Final Exam – Linear Algebra and Image Processing

22 May 2013

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Find the parametric vector form of the general solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & -2 & 6 \\ 2 & 10 & -2 & 1 & -3 & 16 \\ 0 & 0 & 2 & -2 & -4 & -8 \\ -1 & -5 & 2 & -2 & -2 & -17 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 3 \end{bmatrix}.$$

Answer: Row reduce the augmented matrix to get

$$\begin{bmatrix} 1 & 5 & 0 & 0 & -2 & 6 & 3 \\ 2 & 10 & -2 & 1 & -3 & 16 & 1 \\ 0 & 0 & 2 & -2 & -4 & -8 & 6 \\ -1 & -5 & 2 & -2 & -2 & -17 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This translates to

$$\begin{aligned} x_1 + 5x_2 - 2x_5 &= 3 \\ x_3 + x_5 &= 2 \\ x_4 + 3x_5 &= -1 \\ x_6 &= 0. \end{aligned}$$

This gives the parametric vector form

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Question 2

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

(a) Every homogeneous linear system is consistent.
Answer: True: the zero vector is always a solution.

(b) Every linear map $\mathbb{R}^3 \to \mathbb{R}^2$ is onto (surjective).
Answer: False, for example the zero map.

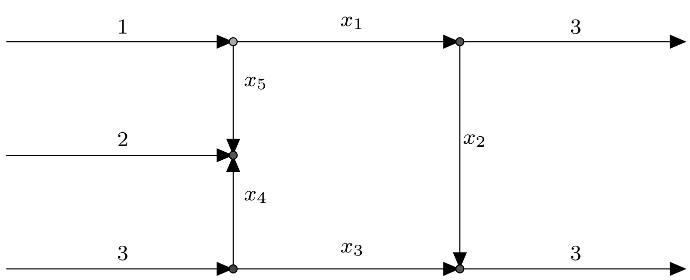
(c) Given an $n \times n$ matrix A, let B denote the matrix obtained from A by interchanging two rows. Then $\det A = \det B$.
Answer: False, $\det A = -\det B$, so eg. take A to be the 2×2 identity matrix.

(d) For any two $n \times n$ matrices A and B, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).
Answer: False, almost any 2×2 counterexample will do.

(e) A linear map $\mathbb{R}^3 \to \mathbb{R}^3$ is onto (surjective) if and only if it is one-to-one (injective).
Answer: True: Injective iff kernel is zero, if and only if image has dimension 3 (since dimension of image + dimension of kernel is 3), iff map is surjective (since target has dimension 3).

Question 3

Consider the following network:



(a) Write down a linear system describing the flow in the network.
Answer:

$$\begin{bmatrix} x_1 + x_5 = 1 \\ x_1 - x_2 = 3 \\ x_2 + x_3 = 3 \\ x_3 + x_4 = 3 \\ x_4 + x_5 = -2 \end{bmatrix}$$

(b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.
Answer:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Write the general solution of the linear system in parametric vector form.
Answer: Not unique. For example:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Question 4

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}, \qquad , B = \begin{bmatrix} k & 1 \\ 0 & 2 \end{bmatrix}.$$

(a) For which real numbers k is $A + B = B + A$?
Answer: For all k.

(b) For which real numbers k is $AB = BA$?
Answer: For no values of k: top left entry of AB is k, and top left entry of BA is $k - 3$.

(c) Find a non-zero 2×2 matrix C such that AC is the 2×2 zero matrix.
Answer: For example, $C = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$.

Question 5

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 3 & 2 & 1 \end{bmatrix}, \qquad , B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 6 & 5 & 4 & 0 & 0 \\ 7 & 5 & 3 & 1 & 1 \\ 5 & -2 & -3 & -4 & 5 \end{bmatrix}.$$

(a) Compute the determinant $\det A$.
Answer: $\det A = 0$.

(b) Compute the determinant $\det B$.
Answer:$\det B = -36$.

(c) Compute the determinant $\det(A^3)$.
Answer: $\det A^3 = (\det A)^3 = 0$.

Question 6

Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 - x_3 \\ -3x_1 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

Write $V = \mathrm{Im}T$ for the image of T, and $W = \ker T$ for the kernel of T.

(a) Write down the standard matrix of T.
Answer: $\begin{bmatrix} 0 & 2 & -1 \\ -3 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Note this matrix is invertible (compute the determinant).

(b) Compute the dimension of the image V.
Answer: The map is surjective , so $\dim V = 3$.

(c) Compute the dimension of the kernel W.
Answer: The map is injective, so $\dim W = 0$.

