Travelling Waves for Adaptive Grid Discretizations of Reaction Diffusion Systems III: Nonlinear Theory

H. J. Hupkes [a,*], E. S. Van Vleck [b]

[a] *Mathematisch Instituut - Universiteit Leiden*
P.O. Box 9512; 2300 RA Leiden; The Netherlands
Email: hhupkes@math.leidenuniv.nl

[b] *Department of Mathematics - University of Kansas*
1460 Jayhawk Blvd; Lawrence, KS 66045; USA
Email: erikvv@ku.edu

Abstract

In this paper we consider a spatial discretization scheme with an adaptive grid for the Nagumo PDE and establish the existence of travelling waves. In particular, we consider the time dependent spatial mesh adaptation method that aims to equidistribute the arclength of the solution under consideration. We assume that this equidistribution is strictly enforced, which leads to the non-local problem with infinite range interactions that we derived in [26]. Using the Fredholm theory developed in [27] we setup a fixed point procedure that enables the travelling PDE waves to be lifted to our spatially discrete setting.

AMS 2010 Subject Classification: 34K31, 37L15.

Key words: travelling waves, adaptive grids, singular perturbations, spatial discretizations.

1 Introduction

Our goal in this paper is to complete the program initiated in [26, 27] to analyze the impact of adaptive discretization schemes on scalar bistable reaction-diffusion PDEs of the form

$$u_t = u_{xx} + g(u). \tag{1.1}$$

In particular, for any discretization distance $h > 0$ and any $j \in \mathbb{Z}$, we write $x_{jh}(t)$ for the time-dependent location of the relevant gridpoint and $U_{jh}(t)$ for the associated approximation for $u\big(x_{jh}(t), t\big)$. We then study the system

$$\begin{array}{lcl} \dot{U}_{jh}(t) & = & \Big[\frac{U_{(j+1)h}(t) - U_{(j-1)h}(t)}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big]\dot{x}_{jh}(t) \\ & & \quad + \frac{2}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big[\frac{U_{(j-1)h}(t) - U_{jh}(t)}{x_{jh}(t) - x_{(j-1)h}(t)} + \frac{U_{(j+1)h}(t) - U_{jh}(t)}{x_{(j+1)h}(t) - x_{jh}(t)}\Big] + g\big(U_{jh}(t)\big), \end{array} \tag{1.2}$$

*Corresponding author.

in which $x(t)$ is defined implicitly by demanding that

$$\big(x_{(j+1)h}(t) - x_{jh}(t)\big)^2 + \big(U_{(j+1)h}(t) - U_{jh}(t)\big)^2 \quad = \quad h^2 \tag{1.3}$$

and imposing the boundary constraint

$$\lim_{j \to -\infty} [x_{jh}(t) - jh] = 0. \tag{1.4}$$

This means that mesh locations are determined dynamically in time so that there is equidistribution of a finite difference approximation of the arclength of the solution of (1.2). We show that this system has solutions of the form

$$U_{jh}(t) = \Phi(x_{jh}(t) + ct), \tag{1.5}$$

which can be interpreted as travelling waves. For concreteness, we will use the cubic nonlinearity

$$g(u) = g_{\mathrm{cub}}(u; a) = u(1-u)(u-a), \qquad\qquad 0 < a < 1 \tag{1.6}$$

throughout this introduction to explain the main ideas.

Travelling waves The pair (Φ, c) that we construct will be close to the travelling wave (Φ_*, c_*) for the PDE (1.1). Using (1.6), this pair must satisfy the travelling wave ODE

$$c\Phi_*' = \Phi_*'' + g_{\mathrm{cub}}(\Phi_*; a), \qquad\qquad \Phi_*(-\infty) = 0, \qquad \Phi_*(+\infty) = 1. \tag{1.7}$$

Such solutions provide a mechanism through which the fitter biological species (corresponding to the deepest well of the potential $-\int g_{\mathrm{cub}}$) can become dominant throughout a spatial domain. For this reason they are sometimes referred to as *invasion waves*.

It is well-known that these waves play an important role in the global dynamics of (1.1). For example, using the comparison principle one can show that these waves are nonlinearly stable under a large class of perturbations [14] and that they determine the spreading speed of localized structures [39]. In addition, they have been used extensively as building blocks to construct general time dependent solutions of reaction-diffusion systems. For example, planar versions of (1.1) support (sharp) travelling corners [4, 16], expansion waves [33], scattering waves [3] and modulated waves [9] that connect periodic travelling waves of nearby frequencies.

Uniform spatial discretizations In order to set the stage, let us return to the lattice differential equation (LDE)

$$\dot{U}_j(t) = \frac{1}{h^2}[U_{j-1}(t) + U_{j+1}(t) - 2U_j(t)] + g_{\mathrm{cub}}\big(U_j(t); a\big), \tag{1.8}$$

which can be used to describe the uniformly discretized approximants $U_j(t) \sim u(jh, t)$. Mathematically speaking, the transition from (1.1) to (1.8) breaks the continuous translational symmetry of the underlying space. Indeed, (1.8) merely admits the discrete group of symmetries $j \mapsto j + k$ with $k \in \mathbb{Z}$. As a consequence, travelling wave solutions

$$U_j(t) = \Phi(jh + ct) \tag{1.9}$$

can no longer be seen as equilibria in an appropriate comoving frame. Instead, they must be treated as periodic solutions modulo the discrete shift symmetry discussed above. The resulting challenges occur frequently in similar discrete settings and general techniques have been developed to overcome them [2, 7, 15].

Direct substitution of (1.9) into (1.8) yields the travelling wave equation

$$c\Phi'(\xi) = \frac{1}{h^2}[\Phi(\xi - h) + \Phi(\xi + h) - 2\Phi(\xi)] + g_{\mathrm{cub}}(\Phi(\xi); a), \tag{1.10}$$

to which we again append the boundary conditions

$$\Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1. \tag{1.11}$$

Due to the presence of the shifted arguments such equations are known as functional differential equations of mixed type (MFDEs). Note that the unbounded second derivative operator in (1.7) has been replaced by a bounded second-difference operator. In addition, the transition $c \to 0$ is now singular since it changes the structure of the equation. As a consequence, there is a fundamental difference between standing and moving wave solutions to (1.8).

In the anti-continuum regime $h \gg 1$, the second-difference operator can be treated as a small perturbation to the remaining ODE. An elegant construction pioneered by Keener [28] allows one to construct standing waves for $a \neq \frac{1}{2}$ that satisfy the boundary conditions (1.11) and block the two stable background states $\Phi \equiv 0$ and $\Phi \equiv 1$ from invading the domain. In particular, the simple geometric condition [27, Eq. (1.5)] is violated in this setting. This phenomenon is often referred to as *pinning* or *propagation failure* and has attracted a considerable amount of attention [1, 8, 10, 11, 18, 22].

In the intermediate $h \sim 1$ regime the shifted terms cannot be handled so easily and one needs to understand the full MFDE. Such equations are ill-posed as initial value problems and hence must be handled delicately. Several important tools have been developed to accomplish this, such as Fredholm theory [29] and exponential dichotomies [17, 31, 34, 35]; see [21] for a detailed overview.

Using a global homotopy argument together with the comparison principle, Mallet-Paret [30] constructed a branch of solutions $\big(\Phi(a), c(a)\big)$ to (1.10) with (1.11), in which $c(a)$ is unique and $\Phi(a)$ is unique up to translation when $c(a) \neq 0$. For the uniform spatial discretization of the FitzHugh-Nagumo PDE [26, Eq. (1.3)], a generalization of Lin's method can be used to establish a version of the exchange Lemma for MFDEs and construct stable travelling pulses [23, 24]. Further results in this area can be found in [5, 6, 28, 30, 40] and the survey [21].

Uniform spatial-temporal discretizations The so-called backward differentiation formula (BDF) are a family of six schemes that can be used for discretizing the temporal derivative in (1.8). These are well-known multistep methods that are appropriate for parabolic PDEs due to their numerical stability properties. As an illustration, we note that the two lowest order schemes prescribe the substitutions

$$\begin{array}{lll} \dot{U}_j(t) & \mapsto & \frac{1}{\Delta t}\big[U_j\big(n\Delta t\big) - U_j\big((n-1)\Delta t\big)\big], \\ \dot{U}_j(t) & \mapsto & \frac{1}{2\Delta t}\big[3U_j\big(n\Delta t\big) - 4U_j\big((n-1)\Delta t\big) + U_j\big((n-2)\Delta t\big)\big], \end{array} \tag{1.12}$$

in which $\Delta t > 0$ denotes the timestep and $t = n\Delta t$. The first scheme is also known as the backward Euler method and has the advantage that is preserves the comparison principle, unlike the five other members of the family.

In [25] we constructed fully discretized travelling wave solutions

$$U_j(n\Delta t) = \Phi(j + nc\Delta t), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1 \tag{1.13}$$

for the coupled map lattices arising from these discretization schemes. The relevant travelling wave equations for the two lowest order schemes can be obtained by making the replacements

$$\begin{array}{lll} c\Phi'(\xi) & \mapsto & \frac{1}{\Delta t}\big[\Phi(\xi) - \Phi(\xi - c\Delta t)\big], \\ c\Phi'(\xi) & \mapsto & \frac{1}{2\Delta t}\big[3\Phi(\xi) - 4\Phi(\xi - c\Delta t) + \Phi(\xi - 2c\Delta t)\big] \end{array} \tag{1.14}$$

in the MFDE (1.10). In the first case we leveraged the comparison principle to obtain global results. We established that the $c(a)$ relation can become multi-valued, which clearly distinguishes the fully-discrete regime from its spatially-discrete counterpart. The same behaviour occurs for the five other

BDF methods, but here we only have results for small $\Delta t > 0$. We remark that related phenomena have been observed in monostable KPP systems [32] in the presence of inhomogeneities.

These non-uniqueness results should be seen as part of the program that was initiated in [11–13] to study the impact of temporal and full discretization schemes on various reaction-diffusion systems. Indeed, these papers studied versions of (1.1) with various smooth and piecewise linear bistable nonlinearities. The authors used adhoc techniques to obtain rigorous, formal and first order information concerning the change in the dynamics of traveling wave solutions. In addition, in [7] the authors considered the forward-Euler scheme and used Poincaré return-maps and topological arguments to obtain the existence of fully-discretized waves.

Computational frame In [26] we showed that the dynamics of the coupled system (1.2)-(1.4) can be reduced to an equivalent system of the form

$$\dot{U}_{kh} = \mathcal{G}\Big(\{U_{jh}\}_{j\le k+1}\Big), \tag{1.15}$$

in which $\mathcal{G}$ is a (convoluted) nonlinear expression that we describe explicitly in §2. In order to appreciate this equation, it is insightful to transform (1.1) into a new coordinate system (θ, t) by demanding $\theta_x = \sqrt{1+u_x^2}$. Indeed, in these new arclength coordinates the transformed functions

$$w(\theta,t) = u\big(x(\theta,t),t\big), \qquad \gamma(\theta,t) = \sqrt{1 - w_\theta(\theta,t)^2} \tag{1.16}$$

satisfy the the nonlocal PDE

$$w_t = \gamma^{-2} w_{\theta\theta} + \gamma^2 g(w) + w_\theta \int_{-} \Big(\gamma^{-4} w_{\theta\theta} + g(w)\Big) w_{\theta\theta}, \tag{1.17}$$

in which we use the notation $[\int_{-} f](\theta) = \int_{-\infty}^{\tau} f(\theta')\, d\theta'$. This coincides with the system that arises by taking the formal $h \downarrow 0$ limit in (1.15).

In [27] we constructed a solution to (1.17) by stretching the PDE waveprofile Φ_* into its arclength parametrized form Ψ_* and writing

$$w(\theta,t) = \Psi_*(\theta + ct). \tag{1.18}$$

Motivated by this observation, the main goal of this paper is to find solutions to (1.15) of the form

$$U_j(t) = \Psi(jh + ct), \qquad \Psi(-\infty) = 0, \qquad \Psi(+\infty) = 1 \tag{1.19}$$

that bifurcate from the pair (Ψ_*, c_*). In particular, we study the travelling wave system

$$c\Psi'(\tau) = \mathcal{G}\Big(\{\Psi(\tau + kh)\}_{k\le 1}\Big) \tag{1.20}$$

posed in terms of the computational coordinate $\tau = jh + ct$ rather than the physical coordinate $\xi = x_j(t) + ct$ appearing in (1.5). We note that the discrete term jh now plays the role of θ.

To appreciate the advantage of this indirect approach, we note that any attempt to use ξ will lead to an equation for the waveprofile Φ with shifts that depend on the waveprofile Φ itself. In particular, the resulting wave equation is a state-dependent MFDE with infinite range interactions. At the moment, even state-dependent delay equations with a finite number of shifts are technically very challenging to analyze, requiring special care in the linearization procedure [38]. Indeed, linearizations typically involve higher order (continuous) derivatives, making it very hard to close fixed-point arguments.

Physical frame It turns out that there is a close relation between the two wave Ansatzes (1.5) and (1.19). In order to see this, let us assume for the moment that we have found a triplet (Φ, c, x) for which x and the function U defined in (1.5) satisfy (1.2) together with (1.3)-(1.4). Let us also assume that for each $\vartheta \in \mathbb{R}$ there is a unique increasing sequence $y_{jh;\vartheta}$ with $y_{0;\vartheta} = \vartheta$ for which

$$\big(\Phi(y_{(j+1)h;\vartheta}) - \Phi(y_{jh;\vartheta})\big)^2 + (y_{(j+1)h;\vartheta} - y_{jh;\vartheta})^2 = h^2 \tag{1.21}$$

holds for all $j \in \mathbb{Z}$. This can be arranged by imposing a-priori Lipschitz bounds on Φ and Φ' and picking $h > 0$ to be sufficiently small. Finally, let us assume for definiteness that $c > 0$ and that the wave outruns the grid in the sense that $\dot{x}_0(t) + c > \epsilon > 0$.

A direct consequence of this inequality is that

$$x_0(T) + cT = x_h(0) \tag{1.22}$$

for some $T > 0$, which implies

$$U_0(T) = U_h(0) = \Phi\big(x_h(0)\big). \tag{1.23}$$

The uniqueness property discussed above hence implies that

$$U_{jh}(T) = \Phi(y_{jh;x_h(0)}) = \Phi\big(x_{(j+1)h}(0)\big) \tag{1.24}$$

for all $j \in \mathbb{Z}$. Since

$$\begin{aligned} \big(x_{(j+1)h}(T) - x_{jh}(T)\big)^2 &= h^2 - \big(U_{(j+1)h}(T) - U_{jh}(T)\big)^2 \\ &= h^2 - \Big(\Phi\big(x_{(j+2)h}(0)\big) - \Phi\big(x_{(j+1)h}(0)\big)\Big)^2 \\ &= \big(x_{(j+2)h}(0) - x_{(j+1)h}(0)\big)^2, \end{aligned} \tag{1.25}$$

we see that in fact

$$x_{jh}(T) + cT = x_{(j+1)h}(0) \tag{1.26}$$

for all $j \in \mathbb{Z}$. Taking the limit $j \to -\infty$, the boundary conditions (1.4) imply that $cT = h$. Exploiting the well-posedness of our dynamics in forward and backward time, we conclude that

$$x_{jh}(t) = x_0(jT + t) + jh \tag{1.27}$$

holds for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Writing $\Psi_x(\vartheta) = x_0(\vartheta/c)$, we hence find

$$x_{jh}(t) - jh = \Psi_x(jh + ct), \tag{1.28}$$

which implies that

$$U_{jh}(t) = \Phi(x_{jh}(t) + ct) = \Phi\big(jh + \Psi_x(jh + ct) + ct\big) \tag{1.29}$$

for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Upon introducing the function

$$\Psi_U(\tau) = \Phi\big(\tau + \Psi_x(\tau)\big), \tag{1.30}$$

this allows us to obtain the representation

$$\big(U_{jh}(t), x_{jh}(t) - jh\big) = \big(\Psi_U(jh + ct), \Psi_x(jh + ct)\big). \tag{1.31}$$

In fact, we show that for arbitrary solutions U to (1.15) for which $U(t_0)$ is close to $\Psi_*(h\mathbb{Z}+\vartheta)$, we indeed have the pointwise inequalities $|\dot{x}(t_0)| < |c|$ whenever c is sufficiently close to c_*. This can be used to show that the coordinate transformation (1.30) can be inverted, allowing us to reconstruct the profile $\Phi(\xi)$ from $\Psi_U(\tau)$.

Fixed-point setup In order to construct our travelling waves, we write $\Psi = \Psi_* + v$ and decompose (1.20) into the form

$$(c - c_*)v' = \mathcal{L}_h v + \mathcal{G}_{\rm nl}\Big(\{v(\tau + kh)\}_{k\leq 1}\Big) + \mathcal{G}\big(\{\Psi_*(\tau + kh)\}_{k\leq 1}\big) - c\Psi_*', \tag{1.32}$$

using the linear operators $\mathcal{L}_h$ that were introduced in [27]. In the limit $h \downarrow 0$, these operators reduce formally to the operator $\mathcal{L}_*$ associated to the linearization (1.17) around the wave (1.18). The singular nature of the transition between (1.15) and (1.17) is fully encoded in the transition between $\mathcal{L}_*$ and $\mathcal{L}_h$, which was studied at length in [27]. As a result, our analysis in this paper can be seen as the construction of a regular fixed point problem. However, there are two main challenges that need to be overcome.

The first complication is that $\mathcal{L}_h$ is not the 'exact' linearization of $\mathcal{G}$, which is far too complicated to handle. Instead, we recover our operator $\mathcal{L}_h$ after several simplification steps, which each introduce h-dependent errors that need to kept under control. In order to achieve this, we reapply the approximation framework developed in [26] in order to systematically bound the global errors that arise by modifying the individual factors of the products that appear in the definition of $\mathcal{G}$.

The second obstacle is that the nonlinearity $\mathcal{G}$ acts on sequences $U : h\mathbb{Z} \to \mathbb{R}$, while the fixed-point problem (1.32) is formulated in terms of functions $v \in L^2$. Since the relevant transitions between supremum and L^2-based norms cost a factor of $h^{-1/2}$, special care must be taken to construct appropriate function spaces that allow uniform bounds for $h \downarrow 0$. This is particularly dangerous for the terms that are quadratic in the second differences of U, which correspond roughly to the $\int_- \gamma^{-4} w_{\theta\theta}^2$ term in (1.17).

In fact, we need to exploit the special structure of $\mathcal{G}$ and take a discrete derivative of (1.32) in order to close our problem. We hence need to obtain estimates on the discrete derivative of the nonlinear residual $\mathcal{G}_{\rm nl}$, which requires an elaborate bookkeeping system.

Outlook We view our work here as a first step towards understanding the impact of adaptive discretization schemes on travelling waves and other patterns that exist for all time. In particular, we believe that the waves constructed here can be seen as a slow manifold for the dynamics of the full system (1.2) with the non-instantaneous gridpoint behaviour

$$\sigma \dot{x}_j = \sqrt{(x_{j+1} - x_j)^2 + (U_{j+1} - U_j)^2} - \sqrt{(x_{j-1} - x_j)^2 + (U_{j-1} - U_j)^2} \tag{1.33}$$

prescribed by the MMPDE5 scheme [20]. Here $\sigma > 0$ is a tunable speed parameter, which we effectively set to zero by passing to (1.3).

Using the Fredholm theory developed in [27] for the operators $\mathcal{L}_h$ one should be able to leverage the ideas in [37] to effectively track the fast grid-dynamics in the $0 < \sigma \ll 1$ regime. A further step in the program would be to also handle temporal discretizations, inspired by the approach developed in [25] that we described above. Finally, we feel that it is important to understand the stability of the discretized waves under the full dynamics of the numerical scheme. To achieve this, one could follow the approach in [36] to transfer information from the operators $\mathcal{L}_h$ to the linearization around the actual adaptive travelling waves constructed in this paper.

We are specially interested here in the pinning phenomenon. Indeed, numerical observations indicate that the set of detuning parameters a for which $c(a) = 0$ shrinks dramatically when using adaptive discretizations. Understanding this in a rigorous fashion would give considerable insight into the *theoretical* benefits of adaptive grids compared to the *practical* benefits of increased performance. Preliminary results in this direction can be found in [19].

Let us emphasize that the application range of our techniques does not appear to be restricted to the scalar problem (1.1) or the specific grid-update scheme (1.33) that we use. Indeed, using the framework developed in [36], it should be possible to perform a similar analysis for the FitzHugh-Nagumo equation PDE and other multi-component reaction-diffusion problems. In addition, any numerical scheme based on the arclength monitor function will share (1.3) as the instantaneous equidistribution limit.

Overview This paper is organized as follows. After formulating our main results in §2, we introduce our notational framework and recap the key contributions from [26, 27] in §3. In §4 we simplify the nonlinear functions that appear as factors in the product structure of $\mathcal{G}$ and obtain estimates on all the resulting errors. These estimates are used in §5 to compute tractable expressions for the linearization of $\mathcal{G}$ and its discrete derivative $\mathcal{G}^+$ around Ψ_* and obtain bounds on the residuals. We conclude in §6 by combining all these ingredients with the theory developed in [27]. In particular, we develop an appropriate fixed-point argument to construct our desired travelling waves.

In order to develop the main story in a reasonably streamlined fashion that focuses on the key ideas, we have chosen to transfer many of the tedious underlying estimates and algebraic manipulations to the appendices. In order to keep this paper as self-contained as possible, these appendices also summarize some of the fundamental auxiliary bounds that were obtained in [26, 27].

Acknowledgements. Hupkes acknowledges support from the Netherlands Organization for Scientific Research (NWO) (grant 639.032.612). Van Vleck acknowledges support from the NSF (DMS-1419047 and DMS-1714195). Both authors wish to thank W. Huang for helpful discussions during the conception and writing of this paper and an anonymous referee for providing valuable feedback.

2 Main results

The main results of this paper concern adaptive-grid discretizations of the scalar PDE

$$u_t = u_{xx} + g(u). \tag{2.1}$$

Throughout the paper, we assume that the nonlinearity g satisfies the following standard bistability condition.

(Hg) The nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is C^3-smooth and has a bistable structure, in the sense that there exists a constant $0 < a < 1$ such that we have

$$g(0) = g(a) = g(1) = 0, \qquad g'(0) < 0, \qquad g'(1) < 0, \tag{2.2}$$

together with

$$g(u) < 0 \text{ for } u \in (0, a) \cup (1, \infty), \qquad g(u) > 0 \text{ for } u \in (-\infty, -1) \cup (a, 1). \tag{2.3}$$

It is well-known that the PDE (2.1) admits a travelling wave solution that connects the two stable equilibria of g [14]. The key requirement in our next assumption is that this wave is not stationary, which can be arranged by demanding $\int_0^1 g(u)\, du \neq 0$.

