Final Exam – Linear Algebra and Image Processing

14 July 2014

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the linear system $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & 1 \\ 2 & -2 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}.$$

Answer: Row reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 1 & 1 \\ 2 & -2 & 4 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & -1 & 1 & 1 \\ 0 & -8 & 6 & 3 & -2 & -3 \\ 0 & -3 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 4 & 0 & -5 \\ 0 & 0 & 6 & 11 & -2 & -19 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 & 0 & 5/2 \\ 0 & 0 & 0 & -1 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & 3 & 5 \\ 0 & 1 & 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & 0 & -4 & -21/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & -1 & -11/2 \\ 0 & 1 & 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & 0 & -4 & -21/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 25/2 \\ 0 & 1 & 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & 0 & -4 & -21/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix}.$$

This gives the solution

$$\begin{aligned} x_1 &= \frac{25}{2} - 5x_5, \\ x_2 &= -6 + 2x_5, \\ x_3 &= -\frac{21}{2} + 4x_5, \\ x_4 &= 4 - 2x_5, \\ x_5 & \text{ is free.} \end{aligned}$$

The parametric vector form is

$$\mathbf{x} = \begin{bmatrix} \frac{25}{2} \\ -6 \\ -\frac{21}{2} \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

Question 2

Consider the following square matrices:

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$.
Answer $\det A = -3$.

(b) Compute the determinant $\det(B)$.
Answer $\det B = 4$.

(c) Compute the determinant $\det(A^3)$ [hint: you should not need to compute the matrix A^3].
Answer $\det(A^3) = (\det A)^3 = (-3)^3 = -27$.

Question 3

To answer this question, you need the extra sheet.
Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.
Answer: Scale does not matter because the transformation is linear. Let's say the corners are at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So the corners of the new triangle are given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

See the sheet for the picture.

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

Answer: Very similar to the above. Let's say the corners are at $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$. So the corners of the new triangle are given by

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

See the sheet for the picture.

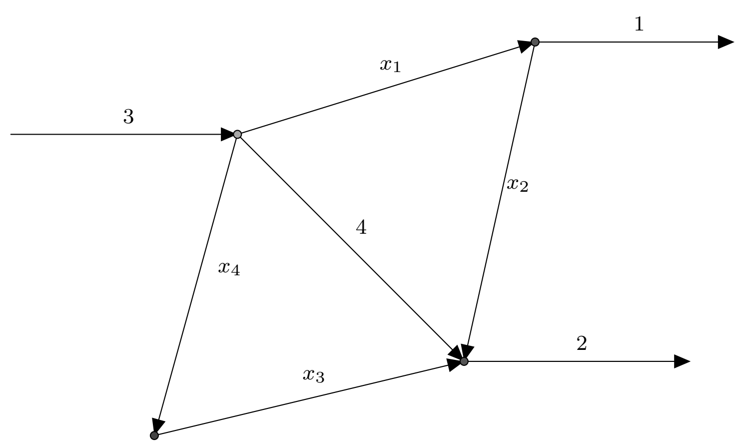
(c) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix AB (so that the standard matrix of T is AB). Draw the image of the triangle in figure (c) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

Answer: We compute $AB = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$. Then we proceed as before. The new corners are at $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}0\\2\end{bmatrix}$ and $\begin{bmatrix}4\\1\end{bmatrix}$. See separate sheet for picture.

(d) Is the effect of the linear transformation given by the matrix AB the same as the effect of the linear transformation given by the matrix BA?
Answer: No, they are not the same (for example, draw the image of the triangle under BA.

Question 4

Consider the following network:



(a) Write down a linear system describing the flow in the network.

Answer: Not unique. For example

$$\begin{bmatrix} x_1 + x_4 = -1 \\ x_1 - x_2 = 1 \\ x_2 + x_3 = -2 \\ x_3 - x_4 = 0 \end{bmatrix}.$$

(b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.
Answer: The augmented matrix of the linear system above is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

The row reduced matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Does there exist a solution with all flows non-negative (≥ 0)?
Answer: No, for example because we have $x_1 + x_4 = -1$.

Question 5

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Answer:

$$A^{-1} = \begin{bmatrix} 3 & 2 & -2 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

Question 6

Let A be the matrix

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}.$$

(a) Show that A has eigenvalues 3 and 2.
Answer: We solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det(A-\lambda I) = (4-\lambda)(1-\lambda)-(-2\cdot 1) = 4-5\lambda+\lambda^2+2 = \lambda^2-5\lambda+6 = (\lambda-3)(\lambda-2) = 0.$$

Hence, $\lambda = 3$ or $\lambda = 2$.

(b) For each eigenvalue in (a) give a basis for the corresponding eigenspace. To calculate the eigenvalues, we solve $(A - 3I)\mathbf{x} = \mathbf{0}$ and $(A - 2I)\mathbf{x} = \mathbf{0}$.

$$A - 3I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

Hence, one eigenvector for eigenvalue 3 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and a basis for the eigenspace corresponding to 3 is $\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

$$A - 2I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence, one eigenvector for eigenvalue 2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and a basis for the eigenspace corresponding to 2 is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

(c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.
Answer: Yes, A is diagonalisable, since it has two distinct eigenvalues. The matrix P has an eigenvector for each eigenvalue as its columns and the diagonal matrix D has the corresponding eigenvalues on the diagonal. So, for example

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

(d) Calculate A^4.
Answer: We use the answer from (c) to get $A^4 = PD^4P^{-1}$. First calculate

$$P^{-1} = \frac{1}{2-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Then

$$A^4 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 162 & 16 \\ 81 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 146 & -130 \\ 65 & -49 \end{bmatrix}.$$