[Hint: Note that $\dim V + \dim W = \dim \mathbb{R}^3$].

Question 7

Exactly one of the following sets of vectors in $\mathbb{R}^4$ is linearly independent:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 17 \\ 2 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(a) Which of the sets S_1, S_2 is linearly independent?
Answer: S_2 is linearly independent - row reducing, we find that every column in the corresponding matrix is a pivot column.

(b) For the set which is linearly independent, expand it to a basis of $\mathbb{R}^4$.

Answer: For example, add in the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and row reduce the resulting matrix to the identity.

Question 8

Let A be the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

(a) Show that $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

Answer:

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-4+3 \\ -4+6-3 \\ 3-3+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

So, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for A with eigenvalue 1.

(b) Calculate all eigenvalues of A.

Answer: We solve $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-6+6\lambda-\lambda+\lambda^2+9) - 4(-4+4\lambda+9) + 3(-12+18+3\lambda) \\ &= -\lambda^3 - 3\lambda^2 + 4 = 0. \end{aligned}$$

We know that $\lambda = 1$ is an eigenvalue from (a), so we divide the polynomial by $\lambda - 1$ to get

$$\det(A - \lambda I) = (\lambda - 1)(\lambda^2 + 4\lambda + 4) = (\lambda - 1)(\lambda + 2) = 0.$$

So, the eigenvalues for A are 1 and -2.

(c) Give a basis for each eigenspace of A.

Answer: A basis for the eigenspace with eigenvalue 1 is

$$\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}.$$

For eigenvalue -2 we find the solution to $A + 2I = 0$.

$$A + 2I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & \frac{3}{4} \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis for the eigenspace with eigenvalue -2 is

$$\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \}.$$

(d) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

Answer: The matrix A is not diagonalisable, since we can find only two linearly independent eigenvectors. The eigenspace for eigenvalue -2 is one dimensional, while -2 has multiplicity 2.

Question 9

Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -3 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 36 \\ 2 \end{bmatrix}.$$

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.
Answer: We reduce the augmented matrix a little:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \\ -2 & 1 & 36 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \\ 0 & -2 & 36 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \\ 0 & 1 & -18 \\ 0 & 1 & 2 \end{bmatrix}.$$

we see that $x_2 = 2$ from the second line and $x_2 = -18$ from the fourth line. Hence, there is no solution.

(b) Find the QR-decomposition of A.
Answer: To find the QR-decomposition, first note that the columns of A are not linearly dependent. We use the Gram-Schmidt process to replace them by two orthonormal vectors: Call the first column of A $\mathbf{x}_1$ and the second column $\mathbf{x}_2$. Then, $\mathbf{v}_1' = \mathbf{x}_1$ and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot v_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

Note that

$$\mathbf{x}_2 \cdot \mathbf{v}_1 = 1 \cdot (-1) + 0 \cdot 1 + +2 \cdot (-3) + (-2) \cdot 1 + 0 \cdot 1 = -9$$

and

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1^2 + 0^2 + 2^2 + (-2)^2 + 0^2 = 9.$$

Hence,

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

We now rescale them so that they have length 1 and these are the columns of Q. Note that $\|\mathbf{v}_1\| = \sqrt{9} = 3$ and $\|\mathbf{v}_2\| = \sqrt{4} = 2$. Hence,

$$Q = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then R is given by

$$R = Q^T A = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -3 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix}.$$

(c) How many least-squares solutions are there?
Answer: Since the columns of A are linearly independent, there exists a unique least-squares solution.

(d) Use your answer from (b) to determine a least-squares solution.
Answer: The least-squares solution is given by $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. First calculate R^{-1}:

$$R^{-1} = \frac{1}{6-0}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then

$$\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 36 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -24 \\ -16 \end{bmatrix} = \begin{bmatrix} -16 \\ -8 \end{bmatrix}.$$

Question 10

Let $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

(a) Show that the matrix A^TA has eigenvalues 8 and 2.
Answer:

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

Then

$$\det(A^TA - \lambda I) = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 2$.

(b) For each of the eigenvalues from (a), give an eigenvector of length 1.
Answer:

$$A - 8I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence, an eigenvector of unit length is $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$A - 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, an eigenvector of unit length here is $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

(c) Give a singular value decomposition of A.
Answer: We are looking for matrices U, Σ and V such that $A = U\Sigma V^T$, with U and V orthogonal matrices and Σ a 2×2 matrix with the singular values of A on its diagonal. So,

$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

For the matrix U we calculate

$$\mathbf{u}_1 = \frac{1}{2\sqrt{2}} A\mathbf{v}_1 = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Take $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.