($H\Phi_*$) There exists a wave speed $c_* \neq 0$ and a profile $\Phi_* \in C^5(\mathbb{R}, \mathbb{R})$ that satisfies the limits

$$\lim_{\xi \to -\infty} \Phi_*(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi_*(\xi) = 1 \tag{2.4}$$

and yields a solution to the PDE (2.1) upon writing

$$u(x, t) = \Phi_*(x + c_* t). \tag{2.5}$$

In [26] we derived an effective equation for the dynamics of the sequence $U(t) : h\mathbb{Z} \to \mathbb{R}$ featuring in the adaptive scheme (1.2)-(1.4) for (2.1) that no longer explictly depends on the location of the gridpoints. In order to formulate this reduced equation, we recall the discrete derivatives

$$\begin{aligned} [\partial^+ U]_{jh} &= h^{-1}\big[U_{(j+1)h} - U_{jh}\big], \\ [\partial^- U]_{jh} &= h^{-1}\big[U_{jh} - U_{(j-1)h}\big], \\ [\partial^0 U]_{jh} &= (2h)^{-1}\big[U_{(j+1)h} - U_{(j-1)h}\big], \end{aligned} \tag{2.6}$$

together with the first-order differences

$$\mathcal{D}^{\diamond\pm}(U) = \frac{\partial^{\pm}U}{\sqrt{1-(\partial^{\pm}U)^2}}, \qquad \mathcal{D}^{\diamond_0}(U) = \frac{2\partial^0 U}{\sqrt{1-(\partial^{+}U)^2}+\sqrt{1-(\partial^{-}U)^2}} \tag{2.7}$$

and the second order analogues

$$\mathcal{D}^{\diamond\diamond_0}(U) = \frac{2}{h}\frac{\mathcal{D}^{\diamond+}(U)-\mathcal{D}^{\diamond-}(U)}{\sqrt{1-(\partial^{+}U)^2}+\sqrt{1-(\partial^{-}U)^2}}, \qquad \mathcal{D}^{\diamond_0;+}(U) = \partial^{+}\mathcal{D}^{\diamond_0}(U). \tag{2.8}$$

This allows us to introduce the auxiliary functions

$$p(U) = \frac{\mathcal{D}^{\diamond+}(U)}{1+\mathcal{D}^{\diamond+}(U)\mathcal{D}^{\diamond_0}(U)}, \qquad q(U) = h^{-1}\ln\big[1+hp(U)\mathcal{D}^{\diamond_0;+}(U)\big], \tag{2.9}$$

which using the notation

$$\big[\sum_{-;h} a\big]_{jh} = h\sum_{k>0} a_{(j-k)h}, \qquad \big[\sum_{+;h} a\big]_{jh} = h\sum_{k>0} a_{(j+k)h} \tag{2.10}$$

allows us to recall the definitions

$$\mathcal{Q}(U) = \sum_{-;h} q(U), \qquad \mathcal{Z}^{\pm}(U) = \exp\big[\pm\mathcal{Q}(U)\big] \tag{2.11}$$

and subsequently write

$$\mathcal{G}(U) = \mathcal{D}^{\diamond\diamond_0}\big(U\big) + g\big(U\big) - \mathcal{D}^{\diamond_0}\big(U\big)\mathcal{Z}^{-}(U)\sum_{-;h} p(U)\mathcal{Z}^{+}(U)\partial^{+}\big[\mathcal{D}^{\diamond\diamond_0}(U)+g(U)\big]. \tag{2.12}$$

These ingredients allow us to formulate the effective reduced system [26, Eq. (2.25)] for the dynamics of (1.2)-(1.4) as

$$\dot{U}(t) = \mathcal{G}\big(U(t)\big), \tag{2.13}$$

which will be the main system that we analyze in this paper.

We recall the arclength parametrization $\xi_*(\tau)$ defined by the identity

$$\mathcal{A}\big(\xi_*(\tau)\big) = \int_0^{\xi_*(\tau)} \sqrt{1+[\partial_{\xi'}\Phi_*(\xi')]^2}\,d\xi' = \tau, \tag{2.14}$$

together with the stretched waveprofile $\Psi_* : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi_*(\tau) = \Phi_*\big(\xi_*(\tau)\big). \tag{2.15}$$

The main result of this paper states that for sufficiently small $h > 0$, the reduced problem (2.13) admits a travelling wave solution

$$U_{jh}(t) = \Psi_h(jh + c_h t) \tag{2.16}$$

with $(\Psi_h, c_h) \approx (\Psi_*, c_*)$ in an appropriate sense. These waves are locally unique up to translation. We note that items (iv) and (v) use the notation $\partial_h^{+} v = h^{-1}[v(\cdot + h) - v(\cdot)]$ for functions v. In addition, we use the shorthands $L^2 = L^2(\mathbb{R};\mathbb{R})$ and $H^1 = H^1(\mathbb{R};\mathbb{R})$, together with the Heaviside sequence $H_{jh} = \mathbf{1}_{j\geq 0}$.

Theorem 2.1 (see §6)**.** *Suppose that (Hg) and (HΦ_*) are satisfied. Then there exists a constant $\delta_h > 0$ together with pairs*

$$(\Psi_h, c_h) \in C^1(\mathbb{R};\mathbb{R}) \times \mathbb{R}, \tag{2.17}$$

defined for $0 < h \leq \delta_h$, such that the following properties are satisfied.

(i) For every $0 < h \le \delta_h$ we have the limits

$$\lim_{\xi \to -\infty} \Psi_h(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Psi_h(\xi) = 1. \tag{2.18}$$

(ii) For every $0 < h \le \delta_h$ we have the strict inequality

$$\sup_{\tau \in \mathbb{R}} |\Psi_h(\tau + h) - \Psi_h(\tau)| < h. \tag{2.19}$$

(iii) For every $0 < h \le \delta_h$, the function $U : \mathbb{R} \to \ell^\infty(h\mathbb{Z}; \mathbb{R})$ defined by

$$U_{jh}(t) = \Psi_h(jh + c_h t) \tag{2.20}$$

satisfies the inclusion

$$t \mapsto U(t) - H \in C^1\big(\mathbb{R}; \ell^2(h\mathbb{Z}; \mathbb{R})\big). \tag{2.21}$$

In addition, the identity (2.13) and the strict inequality $\|\partial^+ U(t)\|_\infty < 1$ both hold for all $t \in \mathbb{R}$.

(iv) We have $\Psi_h - \Psi_* \in H^1$ for every $0 < h \le \delta_h$ and the limit

$$|c_h - c_*| + \|\Psi_h - \Psi_*\|_{H^1} + \big\|\partial_h^+[\Psi_h - \Psi_*]\big\|_{H^1} + \big\|\partial_h^+\partial_h^+\partial_h^+[\Psi_h - \Psi_*]\big\|_{L^2} \to 0 \tag{2.22}$$

holds as $h \downarrow 0$.

(v) Pick any $0 < h \le \delta_h$ and consider a pair $(\widetilde{\Psi}, \tilde{c}) \in L^\infty \times \mathbb{R}$ that has $\widetilde{\Psi} - \Psi_* \in H^1$ with

$$|\tilde{c} - c_*| + \Big\|\widetilde{\Psi} - \Psi_*\Big\|_{H^1} + \Big\|\partial_h^+[\widetilde{\Psi} - \Psi_*]\Big\|_{H^1} + \Big\|\partial_h^+\partial_h^+\partial_h^+[\widetilde{\Psi} - \Psi_*]\Big\|_{L^2} < h^{3/4}. \tag{2.23}$$

Then the function $\widetilde{U} : \mathbb{R} \to \ell^\infty(h\mathbb{Z}; \mathbb{R})$ defined by

$$\widetilde{U}_{jh}(t) = \widetilde{\Psi}_h\big(jh + \tilde{c}t\big) \tag{2.24}$$

satisfies the inclusion

$$t \mapsto \widetilde{U}(t) - H \in C^0\big(\mathbb{R}; \ell^2(h\mathbb{Z}; \mathbb{R})\big), \tag{2.25}$$

together with the strict inequality $\Big\|\partial^+ \widetilde{U}\Big\|_\infty < 1$ for all $t \in \mathbb{R}$. In addition, if $\widetilde{U}$ is a solution to the system (2.13) for all $t \in \mathbb{R}$, then we must have

$$\big(\widetilde{\Psi}(\cdot), \tilde{c}\big) = \big(\Psi_h(\cdot + \vartheta), c_h\big) \tag{2.26}$$

for some $\vartheta \in \mathbb{R}$.

We emphasize that the location of the gridpoints for the waves (2.16) can be determined by using

$$x_{jh}(t) \quad = \quad jh - \sum_{j' < j} \frac{\big(U_{(j'+1)h}(t) - U_{j'h}(t)\big)^2}{\sqrt{h^2 - (U_{(j'+1)h}(t) - U_{j'h}(t))^2} + h}; \tag{2.27}$$

see [26, Thm. 2.3]. In fact, our final result shows how these waves in the computational coordinates can be interpreted as wave-like objects in the original physical coordinates.

Corollary 2.2 (see §6)**.** *Consider the setting of Theorem 2.1. Then there exists a constant $0 < \widetilde{\delta}_h < \delta_h$ so that for all $0 < h \le \widetilde{\delta}_h$ there exist pairs*

$$(\Psi_h^{(x)}, \Phi_h) \in C^1(\mathbb{R}; \mathbb{R}) \times C^1(\mathbb{R}; \mathbb{R}) \tag{2.28}$$

that satisfy the following properties.

(i) Upon writing

$$\begin{array}{rcl} x_{jh}(t) & = & jh + \Psi_h^{(x)}(jh + c_h t), \\ U_{jh}(t) & = & \Psi_h(jh + c_h t), \end{array} \tag{2.29}$$

the adaptive grid equations (1.2) - (1.4) are satisfied for all $t \in \mathbb{R}$.

(ii) For every $t \in \mathbb{R}$ and $j \in \mathbb{Z}$, the functions defined in (2.29) satisfy the relation

$$U_{jh}(t) = \Phi_h\big(x_{jh}(t) + c_h t\big). \tag{2.30}$$

We remark that if (2.16) and (2.30) both hold, simple substitutions yield the identity

$$\begin{array}{rcl} \Psi_h(jh + c_h t) & = & U_{jh}(t) \\ & = & \Phi_h\big(jh + \Psi_h^{(x)}(jh + c_h t) + c_h t\big) \\ & = & \Phi_h\big(jh + c_h t + \Psi_h^{(x)}(jh + c_h t)\big). \end{array} \tag{2.31}$$

In particular, the main assertion in Corollary 2.2 is that the perturbed coordinate transformation

$$\xi_h(\tau) = \tau + \Psi_h^{(x)}(\tau) \tag{2.32}$$

is invertible for sufficiently small $h > 0$, allowing us to transfer the waves back to the original physical framework.

3 Setup and notation

In this section we recall several crucial results and notational conventions introduced in the prequel papers [26, 27]. This will ensure that the current paper can be read reasonably independently. As a preparation, we recall the sequence spaces

$$\begin{array}{rcl} \ell_h^2 & = & \{V : h\mathbb{Z} \to \mathbb{R} \text{ for which } \|V\|_{\ell_h^2}^2 := h \sup_{j\in\mathbb{Z}} |V_{hj}|^2 < \infty\}, \\ \ell_h^\infty & = & \{V : h\mathbb{Z} \to \mathbb{R} \text{ for which } \|V\|_{\ell_h^\infty} := \sup_{j\in\mathbb{Z}} |V_{hj}| < \infty\} \end{array} \tag{3.1}$$

that were introduced in [26, §3.3], together with the higher order norms

$$\begin{array}{rcl} \|V\|_{\ell_h^{2;1}} & = & \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2}, \\ \|V\|_{\ell_h^{2;2}} & = & \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2}, \\ \|V\|_{\ell_h^{2;3}} & = & \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+\partial^+ V\|_{\ell_h^2} \end{array} \tag{3.2}$$

and their counterparts

$$\begin{array}{rcl} \|V\|_{\ell_h^{\infty;1}} & = & \|V\|_{\ell_h^\infty} + \|\partial^+ V\|_{\ell_h^\infty}, \\ \|V\|_{\ell_h^{\infty;2}} & = & \|V\|_{\ell_h^\infty} + \|\partial^+ V\|_{\ell_h^\infty} + \|\partial^+\partial^+ V\|_{\ell_h^\infty}. \end{array} \tag{3.3}$$

For a single fixed $h > 0$ all these norms are naturally equivalent to the ℓ_h^2-norm or the ℓ_h^∞-norm. The point here is that the h^{-1} factor in the definition of ∂^+ introduces a natural scaling that will allow us to formulate h-independent bounds.

In addition, we pick a reference function $U_{\mathrm{ref};*} \in C^2(\mathbb{R}, [0,1])$ that satisfies the properties

$$U_{\mathrm{ref};*}\big((-\infty, -2]\big) = 0, \qquad U_{\mathrm{ref};*}\big([2, \infty)\big) = 1, \qquad 0 \le U'_{\mathrm{ref};*} < 1, \qquad \big|U''_{\mathrm{ref};*}\big| < 1. \tag{3.4}$$

For any $\kappa > 0$, this allows us write

$$U_{\mathrm{ref};\kappa}(\tau) = U_{\mathrm{ref};*}(\kappa\tau) \tag{3.5}$$

and introduce an open subset

$$\mathcal{V}_{h;\kappa} \quad = \quad \{V \in \ell^2_h : \|V\|_{\ell^{2;2}_h} + \|V\|_{\ell^\infty_h} + \|\partial^+\partial^+ V\|_{\ell^\infty_h} < \tfrac{1}{2}\kappa^{-1} \text{ and } \|\partial^+ V\|_{\ell^\infty_h} < 1 - 2\kappa\}. \tag{3.6}$$

We can now recall the affine subset [26, §3.4]

$$\Omega_{h;\kappa} = U_{\mathrm{ref};\kappa}(h\mathbb{Z}) + \mathcal{V}_{h;\kappa} \subset \ell^\infty_h \tag{3.7}$$

that captures the admissable states of the waves that we are interested in and provides adequate control on the necessary difference operators.

Indeed, for each $U \in \Omega_{h;\kappa}$ we have the crucial bound $\|\partial^+ U\|_{\ell^\infty_h} \leq 1 - \kappa$, which ensures that our grid points are well-defined. In addition, the norms $\|\partial^+ U\|_{\ell^{2;1}_h}$, $\|U\|_{\ell^{\infty;2}_h}$ and $\|g(U)\|_{\ell^2_h}$ are all bounded uniformly in $h > 0$. Finally, it is possible to pick $\epsilon_0 > 0$ and $\kappa > 0$ in such a way that for any $0 < h < 1$ and any $v \in H^1$ that has

$$\|v\|_{H^1} + h^{-1/2}\left\|\partial^+ v\right\|_{H^1} < 2\epsilon_0, \tag{3.8}$$

we have the inclusion

$$\big[\Psi_* + v\big](\vartheta + h\mathbb{Z}) \in \Omega_{h;\kappa} \tag{3.9}$$

for all $\vartheta \in [0, h]$. These statements all follow from [26, Prop. 3.1-3.3].

3.1 Linear operators

In [27] we analyzed several important linear operators that will turn out to be closely related to our travelling wave system (1.20). To set the stage, we recall the sequences

$$\gamma_U = \sqrt{1 - (\partial^0 U)^2}, \qquad \partial^{(2)} U = \partial^+\partial^- U, \tag{3.10}$$

which are well defined for any $U \in \Omega_{h;\kappa}$. Following [27, §5], we introduce the linear operators $M_U : \ell^2_h \to \ell^2_h$ that act as

$$M_U[V] = -c_*\gamma_U^{-1}\partial^0 V + 4\gamma_U^{-4}\partial^0 U[\partial^{(2)}U]\partial^0 V + \gamma_U^{-2}\partial^{(2)}V + \gamma_U^2 g'(U)V, \tag{3.11}$$

together with their twisted counterparts $L_U : \ell^2_h \to \ell^2_h$ defined by

$$L_U[V] \quad = \quad c_*\partial^0 V + M_U[V] + \partial^0 U \textstyle\sum_{-;h} \gamma_U^{-2}[\partial^{(2)}U] M_U[V], \tag{3.12}$$

always taking $U \in \Omega_{h;\kappa}$.

A special role is reserved for the discrete derivative $\partial^+ M_U$, which we approximate by the linear operator

$$M^+_{U;\mathrm{apx}}[V] = \gamma_U^2\big(\widetilde{M}_{U;I}[V] + \widetilde{M}_{U;II}[V] + \widetilde{M}_{U;III}[V]\big) - 2\gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V]. \tag{3.13}$$

Based on the computations in [27, Prop. 5.5], this decomposition uses the expressions

$$\begin{aligned}\widetilde{M}_{U;I}[V] \quad &= \quad 4[6\gamma_U^{-8} - 5\gamma_U^{-6}][\partial^{(2)}U]^2\partial^0 V \\ &\qquad +8\gamma_U^{-6}\partial^0 U[\partial^{(2)}U]\partial^{(2)}V \\ &\qquad +g''(U)[\partial^0 U]V + g'(U)\partial^0 V, \\ \widetilde{M}_{U;II}[V] \quad &= \quad -3c_*\gamma_U^{-5}\partial^0 U[\partial^{(2)}U]\partial^0 V - c_*\gamma_U^{-3}\partial^{(2)}V\end{aligned} \tag{3.14}$$

that feature at most second differences of V, together with

$$\widetilde{M}_{U;III}[V] = 4\gamma_U^{-6}\partial^0 U[\partial^+\partial^{(2)}U]\partial^0 V + \gamma_U^{-4}\partial^+\partial^{(2)}V \tag{3.15}$$

which contains third differences that needs to be treated carefully. Several crucial bounds for these operators are collected in Proposition C.1.

We are now ready to recall the linear operator $\mathcal{L}_h : H^1 \to L^2$ that acts as

$$\mathcal{L}_h v \quad = \quad -c_* v' + L_{\Psi_*} v, \tag{3.16}$$

where we are slightly abusing notation. Indeed, recalling the discrete evaluation operator

$$[\mathrm{ev}_\vartheta f]_{jh} = f(\vartheta + jh) \tag{3.17}$$

that 'samples' a function f on the grid $\vartheta + h\mathbb{Z}$, the identity $\mathcal{L}_h v = f$ should be interpreted as the statement that

$$\mathrm{ev}_\vartheta\big[c_* v' + f\big] = L_{\mathrm{ev}_\vartheta \Psi_*}[\mathrm{ev}_\vartheta v] \tag{3.18}$$

holds for each $\vartheta \in [0, h]$. We remark that the right-hand side above is continuous in ℓ^2_h as a function of ϑ as a consequence of (A.6) and the continuity of the translation operator on H^1.

The key purpose of [27] was to construct a quasi-inverse for the operator $\mathcal{L}_h$. Indeed, [27, Thm. 2.3] establishes the existence of two linear maps

$$\beta_h^* : L^2 \to \mathbb{R}, \qquad\qquad \mathcal{V}_h^* : L^2 \to H^1, \tag{3.19}$$

defined for small $h > 0$, so that for each $f \in L^2$ the pair

$$(\beta, v) = \big(\beta_h^* f, \mathcal{V}_h^* f\big) \in \mathbb{R} \times H^1 \tag{3.20}$$

is the unique solution to the problem

$$\mathcal{L}_h v = f + \beta\Psi_*' \tag{3.21}$$

up to a normalization condition that can be used to fix the phase of our constructed wave. The crucial point is that we obtain h-uniform bounds

$$\begin{array}{lcl} |\beta_h^* f| + \|\mathcal{V}_h^* f\|_{H^1} + \big\|\partial_h^+\partial_h^+\mathcal{V}_h^* f\big\|_{L^2} & \leq & K\,\|f\|_{L^2}\,, \\ \big\|\partial_h^+\mathcal{V}_h^* f\big\|_{H^1} + \big\|\partial_h^+\partial_h^+\partial_h^+\mathcal{V}_h^* f\big\|_{L^2} & \leq & K\big[\,\|f\|_{L^2} + \big\|\partial_h^+ f\big\|_{L^2}\,\big], \end{array} \tag{3.22}$$

which will provide the required control on the second and third differences of our travelling wave.

We remark that these difference operators cannot be replaced by the corresponding derivatives, which forces us to develop a rather delicate fixed point argument in §6. In addition, we note that the spectral convergence framework used to obtain (3.22) relies strongly on the inner product structure of L^2, which explains why we do not have L^∞-based estimates (yet). This is the reason that we go to great lengths throughout the paper to work with ℓ^2_h-bounds as much as possible. Indeed, the results in §A show that these mix well with L^2-functions, unlike supremum bounds.

3.2 Error functions

The errors that need to be controlled during our reduction steps arise from various sources that we briefly discuss here. As preparation, we recall the translation operators

$$[T^+ a]_{jh} = a_{(j+1)h}, \qquad [T^- a]_{jh} = a_{(j-1)h} \tag{3.23}$$

and the sums and products

$$S^\pm a = \frac{1}{2}(a + T^\pm a), \qquad\qquad P^\pm a = aT^\pm a. \tag{3.24}$$

These allow us to recall the function

$$\mathcal{E}_{\mathrm{sm}}(U) = h\partial^{-}\Big[\gamma_U^{-4}(2-\gamma_U^2)S^{+}[\partial^{(2)}U]\Big] \tag{3.25}$$

from [26, Eq. (7.28)], which measures the smoothness of U in some sense. Indeed, it becomes small whenever third differences of U can be controlled, which is the case when taking $U = \Psi_*$.

In a similar vein, we introduce the error functions

$$\begin{array}{rcl} \mathcal{E}_{\mathrm{sh};U}(V) & = & h\,\|V\|_{\ell_h^{2;2}}\,, \\ \overline{\mathcal{E}}_{\mathrm{sh};U}(V) & = & h\,\|V\|_{\ell_h^{3;2}} + h\big[\,\|\partial^{+}\partial^{+}\partial^{+}U\|_{\ell_h^{2}} + \|\partial^{+}\partial^{+}\partial^{+}U\|_{\ell_h^{\infty}}\,\big]\,\|V\|_{\ell_h^{2;2}}\,, \end{array} \tag{3.26}$$

which can be used to 'shift' function evaluations back and forth between neighbouring lattice sites. We note in general that overlined symbols will be used for expressions related to $\mathcal{G}^{+}$, which naturally involve higher order differences than those related to $\mathcal{G}$. Indeed, the nonlinearities in our problem will be controlled by the product

$$\begin{array}{rcl} \mathcal{E}_{\mathrm{prod}}(W^{(1)},W^{(2)}) & = & \big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}} \\ & & \quad + \big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{\infty;1}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;1}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}}\,, \end{array} \tag{3.27}$$

together with

$$\begin{array}{rcl} \overline{\mathcal{E}}_{\mathrm{prod};U}(W^{(1)},W^{(2)}) & = & \|\partial^{+}\partial^{+}\partial^{+}U\|_{\ell_h^{\infty}}\Big[\,\big\|W^{(1)}\big\|_{\ell_h^{2;1}}\big\|W^{(2)}\big\|_{\ell_h^{\infty;1}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;1}}\big\|W^{(2)}\big\|_{\ell_h^{2;1}}\,\Big] \\ & & \quad + \big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}} \\ & & \quad + \big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{\infty;2}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;2}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}} \\ & & \quad + \big\|W^{(1)}\big\|_{\ell_h^{2;3}}\big\|W^{(2)}\big\|_{\ell_h^{\infty;1}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;1}}\big\|W^{(2)}\big\|_{\ell_h^{2;3}}\,. \end{array} \tag{3.28}$$

Observe here that the supremum norms are always at least one order smaller than the highest ℓ^2-based norms. In addition, there are no squares of third-differences or products involving only supremum bounds. These facts will turn out to be crucial when passing between sequences and functions in order to apply the estimates (3.22) in §6.

Our final error functions are given by

$$\mathcal{E}_{\mathrm{tw}}(U) \quad = \quad \gamma_U^{-4}\partial^{(2)}U + g(U) - c_*\gamma_U^{-1}\partial^{0}U, \tag{3.29}$$

together with its approximate first difference

$$\begin{array}{rcl} \mathcal{E}^{+}_{\mathrm{tw;apx}}(U) & = & 4\gamma_U^{-6}\partial^{0}US^{+}[\partial^{(2)}U]T^{+}[\partial^{(2)}U] + \gamma_U^{-4}\partial^{+}\partial^{(2)}U \\ & & \quad + g'(U)\partial^{0}U - c_*\gamma_U^{-3}S^{+}[\partial^{(2)}U]. \end{array} \tag{3.30}$$

The relation between these two functions is explored in Proposition C.2. We view both expressions as a measure for the difference between U and the stretched travelling wave Ψ_*. Indeed, upon introducing the notation

$$\gamma_*(\tau) = \sqrt{1-\Psi_*'(\tau)^2}, \tag{3.31}$$

we recall from [27, Eq. (3.4)] that Ψ_* satisfies the ODE

$$c_*\gamma_*^{-1}\Psi_*' \quad = \quad \gamma_*^{-4}\Psi_*'' + g\big(\Psi_*\big), \tag{3.32}$$

which resembles the continuum limit of (3.29). This can be differentiated to yield

$$c_*\gamma_*^{-3}\Psi_*'' \quad = \quad \gamma_*^{-4}\Psi_*''' + 4\gamma_*^{-6}\Psi_*'(\Psi_*'')^2 + g'\big(\Psi_*\big)\Psi_*', \tag{3.33}$$

the natural limit of (3.30).

