Final Exam – Lineaire Algebra 1

10 January 2018

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Consider the matrix A given by

$$A = \begin{bmatrix} -1 & 2 & 2 & -4 \\ 1 & 2 & 2 & 0 \\ 4 & 1 & 2 & -1 \end{bmatrix}.$$

   a) Put the matrix A in reduced echelon form. *Hint: your answer should be one of the following two options:*

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -8 \end{bmatrix}.$$

   b) Is the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ in the subspace of $\mathbb{R}^3$ spanned by the columns of A?

   c) Write the solution set of the homogeneous linear system $A\underline{x} = \underline{0}$ as a span of vectors.

2. Define matrices A and B by

$$A = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ -1 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

   a) Compute the determinant of A.

   b) Show that the inverse of B is $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 1 \\ -1 & -3 & -8 \end{bmatrix}$.

   c) Compute the product AB.

   d) What is the determinant of A^2?

3. To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

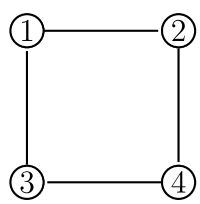
a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B. Draw the image under T of the triangle from figure (a) on figure (b).

c) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A^{-1}. Draw the image under T of the triangle from figure (a) on figure (c).

d) Write down a 2×2 matrix A such that the associated linear transformation will take the triangle in figure (a) to the one in figure(d).

4. Consider the graph G:



a) Write down the Laplacian matrix L of G, ordering the vertices as shown in the picture above.

b) Compute the $(2, 2)$-cofactor of L.

c) How many spanning trees does G have?

d) Write down all the spanning trees of G.

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear function given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - x_2 \\ 3x_1 - x_3 \\ 4x_1 - x_2 - x_3 \end{bmatrix}.$$

(a) Write down the standard matrix A for T.

(b) Is T injective (one-to-one)?

(c) Is T surjective (onto)?

(d) Write down a vector $\underline{v} \in \mathbb{R}^3$ such that $\underline{v}$ together with the columns of A spans the whole of $\mathbb{R}^3$.

6. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

   a) Let A be an $n \times n$ matrix with $n \geq 2$. If A has two rows equal then $\det A = 0$.

   b) If A is an $n \times n$ matrix then the equation $A\underline{x} = \underline{b}$ has at least one solution for every $\underline{b} \in \mathbb{R}^n$.

   c) If A is an invertible 4×4 matrix then $3A$ is invertible and $(3A)^{-1} = \frac{1}{3}A^{-1}$.

   d) Let A be an $n \times n$ matrix. If the equation $A\underline{x} = \underline{0}$ has a non-trivial solution, then A has exactly n-pivot positions.

   e) Performing elementary row operations on a matrix does not change the span of its columns.