Q1a) $A(n)$: $\left[\begin{array}{cccc|c} -3 & 4 & 1 & 1 & -1 \\ 4 & -5 & -1 & 1 & -2 \\ 2 & 1 & 6 & -5 & -3 \end{array}\right]$.

Notice it is enough to compare the 4^{th} colums, so we can ignore the last column to simplify the calculation. So we row-reduce A.

$$\begin{bmatrix} -3 & 4 & 1 & 1 \\ 4 & -5 & -1 & 1 \\ 2 & 1 & 6 & -5 \end{bmatrix} \xrightarrow[r_2 \mapsto r_2 - 2r_3]{r_1 \mapsto r_1 + r_2} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -7 & -13 & 11 \\ 2 & 1 & 6 & -5 \end{bmatrix} \begin{matrix} r_3 \mapsto r_3 - 2r_1 \\ r_2 \mapsto -r_2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 7 & 13 & -11 \\ 0 & 3 & 6 & -9 \end{bmatrix}$$

$$r_2 \mapsto r_2 - 2r_3 \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 3 & 6 & -9 \end{bmatrix} r_3 \mapsto r_3 - 3r_2 \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 3 & -30 \end{bmatrix}$$

$$r_3 \mapsto \frac{r_3}{3} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -10 \end{bmatrix} \begin{matrix} r_2 \mapsto r_2 - r_3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

We see now that the answer must be N.. For safety we carry on:

$$r_1 \mapsto r_1 + r_2: \begin{bmatrix} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

Yes, it is N.

b) As equation:

$$\begin{aligned} x_1 + 19x_4 &= -24 \\ x_2 + 17x_4 &= -21 \\ x_3 - 10x_4 &= 11 \end{aligned}$$

$$\begin{aligned} x_1 &= -24 - 19x_4 \\ x_2 &= -21 - 17x_4 \\ x_3 &= 11 + 10x_4 \\ x_4 &= 0 + 1x_4 \end{aligned}$$

General solution: $\underline{x} = \begin{bmatrix} -24 \\ -21 \\ 11 \\ 0 \end{bmatrix} + t \begin{bmatrix} -19 \\ -17 \\ 10 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

Q2 a) $\underline{a} \cdot \underline{c} = 2 - 2 = 0$, so Yes.

b) $\underline{a} \cdot \underline{b} = 1 - 2 + 1 = 0$, so Yes

c) $\underline{b} \cdot \underline{c} = 2 + 4 + 0 \neq 0$, so No.

d) Since $\{\underline{a}, \underline{b}\}$ is orthogonal, we have the formula

$$\operatorname{proj}_{\operatorname{Span}\{\underline{a},\underline{b}\}} \underline{d} = \frac{\underline{d}\cdot\underline{a}}{\underline{a}\cdot\underline{a}}\,\underline{a} + \frac{\underline{d}\cdot\underline{b}}{\underline{b}\cdot\underline{b}}\,\underline{b}$$

$$= \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{12}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}2\\-4\\2\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix}.$$

~~Q3 a) False, eg. T given by the 1×1 matrix $[2]$, then $T([1]) = [2]$.~~

~~b) True. $T(\underline{u}_1 - \underline{u}_2) = T(\underline{u}_1 + (-\underline{u}_2)) = T(\underline{u}_1) + T(-\underline{u}_2) = T(\underline{u}_1) + T(-1\cdot\underline{u}_2)$~~

~~$= T(\underline{u}_1) + (-1)T(\underline{u}_2) = T(\underline{u}_1) - T(\underline{u}_2)$~~

~~here we use that T is linear.~~

a) False, eg. $\det\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = 1\times1 - 2\times2 = -3$.

True; It has a basis of eigenvectors.

b) ~~False, eg.~~ $A = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$ ~~has only one eigenvalue (2), & is diagonal (hence diagonalisable!)~~.

c) True. Can see from a picture. Precise version:
Let $S = \operatorname{Span}(\underline{a})$. Then $\underline{b} \in S^\perp$ & $\dim S^\perp = 1$, so $S^\perp = \operatorname{Span}(\underline{b})$.

Then $\underline{c} \in (S^\perp)^\perp$ & $\dim (S^\perp)^\perp = 1$. Also, $(S^\perp)^\perp = S$, so $\underline{a}$ & $\underline{c}$ are non-zero vectors in a 1-dimensional vector space, so are scalar multiples.

3 d. True $\underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| . \cos\theta$ where θ is the angle between $\underline{u}$ & $\underline{v}$.