Together with the smoothness term $\mathcal{E}_{\mathrm{sm}}$, the functions (3.29) and (3.30) can be used to define our final remainder terms

$$\begin{aligned}
\mathcal{E}_{\mathrm{rem};U}(V) &= \|V\|_{\ell_h^{2;2}} \Big[\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2} \Big], \\
\overline{\mathcal{E}}_{\mathrm{rem};U}(V) &= \|V\|_{\ell_h^{2;2}} \Big[\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2} \Big] \\
&\quad + \|V\|_{\ell_h^{2;1}} \|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\|_{\ell_h^\infty} .
\end{aligned} \tag{3.34}$$

These are small when taking $U = \Psi_*$ and describe the additional error contributions generated in this paper that cannot be absorbed by the terms in [26].

3.3 Initial approximants for $\mathcal{G}$ and $\mathcal{G}^+$

The expression (2.12) for $\mathcal{G}(U)$ is too convoluted for practical use, featuring third differences and double sums. It hence needs to be simplified, at the cost of introducing error terms. An initial step in this direction was performed in [26, Eq. (6.10)], where we decomposed $\mathcal{G}(U)$ into a number of products featuring nonlinearities from the set

$$\mathcal{S}_{\mathrm{nl;short}} = \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{D}^{\diamond_0;+}, \mathcal{D}^{\diamond_-;+}, \mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}, \tag{3.35}$$

which were all defined in [26, §6] and contain at most second differences. A similar decomposition was obtained for $\mathcal{G}^+(U) = \partial^+\mathcal{G}(U)$ in [26, Eq. (6.16)], but now with nonlinearities from the set

$$\overline{\mathcal{S}}_{\mathrm{nl;short}} = \mathcal{S}_{\mathrm{nl;short}} \cup \{\mathcal{Y}_1^+, \mathcal{Y}_{2b}^+\}, \tag{3.36}$$

together with an explicit third-difference term. In addition, for each of the nonlinearities $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$ we (implicitly) defined an approximation $f_{\mathrm{apx}}(U)$ and an approximate linearization $f_{\mathrm{lin};U}[V]$ in [26, §8].

In fact, the full definitions of the nonlinearities $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$ turn out to be irrelevant for our purposes here, so there is no need to repeat them from [26, §6]. However, we do need to manipulate their approximations, which we therefore evaluate in full here by substituting the relevant expressions from [26, §7] into the definitions [26, Eq. (8.1)-(8.4)]. This yields the approximants

$$\begin{aligned}
\mathcal{X}_{A;\mathrm{apx}}(U) &= \partial^0 U, & \mathcal{Y}_{1;\mathrm{apx}}(U) &= \partial^0 U, \\
\mathcal{X}_{B;\mathrm{apx}}(U) &= S^+[\gamma_U^{-1}]\gamma_U^4, & \mathcal{Y}_{2;\mathrm{apx}}(U) &= \gamma_U^{-4}\partial^{(2)}U + g(U), \\
\mathcal{X}_{C;\mathrm{apx}}(U) &= S^+[\gamma_U^{-1}](\gamma_U^4 - \gamma_U^2), & \mathcal{Y}^+_{1;\mathrm{apx}}(U) &= \gamma_U^{-1}S^+[\partial^{(2)}U]T^+\gamma_U, \\
\mathcal{X}_{D;\mathrm{apx}}(U) &= S^+[\gamma_U\partial^0 U]\partial^0 U, & \mathcal{Y}^+_{2b;\mathrm{apx}}(U) &= \Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U\Big] \\
& & &\quad + c_*\gamma_U^{-3}S^+[\partial^{(2)}U],
\end{aligned} \tag{3.37}$$

together with the approximate linearizations

$$\begin{aligned}
\mathcal{X}_{A;\mathrm{lin};U}[V] &= \partial^0 V + \partial^0 U\big[\textstyle\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big], \\
\mathcal{X}_{B;\mathrm{lin};U}[V] &= S^+\Big[\gamma_U^{-3}\partial^0 U\partial^0 V + \gamma_U^{-1}\big[\textstyle\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big]\Big]\gamma_U^4 \\
&\quad + S^+[\gamma_U^{-1}](-4\gamma_U^2)\partial^0 U\partial^0 V, \\
\mathcal{X}_{C;\mathrm{lin};U}[V] &= S^+\Big[\gamma_U^{-3}\partial^0 U\partial^0 V + \gamma_U^{-1}\textstyle\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big](\gamma_U^4 - \gamma_U^2) \\
&\quad + S^+[\gamma_U^{-1}][2 - 4\gamma_U^2]\partial^0 U\partial^0 V, \\
\mathcal{X}_{D;\mathrm{lin};U}[V] &= S^+[\gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V]\partial^0 U + S^+[\gamma_U\partial^0 U]\partial^0 V \\
&\quad + S^+[\gamma_U\partial^0 U]\partial^0 U\textstyle\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V,
\end{aligned} \tag{3.38}$$

respectively

$$\begin{array}{lcl}\mathcal{Y}_{1;\mathrm{lin};U}[V] & = & \partial^0 V - \partial^0 U\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big],\\ \mathcal{Y}_{2;\mathrm{lin};U}[V] & = & \gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V,\\ \mathcal{Y}^+_{1;\mathrm{lin};U}[V] & = & \big[\gamma_U^{-3}\partial^0 U[S^+\partial^{(2)}U]\partial^0 V + \gamma_U^{-1}S^+\partial^{(2)}V\big]T^+\gamma_U\\ & & \qquad -\gamma_U^{-1}S^+[\partial^{(2)}U]T^+\Big[\gamma_U^{-1}\partial^0 U\partial^0 V + \gamma_U\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big],\\ \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] & = & 4[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^{(2)}U]T^+[\partial^{(2)}U]\partial^0 V\\ & & \qquad +4\gamma_U^{-6}\partial^0 U\Big[T^+[\partial^{(2)}U]S^+[\partial^{(2)}V] + S^+[\partial^{(2)}U]T^+[\partial^{(2)}V]\Big]\\ & & \qquad +g''(U)[\partial^0 U]V + g'(U)\partial^0 V.\end{array} \tag{3.39}$$

The corresponding expressions for the two remaining second-difference operators can be copied from [26, Eq. (7.22)] and read

$$\begin{array}{lclclcl}\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U) & = & \gamma_U^{-3}S^+[\partial^{(2)}U], & \qquad & \mathcal{D}^{\diamond_0;+}_{\mathrm{lin};U}[V] & = & 3\gamma_U^{-5}\partial^0 U S^+[\partial^{(2)}U]\partial^0 V + \gamma_U^{-3}S^+[\partial^{(2)}V],\\ \mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U) & = & \gamma_U^{-3}\partial^{(2)}U, & & \mathcal{D}^{\diamond_-;+}_{\mathrm{lin};U}[V] & = & 3\gamma_U^{-5}\partial^0 U[\partial^{(2)}U]\partial^0 V + \gamma_U^{-3}\partial^{(2)}V.\end{array} \tag{3.40}$$

The expressions above were used in [26, §8.1] to define an initial approximant

$$\mathcal{G}_{\mathrm{apx};I}(U) \;=\; \mathcal{G}_{A;\mathrm{apx};I}(U) + \mathcal{G}_{B;\mathrm{apx};I}(U) + \mathcal{G}_{C;\mathrm{apx};I}(U) + \mathcal{G}_{D;\mathrm{apx};I}(U) \tag{3.41}$$

for $\mathcal{G}(U)$, featuring the four components

$$\begin{array}{lcl}\mathcal{G}_{A;\mathrm{apx};I}(U) & = & \Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U)T^-\big[\mathcal{X}_{A;\mathrm{apx}}(U)\big]\Big]\mathcal{Y}_{2;\mathrm{apx}}(U),\\ \mathcal{G}_{B;\mathrm{apx};I}(U) & = & \mathcal{Y}_{1;\mathrm{apx}}(U)\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U),\\ \mathcal{G}_{\#;\mathrm{apx};I}(U) & = & \mathcal{Y}_{1;\mathrm{apx}}(U)\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big],\end{array} \tag{3.42}$$

for $\# \in \{C, D\}$. In addition, we introduced the approximate linearization

$$\mathcal{G}_{\mathrm{lin};U;I}[V] \;=\; \mathcal{G}_{A;\mathrm{lin};U;I}[V] + \mathcal{G}_{B;\mathrm{lin};U;I}[V] + \mathcal{G}_{C;\mathrm{lin};U;I}[V] + \mathcal{G}_{D;\mathrm{lin};U;I}[V] \tag{3.43}$$

by writing

$$\begin{array}{lcl}\mathcal{G}_{A;\mathrm{lin};U;I}[V] & = & -\mathcal{Y}_{1;\mathrm{lin};U}[V]T^-\big[\mathcal{X}_{A;\mathrm{apx}}(U)\big]\mathcal{Y}_{2;\mathrm{apx}}(U)\\ & & \qquad -\mathcal{Y}_{1;\mathrm{apx}}(U)T^-\big[\mathcal{X}_{A;\mathrm{lin};U}[V]\big]\mathcal{Y}_{2;\mathrm{apx}}(U)\\ & & \qquad +\Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U)T^-\big[\mathcal{X}_{A;\mathrm{apx}}(U)\big]\Big]\mathcal{Y}_{2;\mathrm{lin};U}[V]\end{array} \tag{3.44}$$

and applying the analogous product-rule procedure to obtain $\mathcal{G}_{\#;\mathrm{lin};U;I}[V]$ for $\# \in \{B, C, D\}$; see [26, Eq. (8.6)-(8.7)] and §E.1-§E.3. Treating $\mathcal{G}^+$ in a similar spirit, we defined initial approximants

$$\begin{array}{lcl}\mathcal{G}^+_{\mathrm{apx};I}(U) & = & \mathcal{G}^+_{A'a;\mathrm{apx};I}(U) + \mathcal{G}^+_{A'b;\mathrm{apx};I}(U) + \mathcal{G}^+_{A'c;\mathrm{apx};I}(U)\\ & & \qquad +\mathcal{G}^+_{B';\mathrm{apx};I}(U) + \mathcal{G}^+_{C';\mathrm{apx};I}(U) + \mathcal{G}^+_{D';\mathrm{apx};I}(U),\\ \mathcal{G}^+_{\mathrm{lin};U;I}[V] & = & \mathcal{G}^+_{A'a;\mathrm{lin};U;I}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U;I}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U;I}[V]\\ & & \qquad +\mathcal{G}^+_{B';\mathrm{lin};U;I}[V] + \mathcal{G}^+_{C';\mathrm{lin};U;I}[V] + \mathcal{G}^+_{D';\mathrm{lin};U;I}[V],\end{array} \tag{3.45}$$

in which we have introduced the expressions

$$\begin{array}{rcl} \mathcal{G}^+_{A'a;\mathrm{apx};I}(U) & = & \gamma_U^{-2}\partial^+\partial^{(2)}U, \\ \mathcal{G}^+_{A'b;\mathrm{apx};I}(U) & = & \Big[1-\mathcal{Y}_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{apx}}(U)\Big]\mathcal{Y}^+_{2b;\mathrm{apx}}(U), \\ \mathcal{G}^+_{A'c;\mathrm{apx};I}(U) & = & -\mathcal{Y}^+_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{apx}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx}}(U)\big], \end{array} \tag{3.46}$$

together with

$$\begin{array}{rcl} \mathcal{G}^+_{B';\mathrm{apx};I}(U) & = & \mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U), \\ \mathcal{G}^+_{\#';\mathrm{apx};I}(U) & = & \mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big], \end{array} \tag{3.47}$$

for $\# \in \{C, D\}$. With the sole exception of

$$\mathcal{G}^+_{A'a;\mathrm{lin};U;I}[V] \;\;=\;\; \gamma_U^2\widetilde{M}_{U;III}[V] - 2\gamma_U^{-4}\partial^0U[\partial^+\partial^{(2)}U]\partial^0V, \tag{3.48}$$

all the approximate linearizations in (3.45) can be found by applying the product-rule procedure underpinning (3.44) to the expressions (3.46)-(3.47); see [26, §8.2] and §F.1-F.3.

One of our main aims in [26] was to develop a framework to control the errors that arise by these types of approximations. In particular, we needed to track the propagation of errors on the individual factors of (2.12) through the full sums and exponents. The bounds in [26, Lems. 8.1-8.3] provide a constant $K = K(\kappa) > 0$ so that these errors satisfy

$$\begin{array}{rcl} \|\mathcal{G}(U) - \mathcal{G}_{\mathrm{apx};I}(U)\| & \leq & Kh, \\ \Big\|\mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{apx};I}(U)\Big\|_{\ell^2_h} & \leq & Kh\big[1 + \|\partial^+\partial^+\partial^+U\|_{\ell^2_h}\big] \end{array} \tag{3.49}$$

for all $U \in \Omega_{h;\kappa}$. In addition, the nonlinear residuals

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{nl};U;I}(V) & = & \mathcal{G}(U+V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U;I}[V], \\ \mathcal{G}^+_{\mathrm{nl};U;I}(V) & = & \mathcal{G}^+(U+V) - \mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{lin};U;I}[V] \end{array} \tag{3.50}$$

can be estimated as

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{nl};U;I}(V)\|_{\ell^2_h} & \leq & K\mathcal{E}_{\mathrm{prod}}(V,V) + K\mathcal{E}_{\mathrm{sh};U}(V), \\ \Big\|\mathcal{G}^+_{\mathrm{nl};U;I}(V)\Big\|_{\ell^2_h} & \leq & K\overline{\mathcal{E}}_{\mathrm{prod};U}(V,V) + K\overline{\mathcal{E}}_{\mathrm{sh};U}(V) \end{array} \tag{3.51}$$

for any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

These initial approximants for $\mathcal{G}$ and $\mathcal{G}^+$ are already much easier to work with than (2.12) and enabled us to establish the well-posedness of our reduced system (2.13) in [26]. However, they are still unwieldy on account of the shifts and the sums. In addition, several simplifications can be made that only become apparent when looking at the full combinations (3.41), (3.43) and (3.45). This will be the main focus of §4-§5.

Convention Throughout the remainder of this paper, we use the convention that primed constants (such as C_1', C_2' etc) that appear in proofs are positive and depend only on κ and the nonlinearity g, unless explicitly stated otherwise.

4 Component estimates

The first important task in this paper is to build a bridge between the linear theory described in §3.1 and the approximation framework outlined in §3.3. This requires us to refine the approximants introduced in the latter section. We carry out the first step of this procedure here, focusing our attention on the nonlinearities introduced in (3.35)-(3.36).

In particular, for any $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$ we introduce the further decompositions

$$\begin{aligned} f_{\mathrm{apx}}(U) &= f_{\mathrm{apx;expl}}(U) + f_{\mathrm{apx;sh}}(U) + f_{\mathrm{apx;rem}}(U), \\ f_{\mathrm{lin};U}[V] &= f_{\mathrm{lin};U;\mathrm{expl}}[V] + f_{\mathrm{lin};U;\mathrm{sh}}[V] + f_{\mathrm{lin};U;\mathrm{rem}}[V]. \end{aligned} \tag{4.1}$$

The expressions with the label 'expl' are the actual explicit simplifications that will play a key role in our further computations. The label 'sh' is used for terms which are always small, which we will be able to absorb into the error terms $\mathcal{E}_{\mathrm{sh}}$ and $\overline{\mathcal{E}}_{\mathrm{sh}}$ defined in (3.26). Finally, the label 'rem' is used for remainder terms that are small when using $U = \Psi_*$.

The explicit decompositions (4.1) are provided in §D. Our main contribution here is to summarize the errors that arise in a structured fashion that resembles the main spirit of the framework developed in [26, §7]. This will allow us to replace all the occurrences of f_{apx} and $f_{\mathrm{lin};U}$ in §3.3 by their refinements $f_{\mathrm{apx;expl}}$ and $f_{\mathrm{lin};U;\mathrm{expl}}$, leading to a second round of approximations $\mathcal{G}_{\mathrm{apx};II}$, $\mathcal{G}^+_{\mathrm{apx};II}$, $\mathcal{G}_{\mathrm{lin};U;II}$ and $\mathcal{G}^+_{\mathrm{lin};U;II}$.

In order to achieve this, we define a preferred exponent set

$$Q_{f;\mathrm{pref}} \subset \{2, \infty\} \tag{4.2}$$

for each $f \in \mathcal{S}_{\mathrm{nl;short}}$, together with its counterpart

$$\overline{Q}_{f;\mathrm{pref}} \subset \{2, \infty\} \tag{4.3}$$

for each $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$. This is done in such a way that we can write

$$\begin{aligned} \mathcal{G}_{\mathrm{apx};I}(U) &= \textstyle\sum_{i=1}^N \pi_i\big[f_{\mathrm{apx};i;1}(U), \ldots, f_{\mathrm{apx};i;k_i}(U)\big] \\ \mathcal{G}^+_{\mathrm{apx};I}(U) - \mathcal{G}^+_{A'a;\mathrm{apx};I}(U) &= \textstyle\sum_{i=1}^N \overline{\pi}_i\big[\overline{f}_{\mathrm{apx};i;1}(U), \ldots, \overline{f}_{\mathrm{apx};i;k_i}(U)\big], \end{aligned} \tag{4.4}$$

for a set of bounded multi-linear maps

$$\pi_i : \ell_h^{q_{i;1}} \times \ldots \times \ell_h^{q_{i;k_i}} \to \ell_h^2, \qquad \overline{\pi}_i : \ell_h^{\overline{q}_{i;1}} \times \ldots \times \ell_h^{\overline{q}_{i;k_i}} \to \ell_h^2, \tag{4.5}$$

each defined for $1 \le i \le N$, where we have the inclusions

$$f_{i;j} \in \mathcal{S}_{\mathrm{nl;short}}, \qquad q_{i;j} \in Q_{f_{i;j};\mathrm{pref}}, \qquad \overline{f}_{i;j} \in \overline{\mathcal{S}}_{\mathrm{nl;short}}, \qquad \overline{q}_{i;j} \in \overline{Q}_{\overline{f}_{i;j};\mathrm{pref}}, \tag{4.6}$$

for all $1 \le j \le k_i$. Stated more informally, the ℓ_h^2 norm of $\mathcal{G}_{\mathrm{apx};I}(U)$ can be bounded in terms of products of ℓ_2^q-norms of nonlinearities $f \in \mathcal{S}_{\mathrm{nl;short}}$, where each q is taken from the preferred set of exponents. This is the direct analogue of [26, Cor. 6.4].

A short inspection of the products (3.42), (3.46) and (3.47) readily shows that there is some freedom as to which factors should be measured in ℓ_h^2. In fact, it is possible to put an ℓ_h^2 norm on any chosen factor, at the price of possibly having to flip the exponent of a companion factor that has $2 \in Q_{f;\mathrm{pref}}$ from two to infinity.

This freedom is essential to obtain sharp estimates and hence requires us to deviate from the preferred exponents from time to time. The main focus of [26, §5,§7-8] was to develop a bookkeeping framework to keep track of this procedure. We build on this investment here and follow the spirit of [26, §5.2] to define further exponent sets

$$Q_f \subset \{2, \infty\}, \qquad Q_{f;\mathrm{lin}} \subset \{2, \infty\}, \qquad Q_{f;\mathrm{lin;rem}} \subset \{2, \infty\} \tag{4.7}$$

for each $f \in \mathcal{S}_{\mathrm{nl;short}} \cup \overline{\mathcal{S}}_{\mathrm{nl;short}}$. The first of these contains all values of q for which f_{apx} maps into ℓ^q_h. On the other hand, the set $Q_{f;\mathrm{lin}}$ contains all q for which we need to evaluate the ℓ^q_h-norm of $f_{\mathrm{lin};U;\mathrm{expl}}$ and $f_{\mathrm{lin};U;\mathrm{sh}}$, while $Q_{f;\mathrm{lin;rem}}$ contains these exponents for $f_{\mathrm{lin};U;\mathrm{rem}}$.

In order to illustrate these points, let us consider the example

$$\mathcal{I}_{\mathrm{ex};I;U}[V] \quad = \quad \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U)\mathcal{X}_{B;\mathrm{lin};U}[V]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U), \tag{4.8}$$

which appears (after dropping a shift for notational clarity) as a factor in the component $\mathcal{G}_{B;\mathrm{lin};U;I}$ that needs to be evaluated in the supremum norm; see (E.17). Our goal is to simplify this expression by writing

$$\mathcal{I}_{\mathrm{ex};II;U}[V] \quad = \quad \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx;expl}}(U)\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U), \tag{4.9}$$

noting that $\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}$ is not simplified further; see §D. Exploiting the fact that $\mathcal{Y}_{2;\mathrm{apx;sh}} = 0$, a short computation readily yields the decomposition

$$\mathcal{I}_{\mathrm{ex};I;U}[V] = \mathcal{I}_{\mathrm{ex};II;U}[V] + \mathcal{I}_{\mathrm{ex;rem};a} + \mathcal{I}_{\mathrm{ex;sh};a} + \mathcal{I}_{\mathrm{ex;rem};b}, \tag{4.10}$$

where we have introduced the terms

$$\begin{aligned} \mathcal{I}_{\mathrm{ex;rem};a} &= \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U)\mathcal{X}_{B;\mathrm{lin;rem};U}[V]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U), \\ \mathcal{I}_{\mathrm{ex;sh};a} &= \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U)\mathcal{X}_{B;\mathrm{lin};U;\mathrm{sh}}[V]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U), \\ \mathcal{I}_{\mathrm{ex;rem};b} &= \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx;rem}}(U)\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U). \end{aligned} \tag{4.11}$$

We note here that the preferred exponent sets are defined in (D.12), (D.19) and (D.30) and given by

$$Q_{\mathcal{Y}_2;\mathrm{pref}} = \{2\}, \qquad Q_{\mathcal{X}_B;\mathrm{pref}} = \{\infty\}, \qquad Q_{\mathcal{D}^{\diamond_-;+};\mathrm{pref}} = \{2\}. \tag{4.12}$$

Recalling the remainder function introduced in (3.34), we may readily use these preferred exponents to compute

$$\|\mathcal{I}_{\mathrm{ex;rem};a}\|_{\ell^\infty_h} \le \|\mathcal{Y}_{2;\mathrm{apx}}(U)\|_{\ell^2_h} \|\mathcal{X}_{B;\mathrm{lin;rem};U}[V]\|_{\ell^\infty_h} \left\|\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U)\right\|_{\ell^2_h} \le K\mathcal{E}_{\mathrm{rem};U}(V). \tag{4.13}$$

Here we use property (4.33) below with $q = \infty$ (see (D.36)) together with the a-priori bounds (4.21). For the second term we can use the same properties, but now with $q = 2$ (see (D.36)). In particular, we obtain

$$\|\mathcal{I}_{\mathrm{ex;rem};a}\|_{\ell^\infty_h} \le \|\mathcal{Y}_{2;\mathrm{apx}}(U)\|_{\ell^\infty_h} \|\mathcal{X}_{B;\mathrm{lin;sh};U}[V]\|_{\ell^2_h} \left\|\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U)\right\|_{\ell^2_h} \le K\mathcal{E}_{\mathrm{sh};U}(V). \tag{4.14}$$

Note that this required us to swap the first two exponents, which is made possible by the demand $\infty \in Q_f$ for each $f \in \mathcal{S}_{\mathrm{nl;short}}$; see Proposition 4.1.

