Center Projections for Smooth Difference Equations of Mixed Type

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Abstract. We study a class of mixed type difference equations that enjoy a special smoothening property, in the sense that solutions automatically satisfy an associated functional differential equation of mixed type. Using this connection, a finite dimensional center manifold is constructed that captures all solutions that remain sufficiently close to an equilibrium. The results enable a rigorous analysis of a recently developed model in economic theory, that exhibits periodic oscillations in the interest rates of a simple economy of overlapping generations.

Keywords. mixed type difference equations, life-cycle model, center manifold, Hopf bifurcation, finite dimensional reduction, advanced and retarded arguments.

1 Introduction

The purpose of this paper is to develop a center manifold framework that will enable us to analyze the behaviour of near-equilibrium solutions to a class of nonlinear difference equations of mixed type,

$$F(x_\xi) = 0, \tag{1.1}$$

that enjoy a special smoothening property. In particular, we require that any solution to (1.1) automatically satisfies an associated functional differential equation of mixed type (MFDE), which we will denote by

$$\dot{x}(\xi) = G(x_\xi). \tag{1.2}$$

In the above x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \leq 0$ and $r_{\max} \geq 0$, thus the nonlinearities F and G may depend on advanced and retarded arguments simultaneously.

Our main equation (1.1) should be seen as an infinite dimensional version of a differential-algebraic equation (DAE), i.e., an equation of the form $f(y(\xi), \dot{y}(\xi), \xi) = 0$ that yields an ODE after a finite number of differentiations. Such equations have been studied extensively during the last two decades [1, 4, 5], primarily because they have arisen in many scientific disciplines, including chemical engineering [2, 16], mechanics [12, 22, 23], fluid dynamics [30] and electrical circuit theory [3, 24, 26, 27]. We specially emphasize the applications in the latter area, since the incorporation of time delays into the governing model equations turns out to be an important step towards understanding the dynamical behaviour of many circuits [20]. Inclusion of such delayed arguments in a DAE may lead to equations of the form (1.1).

However, at present our primary motivation for the study of (1.1) comes from the area of economic research, where recent developments have led to models involving such equations. In particular, we mention the work of d'Albis and Véron [8, 6, 7], who have developed several models describing the dynamical features of an economy featuring only a single commodity, that exhibit oscillations which earlier models could only produce by including multiple commodities. This is accomplished by modelling the population as a continuum of individuals that each live for a finite time and act in such a way that their personal welfare is maximized. Such an approach leads to equations of the form (1.1) and (1.2) in a natural fashion. In particular, in [7] the long term behaviour of the capital market is described in terms of (1.1), for an economy in which every participant works during their entire lifetime. The associated MFDE was discussed further in [14] and analyzed numerically in [13].

In this paper we turn our attention to a similar model, which was developed recently and incorporates the effects of retirement on the economy [6]. The chief goal is to study the dynamical behaviour of the interest rate and to rigorously establish the existence of periodic cycles for this rate. This will be accomplished by constructing a smooth local invariant manifold for (1.1) that captures all solutions that remain sufficiently close to an equilibrium and subsequently invoking the Hopf bifurcation theorem. We remark here that from an economic point of view, periodic cycles are in general considered to be very interesting, since they can be readily observed in actual markets. Traditionally, the Hopf theorem has been widely used to establish the existence of such cycles for economic models involving ODEs. The results developed here allow for the statement of such a theorem in the infinite dimensional setting of (1.1), merely in terms of an explicit finite dimensional characteristic function associated to this equation.

We recall that recently a center manifold was constructed for the MFDE (1.2) [14], based upon earlier work by Mallet-Paret [21], Diekmann et al. [9] and Vanderbauwhede et al. [28]. Writing $\overline{x}$ for any equilibrium $G(\overline{x}) = 0$, this construction allows us to relate the dynamics of any sufficiently small solution to the equation

$$\dot{u}(\xi) = DG(\overline{x})u_\xi + (G(\overline{x} + u_\xi) - DG(\overline{x})u_\xi), \tag{1.3}$$

to orbits of a differential equation on a finite dimensional space $\mathcal{N}_G$. This space $\mathcal{N}_G$ contains all the solutions of the linearized equation $\dot{u}(\xi) = DG(\overline{x})u_\xi$ that can be bounded by a polynomial. However, if one attempts to analyze the difference equation (1.1) by using the center manifold construction on the associated MFDE, difficulties arise due to the fact that is is unclear how to lift solutions of (1.2) back to solutions of (1.1). In addition, the structure of the space $\mathcal{N}_G$ will in general differ from $\mathcal{N}_F$, the space of polynomially bounded solutions to $0 = DF(\overline{x})u_\xi$. This implies that extra dynamical behaviour may be observed on the center manifold of (1.2) that is not observed in (1.1). For example, in Section 3 the parameter dependent characteristic equation associated to the MFDE (1.2) admits a double root at $z = 0$, which is absent when studying (1.1) directly. The presence of this ubiquitous double root is troublesome as it adds a resonance to eigenvalues that cross through the imaginary axis as the parameters are varied. The analysis of (1.2) would hence involve studying complicated zero-Hopf bifurcations [10, 11, 17, 18, 19], a situation one would prefer to avoid.

These issues can be resolved by constructing a center manifold directly for (1.1). We will show that the extra smoothness properties provided by (1.2) enable this reduction to be performed and in addition allow us to describe the dynamics on this center manifold by a differential equation, which of course will be related to the nonlinearity G in (1.2). This procedure is performed systematically in Sections 5 and further, mainly in the spirit of [14]. Our main results are formulated in Section 2 and in Section 3 we apply these results to the economic model discussed above.

2 Main Results

Consider the following difference equation of mixed type,

$$0 = Lx_\xi + R(x_\xi), \qquad \xi \in \mathbb{R}, \tag{2.1}$$

in which x is a continuous mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$, the operators L and R are a linear respectively nonlinear map from the state space $X = C([-1, 1], \mathbb{C}^n)$ into $\mathbb{C}^n$ and the state $x_\xi \in X$ is defined by $x_\xi(\theta) = x(\xi + \theta)$ for any $-1 \leq \theta \leq 1$. Notice that in terms of the terminology of (1.1), this means that we have fixed $r_{\min} = -1$ and $r_{\max} = 1$. As a consequence of the Riesz representation theorem, there exists a unique $\mathbb{C}^{n\times n}$-valued normalized function of bounded variation $\mu \in \mathrm{NBV}([-1,1],\mathbb{C}^{n\times n})$, such that for all $\phi \in X$ we have the identity

$$L\phi = \int_{-1}^{1} d\mu(\sigma)\phi(\sigma). \tag{2.2}$$

We recall here that the normalization of μ implies that μ is right-continuous on $(-1,1)$ and satisfies $\mu(-1) = 0$. Throughout this section, the reader may wish to keep in mind the following typical example equation,

$$x(\xi) = \int_{-1}^{1} x(\xi+\sigma)d\sigma + \big(\int_{-1}^{1} x(\xi+\sigma)d\sigma\big)^2. \tag{2.3}$$

As in [14], we will be particularly interested in the following families of Banach spaces during our analysis of (2.1),

$$\begin{array}{lcl} BC_\eta(\mathbb{R},\mathbb{C}^n) & = & \Big\{x \in C(\mathbb{R},\mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}\,|x(\xi)| < \infty\Big\}, \\ BC^1_\eta(\mathbb{R},\mathbb{C}^n) & = & \big\{x \in BC_\eta(\mathbb{R},\mathbb{C}^n) \cap C^1(\mathbb{R},\mathbb{C}^n) \mid \dot{x} \in BC_\eta(\mathbb{R},\mathbb{C}^n)\big\}, \end{array} \tag{2.4}$$

parametrized by $\eta \in \mathbb{R}$, with the standard norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \|\dot{x}\|_\eta$. Notice that for any pair $\eta_2 \geq \eta_1$, there exist continuous inclusions $\mathcal{J}_{\eta_2\eta_1} : BC_{\eta_1}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R},\mathbb{C}^n)$ and $\mathcal{J}^1_{\eta_2\eta_1} : BC^1_{\eta_1}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R},\mathbb{C}^n)$.

In order to construct a center manifold for (2.1), it is essential to consider the homogeneous linear equation

$$0 = Lx_\xi. \tag{2.5}$$

Associated to this system (2.5) one has the characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n\times n}$, given by

$$\Delta(z) = -\int_{-1}^{1} d\mu(\sigma)e^{z\sigma}. \tag{2.6}$$

The minus sign is included here to ensure notational consistency with the characteristic matrix for MFDEs. A value of z such that $\det \Delta(z) = 0$ is called an eigenvalue for the system (2.5). In order to state our main results, we need to impose the following condition on the operator L and the corresponding characteristic matrix Δ.

(HL) There exists a linear operator $M : X \to \mathbb{C}^n$, an integer $\ell > 0$ and constants $\alpha_M, \beta_M \in \mathbb{C}$ with $\beta_M \neq 0$ such that

$$\Delta(z) = \beta_M^{-1}(z-\alpha_M)^{-\ell}\Delta_M(z), \tag{2.7}$$

where $\Delta_M(z)$ is the characteristic matrix corresponding to the homogeneous linear functional differential equation of mixed type $\dot{x}(\xi) = Mx_\xi$.

This condition is related to the fact that we need any solution of the difference equation (2.1) to additionally satisfy a differential equation of mixed type. The operator M should be seen as the linear part of this latter MFDE. For the example equation (2.3) one may conclude that (HL) holds with $\alpha_M = 0$, $\beta_M = 1$ and $\ell = 1$, by computing

$$\Delta(z) = 1 - \frac{1}{z}(e^z - e^{-z}) = \frac{1}{z}(z - e^z + e^{-z}) = \Delta_M(z)/z, \tag{2.8}$$

in which $M\phi = \phi(1) - \phi(-1)$. It is easy to see that this choice for M indeed yields $\dot{x}(\xi) = Mx_\xi$ whenever $Lx_\xi = 0$.

Alternatively, the condition (HL) can be verified directly in terms of the measure $d\mu$ associated to L via (2.2). In particular, we will show in Section 4 that (HL) is equivalent to the following condition, which roughly states that the first non-smooth derivative of μ may only have a jump at zero.

(HL$'$) There exists an integer $\ell > 0$ such that $\mu \in W^{\ell-1,1}_{\text{loc}}([-1,1], \mathbb{C}^{n\times n})$. In addition, there exist a constant $\kappa \neq 0$ and a normalized function of bounded variation $\zeta \in \text{NBV}([-1,1], \mathbb{C}^{n\times n})$, such that

$$D^{\ell-1}\mu(\sigma) = \kappa I H(\sigma) + \textstyle\int_{-1}^{\sigma} \zeta(\tau) d\tau, \qquad -1 \leq \sigma \leq 1, \tag{2.9}$$

in which H denotes the Heaviside function. Finally, for all $1 \leq s \leq \ell - 1$, we have the identity $D^s\mu(\pm 1) = 0$.

Note that when $\ell \geq 2$ in (HL$'$), it follows directly from (2.9) that $\mu \in C^{\ell-2}([-1,1], \mathbb{C}^{n\times n})$, which ensures that the last condition involving $D^s\mu(\pm 1)$ is well-defined.

The following proposition, which will be proved in Section 5, exhibits the finite dimensional space X_0 on which the center manifold will be defined.

Proposition 2.1. *For any homogeneous linear equation (2.5) that satisfies the condition (HL), there exists a finite dimensional linear subspace $X_0 \subset X$ with the following properties.*

(i) Suppose $x \in \bigcap_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (2.5). Then for any $\xi \in \mathbb{R}$ we have $x_\xi \in X_0$.

(ii) For any $\phi \in X_0$, we have $D\phi \in X_0$.

(iii) For any $\phi \in X_0$, there is a solution $x \in \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ of (2.5) that has $x_0 = \phi$. This solution is unique in the set $\bigcup_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ and will be denoted by $E\phi$.

We write Q_0 for the projection operator from X onto X_0, which will be defined precisely in the sequel. Before stating our main result, we introduce two conditions on the nonlinearity $R : X \to \mathbb{C}^n$, which again are related to the MFDE that any solution of (2.1) satisfies.

(HR1) For any $x \in C(\mathbb{R}, \mathbb{C}^n)$, the function $f : \xi \mapsto R(x_\xi)$ satisfies $f \in C^\ell(\mathbb{R}, \mathbb{C}^n)$, where ℓ is as introduced in (HL). In addition, there exist operators $R^{(s)} : X \to \mathbb{C}^n$ for $0 \leq s \leq \ell$, with $R^{(0)} = R$, such that

$$D^s f(\xi) = R^{(s)}(x_\xi), \qquad \text{for } 0 \leq s \leq \ell. \tag{2.10}$$

(HR2) The functions $R^{(s)}$ are C^k-smooth for some integer $k \geq 1$ and all $0 \leq s \leq \ell$. In addition, we have $R^{(s)}(0) = DR^{(s)}(0) = 0$ for all $0 \leq s \leq \ell$.

Theorem 2.2. *Consider the nonlinear equation (2.1) and assume that (HL), (HR1) and (HR2) are satisfied. Then there exists $\gamma > 0$ such that the characteristic equation $\det \Delta_M(z) = 0$ has no roots with $0 < |\mathrm{Re}\, z| < \gamma$. Fix an interval $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ such that $\eta_{\max} > k\eta_{\min}$, with k as introduced in (HR2). Then there exists a mapping $u^* : X_0 \to \bigcap_{\eta > 0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, together with constants $\epsilon > 0$ and $\epsilon^* > 0$, such that the following statements hold.*

(i) For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, the function u^* viewed as a map from X_0 into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is C^k-smooth.

