Final Exam – Linear Algebra and Image Processing (Informatics)

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 & -1 \\ 2 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

in parametric vector form [something like $\mathbf{p} + t\mathbf{u}$, $(t \in \mathbb{R})$].

Question 2

Compute the determinants of the following two matrices:

(a)

$$A = \begin{bmatrix} 5 & 3 & 7 & 2 \\ 1 & 2 & 5 & 5 \\ 4 & 0 & 0 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 & 0 & 0 \\ -1 & 7 & -4 & 0 & 0 & 0 \\ 5 & 3 & 4 & 8 & 1 & 2 \\ 5 & 21 & 234 & 1 & -3 & 2 \\ 5 & 2 & 34 & 0 & 4 & 1 \end{bmatrix}.$$

(c) For those which are invertible, compute also the determinants of their inverses.

Question 3

Consider the 3×3 matrices

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 1 & 2 & 5 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 0 & 3 \\ 4 & 0 & 4 \end{bmatrix}.$$

(a) Is A invertible? If so, find the inverse of A.

(b) Is B invertible? If so, find the inverse of B.

(c) Is the sum $A + B$ invertible?

(d) Is the product $A^{10}B^{1000}$ invertible?

Question 4

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 + x_4 \\ x_1 - x_4 \\ x_1 + 2x_2 + x_3 \end{bmatrix}.$$

(a) Write down the standard matrix for T.

(b) Write down a basis for the kernel of T. Is T injective (one-to-one)? Why?

(c) Write down a basis for the image of T. Is T surjective (onto)? Why?

Question 5

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 2 & 1 & 4 & 3 & 2 \\ 4 & 2 & 12 & 13 & 9 \\ 2 & 1 & 12 & 18 & 14 \\ 6 & 3 & 20 & 27 & 25 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Which of the following pairs of matrices is an LU factorisation of A?

1. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 4 & 3 & 2 \\ 0 & 0 & 4 & 7 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Use your answer to (a) to find a solution to the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Question 6

Let $\mathbb{P}_4$ denote the vector space of polynomials with real coefficients and of degree up to 4 in one variable t.

(a) Is the set $\{1, 1+t, 1+t^2, t+t^2, t^4\}$ linearly independent? Justify your answer.

(b) Does the set $\{1, 1+t, 1+t^2, t+t^2, t^4+t^3, t^3-t^4\}$ span $\mathbb{P}_4$? Justify your answer.

(c) Expand the set $\{t^4, t^2, 1, -1+3t+2t^2+4t^4\}$ to a basis of $\mathbb{P}_4$.

Question 7

Consider the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) Give the characteristic equation of A and deduce that A has eigenvalues 5 and 1.

(b) Give a basis for each eigenspace of A.

(c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

Question 8

Consider the following three vectors:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 14 \\ 3 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Calculate the distance between $\mathbf{u}$ and $\mathbf{v}$.

(c) What is the orthogonal projection of $\mathbf{w}$ on $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$?

Question 9

Let A be the matrix

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 2 & 1 \end{bmatrix}.$$

The matrix A^TA has eigenvalues 9 and 4. The vector $\mathbf{v}_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for eigenvalue 9 and the vector $\mathbf{v}_2' = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector for eigenvalue 4.

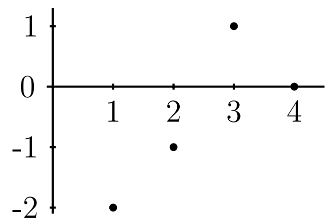
(a) What are the singular values of A?

(b) Calculate the vectors $\mathbf{u}_1' = A\mathbf{v}_1'$ and $\mathbf{u}_2' = A\mathbf{v}_2'$. Find a vector $\mathbf{u}_3'$ that is orthogonal to both $\mathbf{u}_1'$ and $\mathbf{u}_2'$.

(c) Give a singular value decomposition of A.

Question 10

Some experiment gave the following datapoints: $(1, -2)$, $(2, -1)$, $(3, 1)$, $(4, 0)$, see the graph below.



Use the normal equations to find the least squares line that best fits these points.