This swap is also required for the final term, which can be controlled by

$$\|\mathcal{I}_{\mathrm{ex;rem};b}\|_{\ell^\infty_h} \le \|\mathcal{Y}_{2;\mathrm{apx;rem}}(U)\|_{\ell^\infty_h} \|\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^2_h} \left\|\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U)\right\|_{\ell^2_h} \le K\mathcal{E}_{\mathrm{rem};U}(V). \tag{4.15}$$

Indeed, simply using ℓ^∞_h on the middle factor would lead to a contribution proportional to $\|V\|_{\ell^\infty_h}$; see the third line of (D.32). Such a term would lead to problems in §6 and hence is not contained in $\mathcal{E}_{\mathrm{rem};U}$. This scenario is covered in our structural results below by using options (b) from both Proposition 4.2 and 4.3. We feel that this relatively small example already clearly illustrates the benefits of utilizing an abstract bookkeeping scheme instead of direct estimates.

4.1 Summary of estimates

In order to state our results, we introduce the expressions

$$\begin{array}{lclclcl} S_{\mathrm{sh;full}}(U) & = & h, & & S_{\mathrm{sh;2;fix}}(U) & = & 0, \\ \overline{S}_{\mathrm{sh;full}}(U) & = & h\big[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\big], & & \overline{S}_{\mathrm{sh;2;fix}}(U) & = & 0, \end{array} \tag{4.16}$$

together with

$$\begin{array}{lclclcl} S_{\mathrm{rem;full}}(U) & = & \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h}, & & S_{\mathrm{rem;2;fix}}(U) & = & 0, \\ \overline{S}_{\mathrm{rem;full}}(U) & = & S_{\mathrm{rem;full}}(U) + \|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\|_{\ell^2_h}, & & \overline{S}_{\mathrm{rem;2;fix}}(U) & = & 0 \end{array} \tag{4.17}$$

and finally

$$\begin{array}{lcl} S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) & = & \big\|U^{(2)} - U^{(1)}\big\|_{\ell^{2;2}_h} + \big\|U^{(2)} - U^{(1)}\big\|_{\ell^{\infty;1}_h}, \\ S_{\mathrm{diff;2;fix}}(U^{(1)}, U^{(2)}) & = & \big\|U^{(2)} - U^{(1)}\big\|_{\ell^{2;2}_h}. \end{array} \tag{4.18}$$

These expressions are all related to the size of the f_{apx} functions and play a very similar role as the quantities S_{full} and $S_{2;\mathrm{fix}}$ that were defined in [26, §7]. In particular, the 'full' terms correspond to all the exponents that we need to use, while the 'fix' expressions reflect the contributions that are only allowed to be evaluated in ℓ^2_h; see (4.26).

In addition, we recall the quantities

$$\begin{array}{lclclcl} T_{\mathrm{safe}}(V) & = & \|V\|_{\ell^{2;2}_h}, & & \overline{T}_{\mathrm{safe}}(V) & = & T_{\mathrm{safe}}(V), \\ T_{\infty;\mathrm{opt}}(V) & = & \|\partial^+ V\|_{\ell^\infty_h}, & & \overline{T}_{\infty;\mathrm{opt}}(V) & = & T_{\infty;\mathrm{opt}}(V) + \|\partial^+\partial^+ V\|_{\ell^\infty_h} \end{array} \tag{4.19}$$

that are associated to the approximate linearizations f_{lin}. Here $T_{\infty;\mathrm{opt}}$ represents the contributions where the use of the supremum norm is optional, in the sense that they could also be measured in ℓ^2_h. The remaining contributions are all reflected in T_{safe}. We emphasize that the main point of our bookkeeping scheme is to ensure that products of the form $S_{\#;\mathrm{full}} T_{\infty;\mathrm{opt}}$ are never needed, where $\# \in \{\mathrm{sh}, \mathrm{rem}, \mathrm{diff}\}$.

Our main results summarize the structure that the decompositions described in §D will adhere to. Propositions 4.1 and 4.2 state that the approximants $f_{\mathrm{apx};\#}$ are all uniformly bounded and that the full linear approximants $f_{\mathrm{lin};U}$ share the structure and estimates of the nonlinearities in the sets $\mathcal{S}_{\mathrm{nl}} \cup \overline{\mathcal{S}}_{\mathrm{nl}}$ analyzed in [26]. These can be interpreted as the counterparts of [26, Cor. 7.6 and 7.8]. On the other hand, Propositions 4.3-4.4 should be seen as the equivalents of [26, Cor. 7.7], while Propositions 4.5-4.6 are the equivalents of [26, Cor. 7.9].

Proposition 4.1 (see §D)**.** *For every $f \in \mathcal{S}_{\mathrm{nl;short}}$ we have $\infty \in Q_f$ together with*

$$Q_{f;\mathrm{pref}} \subset Q_f \cap Q_{f;\mathrm{lin}} \cap Q_{f;\mathrm{lin;rem}}. \tag{4.20}$$

In addition, there exists $K > 0$ so that for each $q \in Q_f$, the bound

$$\|f_{\mathrm{apx;expl}}(U)\|_{\ell^q_h} + \|f_{\mathrm{apx;sh}}(U)\|_{\ell^q_h} + \|f_{\mathrm{apx;rem}}(U)\|_{\ell^q_h} \le K \tag{4.21}$$

holds for all $h > 0$ and $U \in \Omega_{h;\kappa}$. The same properties hold upon replacing $(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}})$ by $(\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}})$.

Proposition 4.2 (see §D)**.** *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. For any $f \in \mathcal{S}_{\mathrm{nl;short}}$, any $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$ and any $q \in Q_{f;\mathrm{pref}}$, at least one of the following two properties hold true.*

(a) There exists $K > 0$ so that

$$\|f_{\mathrm{lin};U;\#}[V]\|_{\ell^q_h} \le K T_{\mathrm{safe}}(V) \tag{4.22}$$

holds for every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

(b) We have $q = \infty$ and there exists $K > 0$ so that the bounds

$$\begin{array}{rcl} \|f_{\mathrm{lin};U;\#}[V]\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\#}[V]\|_{\ell^\infty_h} & \leq & K T_{\infty;\mathrm{opt}}(V) \end{array} \tag{4.23}$$

hold for every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

The same properties hold upon making the replacement

$$(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}}, T_{\mathrm{safe}}, T_{\infty;\mathrm{opt}}) \mapsto (\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}}, \overline{T}_{\mathrm{safe}}, \overline{T}_{\infty;\mathrm{opt}}). \tag{4.24}$$

Proposition 4.3 (see §D)**.** *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for every $f \in \mathcal{S}_{\mathrm{nl;short}}$, $q \in Q_{f;\mathrm{pref}}$ and $\# \in \{\mathrm{sh}, \mathrm{rem}\}$, we have*

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell^q_h} \quad \leq \quad K S_{\#;\mathrm{full}}(U) \tag{4.25}$$

for any $h > 0$ and $U \in \Omega_{h;\kappa}$.

In addition, if $2 \in Q_{f;\mathrm{pref}}$ then for every $\# \in \{\mathrm{sh}, \mathrm{rem}\}$ at least one of the following two properties hold true.

(a) There exists $K > 0$ so that

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell^2_h} \quad \leq \quad K S_{\#;2;\mathrm{fix}}(U) \tag{4.26}$$

holds for every $h > 0$ and $U \in \Omega_{h;\kappa}$.

(b) There exists $K > 0$ so that

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell^\infty_h} \quad \leq \quad K S_{\#;\mathrm{full}}(U) \tag{4.27}$$

holds for every $h > 0$ and $U \in \Omega_{h;\kappa}$.

The same properties hold upon making the replacement

$$(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}}, S_{\#;\mathrm{full}}, S_{\#;2;\mathrm{fix}}) \mapsto (\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}}, \overline{S}_{\#;\mathrm{full}}, \overline{S}_{\#;2;\mathrm{fix}}). \tag{4.28}$$

Proposition 4.4 (see §D)**.** *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for every $f \in \mathcal{S}_{\mathrm{nl;short}}$, $q \in Q_{f;\mathrm{pref}}$ and $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$, we have*

$$\big\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\big\|_{\ell^q_h} \quad \leq \quad K S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) \tag{4.29}$$

for any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

In addition, if $2 \in Q_{f;\mathrm{pref}}$, then for every $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$ at least one of the following two properties hold true.

(a) There exists $K > 0$ so that

$$\big\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\big\|_{\ell^2_h} \quad \leq \quad K S_{\mathrm{diff};2;\mathrm{fix}}(U^{(1)}, U^{(2)}) \tag{4.30}$$

holds for every $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

(b) There exists $K > 0$ so that

$$\big\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\big\|_{\ell^\infty_h} \quad \leq \quad K S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) \tag{4.31}$$

holds for every $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

Proposition 4.5 (see §D)**.** *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$*. Consider any* $f \in \mathcal{S}_{\mathrm{nl;short}}$ *and any* $\# \in \{\mathrm{sh}, \mathrm{rem}\}$*. Then if* $2 \in Q_{f;\mathrm{pref}}$*, there exists a constant* $K > 0$ *so that*

$$\|f_{\mathrm{lin};U;\#}(V)\|_{\ell^2_h} \leq K\mathcal{E}_{\#;U}(V) \tag{4.32}$$

holds for all $h > 0$*,* $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$ *for which* $U + V \in \Omega_{h;\kappa}$*.*

Otherwise, there exists $q \in \{2, \infty\}$ *together with a constant* $K > 0$ *so that*

$$\|f_{\mathrm{lin};U;\#}(V)\|_{\ell^q_h} \leq K\mathcal{E}_{\#;U}(V) \tag{4.33}$$

holds for all $h > 0$*,* $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$ *for which* $U + V \in \Omega_{h;\kappa}$*. The same properties hold upon making the replacement*

$$(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}}, \mathcal{E}_{\#}) \mapsto (\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}}, \overline{\mathcal{E}}_{\#}). \tag{4.34}$$

Proposition 4.6 (see §D)**.** *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$*. Consider any* $f \in \mathcal{S}_{\mathrm{nl;short}}$ *and any* $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$*. Then if* $2 \in Q_{f;\mathrm{pref}}$*, there exists a constant* $K > 0$ *so that*

$$\left\|f_{\mathrm{lin};U^{(2)};\#}(V) - f_{\mathrm{lin};U^{(1)};\#}(V)\right\|_{\ell^2_h} \leq K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V) \tag{4.35}$$

holds for all $h > 0$*, any pair* $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ *and any* $V \in \ell^2_h$*.*

Otherwise, there exists $q \in \{2, \infty\}$ *together with a constant* $K > 0$ *so that*

$$\left\|f_{\mathrm{lin};U^{(2)};\#}(V) - f_{\mathrm{lin};U^{(1)};\#}(V)\right\|_{\ell^q_h} \leq K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V) \tag{4.36}$$

holds for all $h > 0$*, any pair* $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ *and any* $V \in \ell^2_h$*.*

4.2 Refined approximants for $\mathcal{G}$ and $\mathcal{G}^+$

We now introduce the expressions

$$\{\mathcal{G}_{\#;\mathrm{apx};II}(U),\, \mathcal{G}_{\#;\mathrm{lin};U;II}(U),\, \mathcal{G}^+_{\#';\mathrm{apx};II}(U),\, \mathcal{G}^+_{\#';\mathrm{lin};U;II}(U)\} \tag{4.37}$$

for $\# \in \{A, B, C, D\}$ respectively $\#' \in \{A'b, A'c, B', C', D'\}$ by inspecting the definitions of their predecessors labelled by I in §3.3 (see (3.42), (3.43) and (3.46)) and making the replacements

$$f_{\mathrm{apx}}(U) \mapsto f_{\mathrm{apx;expl}}(U), \qquad f_{\mathrm{lin};U}[V] \mapsto f_{\mathrm{lin};U;\mathrm{expl}}[V] \tag{4.38}$$

for each $f \in \mathcal{S}_{\mathrm{nl;short}} \cup \overline{\mathcal{S}}_{\mathrm{nl;short}}$. The full explicit forms can be found in §E-F, but are not important for our purposes here. We leave the expressions for $A'a$ intact and simply write

$$\mathcal{G}^+_{A'a;\mathrm{apx};II}(U) = \mathcal{G}^+_{A'a;\mathrm{apx};I}(U), \qquad \mathcal{G}^+_{A'a;\mathrm{lin};U;II}[V] = \mathcal{G}^+_{A'a;\mathrm{apx};U;II}[V]. \tag{4.39}$$

Our interest here is in the refined approximants

$$\begin{aligned} \mathcal{G}_{\mathrm{apx};II}(U) &= \mathcal{G}_{A;\mathrm{apx};II}(U) + \mathcal{G}_{B;\mathrm{apx};II}(U) + \mathcal{G}_{C;\mathrm{apx};II}(U) + \mathcal{G}_{D;\mathrm{apx};II}(U) \\ \mathcal{G}^+_{\mathrm{apx};II}(U) &= \mathcal{G}^+_{A'a;\mathrm{apx};II}(U) + \mathcal{G}^+_{A'b;\mathrm{apx};II}(U) + \mathcal{G}^+_{A'c;\mathrm{apx};II}(U) \\ &\quad + \mathcal{G}^+_{B';\mathrm{apx};II}(U) + \mathcal{G}^+_{C';\mathrm{apx};II}(U) + \mathcal{G}^+_{D';\mathrm{apx};II}(U) \end{aligned} \tag{4.40}$$

and the corresponding linearizations

$$\begin{aligned} \mathcal{G}_{\mathrm{lin};U;II}[V] &= \mathcal{G}_{A;\mathrm{lin};U;II}[V] + \mathcal{G}_{B;\mathrm{lin};U;II}[V] + \mathcal{G}_{C;\mathrm{lin};U;II}[V] + \mathcal{G}_{D;\mathrm{lin};U;II}[V] \\ \mathcal{G}^+_{\mathrm{lin};U;II}[V] &= \mathcal{G}^+_{A'a;\mathrm{lin};U;II}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U;II}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] \\ &\quad + \mathcal{G}^+_{B';\mathrm{lin};U;II}[V] + \mathcal{G}^+_{C';\mathrm{lin};U;II}[V] + \mathcal{G}^+_{D';\mathrm{lin};U;II}[V]. \end{aligned} \tag{4.41}$$

In particular, the results below describe the residuals that arise when replacing the initial approximants defined in (3.41), (3.43) and (3.45) by these refined versions. Since our bookkeeping framework has the same overall structure as in [26], we can reuse the analysis developed there in a streamlined fashion.

Lemma 4.7. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with sequences*

$$\mathcal{G}_{\mathrm{apx;sh};a}(U) \in \ell^2_h, \qquad \mathcal{G}_{\mathrm{apx;rem};a}(U) \in \ell^2_h, \qquad \mathcal{G}^+_{\mathrm{apx;sh};a}(U) \in \ell^2_h, \qquad \mathcal{G}^+_{\mathrm{apx;rem};a}(U) \in \ell^2_h, \tag{4.42}$$

defined for every $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the identities

$$\begin{aligned} \mathcal{G}_{\mathrm{apx};I}(U) &= \mathcal{G}_{\mathrm{apx};II}(U) + \mathcal{G}_{\mathrm{apx;sh};a}(U) + \mathcal{G}_{\mathrm{apx;rem};a}(U), \\ \mathcal{G}^+_{\mathrm{apx};I}(U) &= \mathcal{G}^+_{\mathrm{apx};II}(U) + \mathcal{G}^+_{\mathrm{apx;sh};a}(U) + \mathcal{G}^+_{\mathrm{apx;rem};a}(U). \end{aligned} \tag{4.43}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{aligned} \|\mathcal{G}_{\mathrm{apx;sh};a}(U)\|_{\ell^2_h} &\leq K S_{\mathrm{sh;full}}(U) \\ \|\mathcal{G}_{\mathrm{apx;rem};a}(U)\|_{\ell^2_h} &\leq K S_{\mathrm{rem;full}}(U), \end{aligned} \tag{4.44}$$

together with

$$\begin{aligned} \left\|\mathcal{G}^+_{\mathrm{apx;sh};a}(U)\right\|_{\ell^2_h} &\leq K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^+_{\mathrm{apx;rem};a}(U)\right\|_{\ell^2_h} &\leq K\overline{S}_{\mathrm{rem;full}}(U). \end{aligned} \tag{4.45}$$

Proof. Restricting ourselves to $\mathcal{G}$, we consider a single term of the sum (4.4). Dropping the index i, we introduce the corresponding expression

$$\mathcal{I}_\pi(U) = \pi\big[f_{1;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] - \pi\big[f_{1;\mathrm{apx;expl}}(U), \ldots, f_{k;\mathrm{apx;expl}}(U)\big]. \tag{4.46}$$

Recalling the general identity

$$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) - a_1 a_2 a_3 = b_1(a_2 + b_2)(a_3 + b_3) + a_1 b_2 (a_3 + b_3) + a_1 a_2 b_3 \tag{4.47}$$

and its extensions, we write

$$\begin{aligned} \mathcal{I}_{\pi;\#}(U) = \; & \pi\big[f_{1;\mathrm{apx};\#}(U), f_{2;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & + \pi\big[f_{1;\mathrm{apx;expl}}(U), f_{2;\mathrm{apx};\#}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & + \ldots + \pi\big[f_{1;\mathrm{apx;expl}}(U), f_{2;\mathrm{apx;expl}}(U), \ldots, f_{k;\mathrm{apx};\#}(U)\big] \end{aligned} \tag{4.48}$$

for $\# \in \{\mathrm{sh}, \mathrm{rem}\}$ and observe that

$$\mathcal{I}_\pi(U) = \mathcal{I}_{\pi;\mathrm{sh}}(U) + \mathcal{I}_{\pi;\mathrm{rem}}(U). \tag{4.49}$$

We now use (4.21) together with Proposition 4.3 to derive the bound

$$\|\mathcal{I}_{\pi;\#}(U)\|_{\ell^2_h} \leq C_1' S_{\#;\mathrm{full}}(U). \tag{4.50}$$

The desired estimates now follow from the fact that $\mathcal{G}_{\mathrm{apx};I}(U) - \mathcal{G}_{\mathrm{apx};II}(U)$ can be written as a sum of expressions of the form (4.46). □

Lemma 4.8. *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$. *There exists a constant* $K > 0$ *together with linear maps*

$$\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a} \in \mathcal{L}(\ell_h^2, \ell_h^2), \qquad \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};a} \in \mathcal{L}(\ell_h^2, \ell_h^2) \tag{4.51}$$

and their counterparts

$$\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a} \in \mathcal{L}(\ell_h^2, \ell_h^2), \qquad \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};a} \in \mathcal{L}(\ell_h^2, \ell_h^2), \tag{4.52}$$

defined for all $h > 0$ *and* $U \in \Omega_{h;\kappa}$, *so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell_h^2$ *we have the identities*

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{lin};U;I}[V] & = & \mathcal{G}_{\mathrm{lin};U;II}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V], \\ \mathcal{G}^+_{\mathrm{lin};U;I}[V] & = & \mathcal{G}^+_{\mathrm{lin};U;II}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};a}[V]. \end{array} \tag{4.53}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell_h^2$ *we have the bounds*

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V]\|_{\ell_h^2} & \leq & K\mathcal{E}_{\mathrm{sh};U}(V), \\ \|\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V]\|_{\ell_h^2} & \leq & K\mathcal{E}_{\mathrm{rem};U}(V), \end{array} \tag{4.54}$$

together with

$$\begin{array}{rcl} \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};a}[V]\right\|_{\ell_h^2} & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};a}[V]\right\|_{\ell_h^2} & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{4.55}$$

(iii) For every $h > 0$, *any pair* $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ *and any* $V \in \ell_h^2$, *we have the bound*

$$\left\|\mathcal{G}_{\mathrm{lin};U^{(2)};\mathrm{rem};a}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)};\mathrm{rem};a}[V]\right\|_{\ell_h^2} \leq K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \tag{4.56}$$

Proof. Restricting ourselves to $\mathcal{G}$, we again consider a single term of the sum (4.4). Dropping the index i, we introduce the two corresponding expressions

$$\begin{array}{rcl} \mathcal{I}_{\pi;a;U}[V] & = & \pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & & \quad -\pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big], \\ \mathcal{I}_{\pi;b;U}[V] & = & \pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \\ & & \quad -\pi\big[f_{1;\mathrm{lin};U;\mathrm{expl}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big]. \end{array} \tag{4.57}$$

Writing

$$\begin{array}{rcl} \mathcal{I}_{\pi;a;U;\#}[V] & = & \pi\Big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\#}(U), \ldots, f_{k;\mathrm{apx}}(U)\Big] \\ & & \quad + \ldots + \pi\Big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\#}(U)\Big], \\ \mathcal{I}_{\pi;b;U;\#}[V] & = & \pi\big[f_{1;\mathrm{lin};U;\#}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \end{array} \tag{4.58}$$

for $\# \in \{\mathrm{sh}, \mathrm{rem}\}$, we see that

$$\begin{array}{rcl} \mathcal{I}_{\pi;a;U}[V] & = & \mathcal{I}_{\pi;a;U;\mathrm{sh}}[V] + \mathcal{I}_{\pi;a;U;\mathrm{rem}}[V], \\ \mathcal{I}_{\pi;b;U}[V] & = & \mathcal{I}_{\pi;b;U;\mathrm{sh}}[V] + \mathcal{I}_{\pi;b;U;\mathrm{rem}}[V]. \end{array} \tag{4.59}$$

Following the same reasoning used to obtain [26, Eq. (8.15)], we may use Propositions 4.2 and 4.3 to derive the bound

$$\begin{array}{lcl} \|\mathcal{I}_{\pi;a;U;\#}[V]\|_{\ell^2_h} & \leq & C_1'\Big[T_{\mathrm{safe}}(V)S_{\#;\mathrm{full}}(U) + T_{\infty;\mathrm{opt}}(V)S_{\#;2;\mathrm{fix}}(U)\Big] \\ & \leq & C_2'\mathcal{E}_{\#;U}(V). \end{array} \tag{4.60}$$

Indeed, contributions of type $T_{\infty;\mathrm{opt}}(V)S_{\#;\mathrm{full}}(U)$ can be avoided by deviating from the preferred exponents judiciously.

