A note on Néron models, heights and torsion points

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Abstract

We relate the strong uniform boundedness conjecture for abelian varieties to the existence or otherwise of Néron models over higher-dimensional bases, and to the algebraic version of Hain's height jump.

1 Introduction

The strong uniform boundedness conjecture predicts that the size of the group of torsion points on an abelian variety over a global field can be bounded uniformly in the dimension of the abelian variety and the degree of the field (see conjecture 2.1). We propose a conjectural generalisation of a theorem of Silverman and Tate on heights in families of abelian varieties (conjecture 2.2), and show that this conjecture is equivalent to the strong uniform boundedness conjecture (theorem 3.1). Using a recent result of Cadoret and Tamagawa, we can reduce further to the case of families of curves.

We also show that this conjecture holds for any family of abelian varieties which admits a finite-type Néron model over a suitable compactification of the base scheme (lemma 4.1). In the absence of Néron model, we remark on the connection between orders of torsion points and the slow growth of a certain 'jump function' j inspired by asymptotic Hodge theory. Finally, we consider a variant (question 6.1) where torsion points are replaced by 'points of small height'.

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2 Statements of the conjectures

Given a field k, by a *variety over k* we mean an integral separated k-scheme of finite type. We fix a field K, which is either the field of rational numbers or the field $\mathbb{F}_l(T)$ for some prime number l. We fix an algebraic closure $\overline{K}$ of K.

Conjecture 2.1 (The strong uniform boundedness conjecture for abelian varieties)**.** Fix an integer $g \geq 0$. There exists a constant $\epsilon = \epsilon(g) \in \mathbb{Z}_{>0}$ such that for every g-dimensional abelian variety A/K and point $p \in A(K)$, we have that either p is of infinite order, or that the order of p is less than ϵ.

Note that this is equivalent to the usual formulation of the strong uniform boundedness conjecture (see for example [Sil01, conjecture 2.3.2]); by Weil restriction we can obtain a bound for abelian varieties over finite extensions of K which is uniform in the dimension of the variety and the degree of the field. Moreover, since the l-rank of the group of torsion points on an abelian variety of dimension g is bounded by $2g$ for every prime l, a uniform bound on the orders of torsion elements easily implies a uniform bound on the size of the torsion subgroup.

The main result of this note is that 2.1 is equivalent to the following conjecture:

Conjecture 2.2. Let S/K be a variety and let A/S be an abelian scheme. Let $d \geq 1$ be an integer. Let $\sigma \in A(S)$ be a section of infinite order. Define

$$\mathbf{T}(d) = \{p \in S(\bar{K}) | [\kappa(p) : K] \leq d \text{ and } \sigma(p) \text{ is torsion in } A_p(\bar{K})\}.$$

Then $\mathbf{T}(d)$ is not Zariski dense in S.

In the case when the base scheme S has dimension 1, this conjecture is a theorem due to Silverman [Sil83], see also [Tat83], [Lan83], [Cal86], [Gre89], [BHdJ14] for various strengthened versions. In section 6 we will discuss a variant which looks not only at torsion points but at all points of 'small height'.

Using a recent result of Cadoret and Tamagawa [CT13], we can even reduce to the case of families of curves:

Conjecture 2.3. Let S/K be a variety and let C/S be a proper smooth curve with jacobian J/S. Let $d \geq 1$ be an integer. Let $\sigma \in J(S)$ be a section of infinite order. Define

$$\mathbf{T}(d) = \{p \in S(\bar{K}) | [\kappa(p) : K] \leq d \text{ and } \sigma(p) \text{ is torsion in } J_p(\bar{K})\}.$$

Then $\mathbf{T}(d)$ is not Zariski dense in S.

The equivalence of conjecture 2.1 with conjecture 2.3 may be proven in an almost identical fashion to theorem 3.1, after first appealing to the main result of [CT13] to reduce conjecture 2.1 to the case of curves. We omit the details.

3 Proof of the main equivalence

Theorem 3.1. *Conjecture 2.1 is equivalent to conjecture 2.2.*

Remark 3.2. It is possible to simultaneously 'specialise' both conjectures to again obtain equivalent statements; for example, one can restrict conjecture 2.1 to elliptic curves and conjecture 2.2 to families of elliptic curves over base schemes S of dimension at most 2. The proofs are identical and the possible variations numerous, so we do not give further details.

