Final Exam – Lineaire Algebra en Beeldverwerking

17 June 2016

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution to the linear system $A\underline{x} = \underline{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 3 & 2 & -1 & 4 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}.$$

Question 2

Consider the following square matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ -1 & 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 2 & 0 & 0 & 1 \\ 0 & 7 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

a) Compute the determinant $\det(A)$.

b) Compute the determinant $\det(B)$.

c) Compute $\det(A^{-1})$. [Hint: you should not need to compute the matrix A^{-1}].

Question 3

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 0 & -4 & 3 \\ -3 & -5 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

[Hint: after computing A^{-1}, it is a good idea to check that AA^{-1} is the identity matrix].

Question 4

To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the rectangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

c) Write a 2×2 matrix C so that the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix T sends the triangle in figure (c) to the triangle in figure (d). You should write your answer on the extra sheet.

Question 5

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

a) The function $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ sending every vector in $\mathbb{R}^2$ to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is linear.

b) Given an 2×2 matrix A, let B denote the matrix obtained from A by interchanging two rows. It is always true that $\det A = \det B$.

c) For every two 2×2 matrices A and B, we have that $\det(A - B) = \det(A) - \det(B)$.

d) Every homogeneous linear system is consistent (that is, has at least one solution).

e) For every pair A, B of $n \times n$, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).

Question 6

Consider the matrix

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

a) Draw the graph (with 4 vertices) whose Laplacian matrix is the matrix L.

b) Let $L_{1,1}$ be the matrix obtained from L be deleting the first row and the first column. Compute the determinant of $L_{1,1}$.

c) How many spanning trees does the graph in (a) have?

d) Draw all of the spanning trees of the graph.

Question 7

Define a matrix A by

$$A = \begin{bmatrix} -2 & -8 \\ 0 & -4 \end{bmatrix}.$$

a) Show that the eigenvalues of A are -2 and -4.

b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

d) Calculate A^3.

Question 8

Let the matrix A and the vector $\underset{\sim}{b}$ be given by

$$A = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & -3 \\ 1 & 0 & -3 \\ 1 & 2 & -3 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 1 \end{bmatrix}.$$

a) Show that the system $A\underset{\sim}{x} = \underset{\sim}{b}$ is inconsistent (has no solutions).

b) Use the Gram-Schmidt process to turn the columns of A into an orthogonal set.

Question 9

Consider the following three vectors:

$$\underline{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \underline{w} = \begin{bmatrix} 0 \\ 22 \\ 3 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

a) Show that $\underline{u}$ and $\underline{v}$ are orthogonal.

b) Calculate the distance between $\underline{u}$ and $\underline{v}$.

c) Write S for the span of $\underline{u}$ and $\underline{v}$. Find a vector $\hat{\underline{w}}$ in S and $\underline{z}$ in the orthogonal complement $S^{\perp}$ such that $\underline{w} = \hat{\underline{w}} + \underline{z}$ (this $\hat{\underline{w}}$ is the orthogonal projection of $\underline{w}$ on S).

Question 10

1. What does it mean for a set of vectors in $\mathbb{R}^n$ to be *orthogonal*?

2. What does it mean for a set of vectors in $\mathbb{R}^n$ to be *orthonormal*?

3. Recall that an $n \times n$ matrix is called orthogonal if its columns form an orthonormal set. As a function of n, how many $n \times n$ orthogonal matrices with integer entries are there? Hint: try working out the answer for $n = 1$, $n = 2$ first.

Retake Exam – Lineaire Algebra en Beeldverwerking

5 July 2016

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution of the homogeneous equation $A\underline{x} = \underline{0}$ as a span of vectors, where

$$A = \begin{bmatrix} 1 & 5 & 5 & 4 & 0 \\ -3 & 1 & 0 & -1 & -4 \\ 5 & -3 & -1 & -2 & 0 \end{bmatrix}.$$

Question 2

Consider the 3×3 matrices

$$A = \begin{bmatrix} 5 & -5 & -5 \\ 2 & -2 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & -3 \\ -4 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}.$$

a) Is A invertible? If so, find the inverse of A.

b) Is B invertible? If so, find the inverse of B.

c) Is the sum $A + B$ invertible?

d) Is the product A^2B^2 invertible? [Hint: you do not need to compute the matrix AB^{10}.]

Question 3

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 3 & 0 & 3 & 1 \\ 1 & -1 & -5 & -2 \\ 2 & 0 & 5 & 0 \end{bmatrix}.$$

[Hint: after computing A^{-1}, it is a good idea to check that AA^{-1} is the identity matrix].

Question 4

Consider the linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_1 - 3x_2 \end{pmatrix}.$$

In parts (a), (b) and (c), write down the standard matrix of the given transformation $\mathbb{R}^2 \to \mathbb{R}^2$:

a) The transformation sending $\underline{x}$ to $T(\underline{x})$.

b) The transformation sending $\underline{x}$ to $S(T(\underline{x}))$.

c) The transformation sending $\underline{x}$ to $T(S(\underline{x}))$.

Question 6

Let the matrix A and the vector $\underline{b}$ be given by

$$A = \begin{bmatrix} 2 & 0 & -4 & -4 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 2 & -3 & -2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 10 \\ 6 \\ 6 \\ 10 \end{bmatrix}.$$

a) Find a solution to the system $A\underline{x} = \underline{b}$.

b) Use the Gram-Schmidt process to turn the columns of A into an orthogonal set.

Question 7

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) If A, B and C are $n \times n$ matrices with A not the zero matrix, and such that $AB = AC$, then $B = C$.

b) The function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(x + y, x + 1)$ is linear.

c) If an $n \times n$ matrix A has 0 as an eigenvalue, then the null space of A is non-zero.

d) If a finite-dimensional vector space V contains 4 linearly independent vectors, then $\dim V \geq 4$.

e) If $S\colon \mathbb{R}^m \to \mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^p$ are injective functions then the composite $T \circ S$ is also injective.

Question 8

Let A be the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 4 & -5 & 4 \\ 4 & -4 & 3 \end{bmatrix}.$$

a) Find the eigenvalues of A

b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

Question 9

Exactly one of the following sets of vectors in $\mathbb{R}^3$ spans the whole of $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -7 \\ 7 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) Which of the sets S_1, S_2 spans the whole of $\mathbb{R}^3$?

b) For the set which spans the whole of $\mathbb{R}^3$, find a subset of it which is a basis of $\mathbb{R}^3$.

Question 10

Let A be an $n \times n$ matrix. Let $\underline{v}_1$ be an eigenvector with eigenvalue λ_1, and let $\underline{v}_2$ be an eigenvector with eigenvalue λ_2. Assume that $\lambda_1 \neq \lambda_2$.

a) Show that $\underline{v}_1$ and $\underline{v}_2$ are linearly independent.

b) Assume also that A is symmetric. Show that $\underline{v}_1$ and $\underline{v}_2$ are orthogonal.

Question 9

Let A be an invertible $n \times n$ matrix.

1. Show that 0 is not an eigenvalue of A.
2. Let $\underline{v}$ be an eigenvalue of A with eigenvector λ. Show that $\underline{v}$ is an eigenvalue of A^{-1} with eigenvector λ^{-1}.
3. Let B and P be $n \times n$ matrices with P invertible with $A = PBP^{-1}$. Suppose $\underline{v}$ is an eigenvector for A with eigenvalue λ. Show that $P\underline{v}$ is an ... with eigenvalue λ.