In addition, following the arguments used to derive [26, Eq. (8.12)], we may use Proposition 4.5 to obtain the bound

$$\|\mathcal{I}_{\pi;b;U;\#}[V]\|_{\ell^2_h} \quad \leq \quad C_3'\mathcal{E}_{\#;U}(V). \tag{4.61}$$

Writing

$$\begin{array}{lcl} \Delta_{b;i} & = & \pi\Big[f_{1;\mathrm{lin};U^{(2)};\mathrm{rem}}[V] - f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)})\Big], \\ \Delta_{b;ii} & = & \pi\Big[f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)}) - f_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)})\Big] \\ & & + \ldots \\ & & +\pi\Big[f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)}) - f_{k;\mathrm{apx};\mathrm{expl}}(U^{(1)})\Big], \end{array} \tag{4.62}$$

we easily see that

$$\Delta_{b;i} + \Delta_{b;ii} = \mathcal{I}_{\pi;b;U^{(2)};\mathrm{rem}}[V] - \mathcal{I}_{\pi;b;U^{(1)};\mathrm{rem}}[V]. \tag{4.63}$$

Arguing as above, Proposition 4.6 yields

$$\|\Delta_{b;i}\|_{\ell^2_h} \quad \leq \quad C_1'\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \tag{4.64}$$

while Propositions 4.2 and 4.4 imply

$$\begin{array}{lcl} \|\Delta_{b;ii}\|_{\ell^2_h} & \leq & C_2'\Big[T_{\mathrm{safe}}(V)S_{\mathrm{diff};\mathrm{full}}(U^{(1)}, U^{(2)}) + T_{\infty;\mathrm{opt}}(V)S_{\mathrm{diff};2;\mathrm{fix}}(U^{(1)}, U^{(2)})\Big] \\ & \leq & C_3'\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{4.65}$$

Finally, we write

$$\Delta_a = \mathcal{I}_{\pi;a;U^{(2)};\mathrm{rem}}[V] - \mathcal{I}_{\pi;a;U^{(1)};\mathrm{rem}}[V]. \tag{4.66}$$

We note that Δ_a consists of sums of expressions that arise from $\Delta_{b;i}$ and $\Delta_{b;ii}$ after replacing $f_{1;\mathrm{lin};U^{(i)};\mathrm{rem}}$ by $f_{1;\mathrm{lin};U^{(i)}}$ and each occurrence of $f_{j;\mathrm{apx};\mathrm{expl}}$ by an element of the set

$$\{f_{j;\mathrm{apx}}, f_{j;\mathrm{apx};\mathrm{expl}}, f_{j;\mathrm{apx};\mathrm{rem}}\}. \tag{4.67}$$

We can hence again use Propositions 4.2, 4.4 and 4.6 to conclude that $\|\Delta_a\|_{\ell^2_h}$ can be bounded by terms that have already appeared above. The desired bounds now follow from the fact that $\mathcal{G}_{\mathrm{lin};U;I}[V] - \mathcal{G}_{\mathrm{lin};U;II}[V]$ can be written as a sum of expressions of the form $\mathcal{I}_{\pi;a} + \mathcal{I}_{\pi;b}$, together with their obvious permutations. □

5 Estimates for $\mathcal{G}$ and $\mathcal{G}^+$

In this section we exploit the component estimates from §4 to analyze the function $\mathcal{G}$ defined in (2.12), together with its first difference $\mathcal{G}^+$. In particular, recalling the operator L_U defined in (3.12), we introduce our final approximants

$$\begin{array}{lclclcl} \mathcal{G}_{\mathrm{apx}}(U) & = & c_*\partial^0 U, & \qquad & \mathcal{G}_{\mathrm{lin};U}[V] & = & L_U[V], \\ \mathcal{G}^+_{\mathrm{apx}}(U) & = & \partial^+\big[\mathcal{G}_{\mathrm{apx}}(U)\big], & & \mathcal{G}^+_{\mathrm{lin};U}[V] & = & \partial^+\big[\mathcal{G}_{\mathrm{lin};U}[V]\big] \end{array} \tag{5.1}$$

and write

$$\mathcal{G}_{\mathrm{nl};U}(V) = \mathcal{G}(U+V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U}(V) \tag{5.2}$$

together with

$$\mathcal{G}^+_{\mathrm{nl};U}(V) = \partial^+[\mathcal{G}_{\mathrm{nl};U}(V)] = \mathcal{G}^+(U+V) - \mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{lin};U}(V). \tag{5.3}$$

Using the discrete calculus outlined in §A, one may readily verify the identities

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx}}(U) & = & c_* S^+[\partial^{(2)}U], \\ \mathcal{G}^+_{\mathrm{lin};U}[V] & = & c_* S^+[\partial^{(2)}V] + \partial^+\big[M_U[V]\big] + \gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V] \\ & & \quad +S^+[\partial^{(2)}U]T^+\sum_{-;h}\gamma_U^{-2}\partial^{(2)}UM_U[V]. \end{array} \tag{5.4}$$

Our main result quantifies the approximation errors in terms of the functions $\mathcal{E}_{\mathrm{sh};U}$, $\mathcal{E}_{\mathrm{rem};U}$ and $\mathcal{E}_{\mathrm{prod}}$ and their counterparts $\overline{\mathcal{E}}_{\mathrm{sh};U}$, $\overline{\mathcal{E}}_{\mathrm{rem};U}$ and $\overline{\mathcal{E}}_{\mathrm{prod};U}$ defined in (3.26), (3.27), (3.28) and (3.34). For convenience, we also reference the quantities (4.16)-(4.17).

Proposition 5.1. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the following properties hold.*

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have

$$\begin{array}{rcl} \|\mathcal{G}(U) - \mathcal{G}_{\mathrm{apx}}(U)\|_{\ell^2_h} & \leq & K\big[h + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h}\big], \\ & = & KS_{\mathrm{sh;full}} + KS_{\mathrm{rem;full}} \\ \big\|\mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{apx}}(U)\big\|_{\ell^2_h} & \leq & Kh\big[1 + \|\partial^+\partial^+\partial^+U\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+U\|_{\ell^\infty_h}\big] \\ & & \quad +K\big[\,\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h} + \|\partial^+\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h}\big] \\ & = & K\overline{S}_{\mathrm{sh;full}} + K\overline{S}_{\mathrm{rem;full}}. \end{array} \tag{5.5}$$

(ii) For any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$, we have the estimates

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{nl};U}(V)\|_{\ell^2_h} & \leq & K\mathcal{E}_{\mathrm{prod}}(V,V) + K\mathcal{E}_{\mathrm{sh};U}(V) + K\mathcal{E}_{\mathrm{rem};U}(V), \\ \Big\|\mathcal{G}^+_{\mathrm{nl};U}(V)\Big\|_{\ell^2_h} & \leq & K\overline{\mathcal{E}}_{\mathrm{prod};U}(V,V) + Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V) + K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{5.6}$$

(iii) Consider any $h > 0$, $U \in \Omega_{h;\kappa}$ and any pair $(V^{(1)}, V^{(2)}) \in \ell^2_h \times \ell^2_h$ for which the inclusions $U + V^{(1)} \in \Omega_{h;\kappa}$ and $U + V^{(2)} \in \Omega_{h;\kappa}$ both hold. Then we have the Lipschitz estimate

$$\begin{array}{rcl} \big\|\mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)})\big\|_{\ell^2_h} & \leq & K\mathcal{E}_{\mathrm{prod}}(V^{(1)}, V^{(2)} - V^{(1)}) + K\mathcal{E}_{\mathrm{prod}}(V^{(2)}, V^{(2)} - V^{(1)}) \\ & & \quad +K\mathcal{E}_{\mathrm{sh};U}(V^{(2)} - V^{(1)}) + K\mathcal{E}_{\mathrm{rem};U}(V^{(2)} - V^{(1)}). \end{array} \tag{5.7}$$

5.1 Refinement strategy

We recall the refined approximants $\mathcal{G}_{\mathrm{apx};II}(U)$ and $\mathcal{G}_{\mathrm{lin};U;II}[V]$ that we defined in §4.2. The main task in this section is to track the errors that accumulate as we reduce these expressions even further to our relatively simple approximants (5.1). In contrast to the abstract approach in §4, we achieve this in a direct fashion through several explicit computations. Indeed, in §E-F we obtain the following representations.

Proposition 5.2 (see §E-F). *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with sequences*

$$\mathcal{G}_{\mathrm{apx;sh};b}(U) \in \ell^2_h, \qquad \mathcal{G}_{\mathrm{apx;rem};b}(U) \in \ell^2_h, \qquad \mathcal{G}^+_{\mathrm{apx;sh};b}(U) \in \ell^2_h, \qquad \mathcal{G}^+_{\mathrm{apx;rem};b}(U) \in \ell^2_h, \tag{5.8}$$

defined for every $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the identities

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{apx};II}(U) &=& \mathcal{G}_{\mathrm{apx}}(U) + \mathcal{G}_{\mathrm{apx;sh};b}(U) + \mathcal{G}_{\mathrm{apx;rem};b}(U), \\ \mathcal{G}^+_{\mathrm{apx};II}(U) &=& \mathcal{G}^+_{\mathrm{apx}}(U) + \mathcal{G}^+_{\mathrm{apx;sh};b}(U) + \mathcal{G}^+_{\mathrm{apx;rem};b}(U). \end{array} \tag{5.9}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{apx;sh};b}(U)\|_{\ell^2_h} &\leq& K S_{\mathrm{sh;full}}(U) \\ \|\mathcal{G}_{\mathrm{apx;rem};b}(U)\|_{\ell^2_h} &\leq& K S_{\mathrm{rem;full}}(U), \end{array} \tag{5.10}$$

together with

$$\begin{array}{rcl} \left\|\mathcal{G}^+_{\mathrm{apx;sh};b}(U)\right\|_{\ell^2_h} &\leq& K \overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^+_{\mathrm{apx;rem};b}(U)\right\|_{\ell^2_h} &\leq& K \overline{S}_{\mathrm{rem;full}}(U). \end{array} \tag{5.11}$$

Proposition 5.3 (see §E-F). *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with linear maps*

$$\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b} \in \mathcal{L}(\ell^2_h, \ell^2_h), \qquad \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b} \in \mathcal{L}(\ell^2_h, \ell^2_h) \tag{5.12}$$

and their counterparts

$$\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b} \in \mathcal{L}(\ell^2_h, \ell^2_h), \qquad \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b} \in \mathcal{L}(\ell^2_h, \ell^2_h), \tag{5.13}$$

defined for all $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the identities

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{lin};U;II}[V] &=& \mathcal{G}_{\mathrm{lin};U}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b}[V], \\ \mathcal{G}^+_{\mathrm{lin};U;II}[V] &=& \mathcal{G}^+_{\mathrm{lin};U}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b}[V]. \end{array} \tag{5.14}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}[V]\|_{\ell^2_h} &\leq& K \mathcal{E}_{\mathrm{sh};U}(V), \\ \|\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b}[V]\|_{\ell^2_h} &\leq& K \mathcal{E}_{\mathrm{rem};U}(V), \end{array} \tag{5.15}$$

together with

$$\begin{array}{rcl} \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b}[V]\right\|_{\ell^2_h} &\leq& K \overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b}[V]\right\|_{\ell^2_h} &\leq& K \overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{5.16}$$

(iii) For every $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\left\|\mathcal{G}_{\mathrm{lin};U^{(2)};\mathrm{rem};b}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)};\mathrm{rem};b}[V]\right\|_{\ell^2_h} \leq K \mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \tag{5.17}$$

Recalling the initial nonlinear residuals (3.50) together with the expressions (4.42), (4.51) and (4.52), we have hence obtained the decompositions

$$\begin{array}{lcl}\mathcal{G}_{\mathrm{nl};U}(V) & = & \mathcal{G}_{\mathrm{nl};U;I}(V) + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}[V], \\ \mathcal{G}^{+}_{\mathrm{nl};U}(V) & = & \mathcal{G}^{+}_{\mathrm{nl};U;I}(V) + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{rem};a}[V] + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{rem};b}[V] + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{5.18}$$

We now turn towards the Lipschitz bounds for $\mathcal{G}_{\mathrm{nl}}$.

Corollary 5.4. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the estimate*

$$\big\| \mathcal{G}_{\mathrm{lin};U^{(2)}}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)}}[V] \big\|_{\ell^2_h} \leq K \mathcal{E}_{\mathrm{prod}}\big(U^{(2)} - U^{(1)}, V\big) \tag{5.19}$$

holds for all $h > 0$, all $V \in \ell^2_h$ and all pairs $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

Proof. This is a direct restatement of [27, Cor. 5.3]. □

Lemma 5.5. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the estimate*

$$\begin{array}{lcl}\big\| \mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)}) \big\|_{\ell^2_h} & \leq & K \mathcal{E}_{\mathrm{prod}}\big(V^{(2)} - V^{(1)}, V^{(2)} - V^{(1)}\big) + Kh \big\| V^{(2)} - V^{(1)} \big\|_{\ell^{2;2}_h} \\ & & + K \mathcal{E}_{\mathrm{rem};U}\big(V^{(2)} - V^{(1)}\big) + K \mathcal{E}_{\mathrm{prod}}\big(V^{(1)}, V^{(2)} - V^{(1)}\big)\end{array} \tag{5.20}$$

holds for all $h > 0$, all $U \in \Omega_{h;\kappa}$ and all pairs $(V^{(1)}, V^{(2)}) \in \ell^2_h \times \ell^2_h$ for which the inclusions $U + V^{(1)} \in \Omega_{h;\kappa}$ and $U + V^{(2)} \in \Omega_{h;\kappa}$ both hold.

Proof. By definition, we have

$$\mathcal{G}_{\mathrm{nl};U}(V) = \mathcal{G}(U + V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U}[V]. \tag{5.21}$$

In particular, we get

$$\begin{array}{lcl}\mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)}) & = & \mathcal{G}(U + V^{(2)}) - \mathcal{G}_{\mathrm{lin};U}[V^{(2)}] + \mathcal{G}_{\mathrm{lin};U}[V^{(1)}] - \mathcal{G}(U + V^{(1)}) \\ & = & \mathcal{G}\big(U + V^{(1)} + (V^{(2)} - V^{(1)})\big) - \mathcal{G}(U + V^{(1)}) \\ & & \quad - \mathcal{G}_{\mathrm{lin};U}[V^{(2)} - V^{(1)}] \\ & = & \mathcal{G}_{\mathrm{lin};U+V^{(1)}}[V^{(2)} - V^{(1)}] + \mathcal{G}_{\mathrm{nl};U+V^{(1)}}(V^{(2)} - V^{(1)}) \\ & & \quad - \mathcal{G}_{\mathrm{lin};U}[V^{(2)} - V^{(1)}] \\ & = & \mathcal{G}_{\mathrm{nl};U+V^{(1)}}(V^{(2)} - V^{(1)}) \\ & & \quad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}].\end{array} \tag{5.22}$$

For convenience, we write

$$\begin{array}{lcl}\mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}[V] & = & \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b}[V], \\ \mathcal{G}_{\mathrm{lin};U;\mathrm{sh}}[V] & = & \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{5.23}$$

In view of (5.18), we find

$$\begin{aligned}\mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)}) &= \mathcal{G}_{\mathrm{nl};U+V^{(1)};I}(V^{(2)} - V^{(1)}) \\ &\quad + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{rem}}(V^{(2)} - V^{(1)}) + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{sh}}(V^{(2)} - V^{(1)}) \\ &\quad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}] \\ &= \mathcal{G}_{\mathrm{nl};U+V^{(1)};I}(V^{(2)} - V^{(1)}) \\ &\quad + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}(V^{(2)} - V^{(1)}) + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{sh}}(V^{(2)} - V^{(1)}) \\ &\quad + \Big[\mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{rem}} - \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}\Big](V^{(2)} - V^{(1)}) \\ &\quad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}].\end{aligned} \tag{5.24}$$

The desired bound now follows from (3.51), Lemma's 4.7-4.8, Propositions 5.2-5.3 and Corollary 5.4. □

Proof of Proposition 5.1. In view of the expression (5.18), the statements follow from (3.49), (3.51), Lemma's 4.7-4.8, Propositions 5.2-5.3 and Lemma 5.5. □

6 Travelling waves

Formally substituting the travelling wave Ansatz (2.20) into the reduced system (2.13) leads to the nonlocal differential equation

$$c\Psi' = \mathcal{G}(\Psi). \tag{6.1}$$

In this section we set out to construct solutions to this equation for small $h > 0$ that can be written as

$$\Psi = \Psi_* + v, \qquad c = c_* + \tilde{c} \tag{6.2}$$

for pairs $(\tilde{c}, v)$ that tend to zero as $h \downarrow 0$. Care must be taken to ensure that the expression $\mathcal{G}(\Psi)$ is well-defined, but based on our preparations we are able to provide a relatively streamlined fixed-point argument here, which allows us to prove the results stated in §2.

In order to control the size of the perturbation $(\tilde{c}, v) \in \mathbb{R} \times H^1$, we introduce the norms

$$\|(\tilde{c}, v)\|_{\mathcal{Z}_h} = |\tilde{c}| + \|v\|_{H^1} + \big\|\partial_h^+ \partial_h^+ v\big\|_{L^2} \tag{6.3}$$

for $h > 0$ and write $\mathcal{Z}_h$ for the set $\mathbb{R} \times H^1$ equipped with this new norm. Observe that for fixed h this norm is equivalent to the usual one on $\mathbb{R} \times H^1$.

Recalling the discussion at the start of §3, we pick $0 < \kappa < \frac{1}{12}$ and $\epsilon_0 > 0$ in such a way that the inclusion

$$\mathrm{ev}_\vartheta[\Psi_* + v] \in \Omega_{h;\kappa} \tag{6.4}$$

holds for all $0 < h < 1$, all $\vartheta \in [0, h]$ and all $v \in H^1$ that satisfy (3.8). In order to accommodate this, we pick two parameters $\delta > 0$ and $\delta_v^+ > 0$ and introduce the set

$$\begin{aligned}\mathcal{Z}_{h;\delta,\delta_v^+} &= \big\{(\tilde{c}, v) \in \mathcal{Z}_h : \|(\tilde{c}, v)\|_{\mathcal{Z}_h} \le \min\{\delta, \epsilon_0\} \\ &\qquad \text{and } \big\|(0, \partial_h^+ v)\big\|_{\mathcal{Z}_h} \le \min\{\delta_v^+, h^{1/2}\epsilon_0\}\big\}.\end{aligned} \tag{6.5}$$

Since ∂_h^+ is bounded on H^1 and L^2 for each fixed h, we note that this is a closed subset of $\mathcal{Z}_h$.

Substituting (6.2) into (6.1), we obtain

$$\begin{aligned}c_*\Psi_*' + \tilde{c}\Psi_*' + \tilde{c}v' + c_*v' &= \mathcal{G}(\Psi_* + v) \\ &= \mathcal{G}(\Psi_*) + \mathcal{G}_{\mathrm{lin};\Psi_*}[v] + \mathcal{G}_{\mathrm{nl};\Psi_*}(v),\end{aligned} \tag{6.6}$$

which should be interpreted in a sense similar to that of (3.18).

Upon introducing the nonlinearity

$$\mathcal{H}_h(\tilde{c}, v) \quad = \quad \tilde{c}v' - \mathcal{G}_{\mathrm{nl};\Psi_*}(v) \tag{6.7}$$

and inspecting the definitions (3.16) and (5.1), we can rewrite (6.6) as

$$\mathcal{L}_h[v] \quad = \quad \tilde{c}\Psi_*' + \mathcal{H}_h(\tilde{c}, v) + c_*\Psi_*' - \mathcal{G}(\Psi_*). \tag{6.8}$$

Recalling the two solution operators (3.19), we now introduce the map $\mathcal{W}_h : \mathcal{Z}_{h;\delta,\delta_v^+} \to \mathcal{Z}_h$ that acts as

$$\mathcal{W}_h(\tilde{c}, v) \quad = \quad [\beta_h^*, \mathcal{V}_h^*]\Big[\mathcal{H}_h(\tilde{c}, v) + c_*\Psi_*' - \mathcal{G}(\Psi_*)\Big], \tag{6.9}$$

which allows us to recast (6.8) as the fixed point problem

$$(\tilde{c}, v) = \mathcal{W}_h\big(\tilde{c}, v\big). \tag{6.10}$$

In order to show that $\mathcal{W}_h$ is a contraction mapping on $\mathcal{Z}_{h;\delta,\delta_v^+}$, we study the two expressions $\mathcal{H}_h$ and $c_*\Psi_*' - \mathcal{G}(\Psi_*)$ separately in our first results. The sampling bounds from §A play a key role here, as they enable us to extract L^2-based bounds on $\mathcal{G}_{\mathrm{nl};\Psi_*}$ from the sequence estimates obtained in §5.

Notice that the control (6.15) on $\|v\|_{\ell_h^{2;2}}$ would not have been possible using only bounds on (6.3), since L^2-norms cannot directly be turned into ℓ_h^2-norms. This would prevent us from bounding the terms that are quadratic in these second differences. In fact, this is the reason that we needed to obtain such detailed bounds on $\mathcal{G}^+$ in this series of papers. Indeed, the additional third-differences only appear in a linear fashion, which **does** allow us to easily pass between sequences and functions; see (A.18).

Lemma 6.1. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists $K > 0$ so that for any pair $(\delta, \delta_v^+) \in (0,1)^2$ and any $0 < h < 1$ the estimates*

$$\begin{aligned} \|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} &\leq K\big[h\delta + \delta^2 + \delta\delta_v^+\big], \\ \|\partial^+\mathcal{H}_h(\tilde{c}, v)\|_{L^2} &\leq K\big[[\delta + \delta_v^+]^2 + h^{-1/2}\delta[\delta + \delta_v^+] + h[\delta_v + \delta_v^+]\big] \end{aligned} \tag{6.11}$$

hold for each $(\tilde{c}, v) \in \mathcal{Z}_{h;\delta,\delta_v^+}$, while the estimate

$$\big\|\mathcal{H}_h(\tilde{c}^{(2)}, v^{(2)}) - \mathcal{H}_h(\tilde{c}^{(1)}, v^{(1)})\big\|_{L^2} \quad \leq \quad K\big[h^{-1/2}[\delta + \delta_v^+] + h\big]\, \big\|(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)})\big\|_{\mathcal{Z}_h} \tag{6.12}$$

holds for each set of pairs $(\tilde{c}^{(1)}, v^{(1)}) \in \mathcal{Z}_{h;\delta,\delta_v^+}$ and $(\tilde{c}^{(2)}, v^{(2)}) \in \mathcal{Z}_{h;\delta,\delta_v^+}$.