(ii) Suppose for some $\zeta > 0$ that $x \in BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (2.1) with $\sup_{\xi \in \mathbb{R}} |x(\xi)| < \epsilon^*$. Then we have $x = u^*(Q_0 x_0)$. In addition, the function $\Phi : \mathbb{R} \to X_0$ defined by $\Phi(\xi) = Q_0 x_\xi \in X_0$ is of class C^{k+1} and satisfies the ordinary differential equation

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi)), \tag{2.11}$$

in which $A : X_0 \to X_0$ is the linear operator $\phi \mapsto D\phi$ for $\phi \in X_0$. The function $f : X_0 \to X_0$ is C^k-smooth with $f(0) = 0$ and $Df(0) = 0$ and is explicitly given by

$$f(\psi) = Q_0 \chi_\psi, \tag{2.12}$$

in which the state $\chi_\psi \in X$ is given by

$$\chi_\psi(\sigma) = M(u^*(\psi))_\sigma - M(E\psi)_\sigma + \beta_M(\overline{D} - \alpha_M)^\ell R\big((u^*(\psi))_\sigma\big), \tag{2.13}$$

for $\sigma \in [-1, 1]$. Here the expression $\overline{D}^s R(\cdot)$ should be interpreted as $R^{(s)}(\cdot)$ for $0 \leq s \leq \ell$; see also the remark at the end of this section. Finally, we have $x_\xi = \big(u^*(\Phi(\xi))\big)_0$ for all $\xi \in \mathbb{R}$.

(iii) For any $\phi \in X_0$ such that $\sup_{\xi \in \mathbb{R}} |u^*(\phi)(\xi)| < \epsilon^*$, the function $u^*(\phi)$ satisfies (2.1).

(iv) For any continuous function $\Phi : \mathbb{R} \to X_0$ that satisfies (2.11) with $\|\Phi(\xi)\| < \epsilon$ for all $\xi \in \mathbb{R}$, we have that $x = u^*(\Phi(0))$ is a solution of (2.1). In addition, we have $x_\xi = \big(u^*(\Phi(\xi))\big)_0$ for any $\xi \in \mathbb{R}$.

We conclude this section by noting that (2.13) indeed makes sense, since both $u^*(\psi)$ and $E\psi$ are continuous functions on the line, which ensures that the states $(u^*(\psi))_\sigma$ and $(E\psi)_\sigma$ belong to X and depend continuously on $\sigma \in [-1, 1]$. This allows the operators M and $R^{(s)}$ to be applied, yielding a continuous $\mathbb{C}^n$-valued function χ_ψ on $[-1, 1]$, as required.

3 Monetary Cycles with Endogenous Retirement

In this section we illustrate the application range of our results by discussing the overlapping generations economic model developed in [6]. The authors consider a fixed size population of individuals that live for a time $\omega > 1$. The amount of assets that an individual born at time s owns at time t is given by $a(s,t)$, while his income at this time is given by the quantity $e(s,t)$. This quantity satisfies $e(s,t) = 1$ for $t \in [s, s+1]$ and $e(s,t) = 0$ otherwise, i.e., every labourer retires at unit age. Everybody receives interest at the rate $r(t)$ on their assets while consuming $c(s,t)$, which yields the budget constraint

$$\frac{\partial a(s,t)}{\partial t} = r(t)a(s,t) + e(s,t) - c(s,t). \tag{3.1}$$

The utility $u(s)$ as perceived by the generation born at time s is given by

$$u(s) = \int_s^{s+\omega} \frac{c(s,t)^{1-\sigma^{-1}}}{1-\sigma^{-1}} dt \tag{3.2}$$

and everybody acts in such a way that his utility is maximized, subject to both (3.1) and the natural budget constraints $a(s,s) = 0$ and $a(s, s+\omega) \geq 0$. In (3.2) the parameter σ stands for the elasticity of intertemporal substitution and is required to satisfy $\sigma > 0$. The economy features a single, nonstorable consumption good, which we will assume to be produced at exactly the rate required by the consumer market. In terms of our model variables, this means that for all time t the following identity must hold,

$$\int_{t-\omega}^{t} e(\sigma,t)d\sigma = \int_{t-\omega}^{t} c(\sigma,t)d\sigma. \tag{3.3}$$

The rules above are sufficient to fix the dynamical behaviour of the economy and following [6], one easily derives the difference equation $\mathcal{F}(r_t) = 0$ for the interest rate $r(t)$, with $\mathcal{F}$ given by

$$\mathcal{F}(r_t) = 1 - \int_{t-\omega}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+\omega} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds. \tag{3.4}$$

Notice that $r = 0$ is an equilibrium solution of (3.4). The linearization around this equilibrium is given by

$$0 = -\frac{1}{\omega}\int_{t-\omega}^{t}\int_s^{s+1}\int_t^v x(u)du\,dv\,ds + \frac{1-\sigma}{\omega^2}\int_{t-\omega}^{t}\int_s^{s+\omega}\int_t^v x(u)du\,dv\,ds. \tag{3.5}$$

Inserting $x(u) = \exp(zu)$ yields the characteristic function

$$\Delta(z,\sigma,\omega) = -\frac{1}{\omega^2 z^3}[-\omega e^z + (1-\sigma)e^{z\omega} + (\omega e^z - \omega + 1 - \sigma)e^{-z\omega} + (\omega - 2 + 2\sigma) + \sigma\omega^2 z^2]. \tag{3.6}$$

The following result, which was partially proven in [6], shows that the characteristic equation $\Delta(z) = 0$ admits simple roots on the imaginary axis that satisfy the conditions associated with the Hopf bifurcation theorem. The proof is deferred to the end of this section.

Proposition 3.1. *Consider any $\omega > 1$ such that $(\omega-1)^{-1} \notin \mathbb{N}$. There exists an infinite sequence of pairs (σ_k, q_k) parametrized by $k \in \mathbb{N}$, with $\sigma_k > 0$ and $q_k > 0$, such that the following properties are satisfied.*

(i) One has the limits $\sigma_k \to 0$ and $q_k \to \infty$ as $k \to \infty$.

(ii) *The characteristic equation* $\Delta(z,\sigma_k,\omega) = 0$ *has two simple roots at* $z = \pm i q_k$.

(iii) *For all* $k \in \mathbb{N}$ *and* $m \in \mathbb{Z} \setminus \{\pm 1\}$*, the inequality* $\Delta(imq_k, \sigma_k, \omega) \neq 0$ *holds.*

(iv) *For every* $k \in \mathbb{N}$*, the branch of roots* $z(\sigma)$ *of the characteristic equation* $\Delta(z,\sigma,\omega) = 0$ *through* $z = iq_k$ *at* $\sigma = \sigma_k$ *crosses the imaginary axis with positive speed, i.e.,*

$$\operatorname{Re} \frac{D_2 \Delta(iq_k, \sigma_k, \omega)}{D_1 \Delta(iq_k, \sigma_k, \omega)} \neq 0. \tag{3.7}$$

Conversely, if $\omega = 1 + n^{-1}$ *for some* $n \in \mathbb{N}$*, then for all* $\sigma > 0$ *the characteristic equation* $\Delta(z,\sigma,\omega) = 0$ *admits no roots with* $\operatorname{Re} z = 0$.

Fixing any suitable $\omega > 1$ and treating σ as a bifurcation parameter, the result above will allow us to conclude that the algebraic equation (3.4) admits a branch of periodic solutions bifurcating from the equilibrium $r = 0$ at $\sigma = \sigma_k$, for all $k \in \mathbb{N}$. Similarly, any sufficiently small solution of (3.4) with σ near σ_k can be captured on such a branch. To validate this claim using our main theorem in Section 2, we note that twofold differentiation of (3.4) and simplification using the identity (3.4), yields the following mixed type functional differential equation,

$$\begin{array}{rcl} \dot{r}(t) & = & \mathcal{G}(r_t), \quad \text{with} \\ -\sigma \mathcal{G}(r_t) & = & -\sigma^2 r(t)^2 + [\int_t^{t+\omega} e_\sigma(v) dv]^{-1} [e(t+1) - 1] \\ & & -[\int_t^{t+\omega} e_\sigma(v) dv]^{-2} [e_\sigma(t+\omega) - 1] \int_t^{t+1} e(v) dv \\ & & +[\int_{t-\omega}^{t} e_\sigma(v) dv]^{-2} [1 - e_\sigma(t-\omega)] \int_{t-\omega}^{t-\omega+1} e(v) dv \\ & & -[\int_{t-\omega}^{t} e_\sigma(v) dv]^{-1} [e(t-\omega+1) - e(t-\omega)], \end{array} \tag{3.8}$$

in which we have made the abbreviations

$$\begin{array}{rcl} e(w) & = & \exp\big(-\int_t^w r(u) du\big), \\ e_\sigma(w) & = & \exp\big(-(1-\sigma)\int_t^w r(u) du\big). \end{array} \tag{3.9}$$

Linearizing (3.8) around $r = 0$ yields

$$-\sigma \dot{x}(t) = -\frac{1}{\omega} \int_t^{t+1} x(u) du + \frac{1}{\omega} \int_{t-\omega}^{t-\omega+1} x(u) du + \frac{1-\sigma}{\omega^2} \int_t^{t+\omega} x(u) du + \frac{1-\sigma}{\omega^2} \int_t^{t-\omega} x(u) du. \tag{3.10}$$

Inserting $x(u) = \exp(zu)$ and normalizing, we find the characteristic function

$$\begin{array}{rcl} \Delta_M(z,\sigma,\omega) & = & \frac{1}{\sigma\omega^2 z}[\sigma\omega^2 z^2 - \omega(e^z - 1) + \omega(e^z - 1)e^{-\omega z} + (1-\sigma)(e^{\omega z} + e^{-\omega z} - 2)] \\ & = & -\frac{z^2}{\sigma}\Delta(z,\sigma,\omega), \end{array} \tag{3.11}$$

which immediately implies that (HL) is satisfied. Using the expressions above, all the other conditions of Theorem 2.2 can easily be verified as well. Hence upon fixing an appropriate $\omega > 1$ and considering any pair (σ_0, q_0) generated by Proposition 3.1, we can establish the existence of a $2+1$ dimensional center manifold $u^* : X_0 \times \mathbb{R} \to \bigcap_{\eta > 0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ directly for the difference equation (3.4). Here $X_0 = \operatorname{span}(e^{iq_0 \cdot}, e^{-iq_0 \cdot})$ and the extra dimension arises by including the bifurcation parameter $\widetilde{\sigma} =$

$\sigma - \sigma_0$ in the center space. The dynamical behaviour of $\widetilde{\sigma}$ on the center manifold is trivial, while the evolution of $\psi(\xi) = x(\xi)e^{iq_0\cdot} + y(\xi)e^{-iq_0\cdot} \subset X_0$ is governed by the ODE

$$\begin{array}{rcl} \dot{x} & = & iq_0 x + f_1(x, y, \widetilde{\sigma}) + O\big((|x| + |y|)^3 + |\widetilde{\sigma}|\,(|x| + |y|)(|\widetilde{\sigma}| + |x| + |y|)\big), \\ \dot{y} & = & -iq_0 y + f_2(x, y, \widetilde{\sigma}) + O\big((|x| + |y|)^3 + |\widetilde{\sigma}|\,(|x| + |y|)(|\widetilde{\sigma}| + |x| + |y|)\big), \end{array} \tag{3.12}$$

in which the second order terms are given by

$$\begin{array}{rcl} f_1(x, y, \widetilde{\sigma}) & = & -D_2\Delta(iq_0, \sigma_0)D_1\Delta(iq_0, \sigma_0)^{-1}\widetilde{\sigma}x \\ & & +\frac{1}{2}D_1\Delta(iq_0, \sigma_0)^{-1}\big(\alpha_{xx}x^2 + 2\alpha_{xy}xy + \alpha_{yy}y^2\big) \\ & & +\frac{1}{2iq_0}D_1\Delta(iq_0, \sigma_0)^{-1}\Big(\Delta(2iq_0, \sigma_0)(\beta_{xx} - 4\frac{q_0^2}{\sigma_0}\alpha_{xx})x^2 - 2\Delta(0, \sigma_0)\beta_{xy}xy \\ & & \quad -\frac{1}{3}\Delta(-2iq_0, \sigma_0)(\beta_{yy} - 4\frac{q_0^2}{\sigma_0}\alpha_{yy})y^2\Big), \\ f_2(x, y, \widetilde{\sigma}) & = & -D_2\Delta(-iq_0, \sigma_0)D_1\Delta(-iq_0, \sigma_0)^{-1}\widetilde{\sigma}y \\ & & +\frac{1}{2}D_1\Delta(-iq_0, \sigma_0)^{-1}\big(\alpha_{xx}x^2 + 2\alpha_{xy}xy + \alpha_{yy}y^2\big) \\ & & +\frac{1}{2iq_0}D_1\Delta(-iq_0, \sigma_0)^{-1}\Big(\frac{1}{3}\Delta(2iq_0, \sigma_0)(\beta_{xx} - 4\frac{q_0^2}{\sigma_0}\alpha_{xx})x^2 + 2\Delta(0, \sigma_0)\beta_{xy}xy \\ & & \quad -\Delta(-2iq_0, \sigma_0)(\beta_{yy} - 4\frac{q_0^2}{\sigma_0}\alpha_{yy})y^2\Big) \end{array} \tag{3.13}$$

and the quantities α_{xx} through β_{yy} can be calculated by using

$$\begin{array}{rclcrcl} \alpha_{xx} & = & D_1^2\mathcal{F}(0, \sigma_0)(e^{iq_0\cdot}, e^{iq_0\cdot}), & & \beta_{xx} & = & D_1^2\mathcal{G}(0, \sigma_0)(e^{iq_0\cdot}, e^{iq_0\cdot}), \\ \alpha_{xy} & = & D_1^2\mathcal{F}(0, \sigma_0)(e^{iq_0\cdot}, e^{-iq_0\cdot}), & & \beta_{xy} & = & D_1^2\mathcal{G}(0, \sigma_0)(e^{iq_0\cdot}, e^{-iq_0\cdot}), \\ \alpha_{yy} & = & D_1^2\mathcal{F}(0, \sigma_0)(e^{-iq_0\cdot}, e^{-iq_0\cdot}), & & \beta_{yy} & = & D_1^2\mathcal{G}(0, \sigma_0)(e^{-iq_0\cdot}, e^{-iq_0\cdot}). \end{array} \tag{3.14}$$

Using the transversality condition (iv) from Proposition 3.1, it is easily seen that the ODE (3.12) undergoes a Hopf bifurcation at $\sigma = \sigma_0$. This yields a branch of periodic orbits that can be lifted back to periodic solutions of our initial problem (3.4), which establishes our claim above.

Notice that $\Delta_M(z) = 0$ has a double root at $z = 0$ for all valid parameters ω and σ, which arises as an artifact of the differentiation operations needed to derive (3.8). This double root prevents the application of the Hopf bifurcation result developed in [14] to the MFDE (3.8). To give a detailed analysis of the local behaviour of this equation, one would hence have to revert to a complicated normal form reduction. Theorem 2.2 has allowed us to circumvent this difficulty by analyzing (3.4) directly.

