the vector $\mathbf{v} = 2 - 3t^2 + t^3 \in \mathbb{P}_3$. Write $\mathbf{v}$ as a linear combination of the vectors in B.

Consider the set $S = \{1 + t, 1 - t\}$ of vectors in $\mathbb{P}^3$.

c) What is the dimension of the span $\text{Span}(S)$ of the vectors in S?