And $-1 \leq \cos\theta \leq 1$.

e) True. $(\|\underline{u}\| + \|\underline{v}\|)^2 - \|\underline{u} + \underline{v}\|^2 = \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2 - (\underline{u}+\underline{v}) \cdot (\underline{u}+\underline{v})$

$= \|\underline{u}\|^2 + 2\|\underline{u}\|\,\|\underline{v}\| + \|\underline{v}\|^2 - \|\underline{u}\|^2 - 2\underline{u}\cdot\underline{v} - \|\underline{v}\|^2 = 2(\|\underline{u}\|\|\underline{v}\| - \underline{u}\cdot\underline{v})$

≥ 0 by (d).

because row equivalent

Q4a) $Null(A)$ = solution set of $A\underline{x} = \underline{0}$ = solution set of $B\underline{x} = \underline{0}$.

Equations: $x_1 \quad + 9x_3 - 14x_4 = 0$

$x_2 - 6x_3 + 11x_4 = 0$.

Rewrite: $x_1 = -9x_3 + 14x_4$

$x_2 = 6x_3 - 11x_4$

$x_3 = 1x_3 + 0x_4$

$x_4 = 0x_3 + 1x_4$

So

$$Null(A) = Span\left\{ \begin{bmatrix} -9 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ -11 \\ 0 \\ 1 \end{bmatrix} \right\}$$, & this is a basis.

b) $Col(A)$ has basis given by pivot columns, so a basis for $Col(A)$ is

$$\left\{ \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

c) $\ker(T) = \{ \underline{v} \in \mathbb{R}^4 \mid T(\underline{v}) = \underline{0} \}$.

d) $\ker(T) = Null(A)$. Has a basis consisting of 2 vectors by (a),

so $\dim \ker(T) = 2$.

5

Q5) $\det(A-\lambda I_3) = \det\begin{bmatrix} -12-\lambda & -12 & 0 \\ 6 & 5-\lambda & 0 \\ -22 & -28 & 1-\lambda \end{bmatrix}$ 1

4

$= (1-\lambda)\left((-12-\lambda)(5-\lambda)+72\right)$

2 slim 1.

We see already that $\lambda = 1$ 1 is a root, so must be the missing e-value. To be safe, we carry on:

2

$= (1-\lambda)(\lambda^2 + 12\lambda - 5\lambda - 60 + 72)$

$= (1-\lambda)(\lambda^2 + 7\lambda + 12) = (1-\lambda)(\lambda+3)(\lambda+4)$.

b) $A - 1.I_3 = \begin{bmatrix} -13 & -12 & 0 \\ 6 & 4 & 0 \\ -22 & -28 & 0 \end{bmatrix} \to \begin{bmatrix} -13 & -12 & 0 \\ 3 & 2 & 0 \\ -11 & -14 & 0 \end{bmatrix}$

1

5

row reduced echelon 1

$r_3 \mapsto r_3 - r_1$, $r_1 \mapsto -r_1$ $\to$ $\begin{bmatrix} 13 & 12 & 0 \\ 3 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ 0 & 5 & 0 \end{bmatrix}$ 1

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 1

$x_1 = 0$

$x_2 = 0$

x_3 free

So an eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ 1

c) $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} -4 & 3 & 0 \\ 3 & -2 & 0 \\ -1 & 2 & 1 \end{bmatrix}$. (or permute columns).

1

6a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^T \underline{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

ACM: $\left[\begin{array}{cc|c} 3 & 6 & 2 \\ 6 & 14 & 4 \end{array}\right] \rightsquigarrow \begin{bmatrix} 3 & 6 & 2 \\ 0 & 2 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix}$ $v_0 = 2/3$

$v_1 = 0$

least squares line: $y = \frac{2}{3} + 0.x$

Quick solution (b): observe that $\begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$ is a solution to the equation $A^T A \underline{x} = A^T \underline{b}$.

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ $A^T A = \begin{bmatrix} 3 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}$. $A^T \underline{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$

ACM: $\left[\begin{array}{ccc|c} 3 & 6 & 14 & 2 \\ 6 & 14 & 36 & 4 \\ 14 & 36 & 98 & 10 \end{array}\right] \xrightarrow[r_3 \mapsto r_3 - 4r_1]{r_2 \mapsto r_2 - 2r_1} \left[\begin{array}{ccc|c} 3 & 6 & 14 & 2 \\ 0 & 2 & 8 & 0 \\ 2 & 12 & 42 & 2 \end{array}\right]$

$\begin{bmatrix} 3 & 0 & -10 & 2 \\ 0 & 1 & 4 & 0 \\ 2 & 0 & -6 & 2 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & -3 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $a = 4$

$b = -4$

$c = 1$

$\rightsquigarrow y = 4 - 4x + x^2$