We conclude this section by referring the reader to [13], where the branches of periodic solutions to (3.4) constructed above are actually uncovered numerically.

Proof of Proposition 3.1. For convenience, we write $\widetilde{\Delta}(z, \sigma, \omega) = -\omega^2 z^3 \Delta(z, \sigma, \omega)$. First note that $\widetilde{\Delta}(z, \sigma, \omega) = \frac{1}{2}\omega^2(\omega - 1)z^3 + O(z^4)$ around $z = 0$, which implies that $z = 0$ is not a root of $\Delta(z, \sigma_0, \omega)$ for $\omega > 1$. For any $q \in \mathbb{R}$, we write $I(q) = \operatorname{Im}\widetilde{\Delta}(iq, \sigma, \omega)$ and compute

$$I(q) = \omega(\sin\omega q + \sin(1 - \omega)q - \sin q) = 4\omega \sin\frac{\omega q}{2}\sin\frac{(1 - \omega)q}{2}\sin\frac{q}{2}. \tag{3.15}$$

Similarly, writing $R(q) = \operatorname{Re}\widetilde{\Delta}(iq, \sigma, \omega)$, we compute

$$\begin{array}{rcl} R(q) & = & -\omega\cos q + 2(1 - \sigma)(\cos\omega q - 1) + \omega(1 - \cos\omega q) + \omega\cos(1 - \omega)q - \sigma\omega^2 q^2 \\ & = & 2\omega(\sin^2\frac{q}{2} + \sin^2\frac{\omega q}{2} - \sin^2\frac{(1-\omega)q}{2}) - 4\sin^2\frac{\omega q}{2} + \sigma(4\sin^2\frac{\omega q}{2} - \omega^2 q^2). \end{array} \tag{3.16}$$

Notice that for any $l \in \mathbb{N}$ and $q^{(l)} = \frac{2l\pi}{\omega-1}$, we have $\sin\frac{(1-\omega)q^{(l)}}{2} = 0$, while $\sin^2\frac{q^{(l)}}{2} = \sin^2\frac{\omega q^{(l)}}{2}$. This implies that

$$R(q^{(l)}) = 4(\omega-1)\sin^2\frac{q^{(l)}}{2} + \sigma(4\sin^2\frac{q^{(l)}}{2} - (\omega q^{(l)})^2). \tag{3.17}$$

Now assume that $(\omega-1)^{-1} \notin \mathbb{N}$, which implies that $\frac{\pi}{1-\omega} \not\equiv 0 \mod \pi$. There hence exists a strictly increasing sequence of integers $l_k > 0$, parametrized by $k \in \mathbb{N}$, such that $s_k = \sin^2\frac{q^{(l_k)}}{2} > \frac{1}{4}$. Choose $q_k = q^{(l_k)}$ and write

$$\sigma_k = \frac{\omega-1}{\omega^2\frac{q_k^2}{4s_k} - 1} > 0, \tag{3.18}$$

where the last inequality follows from $\omega > 1$ and the fact that $|\sin\theta| < |\theta|$ for $\theta \neq 0$. By construction, we have $\Delta(iq_k, \sigma_k, \omega) = 0$. Suppose that for any $m \in \mathbb{Z} \setminus \{0, \pm 1\}$ we have $\Delta(imq_k, \sigma_k, \omega) = 0$, then using $mq_k = q^{(ml_k)}$ and setting $R(q^{(ml_k)}) = 0$, we find that $\sin^2\frac{q^{(ml_k)}}{2} = s_k m^2 > \frac{1}{4}m^2 > 1$, which is impossible. To prove the claim (iv) involving the derivatives of Δ, note that

$$D_s\Delta(iq_k, \sigma_k, \omega) = \frac{i}{\omega^2 q_k^3} D_s\widetilde{\Delta}(iq_k, \sigma_k, \omega) \tag{3.19}$$

for $s = 1, 2$. In addition,

$$D_2\widetilde{\Delta}(iq_k, \sigma_k, \omega) = 2(1 - \cos\omega q_k) - \omega^2 q_k^2 = 4s_k - \omega^2 q_k^2 < 0. \tag{3.20}$$

It hence suffices to compute

$$\operatorname{Re} D_1\widetilde{\Delta}(iq_k, \sigma_k, \omega) = -\omega\cos q_k + \omega(1-\omega)\cos(1-\omega)q_k + \omega^2\cos\omega q_k = 2\omega(1-\omega)s_k \neq 0. \tag{3.21}$$

We conclude the proof by assuming that $\omega = 1 + n^{-1}$ for some $n \in \mathbb{N}$. Substituting $q = 2l\pi$ into $R(q) = 0$ forces $\sigma < 0$, while the choice $q = \frac{2l\pi}{\omega}$ implies $\sigma = 0$. □

4 Preliminaries

We recall here the definitions of the Fourier transform $\mathcal{F}^+(f)(k) = \widehat{f}(k)$ of a function $f \in L^2(\mathbb{R}, \mathbb{C}^n)$ and the inverse Fourier transform $\mathcal{F}^-(g)(\xi) = \check{g}(\xi)$ for any $g \in L^2(\mathbb{R}, \mathbb{C}^n)$, given by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi) d\xi, \qquad \check{g}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi} g(k) dk. \tag{4.1}$$

We remark here that the integrals above are well-defined only if $f, g \in L^1(\mathbb{R}, \mathbb{C}^n)$. If this is not the case, the integrals have to be understood as integrals in the Fourier sense, i.e., the functions

$$h_n(k) = \int_{-n}^{n} e^{-ik\xi} f(\xi) d\xi \tag{4.2}$$

satisfy $h_n \to \widehat{f}$ in $L^2(\mathbb{R}, \mathbb{C}^n)$ and in addition there is a subsequence $\{n'\}$ such that $h_{n'}(k) \to \widehat{f}(k)$ almost everywhere. We recall that the Fourier transform takes convolutions into products, i.e., $(\widehat{f * g})(k) = \widehat{f}(k)\widehat{g}(k)$ for almost every k.

Now suppose $f : \mathbb{R} \to \mathbb{C}^n$ satisfies $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$. Then for any z with $\operatorname{Re} z > -a$, define the Laplace transform

$$\widetilde{f}_+(z) = \int_0^\infty e^{-z\xi} f(\xi) d\xi. \tag{4.3}$$

Similarly, if $f(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then for any z with $\operatorname{Re} z < b$, define

$$\widetilde{f}_-(z) = \int_0^\infty e^{z\xi} f(-\xi) d\xi. \tag{4.4}$$

The inverse transformation is described in the next result, which can be found in the standard literature on Laplace transforms [29, 7.3-5].

Lemma 4.1. *Let $f : \mathbb{R} \to \mathbb{C}^n$ satisfy a growth condition $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ and suppose that f is of bounded variation on bounded intervals. Then for any $\gamma > -a$ and $\xi > 0$ we have the inversion formula*

$$\frac{f(\xi+) + f(\xi-)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz, \tag{4.5}$$

whereas for $\xi = 0$ we have

$$\frac{f(0+)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz. \tag{4.6}$$

In the remainder of this section we provide some preliminary results regarding the linear operators L and M that appeared in Section 2. We start by showing that (HL) and (HL$'$) are equivalent conditions that automatically provide smoothness properties for functions of the form $\xi \mapsto -Lx_\xi$, which will be encountered frequently in the sequel.

Proposition 4.2. *Recall the linear operator $L : X \to \mathbb{C}^n$ defined by (2.2). The conditions (HL) and (HL$'$) on L with equal values of the integer $\ell > 0$ are equivalent. In addition, when these conditions are satisfied the following properties hold.*

(i) For any $x \in C(\mathbb{R}, \mathbb{C}^n)$, the function f defined by $f(\xi) = -Lx_\xi$ satisfies $f \in C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$.

(ii) There exists a constant $C > 0$ such that for any $x \in C(\mathbb{R}, \mathbb{C}^n)$, we have $|D^s f(\xi)| \le C \|x_\xi\|$ for all $0 \le s \le \ell - 1$, where f is again given by $f(\xi) = -Lx_\xi$.

(iii) If the function $f : \xi \mapsto -Lx_\xi$ associated to any $x \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies $f \in C^\ell(\mathbb{R}, \mathbb{C}^n)$, then we must have $x \in C^1(\mathbb{R}, \mathbb{C}^n)$.

Proof. We first show that (HL$'$) implies (HL) and the properties (i) through (iii) listed in the statement of this result. We proceed by induction on the integer ℓ. Consider therefore an operator $L : X \to \mathbb{C}^n$ with corresponding NBV function μ that satisfies (HL$'$) with $\ell = 1$. Consider any $x \in C(\mathbb{R}, \mathbb{C}^n)$ and let the function $f : \mathbb{R} \to \mathbb{C}^n$ be defined by $f(\xi) = -Lx_\xi$. The identity (2.9) in (HL$'$) now implies that we have $-f(\xi) = \kappa x(\xi) + \int_{\xi-1}^{\xi+1} \zeta(\sigma - \xi) x(\sigma) d\sigma$, from which (i) and (ii) immediately follow. If in fact $f \in C^1(\mathbb{R}, \mathbb{C}^n)$, then differentiation of the above identity yields

$$\kappa D x(\xi) = -Df(\xi) - \zeta(1) x(\xi + 1) + \zeta(-1) x(\xi - 1) + \int_{-1}^{1} d\zeta(\sigma) x(\xi + \sigma), \tag{4.7}$$

showing that $x \in C^1(\mathbb{R}, \mathbb{C}^n)$ and hence establishing (iii). An easy calculation involving integration by parts allows us to establish that also condition (HL) holds, with the corresponding operator $M : X \to \mathbb{C}^n$ given by $M\phi = \kappa^{-1}\big(\zeta(-1)\phi(-1) - \zeta(1)\phi(1) + \int_{-1}^{1} d\zeta(\sigma)\phi(\sigma)\big)$. Indeed, choosing $\alpha_M = 0$ and $\beta_M = -\kappa^{-1}$, we may compute

$$\begin{array}{rcl} -\Delta(z) & = & \int_{-1}^{1} d\mu(\sigma)e^{z\sigma} = \kappa I + \int_{-1}^{1} \zeta(\sigma)e^{z\sigma}d\sigma \\ & = & \kappa I + \frac{1}{z}(\zeta(1)e^{z} - \zeta(-1)e^{-z} - \int_{-1}^{1} d\zeta(\sigma)e^{z\sigma}) \\ & = & \frac{\kappa}{z}(zI - Me^{z\cdot}) = -\beta_M^{-1}z^{-1}\Delta_M(z), \end{array} \tag{4.8}$$

which shows that (2.7) in (HL) is satisfied.

Now let $p > 1$ and consider an operator L with corresponding NBV function μ that satisfies (HL$'$) with $\ell = p$. Observe that (2.9) implies that $D\mu \in L^1_{\mathrm{loc}}([-1,1], \mathbb{C}^{n\times n})$ is also a NBV function. Therefore it induces the operator $L' : X \to \mathbb{C}^n$ given by

$$L'\phi = \int_{-1}^{1} d[D\mu](\sigma)\phi(\sigma) \tag{4.9}$$

and one may easily verify that L' satisfies the condition (HL$'$) with $\ell = p - 1$. In particular, using our induction hypothesis this means that for some operator $M' : X \to \mathbb{C}^n$, condition (HL) with $\ell = p - 1$ is satisfied by L', together with the properties (i) through (iii) listed above.

Now as before, consider an arbitrary $x \in C(\mathbb{R}, \mathbb{C}^n)$ and its corresponding function f given by $f(\xi) = -Lx_\xi$. We may compute

$$\begin{array}{rcl} -Df(\xi) & = & D[\int_{-1}^{1} d\mu(\sigma)x(\xi+\sigma)] = D[\int_{\xi-1}^{\xi+1} D\mu(\sigma-\xi)x(\sigma)d\sigma] \\ & = & D\mu(1)x(\xi+1) - D\mu(-1)x(\xi-1) - \int_{-1}^{1} d[D\mu](\sigma)x(\xi+\sigma) \\ & = & -\int_{-1}^{1} d[D\mu](\sigma)x(\xi+\sigma) = -L'x_\xi, \end{array} \tag{4.10}$$

where the penultimate equality follows from the conditions in (HL$'$) on $D\mu$. Properties (i) through (iii) with $\ell = p$ now follow immediately from the fact that these properties with $\ell = p-1$ are satisfied by L'. To show that L also satisfies the condition (HL), one may compute

$$\begin{array}{rcl} -\Delta(z) & = & \int_{-1}^{1} D\mu(\sigma)e^{z\sigma}d\sigma = \frac{1}{z}\big(D\mu(1)e^{z} - D\mu(-1)e^{-z} - \int_{-1}^{1} d[D\mu](\sigma)e^{z\sigma}\big) \\ & = & -\frac{1}{z}\int_{-1}^{1} d[D\mu](\sigma)e^{z\sigma} = \frac{1}{z}\beta_{M'}^{-1}z^{-p+1}\Delta_{M'}(z). \end{array} \tag{4.11}$$

We now proceed to show that condition (HL) implies (HL$'$). Without loss of generality we will assume $\beta_M = 1$. Using induction on ℓ we will show that if $z \mapsto (z-\alpha)^{-\ell}\Delta_M(z)$ is a holomorphic function, then there exists a NBV function μ that meets the conditions in (HL$'$) and in addition satisfies the identity $-\int_{-1}^{1} d\mu(\sigma)e^{z\sigma} = (z-\alpha)^{-\ell}\Delta_M(z)$. Writing ζ_M for the NBV function associated with M, we introduce the corresponding NBV function $\widetilde{\zeta}$ associated to the operator $\widetilde{M} : X \to \mathbb{C}^n$ given by $\phi \mapsto M\phi - \alpha\phi(0)$. Consider the case $\ell = 1$, write $f(z) = (z-\alpha)^{-1}\Delta_M(z)$ and use repeated integration by parts to compute

$$\begin{array}{rcl} f(z) & = & (z-\alpha)^{-1}(z - \int_{-1}^{1} d\zeta_M(\sigma)e^{z\sigma}) = (z-\alpha)^{-1}(z - \alpha - \int_{-1}^{1} d\widetilde{\zeta}(\sigma)e^{z\sigma}) \\ & = & (z-\alpha)^{-1}e^{z-\alpha}\big(\alpha\int_{-1}^{1} e^{\alpha\sigma}\widetilde{\zeta}(\sigma)d\sigma - \widetilde{\zeta}(1)e^{\alpha}\big) \\ & & +1 - \alpha\int_{-1}^{1} e^{(z-\alpha)\sigma}\int_{-1}^{\sigma} e^{\alpha\tau}\widetilde{\zeta}(\tau)d\tau d\sigma + \int_{-1}^{1} e^{z\sigma}\widetilde{\zeta}(\sigma)d\sigma, \end{array} \tag{4.12}$$

in which we recall the normalization $\widetilde{\zeta}(-1) = 0$. Since f is a holomorphic function, one sees that the following identity must hold,

$$\alpha \int_{-1}^{1} e^{\alpha\sigma} \widetilde{\zeta}(\sigma) d\sigma = \widetilde{\zeta}(1) e^{\alpha}. \tag{4.13}$$

