Retake Exam – Lineaire Algebra 2

9 July 2019

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 questions, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper (no calculator).

1. Consider the matrix A given by

$$A = \begin{bmatrix} -2 & 0 & 0 & 4 \\ -3 & -1 & 1 & 4 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

   a) Put the matrix A in reduced echelon form. *Hint: your answer should be one of the following two options:*

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   b) Write the solution set of the homogeneous linear system $A\underline{x} = \underline{0}$ as a span of vectors.

   c) Can you find three linearly independent vectors in the solution set?

   d) Multiplication by A defines a linear transformation $\mathbb{R}^4 \to \mathbb{R}^3$. Is this linear transformation *injective*?

2. Define matrices A and B by

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -1 & -1 \\ -4 & -3 & -1 \\ -3 & -3 & 0 \end{bmatrix}.$$

   a) Compute the inverse of A.

   b) Compute the determinant of B.

   c) Compute the product AB.

   d) What is the determinant of $-3B$?

3. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

   a) There exists an injective linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

   b) If A and B are $n \times n$ matrices, then $(AB)^2 = A^2B^2$.

   c) The kernel of a linear map is a linear subspace.

   d) If A is a diagonal 2×2 matrix with integer entries, then the eigenvalues of A must be integers.

   e) Let A be an $n \times n$ matrix with $A^T = A^{-1}$, and let $\underline{b} \in \mathbb{R}^n$. Then $\underline{b} \cdot \underline{b} = (A\underline{b}) \cdot (A\underline{b})$.

4. We define several vectors in $\mathbb{R}^3$:

$$\underline{a} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}.$$

a) Is the set $\{\underline{a}, \underline{b}\}$ orthogonal?

b) Is the set $\{\underline{a}, \underline{c}\}$ orthogonal?

c) Is the set $\{\underline{a}, \underline{b}, \underline{c}\}$ orthogonal?

d) Compute the projection of the vector $\underline{d}$ onto the span of $\{\underline{a}, \underline{b}\}$.

5. We define a matrix A by

$$A = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}.$$

a) Show that 1 and -1 are eigenvalues of A.

b) Find a basis for the eigenspace for 1.

c) Find a basis for the eigenspace for -1.

d) Find an invertible matrix P and a diagonal matrix D such that $PDP^{-1} = A$.

e) Compute A^{2019}.

6. Suppose A is an $m \times n$ matrix, $\underline{y}$ a vector of length n, and $\underline{b}$ a vector of length m. Assume they satisfy the equation $A\underline{y} = \underline{b}$.

(a) Show that $\underline{y}$ is a Least Squares Solution to the matrix equation $A\underline{x} = \underline{b}$. [*Hint: recall that a Least Squares Solution is a vector $\underline{x}$ which minimises the distance between $A\underline{x}$ and $\underline{b}$.*]

(b) Show that every Least Squares Solution of the equation $A\underline{x} = \underline{b}$ is an actual solution (i.e. satisfies $A\underline{x} = \underline{b}$).

We define a matrix A and a vector b by

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}.$$

(c) Compute the product $A^T A$.

(d) Compute the product $A^T \underline{b}$.

(e) Compute the least squares solution to the equation $A\underline{x} = \underline{b}$.