Final Exam – Lineaire Algebra 2

23 June 2017

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$$

a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

c) Write a 2×2 matrix C so that the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix T sends the triangle in figure (c) to the triangle in figure (d). You should write your answer on the extra sheet.

Question 2

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

a) The zero vector in $\mathbb{R}^n$ is orthogonal to every other vector in $\mathbb{R}^n$.

b) If $S\colon \mathbb{R}^m \to \mathbb{R}^n$ is injective and $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective, then the composite $T \circ S\colon \mathbb{R}^m \to \mathbb{R}^m$ is injective and surjective.

c) Every homogeneous linear system is consistent.

d) Every invertible matrix has determinant 1.

e) The kernel of a linear map is a linear subspace.

Question 3

Consider the set of vectors in $\mathbb{R}^2$ given by

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ a \end{bmatrix} \right\}$$

where a is a real number.

a) Describe all the values of a for which the set S is *linearly independent*.

b) Describe all the values of a for which the span of the set S equal to the whole of $\mathbb{R}^2$.

[Hint: in parts (a) and (b), if you cannot determine all the values of a, you will still get some points for trying a few values of a].

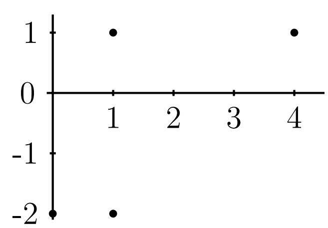
Now consider the set of vectors in $\mathbb{R}^3$ given by

$$T = \left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

c) Is the set T *linearly independent*?

Question 4

An experiment gave the following datapoints: $(1, -2)$, $(4, 1)$, $(1, 1)$, $(0, -2)$, see the graph below.



Use the 'normal equation' $A^T A = A^T \mathbf{b}$ to find the least squares line that best fits these points.

Question 5

Consider the following three vectors:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) Calculate the distance between $\mathbf{u}$ and $\mathbf{v}$.

c) What is the orthogonal projection of $\mathbf{w}$ on $\text{Span}\{\mathbf{u}, \mathbf{v}\}$?

Question 6

Define a sequence of integers F_n by

$$F_1 = 1, F_2 = 1, \text{ and } F_{n+2} = 2F_{n+1} + 3F_n.$$

a) Compute F_3 and F_4.

Define a 2×2 matrix A by

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

b) Show that

$$\begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

We see therefore that $\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $n \geq 1$.

c) Find the eigenvalues of A.

d) For each eigenvalue, find a corresponding eigenvector.

e) Write down a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

f) Compute D^{2016}.

g) Show that $A^{2016} = \frac{1}{4} \begin{bmatrix} 3 + 3^{2016} & 3^{2016} - 1 \\ 3^{2017} - 3 & 3^{2017} + 1 \end{bmatrix}.$

h) Find F_{2017} (you do not need to simplify your answer).