From this it follows that the induction hypothesis is satisfied for the NBV function μ given by

$$\mu(\sigma) = -H(\sigma) - \int_{-1}^{\sigma} \widetilde{\zeta}(\tau) d\tau + \alpha \int_{-1}^{\sigma} e^{-\alpha\tau} \int_{-1}^{\tau} e^{\alpha u} \widetilde{\zeta}(u) du d\tau. \tag{4.14}$$

Now consider an integer $p > 1$ and consider a holomorphic function of the form $f(z) = (z - \alpha)^{-p} \Delta_M(z)$. Assume that our induction hypothesis is satisfied for $\ell = p - 1$, which implies that $(z - \alpha) f(z) = -\int_{-1}^{1} d\nu(\sigma) e^{z\sigma}$ for some NBV function ν that satisfies (HL$'$) at $\ell = p - 1$. We can thus compute

$$\begin{array}{rcl} f(z) & = & -(z-\alpha)^{-1} \int_{-1}^{1} d\nu(\sigma) e^{z\sigma} \\ & = & (z-\alpha)^{-1} e^{z-\alpha} \big(\alpha \int_{-1}^{1} e^{\alpha\sigma} \nu(\sigma) d\sigma - \nu(1) e^{\alpha} \big) \\ & & + \int_{-1}^{1} e^{z\sigma} \nu(\sigma) d\sigma - \alpha \int_{-1}^{1} e^{(z-\alpha)\sigma} \int_{-1}^{\sigma} e^{\alpha\tau} \nu(\tau) d\tau d\sigma. \end{array} \tag{4.15}$$

Again, since f is holomorphic, (4.13) must hold with ν instead of $\widetilde{\zeta}$ and one may readily verify that the induction hypothesis is satisfied at $\ell = p$, for the NBV function

$$\mu(\sigma) = \alpha \int_{-1}^{\sigma} e^{-\alpha\tau} \int_{-1}^{\tau} e^{\alpha u} \nu(u) du d\tau - \int_{-1}^{\sigma} \nu(\tau) d\tau, \tag{4.16}$$

which concludes the proof. $\square$

We now recall the characteristic matrix Δ_M associated to the homogeneous equation $\dot{x}(\xi) = M x_\xi$ that features in condition (HL) and repeat some useful properties of Δ_M that were established in [14].

Lemma 4.3. *Consider any closed vertical strip $S = \{z \in \mathbb{C} \mid \gamma_- \le \operatorname{Re} z \le \gamma_+\}$ and for any $\rho > 0$ define $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. Then there exist $C, \rho > 0$ such that $\det \Delta_M(z) \neq 0$ for all $z \in S_\rho$ and in addition $\left|\Delta_M(z)^{-1}\right| < \frac{C}{|\operatorname{Im} z|}$ for each such z. In particular, there are only finitely many zeroes of $\det \Delta_M(z)$ in S. Furthermore, if $\det \Delta_M(z) \neq 0$ for all $z \in S$, then for any $\alpha \notin S$ the function*

$$R_\alpha(z) = \Delta_M(z)^{-1} - (z - \alpha)^{-1} I \tag{4.17}$$

is holomorphic in an open neighbourhood of S and in addition there exists $C' > 0$ such that $|R_\alpha(z)| \le \frac{C'}{1 + |\operatorname{Im} z|^2}$ for all $z \in S$.

The final result of this section uses Laplace transform techniques to characterize solutions to $-L x_\xi = f$ that have controlled exponential growth at $\pm\infty$.

Proposition 4.4. *Consider the operator L defined by (2.2) and suppose that the condition (HL) is satisfied. Fix constants $\eta_-, \eta_+ \in \mathbb{R}$ and consider any $x \in C(\mathbb{R}, \mathbb{C}^n)$ that satisfies $x(\xi) = O(e^{\eta_\pm \xi})$ as*

$\xi \to \pm\infty$. *Define the function* $f : \xi \mapsto -Lx_\xi$. *Then for any* $\gamma_+ > \eta_+$ *and* $\gamma_- < \eta_-$ *such that the characteristic equation* $\det \Delta_M(z) = 0$ *has no roots with* $\mathrm{Re}\, z = \gamma_\pm$ *and for any* $\xi \in \mathbb{R}$, *we have*

$$\begin{array}{rcl} x(\xi) & = & \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z}\big(K(\xi, z, x) + \Delta(z)^{-1}\widetilde{f}_+(z)\big)dz \\ & & + \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z}\big(K(\xi, z, x) - \Delta(z)^{-1}\widetilde{f}_-(z)\big)dz, \end{array} \tag{4.18}$$

in which $K : \mathbb{R} \times \mathbb{C} \times C(\mathbb{R}, \mathbb{C}^n) \to \mathbb{C}^n$ *is given by*

$$K(\xi, z, x) = \int_\xi^0 e^{-z\tau}x(\tau)d\tau + \Delta(z)^{-1}\int_{-1}^1 d\mu(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau. \tag{4.19}$$

The Laplace transforms $\widetilde{f}_+$ *and* $\widetilde{f}_-$ *are as defined above in (4.3) and (4.4).*

Proof. Note that Proposition 4.2 implies that f shares the growth rate of x at $\pm\infty$. An application of Lemma 4.1 hence shows that

$$\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z}\big(\int_\xi^0 e^{-z\tau}x(\tau)d\tau + \widetilde{x}_+(z)\big)dz. \tag{4.20}$$

Taking the Laplace transform of the identity $-Lx_\xi = f(\xi)$ yields

$$0 = \widetilde{f}_+(z) + \int_{-1}^1 d\mu(\sigma)e^{z\sigma}\big(\widetilde{x}_+(z) + \int_\sigma^0 e^{-z\tau}x(\tau)d\tau\big) \tag{4.21}$$

and thus after rearrangement

$$\widetilde{x}_+(z) = \Delta(z)^{-1}\big(\widetilde{f}_+(z) + \int_{-1}^1 d\mu(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau\big). \tag{4.22}$$

As in [14], a similar argument applied to the function $y(\xi) = x(-\xi)$ completes the proof. □

5 The state space

In this section we study the state space $X = C([-1, 1], \mathbb{C}^n)$ in the spirit of the corresponding treatment for MFDEs employed in [14]. We recall the linear operator $L : X \to \mathbb{C}^n$ defined by (2.2) and define a closed operator $A : \mathcal{D}(A) \subset X \to X$, via

$$\begin{array}{rcl} \mathcal{D}(A) & = & \Big\{\phi \in X \mid \phi \text{ is } C^1\text{-smooth and satisfies } 0 = L\phi = \int_{-1}^1 d\mu(\sigma)\phi(\sigma)\Big\}, \\ A\phi & = & D\phi. \end{array} \tag{5.1}$$

Notice that the domain $\mathcal{D}(A)$ now differs from the corresponding definition in [14] and in addition, A is no longer densely defined. Nevertheless, it is still possible to relate the resolvent of A to the characteristic matrix Δ. We refer to [15] for a general discussion on characteristic matrices for unbounded operators.

Lemma 5.1. *The operator* A *defined in (5.1) has only point spectrum with* $\sigma(A) = \sigma_p(A) = \{z \in \mathbb{C} \mid \det \Delta(z) = 0\}$. *In addition, for* $z \in \rho(A)$, *the resolvent of* A *is given by*

$$(zI - A)^{-1}\psi = e^{\cdot z}K(\cdot, z, \psi), \tag{5.2}$$

in which $K : [-1, 1] \times \mathbb{C} \times X \to \mathbb{C}^n$ *is the appropriate restriction of the operator* K *defined in (4.19).*

Proof. Fix $\psi \in X$ and consider the equation $(zI - A)\phi = \psi$ for $\phi \in \mathcal{D}(A)$, which is equivalent to the system

$$\begin{array}{rcl} D\phi & = & z\phi - \psi, \\ 0 & = & \int_{-1}^{1} d\mu(\sigma)\phi(\sigma). \end{array} \tag{5.3}$$

Suppose that $\det \Delta(z) \neq 0$. Solving the first equation yields

$$\phi(\theta) = e^{\theta z}\phi(0) + e^{\theta z}\int_{\theta}^{0} e^{-z\tau}\psi(\tau)d\tau \tag{5.4}$$

and hence the fullfillment of the second equation requires

$$0 = \int_{-1}^{1} d\mu(\sigma)e^{z\sigma}(\phi(0) + \int_{\sigma}^{0} e^{-z\tau}\psi(\tau)d\tau). \tag{5.5}$$

Thus setting

$$\phi(0) = \Delta(z)^{-1}\int_{-1}^{1} d\mu(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\psi(\tau)d\tau, \tag{5.6}$$

we see that $z \in \rho(A)$. On the other hand, choosing a non-zero $v \in \mathbb{R}^n$ such that $\Delta(z)v = 0$ for some root z of $\det \Delta(z) = 0$, one sees that the function $\phi(\theta) = e^{z\theta}v$ satisfies $\phi \in \mathcal{D}(A)$ and $A\phi = z\phi$. This shows that $z \in \sigma_p(A)$, completing the proof. □

For any pair of reals $\gamma_- < \gamma_+$ such that the characteristic equation $\det \Delta(z) = 0$ has no roots with $\mathrm{Re}\, z = \gamma_\pm$, define the set $\Sigma = \Sigma_{\gamma_-,\gamma_+} = \{z \in \sigma(A) \mid \gamma_- < \mathrm{Re}\, z < \gamma_+\}$. Using Lemma 4.3 it is easy to see that Σ is a finite set. Furthermore, the representation (5.2) implies that $(zI - A)^{-1}$ has a pole of finite order at $z = \lambda_0$ for every $\lambda_0 \in \Sigma$. Standard spectral theory [9, Theorem IV.2.5] now yields the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$ for some closed linear subspace $\mathcal{M}_\Sigma$, together with a spectral projection $Q_\Sigma : X \to \mathcal{M}_\Sigma$, which is explicitly given by

$$Q_\Sigma = \frac{1}{2\pi i}\int_\Gamma (zI - A)^{-1}dz, \tag{5.7}$$

for any Jordan path $\Gamma \subset \rho(A)$ with $\mathrm{int}(\Gamma) \cap \sigma(A) = \Sigma$. The following result gives conditions under which this Dunford integral can be related to the integral representation in (4.18).

Lemma 5.2. *Consider an operator L of the form (2.2) that satisfies (HL). Suppose that $\phi \in C^{\ell-1}([-1,1],\mathbb{C}^n)$ satisfies $LD^s\phi = 0$ for all $0 \leq s \leq \ell - 2$, with ℓ as introduced in (HL). Then the spectral projection $Q_\Sigma\phi$ defined above is given by*

$$(Q_\Sigma\phi)(\theta) = \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\theta z}K(\theta, z, \phi)dz + \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\theta z}K(\theta, z, \phi)dz, \tag{5.8}$$

with K as defined in (4.19).

Proof. For any $\rho > 0$ such that $|\mathrm{Im}\,\lambda| < \rho$ for any $\lambda \in \Sigma$, we introduce the path $\Gamma_\rho = \Gamma_\rho^\uparrow \cup \Gamma_\rho^\leftarrow \cup \Gamma_\rho^\downarrow \cup \Gamma_\rho^\rightarrow$, consisting of the line segments

$$\begin{array}{rclrcl} \Gamma_\rho^\uparrow & = & \mathrm{seg}[\gamma_+ - i\rho, \gamma_+ + i\rho], & \Gamma_\rho^\downarrow & = & \mathrm{seg}[\gamma_- + i\rho, \gamma_- - i\rho], \\ \Gamma_\rho^\leftarrow & = & \mathrm{seg}[\gamma_+ + i\rho, \gamma_- + i\rho], & \Gamma_\rho^\rightarrow & = & \mathrm{seg}[\gamma_- - i\rho, \gamma_+ - i\rho]. \end{array} \tag{5.9}$$

Note that it suffices to show that for every $\theta \in [-1, 1]$, we have

$$\lim_{\rho \to \infty} \int_{\Gamma_\rho^{\rightleftarrows}} e^{\theta z} \int_\theta^0 e^{-z\tau} \phi(\tau) d\tau dz + \int_{\Gamma_\rho^{\rightleftarrows}} e^{\theta z} \Psi(z, \phi) dz = 0, \tag{5.10}$$

with $\Psi(z, \phi)$ given by

$$\Psi(z, \phi) = \beta_M (z - \alpha_M)^\ell \Delta_M(z)^{-1} \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau. \tag{5.11}$$

The first integral in (5.10) can be shown to converge to zero as in [14]. To treat the second integral, use integration by parts to compute

$$\int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau = \frac{1}{z^{\ell-1}} \int_\sigma^0 e^{-z\tau} D^{\ell-1} \phi(\tau) d\tau + \sum_{k=0}^{\ell-2} \frac{1}{z^{k+1}} \big(e^{-z\sigma} D^k \phi(\sigma) - D^k \phi(0) \big). \tag{5.12}$$

Using the fact that $L D^s \phi = 0$ for $0 \le s \le \ell - 2$, we conclude that $\Psi(z, \phi)$ can be rewritten as

$$\begin{array}{rcl} \Psi(z, \phi) & = & \sum_{k=0}^{\ell-2} \frac{D^k \phi(0)}{z^{k+1}} + \\ & & \frac{\beta_M (z - \alpha_M)^{\ell-1}}{z^{\ell-1}} \big(\frac{z - \alpha_M}{z - \alpha} + (z - \alpha_M) R_\alpha(z) \big) \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} D^{\ell-1} \phi(\tau) d\tau, \end{array} \tag{5.13}$$

where $R_\alpha(z) = O(1/|z|^2)$ as $\operatorname{Im} z \to \pm\infty$, uniformly in vertical strips. Ignoring the terms in $\Psi(z, \phi)$ that behave as $O(1/z)$ as $\operatorname{Im} z \to \pm\infty$, it remains to show that

$$\lim_{\rho \to \infty} \int_{\Gamma_\rho^{\rightleftarrows}} e^{\theta z} \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} D^{\ell-1} \phi(\tau) d\tau dz = 0. \tag{5.14}$$

This however can also be established using the arguments in [14]. $\square$

In order to show that $\mathcal{M}_\Sigma$ is finite dimensional, we introduce a new operator $\widehat{A}$ on the larger space $\widehat{X} = \mathbb{C}^n \times X$,

$$\begin{array}{rcl} \mathcal{D}(\widehat{A}) & = & \Big\{ (c, \phi) \in \widehat{X} \mid D\phi \in X \text{ and } c = \phi(0) \Big\}, \\ \widehat{A}(c, \phi) & = & (L\phi + D\phi(0), D\phi). \end{array} \tag{5.15}$$

We write $j : X \to \widehat{X}$ for the canonical continuous embedding $\phi \mapsto (\phi(0), \phi)$. The reader should note that the definition of $\widehat{A}$ given here differs from the corresponding definition in [14]. However, this construction ensures that the part of $\widehat{A}$ in jX is equivalent to A and that the closure of $\mathcal{D}(\widehat{A})$ is given by jX. Hence the spectral analysis of A and $\widehat{A}$ is one and the same. The next result shows that $\Delta(z)$ is a characteristic matrix for $\widehat{A}$, in the sense of [9, Def. IV.4.17].