Proof. The first term in $\mathcal{H}_h$ can be handled by the elementary estimates

$$\begin{aligned} \|\tilde{c}v'\|_{L^2} &\leq \delta^2, \\ \|\tilde{c}\partial^+ v'\|_{L^2} &\leq \delta\,\|\partial^+ v\|_{H^1} \quad \leq \quad \delta\delta_v^+, \end{aligned} \tag{6.13}$$

together with

$$\begin{aligned} \big\|\tilde{c}^{(2)}[v^{(2)}]' - \tilde{c}^{(1)}[v^{(1)}]'\big\|_{L^2} &\leq \big|\tilde{c}^{(2)} - \tilde{c}^{(1)}\big|\,\big\|v^{(2)}\big\|_{H^1} + \big|\tilde{c}^{(1)}\big|\,\big\|v^{(1)} - v^{(2)}\big\|_{H^1} \\ &\leq \delta\,\big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}. \end{aligned} \tag{6.14}$$

Using Corollary A.1 we see that

$$\|v\|_{\ell_h^{2;2}} + \|v\|_{\ell_h^{\infty;1}} \leq C_1'[\delta + \delta_v^+] \tag{6.15}$$

for all $(\tilde{c}, v) \in \mathcal{Z}_{h;\delta,\delta^+}$. For any $\vartheta \in \mathbb{R}$, we may hence exploit Propositions C.3 and 5.1 to obtain the estimate

$$\|\mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v)\|_{\ell^2_h} \quad \leq \quad C_2'\big[\delta + \delta_v^+ + h\big] \, \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}} \, , \tag{6.16}$$

together with

$$\begin{array}{rcl} \big\|\mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v^{(2)}) - \mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v^{(1)})\big\|_{\ell^2_h} & \leq & C_2'\big[\delta + \delta_v^+ + h\big] \, \big\|\mathrm{ev}_\vartheta v^{(1)} - \mathrm{ev}_\vartheta v^{(2)}\big\|_{\ell_h^{2;2}} \\ & & + C_2'\big[\delta + \delta_v^+\big] \, \big\|\mathrm{ev}_\vartheta v^{(1)} - \mathrm{ev}_\vartheta v^{(2)}\big\|_{\ell_h^{\infty;1}} \, . \end{array} \tag{6.17}$$

A second application of Corollary A.1 yields the bound

$$\|v\|_{\ell_h^{2;2}} + \|v\|_{\ell_h^{\infty;2}} \leq C_3' h^{-1/2}[\delta + \delta_v^+]. \tag{6.18}$$

For any $\vartheta \in \mathbb{R}$, we may hence use Propositions C.3 and 5.1 to find

$$\Big\|\mathcal{G}^+_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v)\Big\|_{\ell^2_h} \quad \leq \quad C_4'\big[\delta + \delta_v^+ + h\big] \, \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;3}} + C_4' h^{-1/2}[\delta + \delta_v^+] \, \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}} \, . \tag{6.19}$$

We now apply Lemma A.2 to obtain

$$\begin{array}{rcl} \|\mathcal{G}_{\mathrm{nl};\Psi_*}(v)\|_{L^2} & \leq & C_2'\big[\delta + \delta_v^+ + h\big]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2}\,\big] \\ & \leq & C_2'\big[\delta + \delta_v^+ + h\big]\delta, \\ \Big\|\mathcal{G}^+_{\mathrm{nl};\Psi_*}(v)\Big\|_{L^2} & \leq & C_4'\big[\delta + \delta_v^+ + h\big]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2} + \|\partial^+ v\|_{H^1} + \|\partial^+\partial^+\partial^+ v\|_{L^2}\,\big] \\ & & + C_4' h^{-1/2}[\delta + \delta_v^+]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2}\,\big] \\ & \leq & C_4'\big[\delta + \delta_v^+ + h\big]\big[\delta + \delta_v^+\big] + C_4' h^{-1/2}[\delta + \delta_v^+]\delta. \end{array} \tag{6.20}$$

Using (A.6) we note that

$$\Big\|v^{(2)} - v^{(1)}\Big\|_{\ell_h^{\infty;1}} \leq 2h^{-1/2}\Big\|v^{(2)} - v^{(1)}\Big\|_{H^1}. \tag{6.21}$$

Applying Lemma A.2 once more, we obtain

$$\begin{array}{rcl} \big\|\mathcal{G}_{\mathrm{nl};\Psi_*}(v^{(2)}) - \mathcal{G}_{\mathrm{nl};\Psi_*}(v^{(1)})\big\|_{L^2} & \leq & C_2'\big[\delta + \delta_v^+ + h\big]\Big[\, \big\|v^{(1)} - v^{(2)}\big\|_{H^1} + \big\|\partial^+\partial^+ v^{(1)} - \partial^+\partial^+ v^{(2)}\big\|_{L^2}\,\Big] \\ & & + 2C_2'\big[\delta + \delta_v^+\big]h^{-1/2}\,\big\|v^{(1)} - v^{(2)}\big\|_{H^1}. \end{array} \tag{6.22}$$

The desired bounds follow readily from these estimates. □

Lemma 6.2. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists $K > 0$ so that for each $0 < h < 1$ we have the bounds*

$$\begin{array}{rcl} \|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} & \leq & Kh, \\ \big\|\partial^+\big[c_*\Psi_*' - \mathcal{G}(\Psi_*)\big]\big\|_{L^2} & \leq & Kh. \end{array} \tag{6.23}$$

Proof. Applying Lemma A.2 together with Propositions C.3 and 5.1, we find

$$\|\mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*)\|_{L^2} + \big\|\mathcal{G}^+_{\mathrm{apx}}(\Psi_*) - \mathcal{G}^+(\Psi_*)\big\|_{L^2} \quad \leq \quad C_1' h. \tag{6.24}$$

We now compute

$$\begin{array}{rcl} c_*\Psi_*' - \mathcal{G}(\Psi_*) & = & c_*\Psi_*' - \mathcal{G}_{\mathrm{apx}}(\Psi_*) + \mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*) \\ & = & c_*\Psi_*' - c_*\partial^0\Psi_* + \mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*), \end{array} \tag{6.25}$$

together with

$$\begin{aligned}\partial^+[c_*\Psi_*' - \mathcal{G}(\Psi_*)] &= \partial^+\big[c_*\Psi_*' - \mathcal{G}_{\rm apx}(\Psi_*)\big] + \partial^+\big[\mathcal{G}_{\rm apx}(\Psi_*) - \mathcal{G}(\Psi_*)\big] \\ &= c_*\big[\partial^+\Psi_*]' - c_*\partial^0[\partial^+\Psi_*] + \mathcal{G}^+_{\rm apx}(\Psi_*) - \mathcal{G}^+(\Psi_*).\end{aligned} \tag{6.26}$$

Applying (A.8) we see that

$$\|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} \;\le\; C_2' h\,\|\Psi_*''\|_{L^2} + C_1' h \;\le\; C_3' h, \tag{6.27}$$

together with

$$\|\partial^+[c_*\Psi_*' - \mathcal{G}(\Psi_*)]\|_{L^2} \;\le\; C_2' h\,\|\partial^+\Psi_*''\|_{L^2} + C_1' h \;\le\; C_3' h, \tag{6.28}$$

as desired. □

Utilizing the linear theory from [27] that we outlined in §3.1, we are now in a position to study the full nonlinear term $\mathcal{W}_h$. Our main result subsequently follows in a relatively standard fashion from the uniqueness properties of the contraction mapping theorem.

Lemma 6.3. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for each sufficiently small $h > 0$, the fixed point problem (6.10) posed on the set $\mathcal{Z}_{h;h^{3/4},h^{3/4}}$ has a unique solution.*

Proof. Using the estimates (3.22) together with the a-priori bounds $(h, \delta, \delta_v^+) \in (0,1)^3$, we obtain the inequalities

$$\begin{aligned}\|\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_1'\big[\,\|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} + \|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2}\,\big] \\ &\le C_2'\big[\delta^2 + \delta\delta_v^+ + h\big], \\ \|[0,\partial^+]\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_1'\big[\,\|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} + \|\partial^+\mathcal{H}_h(\tilde{c}, v)\|_{L^2}\,\big] \\ &\quad + C_1'\Big[\,\|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} + \big\|\partial^+\big[c_*\Psi_*' - \mathcal{G}(\Psi_*)\big]\big\|_{L^2}\Big] \\ &\le C_2'\Big[h^{-1/2}\delta[\delta + \delta_v^+] + (\delta + \delta_v^+)^2 + h\Big],\end{aligned} \tag{6.29}$$

together with

$$\begin{aligned}\big\|\mathcal{W}_h\big(\tilde{c}^{(2)}, v^{(2)}\big) - \mathcal{W}_h\big(\tilde{c}^{(1)}, v^{(1)}\big)\big\|_{\mathcal{Z}_h} &\le C_1'\,\big\|\mathcal{H}_h(\tilde{c}^{(2)}, v^{(2)}) - \mathcal{H}_h(\tilde{c}^{(1)}, v^{(1)})\big\|_{L^2} \\ &\le C_2'\big[h^{-1/2}[\delta + \delta_v^+] + h\big]\,\big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}.\end{aligned} \tag{6.30}$$

Picking

$$\delta = \delta_v^+ = h^{3/4}, \tag{6.31}$$

we see that $\delta = \delta_v^+ \le h^{1/2}\epsilon_0$ for all sufficiently small $h > 0$. In addition, we find

$$\begin{aligned}\|\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_2'\big[2h^{3/4} + h^{1/4}\big]\delta, \\ \|[0,\partial^+]\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_2'\Big[2h^{1/4} + 4h^{3/4} + h^{1/4}\Big]\delta,\end{aligned} \tag{6.32}$$

together with

$$\big\|\mathcal{W}_h\big(\tilde{c}^{(2)}, v^{(2)}\big) - \mathcal{W}_h\big(\tilde{c}^{(1)}, v^{(1)}\big)\big\|_{\mathcal{Z}_h} \;\le\; C_2'[2h^{1/4} + h]\,\big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}. \tag{6.33}$$

The result hence follows from the contraction mapping theorem. □

Proof of Theorem 2.1. We write $(\tilde{c}_h, v_h)$ for the unique solution to the fixed point problem (6.10) that is provided by Lemma 6.3. This allows us to define

$$\Psi_h = \Psi_* + v_h, \qquad c_h = c_* + \tilde{c}_h. \tag{6.34}$$

For fixed $h > 0$, we claim that the map

$$\vartheta \mapsto \mathrm{ev}_\vartheta[\Psi_* + v_h] - \mathrm{ev}_0\Psi_* \in \ell_h^2 \tag{6.35}$$

is continous. Indeed, this follows from the smoothness of Ψ_* together with (A.6) and the fact that the translation operator is continuous on H^1. Since the map

$$V \mapsto \mathcal{G}(\Psi_* + V) \in \ell_h^2 \tag{6.36}$$

is continuous on a subset of ℓ_h^2 that contains $\mathrm{ev}_\vartheta v_h$ for all $\vartheta \in [0, h]$, we conclude that

$$\vartheta \mapsto \mathcal{G}\big(\mathrm{ev}_\vartheta \Psi_h\big) \in \ell_h^2 \tag{6.37}$$

is continuous. The travelling wave equation (6.1) now implies the inclusion (2.21).

In a similar fashion, the inclusion (2.25) follows from (A.6) and the continuity of the translation operator on H^1. The remaining statements are a direct consequence of Lemma 6.3. □

We now turn to the proof of Corollary 2.2, which asserts the existence of a waveprofile Φ_h in the original physical coordinates. The key tool for our purpose here is [26, Prop. 4.2], which states that the gridpoints associated to a solution U of (2.13) satisfy

$$\dot{x}(t) = \mathcal{M}\big(U(t)\big). \tag{6.38}$$

Here the sequence $\mathcal{M}$ can be written as

$$\begin{aligned}\mathcal{M}(U) \quad = \quad & -\mathcal{Z}^-(U)T^-[\mathcal{X}_A(U)]\mathcal{Y}_2(U) + \mathcal{Z}^-(U)\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\big[\mathcal{X}_B(U)\big]\mathcal{D}^{\diamond_-;+}(U) \\ & +\mathcal{Z}^-(U)\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\big[\mathcal{X}_C(U)\mathcal{D}^{\diamond_0;+}(U) + \mathcal{X}_D(U)\mathcal{D}^{\diamond_0;+}(U)\big];\end{aligned} \tag{6.39}$$

see [26, Eq. (6.31) and (6.33)] where this function was referred to as $\mathcal{Y}$. Notice the strong resemblance with the structure of (3.42). Indeed, we see that

$$\mathcal{Y}_1(U)\mathcal{M}(U) = \mathcal{Z}^-(U)[\mathcal{G}(U) - \mathcal{Y}_2(U)] \tag{6.40}$$

see also [26, Eq. (6.9)] for comparison. In view of the identities

$$\mathcal{Z}^-_{\mathrm{apx}}(U) = \gamma_U, \qquad \mathcal{Y}_{1;\mathrm{apx}}(U) = \partial^0 U, \qquad \mathcal{G}_{\mathrm{apx}}(U) = c_*\partial^0 U \qquad \mathcal{Y}_{2;\mathrm{apx;expl}}(U) = c_*\gamma_U^{-1}\partial^0 U \tag{6.41}$$

from [26, Eq. (7.29)], (D.1), (5.1) and (D.11), it makes sense to formally factor out $\partial^0 U$ and introduce the approximant

$$\mathcal{M}_{\mathrm{apx}}(U) = \gamma_U c_*(1 - \gamma_U^{-1}) = c_*(\gamma_U - 1). \tag{6.42}$$

This allows us to extract a crucial lower bound for the speed of the gridpoints.

Proof of Corollary 2.2. Upon defining

$$\Psi_h^{(x)} = -\sum_{-;h} \frac{(\partial^+\Psi_h)^2}{\sqrt{1 - (\partial^+\Psi_h)^2} + 1}, \tag{6.43}$$

the identity (2.27) implies that (i) is satisfied. Using [26, Prop. 4.2] we see that

$$\dot{x}_{jh}(t) = c_h[\Psi_h^{(x)}]'(jh + c_h t) = \big[\mathcal{M}\big(\Psi_h(\cdot + c_h t)\big)\big]_{jh}. \tag{6.44}$$

Notice that the terms appearing in (6.39) also all appear in (3.42) after making the replacement $\mathcal{Z}^- \mapsto \mathcal{Y}_1$. The reduction $\mathcal{Z}^- \mapsto \mathcal{Z}^-_{\mathrm{apx}}$ leads to error terms that are covered by the theory in [26]. Since $\mathcal{Z}^-_{\mathrm{apx}}$ does not need to be reduced further, we can follow all the computations in the present paper to obtain the error bound

$$\|\mathcal{M}(U) - \mathcal{M}_{\mathrm{apx}}(U)\|_{\ell^\infty_h} \leq C_1'\big[h + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h}\,\big], \tag{6.45}$$

which is the natural analogue of (5.5). Substituting $U = \Psi_h$ and applying the Lipschitz bounds (C.6), we find

$$\begin{array}{lcl} \|\mathcal{E}_{\mathrm{tw}}(\Psi_h) - \mathcal{E}_{\mathrm{tw}}(\Psi_*)\|_{\ell^2_h} & \leq & C_2'\,\|\Psi_h - \Psi_*\|_{\ell^{2;2}_h} \\ & \leq & C_2'\big[\,\|\Psi_h - \Psi_*\|_{H^1} + \|\partial^+[\Psi_h - \Psi_*]\|_{H^1}\,\big] \\ & \leq & C_2' h^{3/4}. \end{array} \tag{6.46}$$

Using Proposition C.3, we obtain

$$\|\mathcal{E}_{\mathrm{tw}}(\Psi_h)\|_{\ell^2_h} \leq C_3' h^{3/4} \tag{6.47}$$

and hence

$$\|\mathcal{E}_{\mathrm{tw}}(\Psi_h)\|_{\ell^\infty_h} \leq C_3' h^{1/4}. \tag{6.48}$$

In a similar fashion, we may exploit (B.4) to conclude

$$\|\gamma_{\Psi_h} - \gamma_{\Psi_*}\|_{\ell^2_h} \leq C_2' h^{3/4} \tag{6.49}$$

and hence

$$\|\gamma_{\Psi_h} - \gamma_{\Psi_*}\|_{\ell^\infty_h} \leq C_2' h^{1/4}. \tag{6.50}$$

Together, these observations yield the pointwise bound

$$|\mathcal{M}(\Psi_h) - c_*(\gamma_{\Psi_*} - 1)| \leq C_4' h^{1/4}. \tag{6.51}$$

Assuming for clarity that $c_* > 0$, this implies the pointwise inequality

$$\mathcal{M}(\Psi_h) > c_*(\gamma_{\Psi_*} - 1) - C_4' h^{1/4}. \tag{6.52}$$

Since $|c_h - c_*| \leq h^{3/4}$, we find

$$\begin{array}{lcl} c_h\Big[[\Psi_h^{(x)}]' + 1\Big] & > & c_*(\gamma_{\Psi_*} - 1) + c_h - C_4' h^{1/4} \\ & > & c_*\gamma_{\Psi_*} - C_5' h^{1/4}. \end{array} \tag{6.53}$$

Since γ_{Ψ_*} is strictly bounded away from zero, uniformly in h, we conclude that

$$[\Psi_h^{(x)}]'(\tau) > -1 \tag{6.54}$$

for all sufficiently small $h > 0$ and all $\tau \in \mathbb{R}$. This shows that the coordinate transformation (2.32) is invertible, as desired. □

A Discrete calculus

In this appendix we collect several useful identities and bounds from [26, 27] related to the interplay between discrete and continuous calculus. In particular, we state a discrete version of the product rule, provide two summation-by-parts identities and show how taking discrete samples of functions in L^2 and H^1 affects the various norms.

Recalling the notation introduced at the start of §3.1 and §3.2, a short computation yields the basic identities

$$\partial^{(2)}a = \partial^+\partial^- a, \qquad \partial^+\partial^0 a = S^+[\partial^{(2)}a], \tag{A.1}$$

together with the product rules

$$\begin{aligned} \partial^+[ab] &= \partial^+ a T^+ b + a\partial^+ b, \\ \partial^0[ab] &= \partial^0 a T^+ b + T^- a\partial^0 b, \\ \partial^-[ab] &= [\partial^- a]b + \Big[T^- a\Big]\partial^- b, \end{aligned} \tag{A.2}$$

which hold for $a, b \in \ell_h^\infty$. As in [26, §3.1], these can subsequently be used to derive the second-order product rule

$$\partial^{(2)}[ab] = (\partial^{(2)}a)b + \partial^+ a\partial^+ b + \partial^- a\partial^- b + a\partial^{(2)}b. \tag{A.3}$$

Recalling the discrete summation operators (2.10), one can read-off the identities

$$\partial^+\big[\sum_{-;h} a\big]_{jh} = a_{jh}, \qquad \partial^-\big[\sum_{+;h} a\big]_{jh} = -a_{jh} \tag{A.4}$$

for $a \in \ell^1(h\mathbb{Z};\mathbb{R})$. In addition, the discrete summation-by-parts identities

$$\sum_{-;h} b\partial^+ a = aT^- b - \sum_{-;h} a\partial^- b, \qquad \sum_{-;h} bS^+ a = \frac{1}{2}haT^- b + \sum_{-;h} aS^- b \tag{A.5}$$

hold whenever $a, b \in \ell_h^2$; see [26, Eq. (3.13) and (3.15)].

Turning to sampling issues, we repeat the useful estimates [26, Eq. (A.6), (A.4)] which state that

$$\|u\|_{\ell_h^2} \le (2+h)\|u\|_{H^1}, \qquad \left\|\partial_h^\pm u\right\|_{\ell_h^\infty} \le h^{-1/2}\|u'\|_{L^2}, \tag{A.6}$$

for any $u \in H^1$. On the other hand, for any $q \in \{2, \infty\}$ and $u \in W^{1,q}$, we have

$$\left\|\partial_h^\pm u\right\|_{\ell_h^q} \le \|u'\|_{L^q}, \qquad \left\|\partial_h^\pm u\right\|_{L^q} \le \|u'\|_{L^q} \tag{A.7}$$

for any $h > 0$; see [26, Eq. (A.3), (A.13)]. For any $q \in \{2, \infty\}$ and $h > 0$ we also obtain the error estimate

$$\left\|\partial_h^\pm u - u'\right\|_{\ell_h^q} \le h\|u''\|_{L^q} \tag{A.8}$$

whenever $u \in W^{2;q}$ (see [27, Lem. 4.1]), together with

$$\max\{\left\|\partial_h^{(2)}u - u''\right\|_{\ell_h^q}, \left\|\partial_h^{(2)}u(\cdot + h) - u''\right\|_{\ell_h^q}\} \le 2h\|u'''\|_{L^q} \tag{A.9}$$

whenever $u \in W^{3;q}$ (see [27, Cor. 4.2]) and finally

$$\left\|\partial_h^+\partial_h^{(2)}u - u'''\right\|_{\ell_h^q} \le 3h\left\|u^{(iv)}\right\|_{L^q} \tag{A.10}$$

whenever $u \in W^{4;q}$ (see [27, Cor. 4.3]).

We now recall the sampling operator ev_ϑ defined in (3.17). Our final two results here show how to pass back and forth between discrete and continuous estimates.

Corollary A.1 ([26, Cor. A.3])**.** *There exists $K > 0$ so that for any $\vartheta \in \mathbb{R}$, any $v \in H^1$ and any $0 < h < 1$, we have the bounds*

$$\begin{array}{lcl} \|\mathrm{ev}_\vartheta v\|_{\ell_h^\infty} & \leq & K \|v\|_{H^1}, \\ \|\mathrm{ev}_\vartheta v\|_{\ell_h^{\infty;1}} & \leq & K\big[\, \|v\|_{H^1} + \big\|\partial_h^+ v\big\|_{H^1} \big], \\ \|\mathrm{ev}_\vartheta v\|_{\ell_h^{\infty;2}} & \leq & K\big[\, \|v\|_{H^1} + h^{-1/2} \big\|\partial_h^+ v\big\|_{H^1} \big], \end{array} \tag{A.11}$$

together with

$$\begin{array}{lcl} \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;1}} & \leq & K \|v\|_{H^1}, \\ \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}} & \leq & K\big[\, \|v\|_{H^1} + \big\|\partial_h^+ v\big\|_{H^1} \big]. \end{array} \tag{A.12}$$

Lemma A.2 ([26, Lem. A.4])**.** *Consider any $f \in C(\mathbb{R};\mathbb{R})$ and any $g \in H^1$. Then the following properties hold for all $h > 0$.*

(i) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|g\|_\infty \tag{A.13}$$

holds for all $\vartheta \in [0, h]$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_\infty. \tag{A.14}$$

(ii) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|\mathrm{ev}_\vartheta g\|_{\ell_h^{2;2}} \tag{A.15}$$

holds for all $\vartheta \in (0, h)$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_{H^1} + \big\|\partial_h^+ \partial_h^+ g\big\|_{L^2}. \tag{A.16}$$

(iii) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|\mathrm{ev}_\vartheta g\|_{\ell_h^{2;3}} \tag{A.17}$$

holds for all $\vartheta \in [0, h]$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_{H^1} + \big\|\partial_h^+ g\big\|_{H^1} + \big\|\partial_h^+ \partial_h^+ \partial_h^+ g\big\|_{L^2}. \tag{A.18}$$

B The gridspace function γ_U

The gridpoint spacing function

$$\gamma_U = \sqrt{1 - (\partial^0 U)^2} \tag{B.1}$$

plays an important role throughout this paper and was analyzed at length in the prequels [26, 27]. We recall some of these results here and also obtain several novel bounds related to the sums that are evaluated in §E and §F. Recalling the definitions (3.24) for sums and products, we first state some useful identities pertaining to powers of γ_U.

Lemma B.1 ([26, Lem. C.2])**.** *Consider any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ for which $\|\partial^+ U\|_\infty < 1$. Then we have the identities*

$$\begin{array}{rcl} \partial^+[\gamma_U^{-4}] & = & \dfrac{4S^+[\partial^0 U]S^+[\partial^{(2)}U]}{P^+[\gamma_U^2]}\dfrac{S^+[\gamma_U^2]}{P^+[\gamma_U^2]}, \\ \partial^+[\gamma_U^{-2}] & = & \dfrac{2S^+[\partial^0 U]S^+[\partial^{(2)}U]}{P^+[\gamma_U^2]}, \\ \partial^+[\gamma_U^{-1}] & = & \dfrac{S^+[\partial^0 U]S^+[\partial^{(2)}U]}{S^+[\gamma_U]P^+[\gamma_U]}, \\ \partial^+[\gamma_U] & = & -\dfrac{S^+[\partial^0 U]S^+[\partial^{(2)}U]}{S^+\gamma_U}, \\ \partial^+[\gamma_U^2] & = & -2S^+[\partial^0 U]S^+[\partial^{(2)}U]. \end{array} \tag{B.2}$$

Turning to estimates, we first note that

$$\gamma_{U^{(b)}} - \gamma_{U^{(a)}} = -[\gamma_{U^{(a)}} + \gamma_{U^{(b)}}]^{-1}(\partial^0 U^{(a)} + \partial^0 U^{(b)})(\partial^0 U^{(b)} - \partial^0 U^{(a)}) \tag{B.3}$$

holds for any $U^{(a)}, U^{(b)} \in \Omega_{h;\kappa}$; see [26, Eq. (C.4)]. This can be used [26, Cor. D.2] to obtain the Lipschitz bound

$$\|\gamma_{U^{(a)}} - \gamma_{U^{(b)}}\|_{\ell_h^q} \le K \left\|\partial^+ U^{(b)} - \partial^+ U^{(a)}\right\|_{\ell_h^q} \tag{B.4}$$

for $q \in \{2,\infty\}$, where K depends on κ but not on h. In addition, it can be exploited to establish the following approximation errors for various expressions involving γ_U.

