Final Exam – Lineaire Algebra 2

3 June 2019

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 questions, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Consider the matrix A given by

$$A = \begin{bmatrix} 3 & 0 & -3 & 0 \\ 4 & -1 & -1 & 2 \\ -1 & 1 & -2 & -2 \end{bmatrix}$$

   a) Put the matrix A in reduced echelon form. *Hint: your answer should be one of the following two options:*

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   b) Write the solution set of the homogeneous linear system $A\underline{x} = \underline{0}$ as a span of vectors.

   c) What is the dimension of the solution set?

2. We define several vectors in $\mathbb{R}^3$:

$$\underline{a} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

   a) Is the set $\{\underline{a}, \underline{b}\}$ orthogonal?

   b) Is the set $\{\underline{a}, \underline{c}\}$ orthogonal?

   c) Is the set $\{\underline{a}, \underline{b}, \underline{c}\}$ orthogonal?

   d) Compute the projection of the vector $\underline{d}$ onto the span of $\{\underline{a}, \underline{b}\}$.

3. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

   a) For every pair A, B of $n \times n$ matrices, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).

   b) Let A be an $n \times n$ matrix, and let B be the matrix obtained from A by multiplying every entry of A by 2. Then $\det B = 2n \det A$.

   c) If U and V are linear subspaces of $\mathbb{R}^n$, then the intersection $U \cap V$ is a linear subspace of $\mathbb{R}^n$.

   d) If λ is an eigenvalue of an $n \times n$ matrix A, then the set of eigenvectors with eigenvalue λ is equal to the set of non-zero solutions of the matrix equation $(A - \lambda I_n)\underline{x} = \underline{0}$.

   e) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are non-zero vectors in $\mathbb{R}^2$ with $\underline{a}$ orthogonal to $\underline{b}$ and $\underline{b}$ orthogonal to $\underline{c}$, then it must hold that $\underline{a}$ is a scalar multiple of $\underline{c}$.

4. We define matrices A and B by

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the remainder of this question you may use without proof that A and B are row equivalent.

a) Give a basis for the null space $\mathrm{Null}(A)$.

b) Give a basis for the column space $\mathrm{Col}(A)$.

Now let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $T(\underline{x}) = A\underline{x}$.

c) State the definition of the *image* of a linear transformation.

d) Write down a basis of the image $\mathrm{im}(T)$.

5. We define a matrix A by

$$A = \begin{bmatrix} 12 & 5 \\ -30 & -13 \end{bmatrix}.$$

a) Show that 2 and -3 are eigenvalues of A.

b) Find a basis for the eigenspace for 2.

c) Find a basis for the eigenspace for -3.

d) Find an invertible matrix P and a diagonal matrix D such that $PDP^{-1} = A$.

e) Show that

$$A^{2019} = \begin{bmatrix} 6(2^{2018} + 3^{2018}) & 2^{2019} + 3^{2019} \\ -6(2^{2019} + 3^{2019}) & -(2^{2020} + 3^{2020}) \end{bmatrix}.$$

6. We define a matrix A and a vector b by

$$A = \begin{bmatrix} -2 & 1 \\ -2 & 1 \\ 3 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Compute the product $A^T A$.

(b) Compute the product $A^T \underline{b}$.

(c) Compute the least squares solution to the equation $A\underline{x} = \underline{b}$. *Hint: the entries of your answer should be integers.*

An experiment gave the datapoints: $(1,1), (1,0), (4,1), (4,4)$. We want to find the best-fitting curve of the form

$$y = v_0 + v_1\sqrt{x}$$

(here we only allow x to take positive values, and $\sqrt{x}$ means the positive square root).

(d) Write a matrix equation whose solution is the vector $\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$ giving the curve which best fits the data.