Lemma 5.3. *Consider the holomorphic functions $E : \mathbb{C} \to \mathcal{L}(\widehat{X}, \mathcal{D}(\widehat{A}))$ and $F : \mathbb{C} \to \mathcal{L}(\widehat{X}, \widehat{X})$, given by*

$$\begin{array}{rcl} E(z)(c, \psi)(\theta) & = & (c, e^{\theta z} c + e^{\theta z} \int_\theta^0 e^{-z\tau} \psi(\tau) d\tau), \\ F(z)(c, \psi)(\theta) & = & (c - \psi(0) + \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \psi(\tau) d\tau, \psi(\theta)), \end{array} \tag{5.16}$$

in which $\mathcal{D}(\widehat{A})$ is considered as a Banach space with the graph norm. Then $E(z)$ and $F(z)$ are bijective for every $z \in \mathbb{C}$ and we have the identity

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I \end{pmatrix} = F(z)(zI - \widehat{A}) E(z). \tag{5.17}$$

Proof. The bijectivity of $E(z)$ follows as in [14], while the bijectivity of $F(z)$ is almost immediate. The last identity in the statement of the lemma follows easily by using the definition of $\Delta(z)$ and computing

$$(zI - \widehat{A})E(z)(c, \psi) = (\psi(0) - \int_{-1}^{1} d\mu(\sigma)e^{z\sigma}c - \int_{-1}^{1} d\mu(\sigma)e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau}\psi(\tau)d\tau, \psi). \tag{5.18}$$

□

In [14] similar results were obtained for the system $\dot{x}(\xi) = Mx_\xi$. In particular, writing $\Sigma^M = \Sigma^M_{\gamma_-,\gamma_+} = \{z \in \mathbb{C} \mid \det \Delta_M(z) = 0 \text{ and } \gamma_- < \operatorname{Re} z < \gamma_+\}$, the decomposition $X = \mathcal{M}_{\Sigma^M} \oplus \mathcal{R}_{\Sigma^M}$ was obtained, together with a projection $Q^M_\Sigma : X \to \mathcal{M}_{\Sigma^M}$. Using (HL) it is easy to see that $\Sigma \subset \Sigma^M$. In addition, the next result exhibits how the generalized eigenspaces are related.

Proposition 5.4. *Consider the operator L defined in (2.2) and suppose that (HL) holds. Then we have the inclusion $\mathcal{M}_\Sigma \subset \mathcal{M}_{\Sigma^M}$, together with the identity $Q^M_{\Sigma^M} \circ Q_\Sigma = Q_\Sigma$.*

Proof. First recall from [14] that $\Delta_M(z)$ is a characteristic matrix for the operator $\widehat{A}^M : \mathcal{D}(\widehat{A}) \to X$ given by $\widehat{A}^M(c, \phi) = (M\phi, D\phi)$. As in the proof of [9, Theorem IV.4.18], a basis for $\mathcal{M}_\Sigma$ can be constructed using maximal generalized Jordan chains. It hence suffices to show that every such chain for Δ at $z = \lambda$ is also a Jordan chain for Δ_M at the same value of z. Indeed, for any such chain $v_0, \ldots, v_{m-1}$ of length m we have by definition

$$\Delta(z)(v_0 + (z - \lambda)v_1 + \ldots + (z - \lambda)^{m-1}v_{m-1}) = O((z - \lambda)^m), \tag{5.19}$$

which immediately implies that also

$$\begin{aligned} &\Delta_M(z)(v_0 + (z - \lambda)v_1 + \ldots + (z - \lambda)^{m-1}v_{m-1}) \\ &= \beta_M(z - \alpha_M)^\ell \Delta(z)(v_0 + (z - \lambda)v_1 + \ldots + (z - \lambda)^{m-1}v_{m-1}) = O((z - \lambda)^m). \end{aligned} \tag{5.20}$$

The inclusion $\mathcal{M}_\Sigma \subset \mathcal{M}_{\Sigma^M}$ now easily follows, which in turn implies that $Q^M_{\Sigma^M}$ acts as the identity on $\mathcal{M}_\Sigma$, upon which the proof is complete. □

Proof of Proposition 2.1. Choose $\gamma > 0$ such that $\det \Delta_M(z) = 0$ has no roots with $0 < |\operatorname{Re} z| \leq \gamma$ and write $X_0 = \mathcal{M}_{\Sigma_{-\gamma,\gamma}}$, together with $Q_0 = Q_{\Sigma_{-\gamma,\gamma}}$. Consider any $x \in \bigcap_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ that satisfies $0 = Lx_\xi$. Using Proposition 4.2 it follows that $x \in C^1(\mathbb{R}, \mathbb{C}^n)$. However, this implies that also $L\dot{x}_\xi = 0$ for all $\xi \in \mathbb{R}$ and repeated application of this argument shows that in fact $x \in C^\infty(\mathbb{R}, \mathbb{C}^n)$. We can hence combine Proposition 4.4 with Lemma 5.2 to conclude that $Q_0 x_0 = x_0$ and hence by shifting x along the line, $Q_0 x_\xi = x_\xi$ for all $\xi \in \mathbb{R}$. Due to the fact that a basis for X_0 can be constructed using functions of the form $p(\theta)e^{\lambda\theta}$, in which p is a polynomial and $\det \Delta_M(\lambda) = 0$, one sees that any $\phi \in X_0$ can be extended to a solution $x = E\phi$ of $Lx_\xi = 0$ on the line, with $x_0 = \phi$. To see the uniqueness of this extension, suppose that both x^1 and x^2 satisfy $x^1_0 = x^2_0 = \phi$, with $0 = Lx^1_\xi = Lx^2_\xi$ for all $\xi \in \mathbb{R}$. Write $y(\xi) = x^1(\xi) - x^2(\xi)$ for $\xi \geq 0$ and $y(\xi) = 0$ for $\xi < 0$. Then $y \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies $0 = Ly_\xi$, with $y(\xi) = O(e^{\zeta\xi})$ as $\xi \to \pm\infty$ for some $\zeta > 0$, which can be chosen in such a way that there are no roots of $\det \Delta_M(z) = 0$ in the strip $\zeta - \epsilon \leq \operatorname{Re} z \leq \zeta + \epsilon$, for some $\epsilon > 0$. This however implies that for all $\xi \in \mathbb{R}$, we have $y_\xi = Q_{\Sigma_{\zeta-\epsilon,\zeta+\epsilon}} y_\xi = Q_{\{0\}} y_\xi = 0$, i.e., $y = 0$. A similar construction for $\xi \leq 0$ completes the proof. $\square$

6 Linear Inhomogeneous Equations

In this section we study the interplay between the linear inhomogeneous equations

$$\begin{array}{rcl} 0 & = & Lx_\xi + f(\xi), \\ \dot{y}(\xi) & = & My_\xi + g(\xi), \end{array} \tag{6.1}$$

with L as defined in (2.2) and M as in (HL). Associated to these equations we define the linear operators $\Lambda : C(\mathbb{R}, \mathbb{C}^n) \to C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$ and $\Lambda_M : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$\begin{array}{rcl} (\Lambda x)(\xi) & = & -Lx_\xi \\ (\Lambda_M x)(\xi) & = & \dot{x}(\xi) - Mx_\xi. \end{array} \tag{6.2}$$

The operator Λ_M has been extensively studied in [14, 21] and we will use these results to facilitate our treatment of Λ. We will be particularly interested in the spaces

$$W^{\ell,p}(\mathbb{R}, \mathbb{C}^n) = \{x \in L^p(\mathbb{R}, \mathbb{C}^n) \mid D^s x \in L^p(\mathbb{R}, \mathbb{C}^n) \text{ for all } 1 \leq s \leq \ell\}, \tag{6.3}$$

with $p = 2$ or $p = \infty$. In the first result we choose $p = 2$, which enables us to use Fourier transform techniques to define an inverse for Λ on the space $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$. This inverse will turn out to be closely related to the inverse of Λ_M.

Lemma 6.1. *Consider the operator L defined in (2.2) and assume that (HL) is satisfied. Suppose further that the characteristic equation $\det \Delta_M(z) = 0$ has no roots with $\operatorname{Re} z = 0$. Then Λ is a bounded linear isomorphism from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ onto $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$, with*

$$(D - \alpha_M)^\ell \Lambda x = \beta_M^{-1} \Lambda_M x \tag{6.4}$$

for $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$, in which D denotes the differentiation operator. Conversely, suppose $x = \Lambda^{-1} f$ for $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$, then x is given by

$$\widehat{x}(\eta) = \Delta(i\eta)^{-1} \widehat{f}(\eta). \tag{6.5}$$

In addition, there is a representation

$$x(\xi) = \int_{-\infty}^{\infty} G(\xi - s)((D - \alpha_M)^\ell f)(s) ds = \beta_M \big(\Lambda_M^{-1} (D - \alpha_M)^\ell f\big)(\xi), \tag{6.6}$$

with a Greens function G that satisfies $G \in L^p(\mathbb{R}, \mathbb{C}^{n \times n})$ for all $1 \leq p \leq \infty$ and whose Fourier tranform is given by

$$\widehat{G}(\eta) = (i\eta - \alpha_M)^{-\ell} \Delta(i\eta)^{-1} = \beta_M \Delta_M(i\eta)^{-1}. \tag{6.7}$$

The function G decays exponentially at both $\pm\infty$. In particular, fixing $a_- < 0$ and $a_+ > 0$ such that $\det \Delta_M(z) \neq 0$ *for all $a_- \leq \operatorname{Re} z \leq a_+$ and choosing an $\alpha < a_-$, we have the estimate*

$$|G(\xi)| \leq \begin{cases} \beta_M (1 + K(a_-)) e^{a_- \xi} & \text{for all } \xi \geq 0, \\ \beta_M K(a_+) e^{a_+ \xi} & \text{for all } \xi < 0, \end{cases} \tag{6.8}$$

in which

$$K(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |R_\alpha(a + i\omega)| \, d\omega, \tag{6.9}$$

with R_α as introduced in (4.17).

Finally, suppose that f and its derivatives satisfies a growth condition $D^s f(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$ for some $0 < \lambda < -a_-$ and all $0 \leq s \leq \ell$. Then also $x = \Lambda^{-1} f$ satisfies $x(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$, with the same estimate for $\dot{x}$. The analogous statement also holds for $\xi \to -\infty$.

Proof. Suppose that $\Lambda x = 0$ for some $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$. Due to the Sobolev embedding $W^{1,2}(\mathbb{R}, \mathbb{C}^n) \subset C(\mathbb{R}, \mathbb{C}^n) \cap L^\infty(\mathbb{R}, \mathbb{C}^n)$, we know that x is bounded, hence we can apply Proposition 2.1 with $X_0 = \{0\}$ to conclude $x = 0$. Recall the fact that $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ is equivalent to $\eta \mapsto (1 + |\eta| + \ldots + |\eta|^\ell) \widehat{f}(\eta) \in L^2(\mathbb{R}, \mathbb{C}^n)$. The fact that Λ maps $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ into $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ now follows after noting that for some constants C, C' and C'', we have

$$\begin{array}{rcl} (1 + |\eta| + \ldots + |\eta|^\ell) \mathcal{F}^+ \Lambda x(\eta) & = & (1 + |\eta| + \ldots + |\eta|^\ell) \beta_M^{-1} (i\eta - \alpha_M)^{-\ell} \Delta_M(i\eta) \hat{x}(\eta) \\ & \leq & C \Delta_M(i\eta) \hat{x}(\eta) \leq (C' |\eta| + C'') \hat{x}(\eta). \end{array} \tag{6.10}$$

Observe also that the identity $(D - \alpha_M)^\ell \Lambda x = \beta_M^{-1} \Lambda_M x$ follows immediately from

$$\mathcal{F}^+ \big((D - \alpha_M)^\ell \Lambda x \big)(\eta) = (i\eta - \alpha_M)^\ell \Delta(i\eta) \hat{x}(\eta) = \beta_M^{-1} \Delta_M(i\eta) \hat{x}(\eta). \tag{6.11}$$

To show that Λ is invertible, fix any $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ and define $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ by

$$\hat{x}(\eta) = \Delta(i\eta)^{-1} \hat{f}(\eta) = \Delta_M(i\eta)^{-1} \beta_M (i\eta - \alpha_M)^\ell \hat{f}(\eta) = \beta_M \Delta_M(i\eta)^{-1} \mathcal{F}^+ ((D - \alpha_M)^\ell f)(\eta). \tag{6.12}$$

It is clear that indeed $\Lambda x = f$ and the remaining statements now follow easily from this identity together with the theory developed in [14] for the operator Λ_M. $\square$

As in [14], we need to obtain results on the behaviour of Λ on the exponentially weighted spaces

$$W^{\ell,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e^{-\eta \cdot} x(\cdot) \in W^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n) \right\}. \tag{6.13}$$

To ease notation, we introduce the function $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ and $\nu \in \mathbb{R}$. Upon defining a transformed operator $\Lambda_\eta : C(\mathbb{R}, \mathbb{C}^n) \to C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda_\eta x)(\xi) = -\int_{-1}^{1} d\mu(\sigma) e^{-\eta\sigma} x(\xi + \sigma), \tag{6.14}$$

one may easily verify the following identity,

$$\Lambda_\eta e_\eta x = e_\eta \Lambda x. \tag{6.15}$$

The corresponding transformation of the characteristic matrix is given by

$$\Delta_\eta(z) = -\int_{-1}^{1} e^{(z-\eta)\sigma} d\mu(\sigma) = \Delta(z-\eta) = (z-\alpha_{\eta,M})^{-\ell}\Delta_{\eta,M}(z), \tag{6.16}$$

with $\alpha_{\eta,M} = \alpha_M + \eta$ and $\Delta_{\eta,M}(z) = \Delta_M(z-\eta)$.

