Final Exam – Linear Algebra and Image Processing

28 May 2015

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the linear system $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 4 & -2 & 4 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Question 2

Consider the following square matrices:

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 3 & 5 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$.

(b) Compute the determinant $\det(B)$.

(c) Compute $\det(A^2)$. [Hint: you should not need to compute the matrix A^2].

Question 3

Let a linear function $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ 3x_1 + 3x_2 \end{bmatrix}.$$

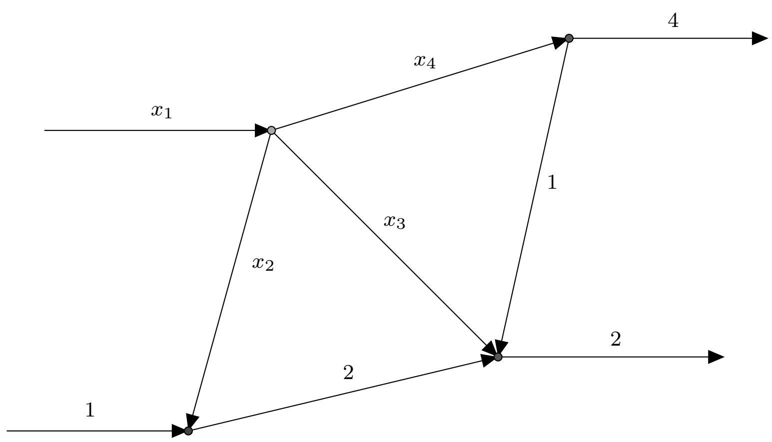
(a) Write down the standard matrix for T.

(b) Is T injective (one-to-one)? Justify your answer.

(c) Is T surjective (onto)? Justify your answer.

Question 4

Consider the following network:



(a) Write down a linear system describing the flow in the network.

(b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

(c) How many possible values are there for the flow along x_3?

Question 5

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

[Hint: after computing A^{-1}, it is a good idea to check that AA^{-1} is the identity matrix].

Question 6

Let A be the matrix

$$A = \begin{bmatrix} 11 & -20 \\ 6 & -11 \end{bmatrix}.$$

(a) Show that A has eigenvalues 1 and -1.

(b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

(c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

(d) Calculate A^3.

Question 7

Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

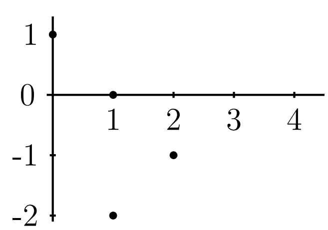
(a) Are the vectors $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

(b) What is the distance between $\mathbf{u}$ and $\mathbf{v}$?

(c) Write $\mathbf{w}$ as the sum of two vectors, $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{z}$ where $\hat{\mathbf{w}}$ is a vector in $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$ and $\mathbf{z}$ is orthogonal to each vector in $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$.

Question 8

An experiment gave the following datapoints: $(0, 1)$, $(1, 0)$, $(1, -2)$ $(2, -1)$, see the graph below.



Use the normal equations to find the least squares line that best fits these points.

Question 9

Let $\mathbb{P}_4$ denote the vector space of polynomials with real coefficients and of degree up to 4 in one variable t.

(a) Is the set $\{t, 1 - t, t - 1, -t + t^2, t^4\}$ linearly independent? Justify your answer.

(b) Does the set $\{t, t+t^2, t^3-3t^4, 2t-t^2, -4t^4-t^3\}$ span $\mathbb{P}_4$? Justify your answer.

(c) Expand the set $\{t^2, t^3+t^2, 3, t+1\}$ to a basis of $\mathbb{P}_4$.

Question 10

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

(a) Every linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ is surjective (onto).

(b) If λ is an eigenvector of an $n \times n$ matrix A, then the set of eigenvectors with eigenvalue λ is equal to the set of non-zero solutions of the matrix equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$.

(c) If an $n \times n$ matrix A has 0 as an eigenvalue, then the null space of A is non-zero.

(d) Suppose V is a vector space which is spanned by a set S containing exactly 5 vectors. Let T be a linearly independent set of 5 vectors in V. Then T is a basis for V.

(e) If A, B and C are non-zero 3×3 matrices and $AB = AC$ then $B = C$.