LA2 Retake 10 July 2018 Solutions.

Q1 a) ACM = $\left[\begin{array}{cccc|c} 3 & 5 & -1 & 2 & 5 \\ -1 & -2 & 4 & -4 & 2 \\ 1 & 2 & -4 & 4 & -2 \end{array}\right]$

b) $r_3 \mapsto r_3 + r_2$
$r_1 \mapsto r_1 + 3r_2$

$$\left[\begin{array}{cccc|c} 0 & -1 & 11 & -10 & 11 \\ -1 & -2 & 4 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r_1 \leftrightarrow r_2$
$r_1 \mapsto -r_1$
$r_2 \mapsto -r_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 4 & -2 \\ 0 & 1 & -11 & 10 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r_1 \mapsto r_1 - 2r_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 18 & -16 & 20 \\ 0 & 1 & -11 & 10 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Ans ~~m~~ is $\underline{M}$.

c) $\left.\begin{array}{l} x_1 + 18x_3 - 16x_4 = 20 \\ x_2 - 11x_3 + 10x_4 = -11 \end{array}\right\}$ $\begin{array}{l} x_1 = 20 - 18x_3 + 16x_4 \\ x_2 = -11 + 11x_3 - 10x_4 \\ x_3 = 0 + 1x_3 + 0x_4 \\ x_4 = 0 + 0.x_3 + 1.x_4 \end{array}$

~~Solution~~ General solution:

$$x = \begin{bmatrix} 20 \\ -11 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -18 \\ 11 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 16 \\ -10 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}.$$

2 a) Standard matrix:

$$A = \begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

b) Rowreduce ~~partially~~:
$$\begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kernel = solutionset of $A\underline{x} = \underline{0}$.

Equations: $x_1 + 4x_3 = 0$ Gen. Sol'n: $\underline{x} = \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} \cdot t$, $t \in \mathbb{R}$

$x_2 - 2x_3 = 0$
$x_3 - x_3 = 0$
$x_4 + 0 \cdot x_3 = 0$.

So kernel = Span $\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Dimesion (kernel) = 1. Not injective, as kernel $\neq 0$.

c) Image = Span of pivot columns.

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} \right\}$$
, & these form a basis.

Dimenson (image) = 3, so is surjective; have 3-dimensional subspace of $\mathbb{R}^3$, so is whole of $\mathbb{R}^3$.

3a) $\det M = 2\det \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} = 2\cdot(6-3) = \underline{6}$

b) $\det A = -\det N = -16$

c) $\det B = \det N = 16$

d) $\det C = \frac{1}{2^{(4)}} \cdot \det N = \frac{16}{16} = 1$

once for each row

4 a) True ; $T(\underline{u}_1 - \underline{u}_2) = T(\underline{u}_1) + T(-\underline{u}_2) = T(\underline{u}_1) - T(\underline{u}_2)$ by linearity.

4 b) False, Eg. $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, Then only e-value is $\lambda = 2$.

But eigenspace $W_2 = \ker(M - 2I_2) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

which has dimension 1, which is < 2.

c) False, eg. $\underline{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{c} = \underline{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

d) False, eg. $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$,

$S = \left\{ \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right\}$.

e) False, eg. $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{v} = -\underline{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, then $\underline{u} \cdot \underline{v} = -1 < 0$.

5 a) $\underline{v}_1 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$.

$$\underline{v}_2 = \underline{a}_2 - \frac{\underline{a}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1$$

$$= \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} - \frac{18}{36} \underline{v}_1 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

$$\underline{v}_3 = \underline{a}_3 - \frac{\underline{a}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \frac{\underline{a}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

$$= \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} - \frac{2}{36} \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} - \frac{-4}{18} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/18 \\ -4/18 \\ 2/18 \end{bmatrix} - \begin{bmatrix} -4/18 \\ 4/18 \\ 16/18 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

So orthogonal set is $\left\{ \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \right\}$.

b) Yes. Orthogonal sets are LI, so ~~....~~ dim(col A) = 3,
so colums must also be LI.

c) Rank = ~~#~~ dim Col A = 3

d) Yes. Row rank = column rank = 3
$\quad\quad \|$
dim(Row A), so rows must be LI.

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6a) $\det(A-\lambda I_2) = \det\begin{pmatrix} 5-\lambda & -6 \\ 4 & -5-\lambda \end{pmatrix}$

$= -(5-\lambda)(5+\lambda) + 24 \quad = -25 + \lambda^2 + 24$

$= \lambda^2 - 1 \quad = (\lambda+1)(\lambda-1).$

roots $\lambda = \pm 1.$

b) $\lambda = 1$: $A - \lambda I_2 = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$

$x_1 - 3/2\, x_2 = 0. \rightarrow 2x_1 = 3x_2$

e-vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = -1$: $A - \lambda I_2 = \begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x_1 - x_2 = 0,$

e-vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

d) $A^{100} = (PDP^{-1})^{100} = P\, D^{100} P^{-1} = PP^{-1} = I_2$

$A^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$