We now wish to use the fact that Λ is invertible as a map from $W^{1,2}(\mathbb{R},\mathbb{C}^n)$ into $W^{\ell,2}(\mathbb{R},\mathbb{C}^n)$ to prove a similar result when considering Λ as a map from $W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ into $W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$. An inverse for Λ will be constructed by writing any $f \in W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ as a sum of functions in $W^{\ell,2}_\zeta(\mathbb{R},\mathbb{C}^n)$ for appropriate values of ζ, on which we can use the inverse of Λ defined in Lemma 6.1. In contrast to the situation in [14], where we merely needed to consider $f \in L^\infty(\mathbb{R},\mathbb{C}^n)$, care has to be taken when splitting the inhomogeneity f to ensure that the components remain sufficiently smooth.

To accomodate this, we choose C^∞-smooth basis functions χ_i for $0 \le i \le \ell-1$ that have compact support contained in $[-1,1]$, such that $D^i\chi_j(0) = \delta_{ij}$ for $0 \le i,j \le \ell-1$. We now define the finite dimensional space $BC_\diamond(\mathbb{R},\mathbb{C}^n) = \operatorname{span}\{\chi_i \mid 0 \le i \le \ell-1\} \subset C^\infty_c(\mathbb{R},\mathbb{C}^n)$ and an operator $\Phi_\diamond : W^{\ell,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n) \to BC_\diamond(\mathbb{R},\mathbb{C}^n)$ by

$$\Phi_\diamond f = \sum_{i=0}^{\ell-1} \chi_i D^i f(0). \tag{6.17}$$

Notice that $D^i\Phi_\diamond f(0) = D^i f(0)$ for all $0 \le i \le \ell-1$, which ensures that we can define the cutoff operators $\Phi_\pm : W^{\ell,1}_{\mathrm{loc}} \to W^{\ell,1}_{\mathrm{loc}}$ via

$$\Phi_+ f(\xi) = H(\xi)(f - \Phi_\diamond f)(\xi) \qquad \Phi_- f(\xi) = (1-H(\xi))(f-\Phi_\diamond f)(\xi), \tag{6.18}$$

where $H(\xi)$ denotes the Heaviside function, i.e., $H(\xi) = 1$ for $\xi \ge 0$ and $H(\xi) = 0$ for $\xi < 0$.

Proposition 6.2. *Consider any $\eta \in \mathbb{R}$ and $\epsilon_0 > 0$ such that there are no roots of $\det\Delta_M(z) = 0$ in the strip $\eta - \epsilon_0 \le \operatorname{Re} z \le \eta + \epsilon_0$. Then the operator Λ is an isomorphism from $W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ onto $W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$. In addition, for any $0 < \epsilon < \epsilon_0$ and any $f \in W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ such that $\Phi_\diamond f = 0$, we have the following integral expression for $x = \Lambda^{-1}f$,*

$$x(\xi) = \frac{1}{2\pi i}\int_{\eta+\epsilon-i\infty}^{\eta+\epsilon+i\infty} e^{\xi z}\Delta(z)^{-1}\widetilde{f}_+(z)dz + \frac{1}{2\pi i}\int_{\eta-\epsilon-i\infty}^{\eta-\epsilon+i\infty} e^{\xi z}\Delta(z)^{-1}\widetilde{f}_-(z)dz, \tag{6.19}$$

where the Laplace transforms $\widetilde{f}_+$ and $\widetilde{f}_-$ are defined as in Section 4. Finally, for any $f \in W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$, we have the following Greens formula for $x = \Lambda^{-1}f$,

$$x(\xi) = e^{\eta\xi}\int_{-\infty}^{\infty} G_{-\eta}(\xi - s)e^{-\eta s}((D-\alpha_M)^\ell f)(s)ds = \beta_M\Lambda_M^{-1}(D-\alpha_M)^\ell f, \tag{6.20}$$

in which $G_{-\eta}$ has exponential decay at both $\pm\infty$ and is given by

$$\mathcal{F}^+ G_{-\eta}(k) = \beta_M\Delta_M(ik+\eta)^{-1}. \tag{6.21}$$

Proof. We first show that we can indeed define an inverse for Λ on the space $W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$. Pick any $0 < \epsilon < \epsilon_0$ and use the cutoff operators introduced above to define $f_\pm \in W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ by

$f_\pm = \Phi_\pm f$ and similarly $f_\diamond = \Phi_\diamond f$. Note that $f_\diamond \in W^{\ell,2}_\eta(\mathbb{R},\mathbb{C}^n)$ and hence we can define $x_\diamond = e_\eta \overline{x}_\diamond \in W^{1,2}_\eta(\mathbb{R},\mathbb{C}^n)$, with $\overline{x}_\diamond = \Lambda^{-1}_{-\eta} e_{-\eta} f_\diamond$. Since $e_{-\eta} f_\diamond \in C^\infty_c(\mathbb{R},\mathbb{C}^n)$, one can use the Greens function representation (6.6) to conclude that also $\overline{x}_\diamond$ and its derivative are uniformly bounded, showing that $x_\diamond \in W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$.

It remains to invert the functions $f_\pm$. To this end, we define

$$\overline{f}_\pm = e_{-(\eta\pm\epsilon)} f_\pm \in W^{\ell,2}(\mathbb{R},\mathbb{C}^n) \cap W^{\ell,\infty}(\mathbb{R},\mathbb{C}^n) \cap W^{\ell,\infty}_{\mp\epsilon}(\mathbb{R},\mathbb{C}^n), \tag{6.22}$$

which allows us to introduce the functions $x_\pm = e_{\eta\pm\epsilon}\overline{x}_\pm$, in which

$$\overline{x}_\pm = \Lambda^{-1}_{-\eta\mp\epsilon}\overline{f}_\pm \in W^{1,2}(\mathbb{R},\mathbb{C}^n) \cap W^{1,\infty}(\mathbb{R},\mathbb{C}^n) \cap W^{1,\infty}_{\mp\epsilon}(\mathbb{R},\mathbb{C}^n), \tag{6.23}$$

where the last two inclusions follow from the Greens function representation of Λ^{-1} in Lemma 6.1. This shows that indeed $x_\pm \in W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ and hence $x = x_\diamond + x_+ + x_-$ satisfies $\Lambda x = f$. The integral expression (6.19) now follows upon applying the substitution $z = \eta \pm \epsilon + ik$ to the equality

$$\mathcal{F}^+\overline{x}_\pm(k) = \Delta^{-1}_{-\eta\mp\epsilon}(ik)\mathcal{F}^+\overline{f}_\pm(k) = \Delta^{-1}(ik+\eta\pm\epsilon)\widetilde{f}_\pm(ik+\eta\pm\epsilon). \tag{6.24}$$

The injectivity of Λ can be shown in exactly the same manner as the corresponding result in Lemma 6.1. Notice that Proposition 4.2 immediately implies that Λ maps $W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ into $W^{\ell-1,\infty}_\eta(\mathbb{R},\mathbb{C}^n) \cap W^{\ell,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)$. To show that the mapping is actually into $W^{\ell,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$, we notice that in a similar fashion as above, any $y \in W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ can be split into $y = y_\diamond + y_+ + y_-$ with $y_\diamond \in C^\infty_c(\mathbb{R},\mathbb{C}^n)$ and $y_\pm \in W^{1,2}_{\eta\pm\epsilon}(\mathbb{R},\mathbb{C}^n)$. Applying Lemma 6.1 to these individual functions and using (6.15), we find that again $(D-\alpha_M)^\ell \Lambda y = \beta_M^{-1}\Lambda_M y \subset L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$, which together with Proposition 4.2 shows that also $D^\ell \Lambda y \in L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$.

Finally, we show that the Greens formula representation (6.6) continues to hold. For convenience, we write $\zeta = \eta + \epsilon$ and note that for any $f \in W^{\ell,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)$ the identity $De_\zeta f = e_\zeta(D+\zeta)f$ implies that

$$\begin{array}{rcl} e_\zeta(D-\alpha_{-\zeta,M})^\ell e_{-(\eta+\epsilon)} f & = & e_{\eta+\epsilon}(D-\alpha_M+\eta+\epsilon)^\ell e_{-\zeta} f = e_\zeta((D-\zeta)-\alpha_M+\zeta)^\ell f \\ & = & (D-\alpha_M)^\ell f. \end{array} \tag{6.25}$$

This allows us to compute

$$\begin{array}{rcl} x_+(\xi) & = & (e_\zeta \Lambda^{-1}_{-\zeta} e_{-\zeta} f_+)(\xi) \\ & = & e^{\zeta\xi}\int_{-\infty}^{\infty} G_{-\zeta}(\xi-s)e^{-\zeta s}(D-\alpha_M)^\ell f_+(s)ds \\ & = & e^{\eta\xi}\int_{-\infty}^{\infty} e^{\epsilon(\xi-s)}G_{-\zeta}(\xi-s)e^{-\eta s}(D-\alpha_M)^\ell f_+(s)ds. \end{array} \tag{6.26}$$

Now noticing that $\mathcal{F}^+ e_\epsilon G_{-\zeta}(k) = \widehat{G}_{-\zeta}(k+i\epsilon)$, we find,

$$\begin{array}{rcl} \mathcal{F}^+ e_\epsilon G_{-\zeta}(k) & = & (ik-\epsilon-\alpha_{-\zeta,M})^{-\ell}\Delta_{-\zeta}(ik-\epsilon)^{-1} \\ & = & (ik-\alpha_M+\eta)^{-\ell}\Delta(ik+\eta)^{-1} = \widehat{G}_{-\eta}(k), \end{array} \tag{6.27}$$

upon which the proof can be completed using similar identities for x_- and $x_\diamond$. $\square$

7 The pseudo-inverse

The goal of this section is to define a pseudo-inverse for the linear inhomogeneous equation $\Lambda x = f$ in the spirit of [14]. However, the construction here will differ from the corresponding construction in [14], due to the fact that we cannot modify the nonlinearities $R^{(s)}$ to become globally Lipschitz continuous in such a way that the differentiation structure in (HR1) is preserved. To bypass this difficulty, we need to decouple the inhomogeneity f from its derivatives, allowing us to replace the vector of functions $(f, Df, \ldots, D^{\ell} f)$ by general vectors $(g_0, \ldots, g_{\ell})$ for which there is no relation between the components. This decoupling should be seen in the context of so-called jet manifolds, which play a role when studying PDEs and DAEs from an algebraic point of view, see e.g. [25].

To formalize this construction, we introduce the product spaces

$$\begin{array}{lcl} L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) & = & \underbrace{L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \times \ldots \times L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)}_{\ell+1 \text{ times}}, \\ BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) & = & \underbrace{BC_{\eta}(\mathbb{R},\mathbb{C}^n) \times \ldots \times BC_{\eta}(\mathbb{R},\mathbb{C}^n)}_{\ell+1 \text{ times}} \end{array} \tag{7.1}$$

and the canonical inclusions $BC^{\ell}_{\eta}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n)$ and $W^{\ell,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \hookrightarrow L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n)$ via $\mathcal{J}x = (x, Dx, \ldots, D^{\ell}x)$. For any $0 \le s \le \ell$ write $\widetilde{D}^s : L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$ for the canonical extensions of the differentiation operators and similarly define $\widetilde{\Phi}_{\diamond} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to C^{\infty}_{c}(\mathbb{R},\mathbb{C}^n)$ and $\widetilde{\Phi}_{\pm} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to L^{\infty,\times(\ell+1)}_{\pm\eta}(\mathbb{R},\mathbb{C}^n)$. Using the explicit representation (6.20) for $\Lambda^{-1} : W^{\ell,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$, we can naturally expand the domain of definition to obtain an operator $\Lambda^{-1} : L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$, given by

$$\Lambda^{-1}\mathbf{f} \quad = \quad \beta_M \Lambda_M^{-1}(\widetilde{D} - \alpha_M)^{\ell}\mathbf{f}, \qquad \mathbf{f} \in L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n). \tag{7.2}$$

We will use the longer notation $\Lambda^{-1}_{(\eta)}$ for this operator whenever we wish to emphasize the η-dependence of the underlying exponentially weighted function spaces.

