LAB solutions 2015.

1). Augmented coefficient matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 1 \\ 4 & -2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 0 & -2 \end{array}\right).$$

Row reduce this to

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} \end{array}\right)$$

So we find

$$\begin{aligned} x_1 &= -\tfrac{1}{2} + \tfrac{1}{2}x_4 \\ x_2 &= -\tfrac{7}{2} + \tfrac{1}{2}x_4 \\ x_3 &= -\tfrac{3}{2} - \tfrac{1}{2}x_4 \\ x_4 &= 0 + 1 \cdot x_4 \end{aligned}$$

So general solution is $\underline{x} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{7}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$

2) $\det A = \det\begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & -4 \end{pmatrix} = 6$

$$\det B = 2\det\begin{pmatrix} 3 & 2 & -1 \\ 0 & 5 & 5 \\ 0 & 3 & 5 \end{pmatrix} = 6\det\begin{pmatrix} 5 & 5 \\ 3 & 5 \end{pmatrix}$$

$$= 30\det\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$$

$$= 30 \cdot 2 = \underline{60}.$$

$$\det(A^2) = \det(A \cdot A) = \det(A) \cdot \det(A)$$

$$= 6^2 = 36.$$

3) a). Standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 3 & 0 \end{bmatrix}.$$

b) Is not injective, because $T\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \underline{0}$.

c) Is not surjective, e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ not in image (or use (b) + dimensions)

Can also explicitly find kernel & image by row-reducing A & looking at null & column spaces.

4) a) $x_1 + 1 = 2 + 4$

$x_1 = x_2 + x_3 + x_4$

$1 + x_2 = 2$

$x_4 = 4 + 1$

$3 + x_3 = 2$.

Tidy up to get:

$x_1 = 5$

$x_1 - x_2 - x_3 - x_4 = 0$

$x_2 = 1$

$x_4 = 5$

$x_3 = -1$.

b) Matrix:

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & - \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \\ 1 & -1 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

Row reduce to
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

c) The system is consistent & x_3 is basic, so exactly 1 solution (or: $x_3 = -1$!)

5) Usual algorithm: row reduce:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap rows; $\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right].$$

Should be A^{-1}. Let's check:

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

6) a) $\det(A-\lambda I_2) = \det\begin{pmatrix} 11-\lambda & -20 \\ 6 & -11-\lambda \end{pmatrix}$

$= (\lambda-11)(\lambda+11) + 120$

$= \lambda^2 - 121 + 120 = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$

b) $\lambda=1$: $A-\lambda I_2 = \begin{pmatrix} 10 & -20 \\ 6 & -12 \end{pmatrix}$. Sol'ns of $\begin{pmatrix} 10 & -20 & 0 \\ 6 & -12 & 0 \end{pmatrix}$.

row reduce: $\to \begin{pmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So x_2 free, say $x_2=1$.

$x_1 = 2x_2 = 2$.

So a basis is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda=-1$: $A-\lambda I_2 = \begin{pmatrix} 12 & -20 \\ 6 & -10 \end{pmatrix}$, $\to \begin{pmatrix} 12 & -20 & 0 \\ 6 & -10 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 3 & -5 & 0 \\ 3 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ x_2 free, say $x_2=1$

$x_1 = \frac{5x_2}{3} = \frac{5}{3}$.

Basis given by $\begin{bmatrix} 5/3 \\ 1 \end{bmatrix}$. Easier: $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

c) Yes, because has 2 distinct eigenvalues.

$P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $P^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Check: $A = PDP^{-1} = \cdots = \begin{bmatrix} 11 & -20 \\ 6 & -11 \end{bmatrix}$.

c) $A^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$.

$D^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$.

So $A^3 = PD^3P^{-1} = PDP^{-1} = A$.

7) a) $\underline{u} \cdot \underline{v} = -1 + 0 + 1 = 0$, so yes.

b) $dist(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \| = \sqrt{\underline{u} - \underline{v} \cdot \underline{u} - \underline{v}}$

$$= \sqrt{\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}} = \sqrt{8}.$$

c) $\hat{w} = \text{Proj}_{\text{Span}\{u,v\}} w$

$$= \left(\frac{u \cdot w}{u \cdot u}\right) u + \left(\frac{v \cdot w}{v \cdot v}\right) v$$

$$= \frac{4}{2} u + \frac{2}{6} v = 2u + \frac{v}{3} = \begin{bmatrix} 2 - \frac{1}{3} \\ 0 - \frac{2}{3} \\ 2 + \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ -2/3 \\ 7/3 \end{bmatrix}.$$

$$z = w - \hat{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 5/3 \\ -2/3 \\ 7/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ -4/3 \end{bmatrix}.$$

Check: $u \cdot z = \frac{4}{3} - \frac{4}{3} = 0$ ✓

$v \cdot z = \frac{-4}{3} + \frac{8}{3} - \frac{4}{3} = 0$ ✓

8) $(0,1)$ $(1,0)$ $(1,-2)$ $(2,-1)$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}. \qquad A^TA = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}.$$

$$A^T\underline{b} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

Solve $(A^TA)\underline{x} = A^T\underline{b}$

$$\left[\begin{array}{cc|c} 4 & 4 & -2 \\ 4 & 6 & -4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 2 & -1 \\ 2 & 3 & -2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 2 & 2 & -1 \\ 0 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$x_1 = \tfrac{1}{2}$$
$$x_2 = -1.$$

$$y = \frac{1}{2} - x.$$

9) a) No; $1-t = -(t-1)$, so there is a non-trivial linear relation.

b) No; The polynomial 1 is not in the span.

c) Add in t^4 (or any vector with a non-zero coefficient of t^4)

10) a) False, eg. the zero map.

b) True. $A\underline{x} = \lambda \underline{x} \iff A\underline{x} = \lambda I_n \underline{x}$

$\iff (A - \lambda I_n)\underline{x} = \underline{0}$.

c) True. let $\underline{v}$ be an eigenvector with eigenvalue 0, then $\underline{v} \neq \underline{0}$ & $A\underline{v} = 0\underline{v} = \underline{0}$.

So $\underline{v} \in \text{Nul}\,A$.

d) True. Existence of $S \Rightarrow \dim V \leq 5$.

e) False. eg. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $AB = AC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.