Lemma B.2 ([26, Lem. D.4])**.** *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimates*

$$\begin{array}{rcl} \left|\partial^+[\gamma_U^2] + 2\partial^0 U S^+[\partial^{(2)}U]\right| & \le & Kh\Big[\left|\partial^{(2)}U\right|^2 + T^+\left|\partial^{(2)}U\right|^2\Big], \\ \left|\partial^+[\gamma_U] + \gamma_U^{-1}\partial^0 U S^+[\partial^{(2)}U]\right| & \le & Kh\Big[\left|\partial^{(2)}U\right|^2 + T^+\left|\partial^{(2)}U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-1}] - \gamma_U^{-3}\partial^0 U S^+[\partial^{(2)}U]\right| & \le & Kh\Big[\left|\partial^{(2)}U\right|^2 + T^+\left|\partial^{(2)}U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-2}] - 2\gamma_U^{-4}\partial^0 U S^+[\partial^{(2)}U]\right| & \le & Kh\Big[\left|\partial^{(2)}U\right|^2 + T^+\left|\partial^{(2)}U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-4}] - 4\gamma_U^{-6}\partial^0 U S^+[\partial^{(2)}U]\right| & \le & Kh\Big[\left|\partial^{(2)}U\right|^2 + T^+\left|\partial^{(2)}U\right|^2\Big]. \end{array} \tag{B.5}$$

Lemma B.3. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left|\partial^+\Big[\frac{\partial^0 U}{\gamma_U}\Big] - \gamma_U^{-3}S^+[\partial^{(2)}U]\right| \le Kh\Big[\left|\partial^{(2)}U\right| + T^+\left|\partial^{(2)}U\right|\Big]. \tag{B.6}$$

Proof. Using $\partial^+\partial^0 U = S^+\partial^{(2)}U$ and the definition (3.10) for γ_U, we compute

$$\begin{array}{rcl} \partial^+\big[\frac{\partial^0 U}{\gamma_U}\big] & = & \partial^+[\gamma_U^{-1}]T^+\partial^0 U + \gamma_U^{-1}\partial^+\partial^0 U \\ & = & \partial^+[\gamma_U^{-1}]\partial^0 U + \mathcal{E}_1(U) + \gamma_U^{-1}\partial^+\partial^0 U \\ & = & \gamma_U^{-3}\partial^0 U S^+[\partial^{(2)}U]\partial^0 U + \mathcal{E}_1(U) + \mathcal{E}_2(U) + \gamma_U^{-1}S^+[\partial^{(2)}U] \\ & = & \gamma_U^{-3}S^+[\partial^{(2)}U] + \mathcal{E}_1(U) + \mathcal{E}_2(U), \end{array} \tag{B.7}$$

in which

$$\begin{array}{rcl} \mathcal{E}_1(U) & = & h\partial^+[\gamma_U^{-1}]\partial^+\partial^0 U, \\ \mathcal{E}_2(U) & = & \Big[\partial^+[\gamma_U^{-1}] - \gamma_U^{-3}\partial^0 U S^+[\partial^{(2)}U]\Big]\partial^0 U. \end{array} \tag{B.8}$$

The estimates (B.5) now yield the bounds

$$|\mathcal{E}_1(U)| + |\mathcal{E}_2(U)| \le C_1' h \Big[\Big|\partial^{(2)} U\Big| + T^+ \Big|\partial^{(2)} U\Big| \Big], \tag{B.9}$$

which establishes (B.6). □

We now continue the discussion from [26, §D] and consider discrete versions of the integral identities

$$\begin{array}{lcl} \int_{-\infty}^{\tau} \frac{u'(\tau')u''(\tau')}{\sqrt{1-u'(\tau')^2}} d\tau' & = & 1 - \sqrt{1 - u'(\tau)^2}, \\ \int_{-\infty}^{\tau} \frac{u'(\tau')v''(\tau')}{\sqrt{1-u'(\tau')^2}} & = & \frac{u'(\tau)v'(\tau)}{\sqrt{1-u'(\tau)^2}} - \int_{-\infty}^{\tau} \frac{u''(\tau')v'(\tau')}{(1-u'(\tau')^2)^{3/2}} \, d\tau'. \end{array} \tag{B.10}$$

Instead of computing the corresponding sums exactly, we obtain useful approximation that are $O(h)$-accurate.

Lemma B.4. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, the two linear expressions*

$$\begin{array}{lcl} S_{A;U}[V] & = & \sum_{-;h} \gamma_U^{-1} [\partial^0 U] \partial^{(2)} V, \\ S_{B:U}[V] & = & \gamma_U^{-1}[\partial^0 U]\partial^0 V - \sum_{-;h} \gamma_U^{-3} [\partial^{(2)} U] \partial^0 V \end{array} \tag{B.11}$$

satisfy the pointwise estimate

$$|S_{B;U}[V] - S_{A;U}[V]| \quad \le \quad Kh\Big[T^- \left|\partial^- V\right| + \left|\partial^- V\right| + \left|\partial^{(2)} V\right| + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+ \partial^+ V\|_{\ell_h^2} \Big] \tag{B.12}$$

for all $V \in \ell_h^2$.

Proof. Using (A.4) we first observe that

$$\left|T^+ S_{A;U}[V] - S_{A;U}[V]\right| = h \left|\partial^+ S_{A;U}[V]\right| \le C_1' \left|\partial^{(2)} V\right|. \tag{B.13}$$

The summation-by-parts identity (A.5) allows us to compute

$$\begin{array}{lcl} T^+ S_{A;U}[V] & = & T^+ \Big[\sum_{-;h} \gamma_U^{-1} [\partial^0 U] \partial^{(2)} V \Big] \\ & = & T^+ \Big[\sum_{-;h} \gamma_U^{-1} [\partial^0 U] \partial^+ \partial^- V \Big] \\ & = & T^+ \Big[T^{-1} \Big[\gamma_U^{-1} \partial^0 U \Big] \partial^- V - \sum_{-;h} \partial^- [\gamma_U^{-1} \partial^0 U] \partial^- V \Big] \\ & = & \gamma_U^{-1}[\partial^0 U] \partial^+ V - \sum_{-;h} \partial^+ [\gamma_U^{-1} \partial^0 U] \partial^+ V. \end{array} \tag{B.14}$$

Upon writing

$$S_{A;U;I}[V] = \gamma_U^{-1}[\partial^0 U]\partial^0 V - \sum_{-;h} \partial^+ [\gamma_U^{-1} \partial^0 U] \partial^0 V, \tag{B.15}$$

we use the identity

$$\partial^+ V - \partial^0 V = \frac{1}{2} h \partial^{(2)} U \tag{B.16}$$

together with (B.6) to obtain

$$\left|S_{A;U;I}[V] - T^+ S_{A;U}[V]\right| \le C_2' h \left|\partial^{(2)} V\right| + C_2' h \left\|\partial^+ \partial^+ V\right\|_{\ell_h^2}. \tag{B.17}$$

We now write

$$S_{A;U;II}[V] = \gamma_U^{-1}[\partial^0 U]\partial^0 V - \sum_{-;h} \gamma_U^{-3} S^+ [\partial^{(2)} U] \partial^0 V, \tag{B.18}$$

which gives

$$S_{A;U;II}[V] - S_{A;U;I}[V] = -\sum_{-;h} \Big[\partial^+[\gamma_U^{-1}\partial^0 U] - \gamma_U^{-3} S^+[\partial^{(2)} U]\Big]\partial^0 V. \tag{B.19}$$

In particular, (B.6) yields

$$|S_{A;U;II}[V] - S_{A;U;I}[V]| \le C_3' h \left\|\partial^+\partial^+ U\right\|_{\ell_h^2} \left\|\partial^0 V\right\|_{\ell_h^2} \le C_4' h \left\|\partial^+ V\right\|_{\ell_h^2}. \tag{B.20}$$

We now transfer the S^+ using the summation-by-parts identity (A.5) to obtain

$$S_{A;U;II}[V] = \gamma_U^{-1}[\partial^0 U]\partial^0 V - \frac{1}{2} h T^-[\gamma_U^{-3}\partial^0 V]\partial^{(2)} U - \sum_{-;h} S^-\left[\gamma_U^{-3}\partial^0 V\right]\partial^{(2)} U. \tag{B.21}$$

We hence see that

$$S_{B;U}[V] - S_{A;U;II}[V] \quad = \quad hT^-\left[\gamma_U^{-3}\partial^0 V\right] + \textstyle\sum_{-;h} h\partial^-\left[\gamma_U^{-3}\partial^0 V\right]\partial^{(2)} U. \tag{B.22}$$

Using the fact that

$$\left\|\partial^-\left[\gamma_U^{-3}\partial^0 V\right]\right\|_{\ell_h^2} \le C_5'\left[\,\left\|\partial^+ V\right\|_{\ell_h^2} + \left\|\partial^+\partial^+ V\right\|_{\ell_h^2}\right] \tag{B.23}$$

the desired estimate follows. □

Lemma B.5. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left|\sum_{-;h} \gamma_U^{-1}[\partial^0 U]\partial^{(2)} U - (1 - \gamma_U)\right| \le Kh. \tag{B.24}$$

Proof. Since $[\gamma_U]_{jh} \to 1$ as $j \to -\infty$, we have

$$\gamma_U - 1 \quad = \quad \textstyle\sum_{-;h} \partial^+ \gamma_U. \tag{B.25}$$

In particular, writing

$$\mathcal{S}_I \quad = \quad \textstyle\sum_{-;h} \gamma_U^{-1}\partial^0 U S^+[\partial^{(2)} U] \tag{B.26}$$

we may use the estimates (B.5) to obtain

$$|\mathcal{S}_I - (1 - \gamma_U)| \quad \le \quad 2Kh \left\|\partial^+\partial^+ U\right\|_{\ell_h^2}^2 \quad \le \quad C_1' h. \tag{B.27}$$

Using the second summation-by-parts identity in (A.5), we can transfer the S^+ to obtain

$$\mathcal{S}_I \quad = \quad \tfrac{1}{2} h\partial^{(2)} U T^-\left[\gamma_U^{-1}\partial^0 U + \textstyle\sum_{-;h} S^-\left[\gamma_U^{-1}\partial^0 U\right]\partial^{(2)} U\right]. \tag{B.28}$$

In particular, writing

$$\mathcal{I} = \mathcal{S}_I - \textstyle\sum_{-;h} \gamma_U^{-1}[\partial^0 U]\partial^{(2)} U, \tag{B.29}$$

we see that

$$\mathcal{I} \quad = \quad \tfrac{h}{2}\partial^{(2)} U T^-\left[\gamma_U^{-1}\partial^0 U\right] - \textstyle\sum_{-;h} \tfrac{h}{2}\partial^-\left[\gamma_U^{-1}\partial^0 U\right]\partial^{(2)} U. \tag{B.30}$$

Using Lemma B.3 we see that

$$|\mathcal{I}| \le C_1' h \left\|\partial^+\partial^+ U\right\|_{\ell_h^2}^2 + C_2' h, \tag{B.31}$$

from which the desired estimate follows. □

C Operator bounds

Our goal here is to establish several crucial bounds on the linear operators and error functions introduced in §3.1-3.2. The errors that arise when approximating $\partial^+ M_U$ and $\partial^+ \mathcal{E}_{\rm tw}$ by $M^+_{U;\rm apx}$ and $\mathcal{E}^+_{\rm tw;apx}$ are of special importance.

Proposition C.1 ([27, Prop. 5.1])**.** *Assume that (Hg) is satisfied and fix $\kappa > 0$. There exists $K > 0$ so that for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the a-priori bounds*

$$\begin{array}{lcl} \|M_U[V]\|_{\ell^2_h} & \leq & K \|V\|_{\ell^{2;2}_h}, \\ \|\partial^+ M_U[V]\|_{\ell^2_h} & \leq & K \|V\|_{\ell^{2;3}_h} + K \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \|\partial^+ V\|_{\ell^2_h}, \\ \|\partial^+ M_U[V] - M_U[\partial^+ V]\|_{\ell^2_h} & \leq & K \|V\|_{\ell^{2;2}_h} + K \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \|\partial^+ V\|_{\ell^2_h}, \end{array} \tag{C.1}$$

together with the estimate

$$\left\|\partial^+ M_U[V] - M^+_{U;\rm apx}[V]\right\|_{\ell^2_h} \leq Kh \|V\|_{\ell^{2;3}_h} + Kh \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \|V\|_{\ell^{2;2}_h}. \tag{C.2}$$

In addition, for any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the Lipschitz bound

$$\|M_{U^{(2)}}[V] - M_{U^{(1)}}[V]\|_{\ell^2_h} \leq K \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{2;2}_h} \|V\|_{\ell^{\infty;1}_h} + K \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{\infty;1}_h} \|V\|_{\ell^{2;2}_h}. \tag{C.3}$$

Proposition C.2. *Assume that (Hg) and ($H\Phi_*$) are satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists $K > 0$ so that for any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the a-priori bounds*

$$\begin{array}{lcl} \|\mathcal{E}_{\rm sm}(U)\|_{\ell^\infty_h} + \|\mathcal{E}_{\rm sm}(U)\|_{\ell^2_h} & \leq & K, \\ \|\mathcal{E}_{\rm tw}(U)\|_{\ell^\infty_h} + \|\mathcal{E}_{\rm tw}(U)\|_{\ell^2_h} & \leq & K \end{array} \tag{C.4}$$

together with the estimate

$$\left\|\partial^+[\mathcal{E}_{\rm tw}(U)] - \mathcal{E}^+_{\rm tw;apx}(U)\right\|_{\ell^\infty_h} + \left\|\partial^+[\mathcal{E}_{\rm tw}(U)] - \mathcal{E}^+_{\rm tw;apx}(U)\right\|_{\ell^2_h} \leq Kh, \tag{C.5}$$

while for any $U^{(1)} \in \Omega_{h;\kappa}$ and $U^{(2)} \in \Omega_{h;\kappa}$ we have the Lipschitz bounds

$$\begin{array}{lcl} \left\|\mathcal{E}_{\rm sm}(U^{(1)}) - \mathcal{E}_{\rm sm}(U^{(2)})\right\|_{\ell^2_h} & \leq & K\Big[\left\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\right\|_{\ell^2_h} + \left\|\partial^+\partial^+ U^{(2)} - \partial^+\partial^+ U^{(1)}\right\|_{\ell^2_h}\Big], \\ \left\|\mathcal{E}_{\rm tw}(U^{(1)}) - \mathcal{E}_{\rm tw}(U^{(2)})\right\|_{\ell^2_h} & \leq & K \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{2;2}_h}. \end{array} \tag{C.6}$$

Proof. The bounds in (C.4) and (C.6) follow directly from $\|h\partial^-\|_{\mathcal{L}(\ell^2_h, \ell^2_h)} \leq 2$, the Lipschitz bounds (B.4), the estimate

$$\|g(U)\|_{\ell^2_h} \leq 4\Big[\sup_{|u| \leq \kappa^{-1}} |g'(u)|\Big] \tag{C.7}$$

from [26, Eq. (3.43)] and the pointwise inequality

$$\left|g(U^{(2)}) - g(U^{(1)})\right| \leq \Big[\sup_{|u| \leq \kappa^{-1}} |g'(u)|\Big] \left|U^{(2)} - U^{(1)}\right|. \tag{C.8}$$

In order to establish (C.5), we compute

$$\partial^+[\mathcal{E}_{\rm tw}(U)] = \partial^+[\gamma_U^{-4}] T^+[\partial^{(2)} U] + \gamma_U^{-4} \partial^+ \partial^{(2)} U + \partial^+[g(U)] - c_* \partial^+[\gamma_U^{-1} \partial^0 U] \tag{C.9}$$

and notice that

$$\partial^+[g(U)] - g'(U)\partial^0 U = \partial^+[g(U)] - g'(U)\partial^+ U + \tfrac{h}{2} g'(U)\partial^{(2)} U. \tag{C.10}$$

Upon estimating

$$\begin{array}{lcl} |\partial^+[g(U)] - g'(U)\partial^+ U| & = & h^{-1}\,|g(U + h\partial^+ U) - g(U) - g'(U)h\partial^+ U| \\ & \leq & \frac{1}{2}\Big[\sup_{|u|\leq \kappa^{-1}} |g''(u)|\Big] h^{-1}\,|h\partial^+ U|^2 \\ & = & \frac{1}{2}h\Big[\sup_{|u|\leq \kappa^{-1}} |g''(u)|\Big]\,|\partial^+ U|^2 , \end{array} \tag{C.11}$$

we can use (B.5) together with (B.6) to obtain the desired bound. □

Proposition C.3. *Assume that (Hg) and (HΦ_*) are satisfied. Then there exists $K > 0$ so that for any $h > 0$ we have the estimates*

$$\begin{array}{lcl} \|\mathcal{E}_{\mathrm{sm}}(\Psi_*)\|_{\ell^\infty_h} + \|\mathcal{E}_{\mathrm{sm}}(\Psi_*)\|_{\ell^2_h} & \leq & Kh, \\ \|\mathcal{E}_{\mathrm{tw}}(\Psi_*)\|_{\ell^\infty_h} + \|\mathcal{E}_{\mathrm{tw}}(\Psi_*)\|_{\ell^2_h} & \leq & Kh, \\ \|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)]\|_{\ell^\infty_h} + \|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)]\|_{\ell^2_h} & \leq & Kh. \end{array} \tag{C.12}$$

Proof. We have $\Psi_* \in W^{3;q}$ for $q \in \{2, \infty\}$, which allows us to apply (A.7) and (A.6) to obtain

$$\|\mathcal{E}_{\mathrm{sm}}(\Psi_*)\|_{\ell^q_h} \;\leq\; C_1' h\,\|\partial^-\partial^+\partial^-\Psi_*\|_{\ell^q_h} \;\leq\; C_1' h\,\|\Psi_*'''\|_{L^q} . \tag{C.13}$$

This yields the first bound.

Since the functions

$$\gamma_* = \sqrt{1 - (\Psi_*')^2}, \qquad\qquad \gamma_{\Psi_*} = \sqrt{1 - (\partial^0\Psi_*)^2} \tag{C.14}$$

are both uniformly bounded away from zero, we have the pointwise estimate

$$\left|\gamma_*^{-1} - \gamma_{\Psi_*}^{-1}\right| + \left|\gamma_*^{-3} - \gamma_{\Psi_*}^{-3}\right| + \left|\gamma_*^{-4} - \gamma_{\Psi_*}^{-4}\right| + \left|\gamma_*^{-6} - \gamma_{\Psi_*}^{-6}\right| \leq C_1' \left|\partial^0\Psi_* - \Psi_*'\right| . \tag{C.15}$$

Exploiting the fact that Ψ_*', Ψ_*'', , γ_*^{-1} $\gamma_{\Psi_*}^{-1}$, $\partial^0\Psi_*$ and $\partial^{(2)}\Psi_*$ are all uniformly bounded, we now see that

$$\begin{array}{lcl} \left\|\gamma_*^{-1}\Psi_*' - \gamma_{\Psi_*}^{-1}\partial^0\Psi_*\right\|_{\ell^q_h} & \leq & C_2' \left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} , \\ \left\|\gamma_*^{-4}\Psi_*'' - \gamma_{\Psi_*}^{-1}\partial^{(2)}\Psi_*\right\|_{\ell^q_h} & \leq & C_2' \big[\left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} + \left\|\partial^{(2)}\Psi_* - \Psi_*''\right\|_{\ell^q_h} \big], \\ \left\|\gamma_{\Psi_*}^{-4}\partial^+\partial^{(2)}\Psi_* - \gamma_*^{-4}\Psi_*'''\right\|_{\ell^q_h} & \leq & C_2' \big[\left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} + \left\|\partial^+\partial^{(2)}\Psi_* - \Psi_*'''\right\|_{\ell^q_h} \big] \end{array} \tag{C.16}$$

for $q \in \{2, \infty\}$.

Since $\Psi_* \in W^{3,2} \cap W^{3,\infty}$, we may apply (A.8) and (A.9) to obtain

$$\left\|\gamma_*^{-1}\Psi_*' - \gamma_{\Psi_*}^{-1}\partial^0\Psi_*\right\|_{\ell^q_h} + \left\|\gamma_*^{-4}\Psi_*'' - \gamma_{\Psi_*}^{-4}\partial^{(2)}\Psi_*\right\|_{\ell^q_h} + \left\|\gamma_*^{-4}\Psi_*''' - \gamma_{\Psi_*}^{-4}\partial^{(2)}\Psi_*\right\|_{\ell^q_h} \;\leq\; C_3' h \tag{C.17}$$

for $q \in \{2, \infty\}$. The travelling wave equation (3.32) allows us to write

$$\mathcal{E}_{\mathrm{tw}}(\Psi_*) \;=\; \gamma_{\Psi_*}^{-4}\partial^{(2)}\Psi_* - \gamma_*^{-4}\Psi_*'' - c_*\gamma_{\Psi_*}^{-1}\partial^0\Psi_* + c_*\gamma_*^{-1}\Psi_*', \tag{C.18}$$

which using (C.17) yields the second bound.

Using the fact that $\Psi_* \in W^{4,2} \cap W^{4,\infty}$, which allows us to apply (A.10), we may argue in a similar fashion as above to conclude

$$\begin{array}{lcl} \big\|\gamma_*^{-6}\Psi_*''\Psi_*'\Psi_*'' - \gamma_{\Psi_*}^{-6}\partial^0\Psi_* S^+[\partial^{(2)}\Psi_*]T^+[\partial^{(2)}\Psi_*]\big\|_{\ell^q_h} & \le & C_4' h, \\ \big\|\gamma_*^{-3}\Psi_*'' - \gamma_{\Psi_*}^{-3} S^+[\partial^{(2)}\Psi_*]\big\|_{\ell^q_h} & \le & C_4' h, \\ \big\|\gamma_*^{-4}\Psi_*''' - \gamma_{\Psi_*}^{-4}\partial^+\partial^{(2)}\Psi_*\big\|_{\ell^q_h} & \le & C_4' h \end{array} \tag{C.19}$$

for $q \in \{2, \infty\}$. The differentiated travelling wave equation (3.33) allows us to write

$$\begin{array}{lcl} \mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*) & = & 4\gamma_{\Psi_*}^{-6}\partial^0\Psi_* S^+[\partial^{(2)}\Psi_*]T^+[\partial^{(2)}\Psi_*] - 4\gamma_*^{-6}\Psi_*''\Psi_*'\Psi_*'' \\ & & +\gamma_{\Psi_*}^{-4}\partial^+\partial^{(2)}\Psi_* - \gamma_*^{-4}\Psi_*''' \\ & & +g'(\Psi_*)\partial^0\Psi_* - g'(\Psi_*)\Psi_*' \\ & & -c_*\gamma_{\Psi_*}^{-3}S^+[\partial^{(2)}\Psi_*] + c_*\gamma_*^{-3}\Psi_*''. \end{array} \tag{C.20}$$

Using (C.19) together with (C.5) we may hence conclude

$$\|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)]\|_{\ell^q_h} \;\le\; \big\|\mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*)\big\|_{\ell^q_h} + \big\|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)] - \mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*)\big\|_{\ell^q_h} \;\le\; C_3' h, \tag{C.21}$$

which yields the third bound. □

D Decompositions for $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$

Our goal here is to provide the explicit decompositions (4.1) for the nonlinearities (3.35) and (3.36). In addition, we validate the bookkeeping claims made in Propositions 4.1-4.6, providing the underpinning for the estimates in §4.2. For efficiency purposes, we combine our treatment of nonlinearities that admit similar bounds.