Pick any $\gamma > 0$ such that there are no roots of $\det \Delta_M(z) = 0$ with $0 < |\mathrm{Re}\, z| \le \gamma$ and fix an $\eta \in (0, \gamma)$. Using the construction above we can define the bounded linear operators $\Lambda^{-1}_{\pm} = \Lambda^{-1}_{(\pm\eta)} : L^{\infty,\times(\ell+1)}_{\pm\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\pm\eta}(\mathbb{R},\mathbb{C}^n)$. As in the proof of Proposition 2.1, we write $X_0 = \mathcal{M}_{\Sigma_{-\gamma,\gamma}}$ for the generalized eigenspace corresponding to roots of $\det \Delta(z) = 0$ on the imaginary axis and $Q_0 = Q_{\Sigma_{-\gamma,\gamma}}$ for the corresponding spectral projection. Similarly, we introduce $X_M = \mathcal{M}_{\Sigma^M_{-\gamma,\gamma}}$ and $Q_M = Q^M_{\Sigma_{-\gamma,\gamma}}$ for the analogues of X_0 and Q_0 associated to the operator Λ_M. Recalling the extension operator $E : X_0 \to BC^1_{\eta}(\mathbb{R},\mathbb{C}^n)$ from Proposition 2.1, we have all the ingredients we need to define the pseudo-inverse $\mathcal{K}_{\eta} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to BC^1_{\eta}(\mathbb{R},\mathbb{C}^n)$. It is given explicitly by the formula

$$\mathcal{K}_{\eta}\mathbf{f} = \Lambda_+^{-1}(\frac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_+\mathbf{f}) + \Lambda_-^{-1}(\frac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_-\mathbf{f}) - EQ_0\mathrm{ev}_0\big[\Lambda_+^{-1}(\frac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_+\mathbf{f}) + \Lambda_-^{-1}(\frac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_-\mathbf{f})\big], \tag{7.3}$$

in which we have introduced the evaluation function $\mathrm{ev}_{\xi} x = x_{\xi} \in X$. Note that by construction we have the identity $Q_0 \mathrm{ev}_0 \mathcal{K}_{\eta}\mathbf{f} = 0$. In addition, from (7.2) together with the inclusion $X_0 \subset X_M$, we see that $\Lambda_M \mathcal{K}_\eta \mathbf{f} = \beta_M (\widetilde{D} - \alpha_M)^\ell \mathbf{f}$. The following result shows that $\mathcal{K}$ is well-behaved on the scale of Banach spaces $BC_\zeta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$.

Lemma 7.1. *Consider any pair $\eta_1, \eta_2 \in \mathbb{R}$ with $0 < \eta_1 < \eta_2 < \gamma$. Then for any $\mathbf{f} \in BC_{\eta_1}^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$, we have*

$$\mathcal{K}_{\eta_1} \mathbf{f} = \mathcal{K}_{\eta_2} \mathbf{f}. \tag{7.4}$$

Proof. We will merely establish (7.4) under the assumption $\mathbf{f} = \widetilde{\Phi}_+ \mathbf{f}$, noting that the remaining components of $\mathbf{f}$ can be treated in a similar fashion. Note that $\overline{h}_+ = e_{-\eta_2} (\widetilde{D} - \alpha_M)^\ell \widetilde{\Phi}_+ \mathbf{f} \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ satisfies a growth condition $\overline{h}_+(\xi) = O(e^{-(\eta_2 - \eta_1)\xi})$ as $\xi \to \infty$ and hence $\overline{x}_+ = \beta_M \Lambda_{-\eta_2, M}^{-1} \overline{h}_+$ shares this growth rate by [14, Corollary 3.3]. This implies that the function $x_+ = e_{\eta_2} \overline{x}_+$ satisfies $x_+ = O(e^{\eta_1 \xi})$ as $\xi \to \infty$. We can hence argue $x_+ \in W_{\eta_1}^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W_{\eta_2}^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$, from which we conclude

$$\Lambda_{(\eta_1)}^{-1} \widetilde{\Phi}_+ \mathbf{f} = \Lambda_{(\eta_2)}^{-1} \widetilde{\Phi}_+ \mathbf{f}, \tag{7.5}$$

which directly implies that also (7.4) holds. □

The final result of this section should be seen as the analogue of Lemma 5.4 in [14]. The conclusions here are however somewhat weaker, due to the fact that in this more general setting we no longer have an automatic interpretation of $\mathcal{K}_\eta \mathbf{f}$ in terms of the operator Λ. The consequences of this fact shall become clear in Section 8, during the analysis of the dynamics on the center manifold.

Lemma 7.2. *For any $\mathbf{f} \in BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ and $\xi_0 \in \mathbb{R}$, define the function $y \in BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$ by*

$$y(\xi) = (\mathcal{K}_\eta \mathbf{f})(\xi + \xi_0) - (\mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot))(\xi). \tag{7.6}$$

Then we have $\Lambda_M y = 0$. In particular, we have the identity

$$(I - Q_M)\mathrm{ev}_{\xi_0} \mathcal{K}_\eta \mathbf{f} = (I - Q_M)\mathrm{ev}_0 \mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot). \tag{7.7}$$

In addition, suppose that for all integers s with $0 \le s \le \ell$ we have

$$(\mathbf{f}_s)_{|J} = D^s g \tag{7.8}$$

for some $g \in C^\ell(J, \mathbb{C}^n)$, with $J = [-1 - |\xi_0|, 1 + |\xi_0|]$. Then in fact $\Lambda y = 0$ and

$$(I - Q_0)\mathrm{ev}_{\xi_0} \mathcal{K}_\eta \mathbf{f} = \mathrm{ev}_0 \mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot). \tag{7.9}$$

Proof. We can no longer as in [14] apply Λ directly to the definition of $\mathcal{K}$. Instead, we introduce the shift operator T_{ξ_0} that acts as $(T_{\xi_0} f)(\xi) = f(\xi + \xi_0)$ and compute

$$\begin{array}{rcl} y & = & x + T_{\xi_0} \Lambda_{(\eta)}^{-1} [\frac{1}{2} \widetilde{\Phi}_\diamond + \widetilde{\Phi}_+] \mathbf{f} - \Lambda_{(\eta)}^{-1} [\frac{1}{2} \widetilde{\Phi}_\diamond + \widetilde{\Phi}_+] T_{\xi_0} \mathbf{f} \\ & & + T_{\xi_0} \Lambda_{(-\eta)}^{-1} [\frac{1}{2} \widetilde{\Phi}_\diamond + \widetilde{\Phi}_-] \mathbf{f} - \Lambda_{(-\eta)}^{-1} [\frac{1}{2} \widetilde{\Phi}_\diamond + \widetilde{\Phi}_-] T_{\xi_0} \mathbf{f} \\ & = & x + \Lambda_{(\eta)}^{-1} \mathbf{g}_+ + \Lambda_{(-\eta)}^{-1} \mathbf{g}_- \end{array} \tag{7.10}$$

for some $x \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$. Using the fact that T_{ξ_0} and $\Lambda_\pm^{-1}$ commute, we can write

$$\begin{array}{lcl} \mathbf{g}_+ & = & T_{\xi_0}[\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]\mathbf{f} - [\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]T_{\xi_0}\mathbf{f} \\ \mathbf{g}_- & = & T_{\xi_0}[\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]\mathbf{f} - [\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]T_{\xi_0}\mathbf{f}. \end{array} \tag{7.11}$$

Now using the identity $I = \widetilde{\Phi}_\diamond + \widetilde{\Phi}_+ + \widetilde{\Phi}_-$, one easily sees that $\mathbf{g}_+ = -\mathbf{g}_-$. Using (7.2) it now easily follows that indeed $\Lambda_M y = 0$.

Now suppose that the differentiability condition (7.8) holds. Since $(T_{\xi_0}\widetilde{\Phi}_\pm \mathbf{f})(\xi) = (\widetilde{\Phi}_\pm T_{\xi_0}\mathbf{f})(\xi) = \mathbf{f}(\xi + \xi_0)$ for all $\xi \ge \max(1, 1 - \xi_0)$ and $\xi \le \min(-1, -1-\xi_0)$, it follows that both $\mathbf{g}_\pm$ have compact support and in addition satisfy $\mathbf{g}_\pm = \mathcal{J} g_*$ for some $g_* \in C^\ell(\mathbb{R}, \mathbb{C}^n)$. In this case the conclusion $\Lambda y = 0$ is immediate from (7.10). □

8 The Center Manifold

We are now in a position in which we can use the pseudo-inverse defined in the previous section to construct a center manifold for the nonlinear equation (2.1). In order to apply the Banach contraction theorem, we consider the set of nonlinearities $R^{(s)}$ for $0 \le s \le \ell$ introduced in condition (HR1) and modify them simultaneously to become globally Lipschitz continuous with a sufficiently small Lipschitz constant. We choose a C^∞-smooth cutoff-function $\chi : [0, \infty) \to \mathbb{R}$ with $\|\chi\|_\infty = 1$ that satisfies $\chi(\xi) = 0$ for $\xi \ge 2$ while $\chi(\xi) = 1$ for $\xi \le 1$. For any $\delta > 0$ we define $\chi_\delta(\xi) = \chi(\xi/\delta)$. We use the projection Q_M defined in the previous section to modify the nonlinearities separately in the hyperbolic and nonhyperbolic directions and define $\mathbf{R}_\delta : X \to \mathbb{C}^{n\times(\ell+1)}$ componentwise by

$$\mathbf{R}_\delta(\phi)_s = \chi_\delta(\|Q_M\phi\|)\chi_\delta(\|(I - Q_M)\phi\|)R^{(s)}(\phi), \qquad 0 \le s \le \ell. \tag{8.1}$$

The fact that we use Q_M instead of Q_0 is motivated by (7.7), which allows us to control the growth of $\xi \mapsto (I - Q_M)\mathrm{ev}_\xi\mathcal{K}$ on the center manifold.

The map $\mathbf{R}_\delta$ induces the map $\widetilde{\mathbf{R}}_\delta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ via $\widetilde{\mathbf{R}}_\delta x(\xi) = \mathbf{R}_\delta x_\xi$. Notice that $\widetilde{\mathbf{R}}_\delta$ is well-defined, since $i_x : \mathbb{R} \to X$ which sends $\xi \mapsto x_\xi$ is a continuous mapping for any continuous x and hence the same holds for $\widetilde{\mathbf{R}}_\delta x = \mathbf{R}_\delta \circ i_x$. The next lemma follows directly from [14, Section 6] and shows that the construction above indeed yields a globally Lipschitz smooth substitution operator $\widetilde{\mathbf{R}}_\delta$.

Lemma 8.1. *For any $\eta \in \mathbb{R}$, the substitution operator $\widetilde{\mathbf{R}}_\delta$ viewed as an operator from $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ into $BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ is globally Lipschitz continuous with Lipschitz constant $\exp(|\eta|)L_\delta$, in which $L_\delta \to 0$ as $\delta \to 0$. In addition, we have $|(\mathbf{R}_\delta\phi)_s| \le 4\delta L_\delta$ for all $\phi \in X$ and integers s with $0 \le s \le \ell$.*

We will apply a fixed point argument to the operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, defined by

$$\mathcal{G}(u, \phi) = E\phi + \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta(u). \tag{8.2}$$

Theorem 8.2. *Consider the system (2.1) and suppose that the conditions (HL), (HR1) and (HR2) are all satisfied. Fix $\gamma > 0$ such that the characteristic equation $\det \Delta_M(z) = 0$ has no roots with $0 < |\mathrm{Re}\, z| < \gamma$ and consider any interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ with $k\eta_{\min} < \eta_{\max}$, where k is as defined in (HR2). Then there exist constants $0 < \epsilon < \delta$ such that*

(i) For all $\eta \in [\eta_{\min}, \eta_{\max}]$ and for any $\phi \in X_0$, the fixed point equation $u = \mathcal{G}(u, \phi)$ has a unique solution $u = u^*_\eta(\phi) \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$.

(ii) For any pair $\eta_{\min} \leq \eta_1 < \eta_2 \leq \eta_{\max}$, we have that $u^*_{\eta_2} = \mathcal{J}^1_{\eta_2\eta_1} u^*_{\eta_1}$.

(iii) For all $\xi \in \mathbb{R}$ and all $\phi \in X_0$, we have

$$\left\|(I - Q_M)\mathrm{ev}_\xi u^*_\eta(\phi)\right\| < \delta. \tag{8.3}$$

(iv) For any $\phi \in X_0$ with $\|\phi\| < \epsilon$, we have for all $-2 \leq \theta \leq 2$ that

$$\left\|Q_M \mathrm{ev}_\theta u^*_\eta(\phi)\right\| < \delta. \tag{8.4}$$

(v) For all $\eta \in (k\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is of class C^k and admits the Taylor expansion

$$u^*_\eta(\phi) = E\phi + \frac{1}{2}\mathcal{K}_\eta D^2\mathbf{R}_\delta(0)(\mathrm{ev}_\xi E\phi, \mathrm{ev}_\xi E\phi) + o(\|\phi\|^2), \tag{8.5}$$

if $k \geq 2$, in which $\mathcal{K}_\eta$ acts on the variable ξ.

Proof. (i) First note that as in [14] we can use the Greens function representation (6.20) to uniformly bound $\|\mathcal{K}_\eta\|$ for $\eta \in [\eta_{\min}, \eta_{\max}]$, hence it is possible to choose δ in such a way that for all such η we have

$$\exp(\eta) L_\delta \|\mathcal{K}_\eta\| < \frac{1}{4}. \tag{8.6}$$

This ensures that $\mathcal{G}(\cdot, \phi)$ is Lipschitz continuous with constant $\frac{1}{4}$. Since $\mathcal{G}(\cdot, \phi)$ leaves the ball with radius ρ in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ invariant when $\|E\|_\eta \|\phi\| < \frac{\rho}{2}$, the mapping $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ can be defined using the contraction mapping theorem. By computing

$$\begin{array}{rcl} \left\|u^*_\eta(\phi_1) - u^*_\eta(\phi_2)\right\|_{BC^1_\eta} & \leq & \|E\|_\eta \|\phi_1 - \phi_2\| + \|\mathcal{K}_\eta\| \exp(\eta) L_\delta \left\|u^*_\eta(\phi_1) - u^*_\eta(\phi_2)\right\|_{BC^1_\eta} \\ & \leq & \|E\|_\eta \|\phi_1 - \phi_2\| + \frac{1}{4}\left\|u^*_\eta(\phi_1) - u^*_\eta(\phi_2)\right\|_{BC^1_\eta}, \end{array} \tag{8.7}$$

it is clear that u^*_η is in fact Lipschitz continuous.

(ii) Observing that $\left|\mathbf{R}_\delta\big(\mathrm{ev}_\xi u^*_\eta(\phi)\big)_s\right| \leq 4\delta L_\delta$ for all $0 \leq s \leq \ell$, Lemma 7.1 implies that $\mathcal{K}_{\eta_2}\widetilde{\mathbf{R}}_\delta(u^*_{\eta_1}(\phi)) = \mathcal{K}_{\eta_1}\widetilde{\mathbf{R}}_\delta(u^*_{\eta_1}(\phi))$, from which the result follows immediately.

