Solutions

1) Row-reduce the ACM to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 8 \\ 0 & 1 & 0 & 3 & -9 \\ 0 & 0 & 1 & -1 & 4 \end{array}\right]$

$\Rightarrow \quad \underline{x} = \begin{bmatrix} 8 \\ -9 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$

2) $\det A = 27$, $\det B = -21$, $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{27}$

3) Usual algorithm, $A^{-1} = \begin{bmatrix} 5 & -4 & 11 \\ -4 & 3 & -9 \\ -5 & 4 & -12 \end{bmatrix}$

4) a, b: omitted. c: $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unique sol'n. Eg. write $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $C\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$C\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

& solve for a, b, c, d.

5) a) False. eg. if linear then $f(\underline{0}) = \underline{0}$

b) False. $\det B = -\det A$

c) False, eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

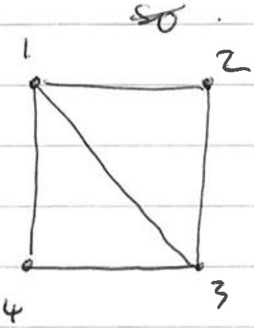
d) True; the zero vector is a sol'n.

e) False, eg. $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

~~6) $\det(A - \lambda I) = -\lambda(\lambda^2 + 2\lambda + 1) = -\lambda(\lambda+1)^2$~~

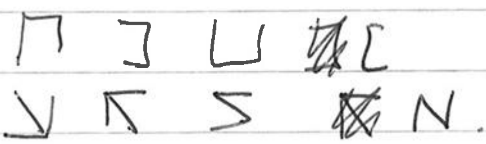
so

6) a)



b) $\det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 8$. c) 8 spanning trees.

d)



1 per 2 computational mistakes (-1 if 1 mistake, etc)

7) a) 2 $\det(A-\lambda I) = (\lambda+2)(\lambda+4)$.

b) 4 $\begin{bmatrix}1\\0\end{bmatrix}$ for $\lambda=-2$, $\begin{bmatrix}4\\1\end{bmatrix}$ for $\lambda=-4$.

c) 3 Yes; has 2 distinct e-values.

$D = \begin{pmatrix}-2 & 0\\ 0 & -4\end{pmatrix}$ $P = \begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}$

d) 1 $A^{34} = (PDP^{-1})^{34} = PD^{34}P^{-1} = \begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}\begin{pmatrix}-8 & 0\\ 0 & -64\end{pmatrix}\begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}$

$= \begin{pmatrix}-8 & -224\\ 0 & -64\end{pmatrix}$

8a) 4 eg Row reduce, last column is a pivot

b) 6 $\underline{v}_1 = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$ $\underline{w}_1 := \underline{v}_1$

$\underline{w}_2 := \underline{v}_2 - \frac{\underline{v}_1\cdot\underline{v}_2}{\underline{v}_1\cdot\underline{v}_1}\underline{v}_1 = \begin{bmatrix}-2\\0\\0\\2\end{bmatrix} - \left(\frac{2+2}{4}\right)\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$

$\underline{w}_3 = \underline{v}_3 - \frac{\underline{w}_1\cdot\underline{v}_3}{\underline{w}_1\cdot\underline{w}_1}\underline{w}_1 + \frac{\underline{w}_2\cdot\underline{v}_3}{\underline{w}_2\cdot\underline{w}_2}\underline{w}_2$

$= \underline{v}_3 + \frac{3}{2}\underline{w}_1 + \frac{3}{2}\underline{w}_2 = \begin{bmatrix}0\\-3\\-3\\0\end{bmatrix}$.

9a) 2 $\underline{u}\cdot\underline{v} = \cdots = 0$.

b) 5 $\mathrm{dist}(\underline{u},\underline{v}) = \sqrt{\underline{u}-\underline{v}\cdot\underline{u}-\underline{v}} = \sqrt{29}$.

c) 6 $\hat{\underline{w}} = \underline{w} - \underline{u}\left(\frac{\underline{w}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\right) - \underline{v}\left(\frac{\underline{w}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\right) = \begin{bmatrix}1\\0.5\\-1\\-1.5\\-4\end{bmatrix}$ $\underline{z} = \underline{w} - \hat{\underline{w}} = \begin{bmatrix}-1\\2.1\\2.5\\7\\1.5\\3\end{bmatrix}$.

10 a) 3 $\underline{u}\cdot\underline{v} = 0 \quad \forall \underline{u}\neq\underline{v}$

b) 3 orthogonal + $\underline{u}\cdot\underline{u} = 1 \quad \forall \underline{u}$.

c) 4 $2^n \cdot n!$ Proof Each column has entries in $\mathbb{Z}$ & $\underline{v}\cdot\underline{v}=1$, so $\underline{v} = \pm \underline{e}_i$ some i.

So • choose a sign for each column $(\rightsquigarrow 2^n)$

• n choices for first $\underline{e}_i$, $n-1$ for second, etc. $(\rightarrow n!)$.