It is easy to see that conjecture 2.1 implies conjecture 2.2; we must prove the converse.

Lemma 3.3. *Assume conjecture 2.2. Let S/K be a variety, and let A/S be an abelian scheme with a section $\sigma \in A(S)$. Fix $d \in \mathbb{Z}_{>0}$. Then there exists an integer $c = c(d) > 0$ such that for every L/K of degree at most d and every point $s \in S(L)$, either $\sigma(s)$ has infinite order, or $\sigma(s)$ is torsion of order less than c.*

Proof. We proceed by induction on the dimension of S. If $\dim S = 0$ then S has only finitely many $\overline{K}$-points, so the result is immediate.

In general, we fix an integer $\delta > 0$ and assume the lemma holds for every variety S of dimension less than δ. Now let S have dimension δ. If σ is torsion (say of order c_0) then for every $s \in S(\overline{K})$ it holds that $c_0\sigma(s) = 0$, and we are done. As such, we may and do assume that σ has infinite order. We apply conjecture 2.2 to obtain a proper closed subscheme $Z \subsetneq S$ such that

$$\{p \in S(L) | [L : K] \leq d \text{ and } \sigma(p) \text{ is torsion in } A_p(L)\} \subseteq Z(\overline{K}).$$

Now Z has only finitely many irreducible components, and each has dimension less than δ. We are done by the induction hypothesis. □

Definition 3.4. Fix an integer $g > 0$. Let $\mathscr{A}_g$ denote the moduli stack of PPAS of dimension g. Then $\mathscr{A}_g$ is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}$. Given also an integer $n \geq 0$, let $\mathscr{A}_{g,n}$ denote the moduli stack of PPAS of dimension g, with a collection of n ordered marked sections (not assumed distinct)

We have natural maps $\varphi_n \colon \mathscr{A}_{g,n+1} \to \mathscr{A}_{g,n}$ given by forgetting the last section. The map φ_n has n natural sections τ_i, given by 'doubling up' the sections σ_i. One checks without difficulty that $(\varphi_n \colon \mathscr{A}_{g,n+1} \to \mathscr{A}_{g,n}, \tau_1, \cdots, \tau_n)$ is the 'universal PPAS with n marked sections'. In particular, each $\mathscr{A}_{g,n}$ is a Deligne-Mumford stack, separated and of finite type over $\mathbb{Z}$.

Proof of theorem 3.1. We begin by remarking that conjecture 2.1 for PPASs implies conjecture 2.1 for all abelian varieties. Indeed, fix $g \geq 0$, and let $\epsilon'(g)$ be the constant from conjecture 2.1 for PPASs. Now given any abelian variety A of dimension g, we have that $A^4 \times \hat{A}^4$ is principally polarised (Zarhin's trick) and has dimension $8g$. As such, the rational torsion in $A^4 \times \hat{A}^4$ has order bounded by $\epsilon'(8g)$, so the same holds for A.

Now we assume conjecture 2.2 holds, and we prove conjecture 2.1 for PPASs. Consider the universal map $\mathscr{A}_{g,2} \to \mathscr{A}_{g,1}$ with its tautological section σ_1. Let $f\colon S \to \mathscr{A}_{g,1}$ be a finite flat (hence locally free as $\mathscr{A}_{g,1}$ locally noetherian) morphism from a scheme to $\mathscr{A}_{g,1}$ (which exists by [LMB00, theorem 16.6]).

'Finite' and 'locally free' both satisfy fpqc descent, and so make sense for a representable morphism of stacks. Similarly, the degree at a point in the target makes sense, and is locally constant. If the target is quasi-compact (say over $\mathbb{Z}$) then so is the source, in which case a finite locally free morphism has bounded degree. The morphism f above satisfies these hypotheses, and so abusing notation we write $\deg f$ for a bound on the degree of f. This is perhaps unnecessarily complicated, but it avoids worrying about whether a connected stack admits a finite flat cover by a connected scheme.

