1) $\begin{pmatrix} 1 & 5 & 5 & 4 & 0 \\ -3 & 1 & 0 & -1 & -4 \\ 5 & -3 & -1 & -20 & \end{pmatrix} \to \begin{bmatrix} 1 & 5 & 5 & 4 & 0 \\ 3 & -1 & 0 & 1 & 4 \\ -1 & -1 & -1 & \cancel{-5}\ {-4} & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 5 & 4 & 0 \\ 0 & -4 & -3 & -14 & -20 \\ 0 & 4 & 4 & \cancel{\ }0 & -8 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 5 & 5 & 4 & 0 \\ 0 & 0 & 1 & -15 & -28 \\ 0 & 4 & 4 & 0 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \cancel{8}4 & 8 \\ 0 & 4 & 4 & 0 & -8 \\ 0 & 0 & 1 & -15 & -28 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 1 & \cancel{8}\,4 & 8 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -15 & -28 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \cancel{4}4 & 10 \\ 0 & 1 & 0 & 15 & \cancel{30}26 \\ 0 & 0 & 1 & -15 & -28 \end{bmatrix}$

$x_1 \quad +\cancel{4}4x_4 + 10x_5 = 0$

$x_2 \quad +15x_4 + 26x_5 = 0$

$x_3 - 15x_4 - 28x_5 = 0$

$$\begin{array}{l} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \\ x_5 = \end{array} \begin{array}{c} x_4 \\ \begin{pmatrix} -4 \\ -15 \\ 15 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{c} x_5 \\ \begin{pmatrix} -10 \\ -26 \\ 28 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

(span of these two.

2) a) $\det A = 10 \det \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = 0.$ Not inv.

b) ~~det B~~ $\left[\begin{array}{ccc|ccc} 5 & -1 & -3 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 5 & -1 & -3 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 0 & -1 & 7 & -4 & -5 & 0 \\ 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -7 & 4 & 5 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 2 \\ 0 & 1 & 0 & 11 & 12 & 7 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ $\quad 2\det \begin{bmatrix} 0 & -11 & 7 \\ 1 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix} = -2(-55+7) \neq 0$

B^{-1} $\quad$ // $\quad$ so Yes

c. $\det(A+B) = \det \begin{bmatrix} 10 & -6 & -8 \\ -2 & -1 & 3 \\ 0 & -1 & 5 \end{bmatrix} = -\det \begin{bmatrix} 0 & -11 & 7 \\ -2 & -1 & 3 \\ 0 & -1 & 5 \end{bmatrix} = -\det \begin{bmatrix} 0 & -11 & 7 \\ -2 & 0 & -2 \\ 0 & -1 & 5 \end{bmatrix}$

d) No. det = 0.

3) a) $\begin{bmatrix} 1 & 3 \\ 9 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 9 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & -6 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 3 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}$

4. Row red: $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + 2x_3 = 0$

$x_2 + 3x_3 = 0$ $\qquad \underline{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} x_3$

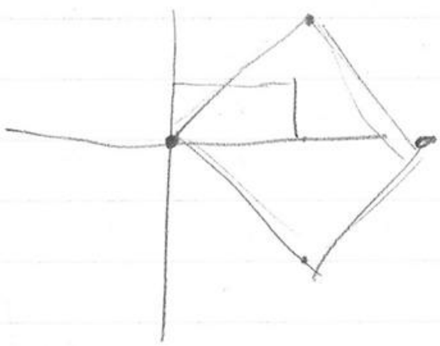
Basis of kernel $= \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

Basis of image $= \{$pivot columns$\}$

$= \left\{ \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

5. $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$

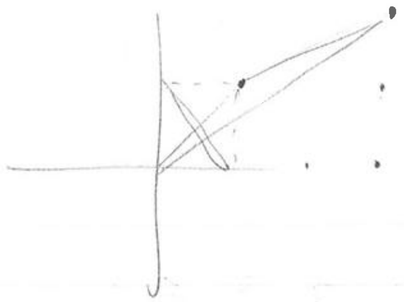
a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 4 \\ 0 \end{bmatrix}$



b) $\begin{bmatrix}0\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} \mapsto \begin{bmatrix}3\\2\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\1\end{bmatrix}$

c) $C\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad C = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$C\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}2a\\2c\end{bmatrix}$ $\quad a=0,\ c=-1$

$C\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}a+2b\\c+2d\end{bmatrix}$ $\quad b=1,\ d=0$

$C = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$

6a) No, eg. $n=2$, lots of egs.

b) No, eg $T(\underline{0}) \neq \underline{0}$

c) ~~NB. Eg. $n=1$, $A=[0]$, Nullspace~~

c) Yes, $\exists \underline{v} \neq \underline{0}$ s.t. $A\underline{v} = \lambda\underline{v} = \underline{0}$, so $\underline{v} \in$ Nullspace.

d) Yes. Any basis is ≥ 4 elts, or can be expanded to a basis.

e) Yes. Say $TS(\underline{v}) = 0$. Then $S(\underline{v}) = 0$ (since T inj)

so $\underline{v} = 0$ (since S inj.)

