Retake Exam – Lineaire Algebra en Beeldverwerking

5 July 2016

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Write the general solution of the homogeneous equation $A\underline{x} = \underline{0}$ as a span of vectors, where

$$A = \begin{bmatrix} 1 & 5 & 5 & 4 & 0 \\ -3 & 1 & 0 & -1 & -4 \\ 5 & -3 & -1 & -2 & 0 \end{bmatrix}.$$

Question 2

Consider the 3×3 matrices

$$A = \begin{bmatrix} 5 & -5 & -5 \\ 2 & -2 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & -3 \\ -4 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}.$$

a) Is A invertible? If so, find the inverse of A.

b) Is B invertible? If so, find the inverse of B.

c) Is the sum $A + B$ invertible?

d) Is the product A^2B^2 invertible? [Hint: you do not need to compute the matrix A^2B^2].

Question 3

Consider the linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix},$$

and the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_1 - 3x_2 \end{pmatrix}.$$

In parts (a), (b) and (c), write down the standard matrix of the given transformation $\mathbb{R}^2 \to \mathbb{R}^2$:

a) The transformation sending $\underline{x}$ to $T(\underline{x})$.

b) The transformation sending $\underline{x}$ to $S(T(\underline{x}))$.

c) The transformation sending $\underline{x}$ to $T(S(\underline{x}))$.

Question 4

Consider the linear transformation $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$\begin{pmatrix} 3 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}.$$

a) Find a basis of the kernel of S.

b) Find a basis of the image of S.

Question 5

To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the rectangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

c) Write a 2×2 matrix C so that the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix T sends the triangle in figure (c) to the triangle in figure (d). You should write your answer on the extra sheet.

Question 6

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) If A, B and C are $n \times n$ matrices with A not the zero matrix, and such that $AB = AC$, then $B = C$.

b) The function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(x + y, x + 1)$ is linear.

c) If an $n \times n$ matrix A has 0 as an eigenvalue, then the null space of A is non-zero.

d) If a finite-dimensional vector space V contains 4 linearly independent vectors, then $\dim V \geq 4$.

e) If $S\colon \mathbb{R}^m \to \mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^p$ are injective functions then the composite $T \circ S$ is also injective.

Question 7

Let the matrix A and the vector $\underline{b}$ be given by

$$A = \begin{bmatrix} 2 & 0 & -4 & -4 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 2 & -3 & -2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 10 \\ 6 \\ 6 \\ 10 \end{bmatrix}.$$

a) Find a solution to the system $A\underline{x} = \underline{b}$.

b) Use the Gram-Schmidt process to turn the columns of A into an orthogonal set.

Question 8

Let A be the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 4 & -5 & 4 \\ 4 & -4 & 3 \end{bmatrix}.$$

a) Find the eigenvalues of A

b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

Question 9

Exactly one of the following sets of vectors in $\mathbb{R}^3$ spans the whole of $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -7 \\ 7 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) Which of the sets S_1, S_2 spans the whole of $\mathbb{R}^3$?

b) For the set which spans the whole of $\mathbb{R}^3$, find a subset of it which is a basis of $\mathbb{R}^3$.

Question 10

Let A be an invertible $n \times n$ matrix.

1. Show that 0 is not an eigenvalue of A.
2. Let $\underline{v}$ be an eigenvalue of A with eigenvector λ. Show that $\underline{v}$ is an eigenvalue of A^{-1} with eigenvector λ^{-1}.
3. Let B and P be $n \times n$ matrices with P invertible and with $B = PAP^{-1}$. Suppose $\underline{v}$ is an eigenvector for A with eigenvalue λ. Show that $P\underline{v}$ is an eigenvector for B with eigenvalue λ.