Final Exam – Linear Algebra and Image Processing

14 July 2014

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the linear system $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & 1 \\ 2 & -2 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}.$$

Question 2

Consider the following square matrices:

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$.

(b) Compute the determinant $\det(B)$.

(c) Compute the determinant $\det(A^3)$ [hint: you should not need to compute the matrix A^3].

Question 3

To answer this question, you need the extra sheet.
Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

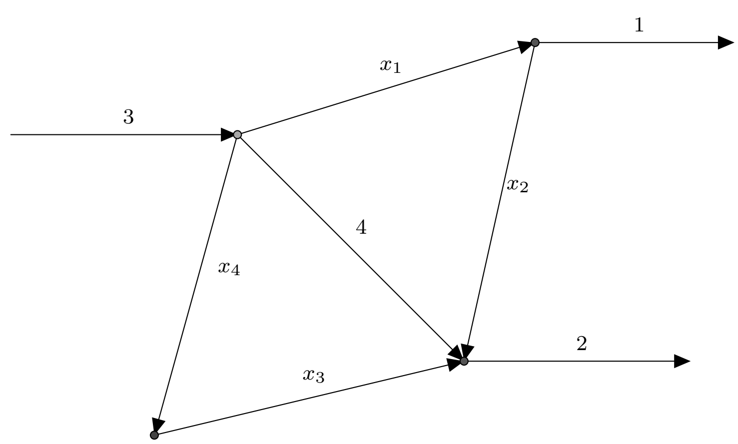
(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

(c) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix AB (so that the standard matrix of T is AB). Draw the image of the triangle in figure (c) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

(d) Is the effect of the linear transformation given by the matrix AB the same as the effect of the linear transformation given by the matrix BA?

Question 4

Consider the following network:



(a) Write down a linear system describing the flow in the network.

(b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

(c) Does there exist a solution with all flows non-negative (≥ 0)?

Question 5

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

[Hint: after computing A^{-1}, it is a good idea to check that AA^{-1} is the identity matrix].

Question 6

Let A be the matrix

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}.$$

(a) Show that A has eigenvalues 3 and 2.

(b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

(c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

(d) Calculate A^4.

Question 7

Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix}.$$

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) What is the distance between $\mathbf{u}$ and $\mathbf{v}$?

(c) Write $\mathbf{w}$ as the sum of two vectors, $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{z}$ where $\hat{\mathbf{w}}$ is a vector in $\mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$ and $\mathbf{z}$ is orthogonal to each vector in $\mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$.

Question 8

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief justification if it is true).

(a) If two matrices have the same number of rows, then they are row equivalent.

(b) Every linear map $\mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (injective).

(c) Let $V = \{\mathbf{0}\}$ be the zero vector space. Then every function $V \to \mathbb{R}^2$ is linear.

(d) If a finite-dimensional vector space V contains 5 linearly independent vectors, then $\dim V \geq 5$.

(e) For any two 2×2 matrices A and B, we have that $\det(A + B) = \det(A) + \det(B)$.

Question 9

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 1 \end{bmatrix}.$$

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(b) Use the Gram-Schmidt process to turn the columns of A into an orthonormal set.

(c) Give a QR-decomposition of A.

Question 10

Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

(a) Calculate $A^T A$ (here A^T denotes the transpose of A).

(b) Show that $A^T A$ has eigenvalues 12, 10 and 0.

(c) For each of the eigenvalues of $A^T A$ give an eigenvector of length 1.

(d) Give a singular value decomposition of A.