7a) Row red

$$\left[\begin{array}{cccc|c} 2 & 0 & -4 & -4 & 10\\ 0 & 0 & -1 & -3 & 6\\ 0 & 0 & 0 & -3 & 6\\ 0 & 2 & -3 & -2 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & -2 & 5\\ 0 & 2 & -3 & -2 & 10\\ 0 & 0 & 1 & 3 & -6\\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc} 1 & 0 & -2 & 0 & 1\\ 0 & 2 & -3 & 0 & 6\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 3\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

~~$x_4 = 2x$~~ $\quad \underline{x} = \begin{bmatrix}1\\3\\0\\-2\end{bmatrix}$ $\quad$ Check! $\quad$ (unique sol'n).

$2 + 8 = 10$ ✓

$= 6$

$6 + 4 = 10$ ✓

b) $\underline{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} -4 \\ -1 \\ 0 \\ -3 \end{bmatrix}$ $\underline{v}_4 = \begin{bmatrix} -4 \\ -3 \\ -3 \\ 2 \end{bmatrix}$.

$\cancel{\underline{u}_1 = \underline{v}_1}$

$\underline{u}_1 = \underline{v}_1$

$\underline{u}_2 = \underline{v}_2 - \frac{\overset{0}{u_1 \cdot v_2}}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1 = \underline{v}_2$

$\underline{u}_3 = \underline{v}_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} \cdot \underline{u}_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} \cdot u_2$

$= \begin{bmatrix} -4 \\ -1 \\ 0 \\ -3 \end{bmatrix} - \left(\frac{-8}{4}\right) \underline{u}_1 - \left(\frac{-6}{4}\right) \underline{u}_2$

$= \begin{bmatrix} -4 \\ -1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \underline{u}_3$.

$\underline{u}_4 = \underline{v}_4 - \frac{u_1 v_4}{u_1 u_1} u_1 - \frac{u_2 v_4}{u_2 u_2} \underline{u}_2 - \frac{u_3 v_4}{u_3 u_3} \underline{u}_3$

$= \begin{bmatrix} -4 \\ -3 \\ -3 \\ 2 \end{bmatrix} - \left(\frac{-8}{4}\right) \underline{u}_1 - \left(\frac{4}{4}\right) \underline{u}_2 - \left(\frac{3}{1}\right) \underline{u}_3$

$= \begin{bmatrix} -4 \\ -3 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} = \underline{u}_4$.

8a) $\det \begin{bmatrix} -\lambda & -1 & 1 \\ 4 & -5-\lambda & 4 \\ 4 & -4 & 3-\lambda \end{bmatrix} = -\lambda\left((\lambda+5)(\lambda-3) + 16\right) + (12 - 4\lambda - 16)$
$+ (-16 + 20 + 4\lambda)$

$= -\lambda(\lambda^2 + 2\lambda + 1) \cancel{-4\lambda - 4 + 4\lambda + 4}$
$= -\lambda(\lambda+1)^2$

e-values $0, -1$.

b $\lambda = 0$: $\begin{bmatrix} 0 & -1 & 1 \\ 4 & -5 & 4 \\ 4 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad x_1 - \frac{1}{4}x_3 = 0$

$\qquad x_2 - x_3 = 0$

basis $= \left\{ \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \right\}$

$\lambda = -1$: $\begin{bmatrix} 1 & -1 & 1 \\ 4 & -4 & 4 \\ 4 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0.$ $\qquad x_1 = x_2 - x_3$ $\quad \cancel{x_1 = x_3 - x_2}$

basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

9) eg. $\det \begin{pmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ -1 & 1 & 2 \end{pmatrix} = -(-4-2) - 2(2+2) + 1(1-2)$

$= 6 - 8 - 1 = -3 \neq 0.$

So S_2 spans, & first 3 cols are a basis.

10 a) Say $A\underline{v} = 0 \cdot \underline{v} = \underline{0}$. Then A invertible so $\underline{v} = 0$.

b) $A\underline{v} = \lambda\underline{v}$. Mult. by A^{-1}: $\begin{array}{ccc} A^{-1}A\underline{v} & = & A^{-1}\lambda\underline{v} \\ \parallel & & \parallel \\ \underline{v} & & \lambda A^{-1}\underline{v}. \end{array}$

c) $A\underline{v} = \lambda\underline{v}$. $\qquad BP\underline{v} = PAP^{-1}P\underline{v} = PA\underline{v} = P\lambda\underline{v} = \lambda P\underline{v}$