Question 7

Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix}.$$

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
Answer: The dot product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 2 \cdot (-1) + 1 \cdot 0 = 2 - 2 = 0.$$

Hence, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) What is the distance between $\mathbf{u}$ and $\mathbf{v}$?
Answer: The distance between $\mathbf{u}$ and $\mathbf{v}$ is equal to the length of the vector $\mathbf{u} - \mathbf{v}$. This is the vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Hence,

$$dist(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}.$$

(c) Write $\mathbf{w}$ as the sum of two vectors, $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{z}$ where $\hat{\mathbf{w}}$ is a vector in Span($\{\mathbf{u}, \mathbf{v}\}$) and $\mathbf{z}$ is orthogonal to each vector in Span($\{\mathbf{u}, \mathbf{v}\}$).
Answer: The vector $\hat{\mathbf{w}}$ is the orthogonal projection of $\mathbf{w}$ onto Span($\{\mathbf{u}, \mathbf{v}\}$). This is given by

$$\begin{aligned} \hat{\mathbf{w}} &= \frac{\mathbf{w} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} + \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \\ &= \frac{-8 + 4 - 5}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{-4 - 2 + 0}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

The vector $\mathbf{z}$ can then be found by

$$\mathbf{z} = \mathbf{w} - \hat{\mathbf{w}} = \begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}.$$

Question 8

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief justification if it is true).

(a) If two matrices have the same number of rows, then they are row equivalent.
Answer: False, for example the systems with augmented matrices

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Every linear map $\mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (injective).
Answer: False, for example the zero map is not injective.

(c) Let $V = \{\mathbf{0}\}$ be the zero vector space. Then any function $V \to \mathbb{R}^2$ is linear.
Answer: False: such a map is linear if and only if it maps $\mathbf{0}$ to the zero vector in $\mathbb{R}^2$.

(d) If a finite-dimensional vector space V contains 5 linearly independent vectors, then $\dim V \geq 5$.
Answer: True; any linearly independent set can be expanded to a basis.

(e) For any two 2×2 matrices A and B, we have that $\det(A + B) = \det(A) + \det(B)$.
Answer: False, for example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Question 9

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 1 \end{bmatrix}.$$

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.
Answer: Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 4 & 2 \\ 1 & 4 & -2 & 2 \\ 1 & 4 & 2 & 6 \\ 1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 5 & -6 & 0 \\ 0 & 5 & -2 & 4 \\ 0 & 0 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

This last line gives $0x_1 + 0x_2 + 0x_3 = 3$, so the system is inconsistent.

(b) Use the Gram-Schmidt process to turn the columns of A into an orthonormal set.
Answer: Call the columns of A $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Put $\mathbf{v}_1 = \mathbf{x}_1$. Then

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1.$$

Since $\mathbf{x}_2 \cdot \mathbf{v}_1 = -1 + 4 + 4 - 1 = 6$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1 + 1 + 1 + 1 = 4$, this gives

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix}.$$

Use the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ instead for further calculations. Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2.$$

Note that $\mathbf{x}_3 \cdot \mathbf{v}_1 = 4-2+2 = 4$, $\mathbf{x}_3 \cdot \mathbf{v}_2 = -4-2+2 = -4$ and $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1+1+1+1 = 4$. So,

$$\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}.$$

Now we need to normalise these vectors to get the orthonormal set:

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}.$$

(c) Give a QR-decomposition of A.
Answer: The orthonormal set from (b) are the columns of Q:

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

For the matrix R we compute $Q^T A$:

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Question 10

Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

(a) Calculate $A^T A$.
Answer:

$$A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}.$$

(b) Show that $A^T A$ has eigenvalues 12, 10 and 0.
Answer: Solve $det(A^T A - \lambda I) = 0$:

$$\begin{aligned} det(A^T A - \lambda I) &= (10-\lambda)\big((10-\lambda)(2-\lambda) - 16\big) + 2(0 - 2(10-\lambda)) \\ &= (10-\lambda)\big((10-\lambda)(2-\lambda) - 20\big) \\ &= (10-\lambda)(\lambda^2 - 12\lambda + 20 - 20) \\ &= (10-\lambda)\lambda(\lambda - 12) = 0. \end{aligned}$$

So, the eigenvalues of $A^T A$ are $\lambda_1 = 12$, $\lambda_2 = 10$ and $\lambda_3 = 0$.

(c) For each of the eigenvalues of A^TA give an eigenvector of length 1.
Answer:

$$A - 12I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and an eigenvector of length 1 then is $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$A - 10I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

An eigenvector is $\begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$ and an eigenvector of length 1 is $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$.

$$A - 0I = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 2 \\ 0 & -20 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

An eigenvector is $\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$ and an eigenvector of length 1 is $\frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$.

(d) Give a singular value decomposition of A.
Answer: The non-zero singular values of A are $\sqrt{12}$ and $\sqrt{10}$, so

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}.$$

The matrix V has the unit eigenvectors from (c) as its columns, hence

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}.$$

Call the columns of V $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Then the column of the matrix U are given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{12}} A\mathbf{v}_1 = \frac{1}{\sqrt{12}} \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{12}} \frac{1}{\sqrt{6}} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}} A\mathbf{v}_2 = \frac{1}{\sqrt{10}} \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{10}} \frac{1}{\sqrt{5}} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

So, the matrix U becomes

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then $A = U\Sigma V^T$.