LA1 exam Jan 2018 Solutions.

1a)
$$\begin{bmatrix} -4 & -2 & 1 & -1 \\ 3 & 2 & 1 & -3 \\ 1 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 3 & 2 & 1 & -3 \\ -4 & -2 & 1 & -1 \end{bmatrix} \xrightarrow{\substack{r_2 \mapsto r_2 - 3r_1 \\ r_3 \mapsto r_3 + 4r_1}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 7 & -3 \\ 0 & 2 & -7 & -1 \end{bmatrix}$$

$$\xrightarrow{\substack{r_3 \mapsto r_3 + 2r_2 \\ r_2 \mapsto -r_2}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 7 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

b) Yes. M is the ACM of the matrix equation $A\underline{x} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$, & last column is not a pivot, so ~~the~~ system is consistent.

c) No. A has a pivot in every column, so no free variables, so non non-trivial solution.

2 a) $\det A = \det\begin{pmatrix} 0 & 1 & 1 & -1 \\ 2 & 0 & 0 & 1 \\ 1 & 3 & -2 & 1 \\ -3 & 0 & 2 & -3 \end{pmatrix} = \det\begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & 0 & 0 & 1 \\ 1 & 3 & -2 & 1 \\ 0 & 3 & 0 & -1 \end{bmatrix}$

$= \det\begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 2 \\ 0 & 3 & 0 & -1 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 3 & 0 & -1 \end{bmatrix} + \det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}$

$= 4\det\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 3 & 0 & -1 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$

$= 4\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 3 & 0 & -1 \end{bmatrix} - (-3) = 4\det\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} + 3 = -4 + 3 = \underline{\underline{-1}}$

b) $\det B = \det\begin{bmatrix} -2 & 3 & 2 & 0 \\ -1 & 3 & 2 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -4 & -3 & 2 \end{bmatrix} = \det\begin{bmatrix} 0 & -3 & -2 & 2 \\ -1 & 3 & 2 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -4 & -3 & 2 \end{bmatrix}$

$= \det\begin{bmatrix} -3 & -2 & 2 \\ -2 & 1 & 4 \\ -4 & -3 & 2 \end{bmatrix} = \det\begin{bmatrix} -3 & -2 & 2 \\ -2 & 1 & 4 \\ -1 & -1 & 0 \end{bmatrix}$

$= -\det\begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix} + \det\begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix} = 10 - 8 = \underline{\underline{2}}$

c) $\det A^{-1} = \frac{1}{\det A} = \underline{\underline{-1}}$

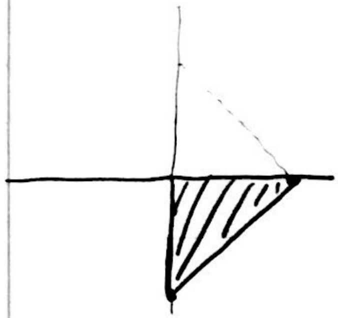
d) $\det(A^2 B^3) = (\det A)^2 (\det B)^3 = \underline{\underline{8}}$

3) $A = \cancel{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

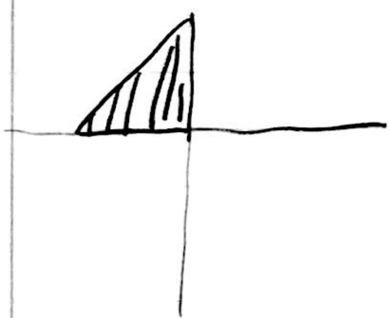
$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

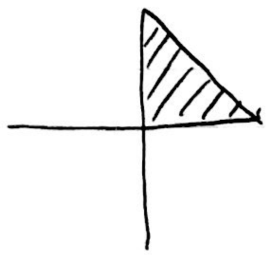
a)



b) A rotates $90°$ clockwise, so A^3 rotates $270°$ clockwise, so 98 counter clockwise.

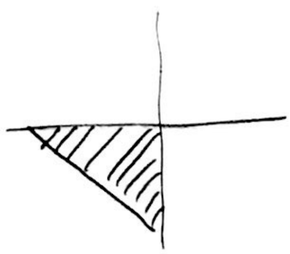


c) $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, does nothing:



d) $A^{2018} = A^{2+4\cdot 504} = A^2 \cdot (A^4)^{504} = A^2 \cdot (I_2)^{504} = A^2$,

rotates $180°$:



3) See extra sheet.

(4) (8)

4) a) $x_1 + 5 = x_2$

$x_2 = 2 + x_3$

$x_3 = x_4 + 7$

$x_4 + 4 = x_1$

(optional: $5 + 4 = 2 + 7$)

b)
$$\begin{aligned} x_1 - x_2 \quad\quad\quad &= -5 \\ x_2 - x_3 \quad\quad &= 2 \\ x_3 - x_4 &= 7 \\ x_1 \quad\quad\quad - x_4 &= 4 \end{aligned}$$

ACM:
$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 7 \\ 1 & 0 & 0 & -1 & 4 \end{array}\right]$$

$r_4 \mapsto r_4 - (r_1 + r_2 + r_3)$
$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r_2 \mapsto r_2 + r_3$

then $r_1 \mapsto r_1 + r_2$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & -1 & 9 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
← Row reduced form of ACM

c) General sol'n:

$x_1 = 4 + x_4$

$x_2 = 9 + x_4$

$x_3 = 7 + x_4$

$x_4 = \text{free}$

So choosing $x_4 > 0$ gives a positive solution,

eg $x_1 = 5$

$x_2 = 10$

$x_3 = 8$

$x_4 = 1$

5) a) Row reduce:

$$\begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix} \to \begin{bmatrix} 0 & 1+2a \\ -1 & a \end{bmatrix} \to \begin{bmatrix} 1 & -a \\ 0 & 1+2a \end{bmatrix}.$$

So lin. indep $\iff$ no free variables in $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & a & 0 \end{array}\right]$

$\iff$ every column of $\begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ is a pivot

$\iff 1+2a \neq 0.$

$\iff a \neq -\frac{1}{2}.$

b) Spans $\mathbb{R}^2 \iff$ every row contains a pivot

$\iff 1+2a \neq 0$

$\iff a \neq -\frac{1}{2}.$

c) Rowreduce:

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Not every row contains a pivot,

so T does not span $\mathbb{R}^3$.

6) a) True. $\cancel{(A^2)A} = \cancel{2A}$ $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = \underline{0}+\underline{0}=\underline{0}$.

b) False, eg $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

c) False, $\det(2A) = 2^4 \det A \neq 2\det A$ since $\det A \neq 0$.

d) False, eg. zero map.

e) False. Suppose it exists, given by a matrix $A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$

Then A has only two columns, so at most two pivot positions, so cannot have a pivot in every row.