Final Exam – Linear Algebra and Image Processing (Informatica)

22 May 2013

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 & -1 \\ 2 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

in parametric vector form [something like $\mathbf{p} + t\mathbf{u}$, $(t \in \mathbb{R})$].

Anwser: Row reduce the augmented coefficient matrix. The solution is not unique, but one possibility is

$$\begin{bmatrix} 1 \\ -5 \\ 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Question 2

Compute the determinants of the following two matrices:

(a)

$$A = \begin{bmatrix} 5 & 3 & 7 & 2 \\ 1 & 2 & 5 & 5 \\ 4 & 0 & 0 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}.$$

Answer: Expand along 3rd row, get 108.

(b)

$$B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 & 0 & 0 \\ -1 & 7 & -4 & 0 & 0 & 0 \\ 5 & 3 & 4 & 8 & 1 & 2 \\ 5 & 21 & 234 & 1 & -3 & 2 \\ 5 & 2 & 34 & 0 & 4 & 1 \end{bmatrix}.$$

Answer: Expand along top row, then next row, then next row, then left with a 3×3 determinant, which is easy to compute. The answer is $120 \times 81 = 9720$.

(c) For those which are invertible, compute also the determinants of their inverses.
Answer: Recall that $\det M^{-1} = 1/\det M$. Hence $\det A^{-1} = 1-/133$, and $\det B^{-1} = 1/9720$.

Question 2

Consider the 3×3 matrices

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 1 & 2 & 5 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 0 & 3 \\ 4 & 0 & 4 \end{bmatrix}.$$

(a) Is A invertible? If so, find the inverse of A.
Answer: Yes, $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -7 & -18 \\ -2 & 3 & 7 \end{bmatrix}$.

(b) Is B invertible? If so, find the inverse of B.
Answer: Yes, $B^{-1} = \frac{1}{4}\begin{bmatrix} 0 & -4 & 3 \\ 4 & 24 & -16 \\ 0 & 4 & -2 \end{bmatrix}$.

(c) Is the sum $A + B$ invertible?
Answer: Yes: we compute its determinant and get 134 which is non-zero.

(d) Is the product $A^{10}B^{1000}$ invertible?
Answer: Yes: a product of invertible matrices is invertible.

Question 4

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 + x_4 \\ x_1 - x_4 \\ x_1 + 2x_2 + x_3 \end{bmatrix}.$$

(a) Write down the standard matrix for T.
Answer: $A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$.

(b) Write down a basis for the kernel of T.
Answer: Row reduce A, and apply standard method for writing the solution set of a homogeneous linear system as a span. Alternatively, note that the kernel must have

dimension 1, and that $v = \begin{bmatrix} 1 \\ 4 \\ -9 \\ 1 \end{bmatrix}$ is a non-zero element, and therefore spans the kernel.
Not injective, since Kernel non-zero.

(c) Write down a basis for the image of T.
Answer: Is surjective, for example because the determinant of the matrix consisting of the first three columns of A has non-zero determinant, so these columns alone span $\mathbb{R}^3$. Any basis of $\mathbb{R}^3$ will do. Alternatively, remember that the image is the column space, and the pivor columns give a basis fo the column space.

Question 5

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 2 & 1 & 4 & 3 & 2 \\ 4 & 2 & 12 & 13 & 9 \\ 2 & 1 & 12 & 18 & 14 \\ 6 & 3 & 20 & 27 & 25 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Which of the following pairs of matrices is an LU factorisation of A?

1. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 4 & 3 & 2 \\ 0 & 0 & 4 & 7 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Answer: (2): check that $LU = A$.

(b) Use your answer to (a) to find a solution to the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Answer: First solve $Ly = b$, finding that $y = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Then solve $Ux = y$; the solution is non-unique, but a likely candidate is $x = \begin{bmatrix} 0 \\ 11 \\ -17/4 \\ 3 \\ -1 \end{bmatrix}$.