Write $A = \mathscr{A}_{g,2} \times_{\mathscr{A}_{g,1}} S$ (an abelian scheme over S), and write $\sigma \in A(S)$ for the base change of σ_1. Let $c := c(\deg f)$ be the constant from lemma 3.3, so for every L/K of degree at most $\deg f$ and every $s \in S(L)$ we have that $\sigma(s)$ has infinite order or order less than c.

Now let B/K a PPAS of dimension g and $p \in B(K)$ a torsion point. We claim that p has order less than c. The pair (B, p) is a point in $\mathscr{A}_{g,1}(K)$, and the universal property yields a pull-back

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathscr{A}_{g,2} \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \mathscr{A}_{g,1} \end{tikzcd}$$

such that σ_1 pulls back to p. There exists an extension L/K of degree at most $\deg f$ and an L-point $b \in S(L)$ such that $f(b) = (B, p) \in \mathscr{A}_{g,1}(K)$. We obtain by base-change an abelian variety $A_b/\operatorname{Spec} L$ and a section $\sigma_b \in A_b(\operatorname{Spec} L)$, such that the following diagram is a pull-back:

$$\begin{tikzcd} A_b \arrow[r] \arrow[d] & B \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

such that σ_b arises by base-change of p. Now the section $\sigma_b \in A_b(\operatorname{Spec} L)$ is torsion (because p is torsion), and so has order less than c (by the lemma). Hence p has order less than c and we are done. $\square$

4 Proof of conjecture 2.2 when a Néron model exists

The main result of this section can be more succinctly expressed as 'conjecture 2.2 holds if a Néron model of the base exists after suitable compactification and alteration of S'. A more precise statement and proof follow.

First, some notation: if $K = \mathbb{Q}$ then set $\Lambda = \operatorname{Spec}\mathbb{Z}$, and if $K = \mathbb{F}_p(T)$ then set $\Lambda = \mathbb{P}^1_{\mathbb{F}_p}$.

Lemma 4.1. *Let A/S be a PPAS, and $\sigma \in A(S)$ a section of infinite order. Suppose we have*

1. *a proper flat morphism $\mathcal{S} \to \Lambda$;*
2. *a dense open subscheme $U \subseteq \mathcal{S}$;*
3. *an alteration $f\colon U \times_\Lambda K \to S$;*
4. *an abelian scheme $\mathcal{A}/U$;*
5. *a section $\bar{\sigma} \in \mathcal{A}(U)$;*
6. *an isomorphism $\mathcal{A} \times_\Lambda K \to f^*A$ such that $f^*\sigma = \bar{\sigma}$;*
7. *a (finite type) Néron model $\mathscr{A}/\mathcal{S}$ for $\mathcal{A}$ with fibrewise-connected-component $\mathscr{A}^0$ semiabelian.*

Define

$$\mathbf{T}(d) = \{p \in S(\bar{K}) | [\kappa(p) : K] \leq d \text{ and } \sigma(p) \text{ is torsion in } A_p(\bar{K})\}.$$

Then $\mathbf{T}(d)$ is not Zariski dense in S.

Remark 4.2. The list of assumptions may seem rather extensive. However, note that assumptions (1) through (6) can always be easily arranged, even in very many non-isomorphic ways. It is condition (7) (the existence of the Néron model) that is very restrictive and, by [Hol14], cannot in general be arranged.

Proof. We begin with some simple reductions. It is clear that the conclusion is not affected by making alterations of S, so we will assume to simplify the notation that the alteration f is an isomorphism. We can freely alter $\mathcal{S}$ without affecting the other assumptions, so we may assume by [dJ96] that $\mathcal{S}$ is regular. We know that some multiple of $\bar{\sigma}$ is contained in the fibrewise-connected-component $\mathscr{A}^0$, so to simplify the notation we will assume that $\bar{\sigma}$ itself extends across $\mathscr{A}^0$, and we will simply write σ for the extension.