(iii) If $\delta > 0$ is chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$

$$\|I - Q_M\| \exp(\eta_0) L_\delta < (4\|\mathcal{K}_{\eta_0}\|)^{-1}, \tag{8.8}$$

then we can use Lemma 7.2 and Lemma 7.1 to compute

$$\begin{array}{rcl} \left\|(I-Q_M)\mathrm{ev}_\xi u_\eta^*(\phi)\right\| & = & \left\|(I-Q_M)\mathrm{ev}_\xi \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi))\right\| = \left\|(I-Q_M)\mathrm{ev}_0 \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi))(\cdot+\xi)\right\| \\ & \leq & \|I-Q_M\|\exp(\eta_0)\,\|\mathcal{K}_{\eta_0}\|\,4\delta L_\delta < \delta. \end{array} \tag{8.9}$$

(iv) If $\delta > 0$ and $\epsilon > 0$ are chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$

$$\begin{array}{lcl} (8\,\|\mathcal{K}_{\eta_0}\|)^{-1} & > & L_\delta(\exp(3\eta_0) + \|I-Q_M\|\exp(\eta_0)) \quad \text{and} \\ \frac{1}{2}\delta & > & \epsilon\exp(3\eta_0)\,\|E\|_{\eta_0}\,, \end{array} \tag{8.10}$$

then we can compute

$$Q_M\mathrm{ev}_\theta u_\eta^*(\phi) = \mathrm{ev}_\theta E\phi + \mathrm{ev}_\theta\mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi)) - (I-Q_M)\mathrm{ev}_0\mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta\big(u_\eta^*(\phi)(\theta+\cdot)\big) \tag{8.11}$$

and hence

$$\begin{array}{rcl} Q_M\mathrm{ev}_\theta u_\eta^*(\phi) & \leq & \exp(2\eta_0)\exp(\eta_0)\,\|E\|_{\eta_0}\,\|\phi\| \\ & & +4\delta L_\delta\big(\exp(2\eta_0)\exp(\eta_0)\,\|\mathcal{K}\|_{\eta_0} + \|I-Q_M\|\exp(\eta_0)\,\|\mathcal{K}\|_{\eta_0}\big) < \delta. \end{array} \tag{8.12}$$

(v) Notice that item (iii) ensures that u_η^* maps precisely into the region on which the modification of R in the infinite dimensional hyperbolic direction is trivial, which means that $\mathbf{R}_\delta$ is C^k-smooth in this region. This fact ensures that we can follow the approach in [14] to prove that u^* is in fact C^k-smooth, in the sense defined above.

□

In order to show that u_η^* behaves appropriately under translations, we need to be able to control the size of the center part of $u_\eta^*(\phi)$, as is made precise in the next result.

Lemma 8.3. *Consider the setting of Theorem 8.2 and let $\phi \in X_0$. Consider any $\xi_0 \in \mathbb{R}$ such that $\left\|Q_M\mathrm{ev}_\xi u_\eta^*(\phi)\right\| < \delta$ for all $-1-|\xi_0| \leq \xi \leq 1+|\xi_0|$. Then the following identity holds,*

$$u_\eta^*(\phi)(\xi_0+\cdot) = \big[u_\eta^*\big(Q_0\mathrm{ev}_{\xi_0}u_\eta^*(\phi)\big)\big](\cdot). \tag{8.13}$$

Proof. Due to item (iii) of Theorem 8.2, we have $\left\|(I-Q_M)\mathrm{ev}_\xi u_\eta^*(\phi)\right\| < \delta$ for all $\xi \in \mathbb{R}$. From (HR1), the definition of $\mathbf{R}_\delta$ in (8.1) and the condition in the statement of the lemma, it now follows that $\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi))_{|J} = \mathcal{J}g$ for some $g \in C^\ell(J,\mathbb{C}^n)$, where J denotes the interval $J = [-1-|\xi_0|\,, 1+|\xi_0|]$. We can hence apply Lemma 7.2 to conclude that the function

$$y(\xi) = E\phi(\xi_0+\xi) + \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi))(\xi_0+\xi) - \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta\big(u_\eta^*(\phi)(\xi_0+\cdot)\big)(\xi) \tag{8.14}$$

satisfies $\Lambda y = 0$, with $y = E\psi$ for $\psi = Q_0\mathrm{ev}_{\xi_0}u_\eta^*(\phi)$. Upon calculating

$$\begin{array}{rcl} \mathcal{G}\big(u_\eta^*(\phi)(\xi_0+\cdot),\psi\big)(\xi) & = & y(\xi) + \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta\big(u_\eta^*(\phi)(\xi_0+\cdot)\big)(\xi) \\ & = & E\phi(\xi_0+\xi) + \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta(u_\eta^*(\phi))(\xi_0+\xi) = u_\eta^*(\phi)(\xi_0+\xi), \end{array} \tag{8.15}$$

the conclusion follows from the uniqueness of fixed points for $\mathcal{G}$. □

We are now ready to construct the ODE that describes the dynamics on the center manifold. Note that in contrast to the situation in [14], this is no longer possible globally, but upon combining the results in Lemma 8.3 and item (iv) of Theorem 8.2 the ODE can at least be defined locally. Nevertheless, the next result will turn out to be strong enough to lift sufficiently small solutions of the ODE (8.16) back to solutions of (2.1).

Proposition 8.4. *Consider for any $\phi \in X_0$ the function $\Phi : \mathbb{R} \to X_0$, given by $\Phi(\xi) = Q_0 \mathrm{ev}_\xi u_\eta^*(\phi)$. Suppose that for some $\xi_0 > 0$ we have $\|\Phi(\xi)\| < \epsilon$ for all $\xi \in (-\xi_0, \xi_0)$. Then Φ satisfies the following ODE on the interval $(-\xi_0, \xi_0)$,*

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f_\delta(\Phi(\xi)). \tag{8.16}$$

Here the function $f_\delta : X_0 \to X_0$ is C^k-smooth and is explicitly given by

$$f_\delta(\psi) = Q_0\big(M \mathrm{ev}_\theta(u_\eta^*(\psi) - E\psi) + \beta_M(\overline{D} - \alpha_M)^\ell \mathbf{R}_\delta(\mathrm{ev}_\theta u_\eta^*(\psi))\big), \tag{8.17}$$

in which the projection Q_0 is taken with respect to the variable θ and the expression $\overline{D}^s \mathbf{R}_\delta(\cdot)$ should be read as $\mathbf{R}_\delta(\cdot)_s$. Finally, we have $f_\delta(0) = 0$ and $Df_\delta(0) = 0$.

Proof. Notice first that Φ is a continuous function, since $\xi \mapsto \mathrm{ev}_\xi u_\eta^*(\phi)$ is continuous. We calculate

$$\begin{array}{rcl} \dot{\Phi}(\xi)(\sigma) & = & \lim_{h\to 0} \frac{1}{h}(\Phi(\xi+h)(\sigma) - \Phi(\xi)(\sigma)) \\ & = & \lim_{h\to 0} \frac{1}{h}\big([Q_0 \mathrm{ev}_{\xi+h} u_\eta^*(\phi)](\sigma) - [Q_0 \mathrm{ev}_\xi u_\eta^*(\phi)](\sigma)\big) \\ & = & \big[Q_0[Du_\eta^*(\phi)](\xi + \cdot)\big](\sigma), \end{array} \tag{8.18}$$

where the continuity of the projection Q_0, together with the fact that $\mathcal{K}_\eta$ maps into $C^1(\mathbb{R}, \mathbb{C}^n)$, was used in the last step. Using the definition of $\mathcal{K}_\eta$, we compute

$$[Du_\eta^*(\phi)](\xi + \theta) = M \mathrm{ev}_{\xi+\theta} u_\eta^*(\phi) + \beta_M(\overline{D} - \alpha_M)^\ell \mathbf{R}_\delta\big(\mathrm{ev}_{\xi+\theta} u_\eta^*(\phi)\big). \tag{8.19}$$

Assume for the moment that for all $\xi \in (-\xi_0, \xi_0)$ and all $-1 \le \theta \le 1$ we have that $\mathrm{ev}_{\xi+\theta} u_\eta^*(\phi) = \mathrm{ev}_\theta u_\eta^*(\psi)$, where $\psi = \Phi(\xi)$. Then the ODE (8.16) follows upon noting that

$$Q_0(M \mathrm{ev}_\theta E\psi) = Q_0(D\psi(\theta)) = Q_0((A\psi)(\theta)) = A\psi, \tag{8.20}$$

in which Q_0 acts on the variable θ. The fact that f is C^k-smooth follows from the fact that the C^k-smooth function $u_\eta^* : X_0 \to BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$ maps into a region on which $\widetilde{\mathbf{R}}_\delta$ is itself C^k-smooth by part (iii) of Theorem 8.2. It is easy to see that $f_\delta(0) = 0$ and from (HR2) and the Taylor expansion (8.5), it follows that $Df_\delta(0) = 0$. The fact that Φ is C^{k+1}-smooth follows from repeated differentiation of (8.16).

To conclude the proof, write $\phi = \Phi(0)$ and notice that $\|\phi\| < \epsilon$, which by (iv) of Theorem 8.2 implies that $\left\|Q_M \mathrm{ev}_\xi u_\eta^*(\phi)\right\| < \delta$ for all $-2 \le \xi \le 2$. This allows us to apply Lemma 8.3 to conclude that for all $-1 \le \xi' \le 1$ and all $\theta \in \mathbb{R}$, we have

$$\mathrm{ev}_{\xi'+\theta} u_\eta^*(\phi) = \mathrm{ev}_\theta u_\eta^*\big(Q_0 \mathrm{ev}_{\xi'} u_\eta^*(\phi)\big) = \mathrm{ev}_\theta u_\eta^*(\Phi(\xi')). \tag{8.21}$$

Since also $\|\Phi(\xi')\| < \epsilon$ for all $-\xi_0 \le \xi' \le \xi_0$, the above identity implies that also $\left\|Q_M \mathrm{ev}_\xi u^*_\eta(\phi)\right\| < \delta$ for all $-\min(\xi_0, 1) - 2 \le \xi \le \min(\xi_0, 1) + 2$, implying (8.21) for all $-\min(\xi_0, 1) - 1 \le \xi' \le 1 + \min(\xi_0, 1)$. Repeating this procedure a sufficient number of times ensures that in fact (8.21) holds for all $\xi' \in (-\xi_0, \xi_0)$, as required. □

Proof of Theorem 2.2. We choose $\delta > \epsilon > 0$ as in the statement of Theorem 8.2 and fix the constant $\epsilon^* > 0$ such that $\epsilon^* \max(\|Q_M\|, \|Q_0\|, \|I - Q_M\|) < \epsilon$. Pick any $\eta \in (k\eta_{\min}, \eta_{\max}]$ and write $u^* = u^*_\eta$.

(i) This follows from Theorem 8.2 together with $u^* = u^*_\zeta = \mathcal{J}^1_{\zeta\eta_{\min}} u^*_{\eta_{\min}}$ for any $\zeta \in (k\eta_{\min}, \eta_{\max}]$.

(ii) The conditions (HR1) and (HR2) together with (i) imply that f is C^k-smooth with $f(0) = Df(0) = 0$. Since $\xi \mapsto x_\xi$ maps into the subset of X on which $\mathbf{R}$ and $\mathbf{R}_\delta$ agree, it is easy to see that $\mathcal{G}(x, Q_0 x_0) = x$ which due to the uniqueness of fixed points immediately implies $x = u^*(Q_0 x_0)$. An application of Lemma 8.3 shows that indeed $x_\xi = \mathrm{ev}_0 u^*(\Phi(\xi))$. Note that for all $\xi \in \mathbb{R}$ we have $\|\Phi(\xi)\| < \epsilon$, which implies that Φ satisfies the ODE (8.16) on the line. It hence suffices to show f and f_δ agree on all $\Phi(\xi)$. This however follows immediately from the fact that $\|Q_M \mathrm{ev}_\theta u^*(\Phi(\xi))\| = \|Q_M x_{\xi+\theta}\| < \epsilon < \delta$.

(iii) This is clear from the fact that $\xi \mapsto \mathrm{ev}_\xi u^*(\phi)$ maps into the subset of x on which $\mathbf{R}$ and $\mathbf{R}_\delta$ agree.

(iv) Define the function $\Psi(\xi) = Q_0 \mathrm{ev}_\xi u^*(\Phi(0))$. Since $\|\Psi(0)\| = \|\Phi(0)\| < \epsilon$, there exists an interval $(-\xi_0, \xi_0)$ with $\xi_0 > 0$, on which the ODE (8.16) is satisfied for Ψ. However, since f and f_δ agree on the set $\{\phi \in X_0 \mid \|\phi\| < \epsilon\}$ and both nonlinearities are Lipschitz continuous, we can conclude that in fact (8.16) is satisfied on the line, with $\Psi(\xi) = \Phi(\xi)$ for all $\xi \in \mathbb{R}$. Thus defining $x = u^*(\Phi(0))$, we have by construction that $\Phi(\xi) = Q_0 x_\xi$. It remains to show that $\|Q_M x_\xi\| < \delta$ for all $\xi \in \mathbb{R}$ and $x_\xi = \mathrm{ev}_0 u^*(\Phi(\xi))$. Writing $\phi = \Phi(0)$, note that $\|\phi\| < \epsilon$ which implies that $\|Q_M \mathrm{ev}_\xi u^*(\phi)\| < \delta$ for all $-2 \le \xi \le 2$. This allows us to apply Lemma 8.3 to conclude that for all $-1 \le \xi' \le 1$ and all $\theta \in \mathbb{R}$, we have

$$\mathrm{ev}_{\xi'+\theta} u^*_\eta(\phi) = \mathrm{ev}_\theta u^*_\eta\big(Q_0 \mathrm{ev}_{\xi'} u^*_\eta(\phi)\big) = \mathrm{ev}_\theta u^*_\eta(\Psi(\xi')) = \mathrm{ev}_\theta u^*_\eta(\Phi(\xi')). \tag{8.22}$$

Arguing as in Proposition 8.4 we can extend the conclusions above to $\xi \in \mathbb{R}$ and $\xi' \in \mathbb{R}$, which concludes the proof.

□

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