Question 6

Let $\mathbb{P}_4$ denote the vector space of polynomials with real coefficients and of degree up to 4 in one variable t.

(a) Is the set $\{1, 1+t, 1+t^2, t+t^2, t^4\}$ linearly independent?
Answer: No, there is a non-trivial linear relation between the elements.

(b) Does the set $\{1, 1+t, 1+t^2, t+t^2, t^4+t^3, t^3-t^4\}$ span $\mathbb{P}_4$?
Answer: Yes: we can write $1, t, t^2, t^3$ and t^4 as linear combinations of the given elements, and therefore we can write every polynomial as a linear combination of the given elements.

(c) Expand the set $\{t^4, t^2, 1, -1+3t+2t^2+4t^4\}$ to a basis of $\mathbb{P}_4$.
Answer: The set is already linearly independent. Add in the vector t^3 to get a basis (any vector containing a non-zero multiple of t^3 will do).

Question 7

Consider the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) Give the characteristic equation of A and deduce that A has eigenvalues 5 and 1.
Answer: We solve $\det\,(A-\lambda I) = 0$:

$$\det\,(A-\lambda I) = (3-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 0 \\ 0 & 5-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (3-\lambda)(3-\lambda)(5-\lambda) + 2(-2)(5-\lambda) \\ &= (5-\lambda)\big((3-\lambda)^2 - 4\big) \\ &= (5-\lambda)(\lambda^2 - 6\lambda + 5) = (5-\lambda)(\lambda-5)(\lambda-1) = 0. \end{aligned}$$

Hence, the eigenvalues of A are 5 and 1.

(b) Give a basis for each eigenspace of A.
Answer:

$$A - 5I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, a basis for the eigenspace of eigenvalue 5 is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$A - I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, a basis for the eigenspace of eigenvalue 1 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.
Answer: Yes, A is diagonalisable, because we have 3 linearly independent eigenvectors. The matrix P has such eigenvectors as columns and the matrix D has the corresponding eigenvalues on the diagonal. So, take for example

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 8

Consider the following three vectors:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 14 \\ 3 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
Answer: We calculate the dot product. If it is 0, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-4) + 0 \cdot 1 + 1 \cdot (-1) + 2 \cdot 3 + 1 \cdot 0 + (-1) \cdot 1 = -4 - 4 + 6 - 1 = 0.$$

So, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Calculate the distance between $\mathbf{u}$ and $\mathbf{v}$.
Answer: The distance between $\mathbf{u}$ and $\mathbf{v}$ is defined as the length of the vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1+4 \\ 0-1 \\ 1+1 \\ 2+3 \\ 1-0 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}.$$

The length of this vector is

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{5^2 + 1^2 + 2^2 + 1^2 + 1^2 + 2^4} = \sqrt{36} = 6.$$

So, $\text{dist}(\mathbf{u}, \mathbf{v}) = 6$.

(c) What is the orthogonal projection of $\mathbf{w}$ on Span$\{\mathbf{u}, \mathbf{v}\}$? **Answer:** Since $\{\mathbf{u}, \mathbf{v}\}$ is an orthogonal basis for Span$\{\mathbf{u}, \mathbf{v}\}$, the orthogonal projection of $\mathbf{w}$ on Span$\{\mathbf{u}, \mathbf{v}\}$ is given by

$$\hat{\mathbf{w}} = \frac{\mathbf{w}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{16}{8}\begin{bmatrix}1\\0\\1\\2\\1\\-1\end{bmatrix} + \frac{28}{28}\begin{bmatrix}-4\\1\\1\\3\\0\\-1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\\1\\2\\-1\end{bmatrix}.$$

Question 9 (10 points)

Let A be the matrix

$$A = \begin{bmatrix}0 & 2\\-2 & 0\\2 & 1\end{bmatrix}.$$

The matrix A^TA has eigenvalues 9 and 4. The vector $\mathbf{v}_1' = \begin{bmatrix}2\\1\end{bmatrix}$ is an eigenvector for eigenvalue 9 and the vector $\mathbf{v}_2' = \begin{bmatrix}1\\-2\end{bmatrix}$ is an eigenvector for eigenvalue 4.