Since $\mathcal{A}$ is isomorphic to $\mathcal{A}^\vee$ we see that the latter also admits a finite type Néron model over $\mathcal{S}$, denoted $\mathscr{A}^\vee$ and with fibrewise-connected-component $\mathscr{A}^{\vee,0}$, and the polarisation extends to an isomorphism $\mathscr{A} \to \mathscr{A}^\vee$. Write $\mathcal{P}$ for the Poincare bundle on $\mathcal{A} \times_U \mathcal{A}^\vee$. By [MB85b, Definition II.1.2.7 and Theorem II.3.6] we find that $\mathcal{P}$ extends uniquely as a biextension over $\mathscr{A}^0 \times_{\mathcal{S}} \mathscr{A}^{\vee,0}$. Write $\mathscr{L}$ for the pullback to $\mathscr{A}^0$ of the extended Poincare bundle via the polarisation, as a rigidified line bundle.

At any Archimedean place of K, the restricted rigidified bundle $\mathscr{L}_{\mathcal{A}}$ admits a unique hermitian metric with translation-invariant first-chern class and such that the rigidifcation is an isometry (see [MB85a]). By [Gre89] we find that this metric admits a (unique) continuous extension to $\mathscr{L}$ on the whole of $\mathscr{A}^0$. From now on we will assume that the rigidified bundle $\mathscr{L}$ is equipped with this metric at each Archimedean place.

The pullback of $\mathscr{L}$ along σ gives a hermitian metrised line bundle on $\mathcal{S}$. Write

$$\mathrm{h}_\sigma \colon \mathcal{S}(\bar{K}) \to \mathbb{R}_{\geq 0}$$

for the height on $\bar{K}$-points induced by $\sigma^*\mathscr{L}$. Then we see by [MB85a] that this function coincides with the function

$$\begin{aligned} \hat{\mathrm{h}}_\sigma \colon \mathcal{S}(\bar{K}) &\to \mathbb{R}_{\geq 0} \\ p &\mapsto \hat{\mathrm{h}}_p(\sigma(p)), \end{aligned}$$

where $\hat{\mathrm{h}}_p$ denotes the Néron-Tate height in the fibre A_p with respect to the given principal polarisation. If we can show that some positive tensor power of the line bundle $\sigma^*\mathscr{L}_K$ on $\mathcal{S}_K$ induces a non-constant map from $\mathcal{S}_K$ to some projective space over K, then the non-Zariski-density of the set $\mathbf{T}(d)$ follows from the Northcott property applied to this projective space.

To show that some positive tensor power of $\sigma^*\mathscr{L}_K$ induces a non-constant map, we will use that σ has infinite order. Pick a proper curve $Z \subseteq \mathcal{S}_K$ such that the restriction $\sigma_Z \in \mathcal{A}_Z(Z)$ still has infinite order. Then since the Néron-Tate height is non-degenerate and is given by the line bundle $\sigma^*\mathscr{L}_K|_Z$, we know that $\sigma^*\mathscr{L}_K|_Z$ must have positive degree. Since Z is a curve over K, this means some tensor power is very ample. If we use the same tensor power to define a map from $\mathcal{S}_K$ to some projective space then this map cannot contract the curve Z, in particular it is non-constant. $\square$

5 Conjecture 2.2 in the absence of a Néron model; height jumping

We know from [Hol14] that Néron models do *not* in general exist even after alteration of the base. As such, perhaps lemma 4.1 may be seen as motivation for analysing in depth obstructions to the existence of Néron models.

The proof of lemma 4.1 may be summarised as saying that the existence of a Néron model implies that the function

$$\hat{\mathrm{h}}_\sigma \colon S(\bar{K}) \to \mathbb{R}_{\geq 0}$$

sending a $\bar{K}$-point in S to the height σ in the fibre over it can be expressed as a height on S with respect to a certain metrised line bundle. In particular, this function is a Weil height, and the result then follows from the Northcott property and the fact that σ has infinite order.

If a Néron model does not exist, then this function $\hat{\mathrm{h}}_\sigma$ cannot in general be expressed as the height with respect to some metrised line bundle. In [BHdJ14], we give a new proof of conjecture 2.2 for S of dimension 1 by showing that the function $\hat{\mathrm{h}}_\sigma$ is still a Weil height even when there is no Néron model; we do this by showing that $\hat{\mathrm{h}}_\sigma$ can be written as a difference $\mathrm{h}_{\mathscr{L}} - j$ where $\mathrm{h}_{\mathscr{L}}$ is a height on S with respect to a suitable metrised line bundle, and j is a bounded function.

