Retake Exam – Linear Algebra and Image Processing

7 July 2015

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ as a span of vectors, where

$$A = \begin{bmatrix} 4 & 0 & 8 & 12 \\ -1 & -1 & -6 & -4 \\ 2 & -1 & 0 & 5 \end{bmatrix}.$$

Question 2

Consider the following square matrices:

$$A = \begin{bmatrix} 4 & 0 & 8 & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 3 & 4 & -1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$.

(b) Compute the determinant $\det(B)$.

(c) Compute $\det(AB)$. [Hint: you should not need to compute the matrix AB].

Question 3

Let a linear function $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 + 3x_3 \\ -x_1 + 4x_2 \\ 2x_1 + 1x_2 + 4x_3 \end{bmatrix}.$$

(a) Write down the standard matrix for T.

(b) Compute the determinant of the standard matrix.

(c) Let C be a cube in $\mathbb{R}^3$ with side lengths 1 (and so with volume 1). What is the volume of the image $T(C)$?

Question 4

To answer this question, you need the extra sheet.
Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.$$

(a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

(c) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix AB (so that the standard matrix of T is AB). Draw the image of the triangle in figure (c) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

(d) Is the effect of the linear transformation given by the matrix AB the same as the effect of the linear transformation given by the matrix BA?

Question 5

Find the inverse of the following matrix:

$$A = \begin{bmatrix} -2 & 2 & 3 & -2 \\ 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

[Hint: after computing A^{-1}, it is a good idea to check that AA^{-1} is the identity matrix].

Question 6

Let A be the matrix

$$A = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

Question 7

Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

(a) Are the vectors $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

(b) What is the distance between $\mathbf{u}$ and $\mathbf{v}$?

(c) Find a vector $\mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{w}$ has length $\sqrt{21}$ and is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Question 8

Consider the following set S of vectors in $\mathbb{R}^3$:

$$S = \{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} \}.$$

(a) Is the set S linearly independent?

(b) Does the set S span $\mathbb{R}^3$?

(c) What is the dimension of the span of S?

Question 9

Find a basis of $\mathbb{R}^4$ which contains both of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 5 \end{bmatrix}.$$

Question 10

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

(a) If A is an $n \times n$ matrix and A^2 is invertible, then A is invertible.

(b) If A is a diagonal 2×2 matrix with integer entries, then the eigenvalues of A must be integers.

(c) Let A be an $n \times n$ matrix with $A^T = A^{-1}$, and let $\mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{b} \cdot \mathbf{b} = (A\mathbf{b}) \cdot (A\mathbf{b})$.

(d) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and let $\{\mathbf{u}, \mathbf{v}\}$ be a basis of $\mathbb{R}^2$. Then $\{T(\mathbf{u}), T(\mathbf{v})\}$ must be a basis of $\mathbb{R}^2$.

(e) If A and B are $n \times n$ matrices, then $(AB)^2 = A^2B^2$.