(a) What are the singular values of A?
Answer: The singular values of A are the roots of the eigenvalues of A^TA, hence $\sigma_1 = 3$ and $\sigma_2 = 2$.

(b) Calculate the vectors $\mathbf{u}_1' = A\mathbf{v}_1'$ and $\mathbf{u}_2' = A\mathbf{v}_2'$. Find a vector $\mathbf{u}_3'$ that is orthogonal to both $\mathbf{u}_1'$ and $\mathbf{u}_2'$.
Answer:

$$\mathbf{u}_1' = A\mathbf{v}_1' = \begin{bmatrix}0 & 2\\-2 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\-4\\5\end{bmatrix}.$$

$$\mathbf{u}_2' = A\mathbf{v}_2' = \begin{bmatrix}-4\\-2\\0\end{bmatrix}.$$

For $\mathbf{u}_3'$, one can take for example the vector $\begin{bmatrix}1\\-2\\-2\end{bmatrix}$. It is easy to check that both $\mathbf{u}_1'\cdot\mathbf{u}_3' = 0$ and $\mathbf{u}_2'\cdot\mathbf{u}_3' = 0$.

(c) Give a singular value decomposition of A.
Answer: We want to write $A = U\Sigma V^T$. The matrix Σ is the matrix that has the same size as A and has the singular values on the diagonal:

$$\Sigma = \begin{bmatrix}3 & 0\\0 & 2\\0 & 0\end{bmatrix}.$$

The matrix V has the unit eigenvectors for A^TA as its columns. The vectors $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are eigenvectors, but they do not have length 1: $\|\mathbf{v}_1'\| = \sqrt{2^2+1^2} = \sqrt{5}$ and $\|\mathbf{v}_2'\| = \sqrt{1+4} = \sqrt{5}$. Hence, the matrix V is given by

$$V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}.$$

To find the matrix U, we use the answer from (d). The vectors $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ form an orthogonal set, but they do not have length 1.

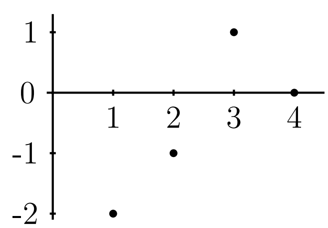
$$\begin{aligned} \|\mathbf{u}_1'\| &= \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}, \\ \|\mathbf{u}_2'\| &= \sqrt{16+4} = 2\sqrt{5}, \\ \|\mathbf{u}_3'\| &= \sqrt{1+4+4} = \sqrt{9} = 3. \end{aligned}$$

Then

$$U = \begin{bmatrix} \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{3\sqrt{5}} & 0 & -\frac{2}{3} \end{bmatrix}.$$

Question 10

Some experiment gave the following datapoints: $(1,-2)$, $(2,-1)$, $(3,1)$, $(4,0)$, see the graph below.



Use the normal equations to find the least squares line that best fits these points.

Answer: Let $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ be the design matrix, $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ be the parameter vector and $\mathbf{y} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ be the observation vector. We want a least squares solution to the system $X\beta = \mathbf{y}$.

The normal equations are $X^TX\beta = X^T\mathbf{y}$. First calculate X^TX and its inverse:

$$X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix},$$

$$(X^TX)^{-1} = \frac{1}{4 \cdot 30 - 10 \cdot 10} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix}.$$

Since X^TX is invertible, we know that a unique least squares solution exists. Now calculate X^Ty:

$$X^Ty = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Then the least squares solution is

$$\hat{\beta} = (X^TX)^{-1}X^T\mathbf{y} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{4}{5} \end{bmatrix}.$$

Hence, an equation for the line we are looking for is

$$y = \beta_1 + \beta_2 x = -\frac{5}{2} + \frac{4}{5}x.$$