This function j is closely related to the 'height jump' of asymptotic hodge theory (see [Hai13]). In more general situations, one can still write $\hat{\mathrm{h}}_\sigma = \mathrm{h}_{\mathscr{L}} - j$ with $\mathrm{h}_{\mathscr{L}}$ a height on S with respect to a metrised line bundle and j a 'height jump' (see [BHdJ14] for details in the case of jacobians), but unfortunately this function j is no longer bounded. On the other hand, if j grows slowly enough with $\mathrm{h}_{\mathscr{L}}$ then conjecture 2.2 and hence conjecture 2.1 would follow.

6 Sparse small points and a conjecture of Silverman

We can consider an analogue of conjecture 2.2 where we look not only at torsion points but at all points of small Néron-Tate height:

Question 6.1. Let S/K be a variety and A/S a PPAS. Let $d \geq 1$ be an integer. Let $\sigma \in A(S)$ be a section of infinite order. Given $\epsilon \geq 0$, define

$$\mathbf{T}_\epsilon(d) = \{p \in S(\bar{K}) | [\kappa(p) : K] \leq d \text{ and } \hat{\mathrm{h}}(\sigma(p)) \leq \epsilon\}.$$

Does there exists $\delta = \delta(A/S, d) > 0$ such that for all $\delta > \epsilon \geq 0$ the set $\mathbf{T}_\epsilon(d)$ is not Zariski dense in S?

It is clear that a positive answer to this question would imply conjecture 2.2. If we knew whether or not the converse implication held, this might by useful as a guide to what techniques could be applied in trying to prove conjecture 2.2 - for example, will (good) estimates on the height suffice to prove non-Zariski-density?

To suggest that question 6.1 is not completely unreasonable, we show that a conjecture of Lang would imply a positive answer question 6.1 for elliptic curves and $d = 1$. Moreover, this conjecture of Lang is in fact a theorem (see [HS88]) over global function fields, yielding an unconditional positive answer to question 6.1 for elliptic curves and $d = 1$ over global function fields.

We begin by recalling Lang's conjecture:

Conjecture 6.2. (Lang, [Lan78] or [HS00, conjecture F.3.4 (a)]). Fix a global field k. There exists a constant $c = c(k) > 0$ such that for all elliptic curves A/k and all non-torsion points $a \in A(k)$, we have

$$\hat{\mathrm{h}}(a) \geq c \cdot \log N_{k/K} \Delta_{E/k}.$$

Here $\Delta_{E/k}$ is the discriminant, and $N_{k/K}$ denotes the norm down to either the rationals or $\mathbb{F}_q(T)$ for a suitable q.

Lemma 6.3. *If* $\operatorname{char} K = 0$ *then assume conjecture 6.2 holds. Question 6.1 has a positive answer assuming that* A/S *is a family of elliptic curves and that* $d = 1$.

Proof. Let Σ denote the finite set of elliptic curves over K with everywhere good reduction, and let $b > 0$ denote the smallest height of a non-torsion K-point appearing on any curve in Σ, or set $b = 1$ if no such exists. Let c be the constant from Lang's conjecture, and let m denote the infimum of the values taken by the expression $c \cdot \log \Delta_{E/K}$ as E runs over all elliptic curves over K with at least one place of bad reduction; this infimum is achieved (since there are only finitely many curves of bounded discriminant) and is positive (by our bad-reduction assumption).

Then setting $\delta = \min(b, m)$ we find for all $\delta > \epsilon \geq 0$ we have

$$\mathbf{T}_\epsilon(1) = \mathbf{T}(1).$$

Now conjecture 2.1 is known for elliptic curves over the rationals by work of Mazur [Maz77], [Maz78], and over global function fields by various authors (see eg. [Poo07]), so we know that conjecture 2.2 holds in our situation, hence $\mathbf{T}(1)$ is not Zariski dense in S. $\square$

In general, a positive answer to question 6.1 might be expected to follow from conjecture 2.2 and very good lower bounds on the heights of non-torsion points.

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