D.1 Decompositions for $\mathcal{Y}_1$ and $\mathcal{X}_A$

Recalling the definitions

$$\begin{array}{lclclcl} \mathcal{Y}_{1;\mathrm{apx}}(U), & = & \partial^0 U & \quad & \mathcal{Y}_{1;\mathrm{lin};U}[V] & = & \partial^0 V - \partial^0 U\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big], \\ \mathcal{X}_{A;\mathrm{apx}}(U) & = & \partial^0 U, & & \mathcal{X}_{A;\mathrm{lin};U}[V] & = & \partial^0 V + \partial^0 U\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big] \end{array} \tag{D.1}$$

from (3.37) and (3.39), we realize the splittings (4.1) by writing

$$\begin{array}{lclclcl} \mathcal{Y}_{1;\mathrm{apx;expl}}(U) & = & \partial^0 U, & \quad & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V] & = & \partial^0 V, \\ \mathcal{Y}_{1;\mathrm{apx;sh}}(U) & = & 0, & & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{Y}_{1;\mathrm{apx;rem}}(U) & = & 0, & & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{rem}}[V] & = & -\partial^0 U\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big], \end{array} \tag{D.2}$$

together with

$$\begin{array}{lclclcl} \mathcal{X}_{A;\mathrm{apx;expl}}(U) & = & \partial^0 U, & \quad & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V] & = & \partial^0 V, \\ \mathcal{X}_{A;\mathrm{apx;sh}}(U) & = & 0, & & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{X}_{A;\mathrm{apx;rem}}(U) & = & 0, & & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{rem}}[V] & = & \partial^0 U\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big]. \end{array} \tag{D.3}$$

In addition, for both nonlinearities $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$ we introduce the exponent sets

$$Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = Q_f = Q_{f;\mathrm{lin}} = Q_{f;\mathrm{lin;rem}} = \{2, \infty\}. \tag{D.4}$$

Lemma D.1. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. Then there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{lllll} \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^2_h} & \leq & K\|\partial^+ V\|_{\ell^2_h} & \leq & KT_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^\infty_h} & \leq & K\|\partial^+ V\|_{\ell^2_h} & \leq & KT_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^2_h} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^2_h} & \leq & K\|\partial^+ V\|_{\ell^2_h} & \leq & KT_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^\infty_h} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^\infty_h} & \leq & K\|\partial^+ V\|_{\ell^\infty_h} & \leq & KT_{\infty;\mathrm{opt}}(V) \end{array} \tag{D.5}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. The bounds follow from inspection. □

Lemma D.2. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|f_{\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^2_h} + \|f_{\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^\infty_h} \quad \leq \quad K. \tag{D.6}$$

(ii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{lll} \big\|f_{\mathrm{apx};\mathrm{expl}}(U^{(1)}) - f_{\mathrm{apx};\mathrm{expl}}(U^{(2)})\big\|_{\ell^2_h} & \leq & K\big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell^2_h} \\ & \leq & KS_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big), \\ \big\|f_{\mathrm{apx};\mathrm{expl}}(U^{(1)}) - f_{\mathrm{apx};\mathrm{expl}}(U^{(2)})\big\|_{\ell^\infty_h} & \leq & K\big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell^\infty_h} \\ & \leq & KS_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big). \end{array} \tag{D.7}$$

Proof. These estimates follow by inspection. □

Lemma D.3. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\begin{array}{lll} \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^2_h} & \leq & K\|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h}\|\partial^+ V\|_{\ell^2_h} \\ & \leq & K\mathcal{E}_{\mathrm{rem};U}(V). \end{array} \tag{D.8}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bounds

$$\begin{array}{lll} \big\|f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V]\big\|_{\ell^2_h} & = & 0, \\ \big\|f_{\mathrm{lin};U^{(1)};\mathrm{rem}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{rem}}[V]\big\|_{\ell^2_h} & \leq & K\|\partial^+ V\|_{\ell^2_h}\Big[\big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell^2_h} \\ & & \qquad + \big\|\partial^{(2)} U^{(1)} - \partial^{(2)} U^{(2)}\big\|_{\ell^2_h}\Big] \\ & \leq & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{D.9}$$

Proof. Recalling the Lipschitz bound (C.6), the estimates follow by inspection. □

D.2 Decomposition for $\mathcal{Y}_2$

Recalling the definitions

$$\mathcal{Y}_{2;\mathrm{apx}}(U) \;=\; \gamma_U^{-4}\partial^{(2)}U + g(U), \qquad \mathcal{Y}_{2;\mathrm{lin};U}[V] \;=\; \gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V \tag{D.10}$$

from (3.37) and (3.39) and $\mathcal{E}_{\mathrm{tw}}$ from (3.29), we realize the splittings (4.1) by writing

$$\begin{array}{rclrcl} \mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U) & = & c_*\gamma_U^{-1}\partial^0 U, & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V] & = & \gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V, \\ \mathcal{Y}_{2;\mathrm{apx};\mathrm{sh}}(U) & = & 0, & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U) & = & \mathcal{E}_{\mathrm{tw}}(U), & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{rem}}[V] & = & 0. \end{array} \tag{D.11}$$

In addition, we introduce the sets

$$Q_{\mathcal{Y}_2;\mathrm{pref}} = \overline{Q}_{\mathcal{Y}_2;\mathrm{pref}} = \{2\}, \tag{D.12}$$

together with

$$Q_{\mathcal{Y}_2} = \{2, \infty\}, \qquad Q_{\mathcal{Y}_2;\mathrm{lin};\mathrm{rem}} = Q_{\mathcal{Y}_2;\mathrm{lin}} = \{2\}. \tag{D.13}$$

Lemma D.4. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}_2$. Then there exists a constant $K > 0$ so that the bound*

$$\|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell_h^2} \;\leq\; K\|V\|_{\ell_h^{2;2}} \;\leq\; KT_{\mathrm{safe}}(V) \tag{D.14}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$.

Proof. This follows from Proposition C.1. □

Lemma D.5. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\|_{\ell_h^2} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\|_{\ell_h^\infty} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell_h^2} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell_h^\infty} \;\leq\; K. \tag{D.15}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{rcccl} \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell_h^2} & \leq & \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} & \leq & S_{\mathrm{rem;full}}(U), \\ \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell_h^\infty} & \leq & \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} & \leq & S_{\mathrm{rem;full}}(U). \end{array} \tag{D.16}$$

(iii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{rcl} \left\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}) - \mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)})\right\|_{\ell_h^2} & \leq & K\left\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\right\|_{\ell_h^2} \\ & \leq & KS_{\mathrm{diff};2;\mathrm{fix}}\big(U^{(1)}, U^{(2)}\big), \\ \left\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U^{(1)}) - \mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U^{(2)})\right\|_{\ell_h^2} & \leq & K\left\|U^{(1)} - U^{(2)}\right\|_{\ell_h^{2;2}} \\ & \leq & KS_{\mathrm{diff};2;\mathrm{fix}}\big(U^{(1)}, U^{(2)}\big). \end{array} \tag{D.17}$$

Proof. Recalling (C.6), these bounds follow by inspection. □

Lemma D.6. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bound*

$$\begin{aligned} \left\| \mathcal{Y}_{2;\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - \mathcal{Y}_{2;\mathrm{lin};U^{(2)};\mathrm{expl}}[V] \right\|_{\ell_h^2} &\leq K \left\| U^{(1)} - U^{(2)} \right\|_{\ell_h^{2;2}} \|V\|_{\ell_h^{\infty;1}} \\ &\qquad + K \left\| U^{(1)} - U^{(2)} \right\|_{\ell_h^{\infty;1}} \|V\|_{\ell_h^{2;2}} \\ &\leq K \mathcal{E}_{\mathrm{prod}}(U^{(1)} - U^{(2)}, V) \end{aligned} \tag{D.18}$$

for any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. This bound follows directly from (B.4) and [27, Prop 5.1]. □

D.3 Decomposition for $\mathcal{D}^{\diamond_0;+}$ and $\mathcal{D}^{\diamond_-;+}$

For both functions $f \in \{\mathcal{D}^{\diamond_0;+}, \mathcal{D}^{\diamond_-;+}\}$ we write $f_{\mathrm{apx;sh}}(U) = f_{\mathrm{apx;rem}}(U) = 0$ and $f_{\mathrm{lin;sh}}(U) = f_{\mathrm{lin;rem}}(U) = 0$ and introduce the exponent sets

$$Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = Q_{f;\mathrm{lin}} = Q_{f;\mathrm{lin;rem}} = \{2\}, \qquad Q_f = \{2, \infty\}. \tag{D.19}$$

Besides the Lipschitz estimates below, all the estimates we require here can be found in [26, Prop. 7.3].

Lemma D.7. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{D}^{\diamond_0;+}, \mathcal{D}^{\diamond_-;+}\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{aligned} \left\| f_{\mathrm{apx;expl}}(U^{(1)}) - f_{\mathrm{apx;expl}}(U^{(2)}) \right\|_{\ell_h^2} &\leq K \big[\left\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \right\|_{\ell_h^2} + \left\| \partial^{(2)} U^{(1)} - \partial^{(2)} U^{(2)} \right\|_{\ell_h^2} \big] \\ &\leq K S_{\mathrm{diff;2;fix}}\big(U^{(1)}, U^{(2)}\big). \end{aligned} \tag{D.20}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\begin{aligned} \left\| f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V] \right\|_{\ell_h^2} &\leq K \big[\|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2} \big] \left\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \right\|_{\ell_h^\infty} \\ &\qquad + K \|\partial^+ V\|_{\ell_h^\infty} \left\| \partial^+\partial^+ U^{(1)} - \partial^+\partial^+ U^{(2)} \right\|_{\ell_h^2} \\ &\leq K \mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{aligned} \tag{D.21}$$

Proof. These bounds follow by inspecting the definitions (3.40). □

D.4 Decompositions for $\mathcal{X}_B$, $\mathcal{X}_C$ and $\mathcal{X}_D$

Recalling the definitions

$$\begin{aligned} \mathcal{X}_{B;\mathrm{apx}}(U) &= S^+[\gamma_U^{-1}]\gamma_U^4, \\ \mathcal{X}_{C;\mathrm{apx}}(U) &= S^+[\gamma_U^{-1}](\gamma_U^4 - \gamma_U^2), \\ \mathcal{X}_{D;\mathrm{apx}}(U) &= S^+[\gamma_U \partial^0 U]\partial^0 U \end{aligned} \tag{D.22}$$

from (3.37) together with

$$\begin{array}{lll}\mathcal{X}_{B;\mathrm{lin};U}[V] & = & S^+\Big[\gamma_U^{-3}\partial^0 U\partial^0 V + \gamma_U^{-1}\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big]\Big]\gamma_U^4 \\ & & \quad +S^+[\gamma_U^{-1}](-4\gamma_U^2)\partial^0 U\partial^0 V, \\ \mathcal{X}_{C;\mathrm{lin};U}[V] & = & S^+\Big[\gamma_U^{-3}\partial^0 U\partial^0 V + \gamma_U^{-1}\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big](\gamma_U^4 - \gamma_U^2) \\ & & \quad +S^+[\gamma_U^{-1}][2-4\gamma_U^2]\partial^0 U\partial^0 V, \\ \mathcal{X}_{D;\mathrm{lin};U}[V] & = & S^+[\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V]\partial^0 U + S^+[\gamma_U\partial^0 U]\partial^0 V \\ & & \quad +S^+[\gamma_U\partial^0 U]\partial^0 U\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\end{array} \tag{D.23}$$

from (3.38), we realize the splittings (4.1) by writing

$$\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U) = T^+[\gamma_U^3], \qquad \mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V] = -3T^+\Big[\gamma_U\partial^0 U\partial^0 V\Big], \tag{D.24}$$

together with

$$\begin{array}{lllll}\mathcal{X}_{C;\mathrm{apx};\mathrm{expl}}(U) & = & -\mathcal{X}_{D;\mathrm{apx};\mathrm{expl}}(U) & = & \gamma_U(\gamma_U^2-1), \\ \mathcal{X}_{C;\mathrm{lin};U;\mathrm{expl}}[V] & = & -\mathcal{X}_{D;\mathrm{lin};U;\mathrm{expl}}[V] & = & \gamma_U^{-1}(1-3\gamma_U^2)\partial^0 U\partial^0 V\end{array} \tag{D.25}$$

for the explicit terms. The shift terms are given by

$$\begin{array}{lll}\mathcal{X}_{B;\mathrm{apx};\mathrm{sh}}(U) & = & -hS^+[\gamma_U^{-1}]\partial^+[\gamma_U^4] - \frac{h}{2}\partial^+[\gamma_U^{-1}]T^+[\gamma_U^4], \\ \mathcal{X}_{C;\mathrm{apx};\mathrm{sh}}(U) & = & \frac{h}{2}\partial^+[\gamma_U^{-1}]\gamma_U^2(\gamma_U^2-1), \\ \mathcal{X}_{D;\mathrm{apx};\mathrm{sh}}(U) & = & \frac{h}{2}\partial^+[\gamma_U\partial^0 U]\partial^0 U,\end{array} \tag{D.26}$$

together with

$$\begin{array}{lll}\mathcal{X}_{B;\mathrm{lin};U;\mathrm{sh}}[V] & = & -\frac{h}{2}\partial^+\Big[\gamma_U^{-3}\partial^0 U\partial^0 V\Big]\gamma_U^4 \\ & & \quad -hS^+[\gamma_U^{-1}]\partial^+[-4\gamma_U^2\partial^0 V] - h\partial^+[\gamma_U^{-1}]T^+\big[-2\gamma_U^2\partial^0 V\big], \\ \mathcal{X}_{C;\mathrm{lin};U;\mathrm{sh}}[V] & = & \frac{h}{2}\partial^+\big[\gamma_U^{-3}\partial^0 U\partial^0 V\big]\gamma_U^2(\gamma_U^2-1) \\ & & \quad +\frac{h}{2}\partial^+\big[\gamma_U^{-1}\big][2-4\gamma_U^2]\partial^0 U\partial^0 V, \\ \mathcal{X}_{D;\mathrm{lin};U;\mathrm{sh}}[V] & = & \frac{h}{2}\partial^+\big[\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V\big]\partial^0 U + \frac{h}{2}\partial^+[\gamma_U\partial^0 U]\partial^0 V,\end{array} \tag{D.27}$$

while the remainder terms are given by

$$\mathcal{X}_{B;\mathrm{apx};\mathrm{rem}}(U) = \mathcal{X}_{C;\mathrm{apx};\mathrm{rem}}(U) = \mathcal{X}_{D;\mathrm{apx};\mathrm{rem}}(U) = 0, \tag{D.28}$$

together with

$$\begin{array}{lll}\mathcal{X}_{B;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+\Big[\gamma_U^{-1}\big[\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big]\Big]\gamma_U^4, \\ \mathcal{X}_{C;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+\big[\gamma_U^{-1}\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big](\gamma_U^4-\gamma_U^2), \\ \mathcal{X}_{D;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+[\gamma_U\partial^0 U]\partial^0 U\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V.\end{array} \tag{D.29}$$

In addition, for any $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$ we introduce the exponent sets

$$Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = \{\infty\}, \tag{D.30}$$

together with

$$Q_f = \{\infty\}, \qquad Q_{f;\mathrm{lin}} = \{2,\infty\}, \qquad Q_{f;\mathrm{lin};\mathrm{rem}} = \{\infty\}. \tag{D.31}$$

Lemma D.8. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. Then there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{lllll} \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell_h^\infty} & \le & K\,\|\partial^+ V\|_{\ell_h^2} & \le & KT_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell_h^2} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell_h^2} & \le & K\,\|\partial^+ V\|_{\ell_h^2} & \le & KT_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell_h^\infty} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell_h^\infty} & \le & K\,\|\partial^+ V\|_{\ell_h^\infty} & \le & KT_{\infty;\mathrm{opt}}(V) \end{array} \tag{D.32}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$.

Proof. These bounds follow by inspection. □

Lemma D.9. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|f_{\mathrm{apx;expl}}(U)\|_{\ell_h^\infty} + \|f_{\mathrm{apx;sh}}(U)\|_{\ell_h^\infty} \;\le\; K. \tag{D.33}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\|f_{\mathrm{apx;sh}}(U)\|_{\ell_h^\infty} \;\le\; Kh \;\le\; KS_{\mathrm{sh;full}}(U). \tag{D.34}$$

(iii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{lll} \big\|f_{\mathrm{apx;expl}}(U^{(1)}) - f_{\mathrm{apx;expl}}(U^{(2)})\big\|_{\ell_h^\infty} & \le & K\,\big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell_h^\infty} \\ & \le & KS_{\mathrm{diff;full}}\big(U^{(1)}, U^{(2)}\big), \\ \big\|f_{\mathrm{apx;sh}}(U^{(1)}) - f_{\mathrm{apx;sh}}(U^{(2)})\big\|_{\ell_h^\infty} & \le & K\,\big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell_h^\infty} \\ & \le & KS_{\mathrm{diff;full}}\big(U^{(1)}, U^{(2)}\big). \end{array} \tag{D.35}$$

Proof. These bounds follow from the discrete derivative expressions in Lemma B.1 and the Lipschitz bounds for γ_U in (B.4). □

Lemma D.10. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell_h^2$, we have the bounds

$$\begin{array}{lllll} \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell_h^2} & \le & Kh\big[\,\|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2}\,\big] & \le & K\mathcal{E}_{\mathrm{sh};U}(V), \\ \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell_h^\infty} & \le & K\,\|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2}\,\|\partial^+ V\|_{\ell_h^2} & \le & K\mathcal{E}_{\mathrm{rem};U}(V). \end{array} \tag{D.36}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}^2$ and any $V \in \ell_h^2$, we have the bounds

$$\begin{array}{lll} \big\|f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V]\big\|_{\ell_h^2} & \le & K\,\big\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\big\|_{\ell_h^\infty}\,\|\partial^+ V\|_{\ell_h^2} \\ & \le & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \\ \big\|f_{\mathrm{lin};U^{(1)};\mathrm{sh}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{sh}}[V]\big\|_{\ell_h^2} & \le & K\,\big\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\big\|_{\ell_h^\infty}\,\|\partial^+ V\|_{\ell_h^2} \\ & \le & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \\ \big\|f_{\mathrm{lin};U^{(1)};\mathrm{rem}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{rem}}[V]\big\|_{\ell_h^\infty} & \le & K\,\|\partial^+ V\|_{\ell_h^2}\,\Big[\,\big\|U^{(1)} - U^{(2)}\big\|_{\ell_h^{2;2}} \\ & & \qquad + \big\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\big\|_{\ell_h^\infty}\,\Big] \\ & \le & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{D.37}$$

Proof. Recalling the Lipschitz bounds (C.6), these estimates follow from inspection. □

D.5 Decomposition for $\mathcal{Y}_1^+$

Recalling the definitions

$$\begin{array}{rcl} \mathcal{Y}^+_{1;\mathrm{apx}}(U) & = & \gamma_U^{-1} S^+[\partial^{(2)}U]T^+\gamma_U, \\ \mathcal{Y}^+_{1;\mathrm{lin};U}[V] & = & \big[\gamma_U^{-3}\partial^0 U[S^+\partial^{(2)}U]\partial^0 V + \gamma_U^{-1}S^+\partial^{(2)}V\big]T^+\gamma_U \\ & & \quad -\gamma_U^{-1}S^+[\partial^{(2)}U]T^+\Big[\gamma_U^{-1}\partial^0 U\partial^0 V + \gamma_U \sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big]. \end{array} \tag{D.38}$$

from (3.37) and (3.39), we realize the splittings (4.1) by writing

$$\begin{array}{rcl} \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U) & = & \partial^{(2)}U, \\ \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U) & = & \frac{h}{2}\partial^+\partial^{(2)}U + h\partial^+[\gamma_U]\gamma_U^{-1}S^+[\partial^{(2)}U], \\ \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{rem}}(U) & = & 0, \end{array} \tag{D.39}$$

together with

$$\begin{array}{rcl} \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{expl}}[V] & = & S^+[\partial^{(2)}V], \\ \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{sh}}[V] & = & h\gamma_U^{-3}\partial^+[\gamma_U]\partial^0 U S^+[\partial^{(2)}U]\partial^0 V \\ & & \quad +h\gamma_U^{-1}\partial^+[\gamma_U]S^+[\partial^{(2)}V] \\ & & \quad -h\gamma_U^{-1}S^+[\partial^{(2)}U]\partial^+\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big], \\ \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{rem}}[V] & = & \gamma_U^{-1}S^+[\partial^{(2)}U]T^+\Big[\gamma_U\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big]. \end{array} \tag{D.40}$$

Notice that we have eliminated the $T^+[\partial^{(2)}U]$ term in the explicit expressions, while keeping the $T^+[\partial^{(2)}V]$ dependency. This inconsistency is deliberate as it will help us to make a useful substitution in the sequel.

In addition, we introduce the sets

$$\overline{Q}_{\mathcal{Y}_1^+;\mathrm{pref}} = \{2,\infty\}, \tag{D.41}$$

together with

$$Q_{\mathcal{Y}_1^+} = \{2,\infty\}, \qquad Q^A_{\mathcal{Y}_1^+;\mathrm{lin}} = Q^B_{\mathcal{Y}_1^+;\mathrm{lin}} = \{2,\infty\}. \tag{D.42}$$

Lemma D.11. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}_1^+$. Then there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{rcl} \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^2_h} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^2_h} & \leq & K\big[\,\|\partial^+V\|_{\ell^2_h} + \|\partial^+\partial^+V\|_{\ell^2_h}\,\big] \quad \leq \quad K\overline{T}_{\mathrm{safe}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^\infty_h} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^\infty_h} & \leq & K\big[\,\|\partial^+V\|_{\ell^\infty_h} + \|\partial^+\partial^+V\|_{\ell^\infty_h}\,\big] \quad \leq \quad K\overline{T}_{\infty;\mathrm{opt}}(V), \\ \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^2_h} + \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^\infty_h} & \leq & K\,\|\partial^+V\|_{\ell^2_h} \quad \leq \quad K\overline{T}_{\mathrm{safe}}(V) \end{array} \tag{D.43}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. These bounds follow by inspection. □

Lemma D.12. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\Big\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\Big\|_{\ell^2_h} + \Big\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\Big\|_{\ell^\infty_h} + \Big\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\Big\|_{\ell^2_h} + \Big\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\Big\|_{\ell^\infty_h} \leq K. \tag{D.44}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\begin{array}{lllll} \left\| \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^2_h} & \leq & Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}] & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\| \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^\infty_h} & \leq & Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}] & \leq & K\overline{S}_{\mathrm{sh;full}}(U). \end{array} \tag{D.45}$$

Proof. These bounds follow from the discrete derivative identities in Lemma B.1. □

Lemma D.13. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bounds*

$$\begin{array}{lllll} \left\| \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{sh}}[V] \right\|_{\ell^2_h} & \leq & Kh\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \,\big] & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\| \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{rem}}[V] \right\|_{\ell^2_h} & \leq & K \, \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h} \, \|\partial^+ V\|_{\ell^2_h} & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V) \end{array} \tag{D.46}$$

for any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. These estimates follow by inspection. □

D.6 Decomposition for $\mathcal{Y}^+_{2b}$

Recalling the definitions

$$\begin{array}{lll} \mathcal{Y}^+_{2b;\mathrm{apx}}(U) & = & \Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U\Big] + c_*\gamma_U^{-3}S^+[\partial^{(2)}U], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] & = & 4[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^{(2)}U]T^+[\partial^{(2)}U]\partial^0 V \\ & & +4\gamma_U^{-6}\partial^0 U\Big[T^+[\partial^{(2)}U]S^+[\partial^{(2)}V] + S^+[\partial^{(2)}U]T^+[\partial^{(2)}V]\Big] \\ & & +g''(U)[\partial^0 U]V + g'(U)\partial^0 V \end{array} \tag{D.47}$$

from (3.37) and (3.39), we realize the first splitting in (4.1) by writing

$$\begin{array}{lll} \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U) & = & \Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U\Big] + c_*\gamma_U^{-3}\partial^{(2)}U, \\ \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{sh}}(U) & = & \frac{1}{2}c_* h\gamma_U^{-3}\partial^+[\partial^{(2)}U], \\ \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{rem}}(U) & = & 0. \end{array} \tag{D.48}$$

The second splitting (4.1) is obtained implicitly by recalling the definition (3.15) and writing

$$\begin{array}{lll} \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V] & = & \gamma_U^{-2}\partial^+\big[M_U[V]\big] + 2\gamma_U^{-4}\partial^0 U[\partial^{(2)}U]M_U[V] - \widetilde{M}_{U;III}[V] \\ & & +c_*\Big[3\gamma_U^{-5}\partial^0 U[\partial^{(2)}U]\partial^0 V + \gamma_U^{-3}S^+[\partial^{(2)}V]\Big], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh}}[V] & = & \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] - \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{rem}}[V] & = & 0. \end{array} \tag{D.49}$$

In addition, we introduce the sets

$$\overline{Q}_{\mathcal{Y}^+_{2b};\mathrm{pref}} = Q_{\mathcal{Y}^+_{2b};\mathrm{lin}} = Q_{\mathcal{Y}^+_{2b};\mathrm{lin};\mathrm{rem}} = \{2\}, \qquad Q_{\mathcal{Y}^+_{2b}} = \{2, \infty\}. \tag{D.50}$$

Lemma D.14. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}^+_{2b}$. Then there exists a constant $K > 0$ so that the bound*

$$\|f_{\mathrm{lin};U;\mathrm{expl}}[V]\|_{\ell^2_h} + \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^2_h} \;\leq\; K\,\|V\|_{\ell^{2;2}_h} \;\leq\; K\overline{T}_{\mathrm{safe}}(V) \tag{D.51}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. The bound follows by inspection. □

Lemma D.15. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{expl}}(U) \right\|_{\ell^2_h} + \left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^2_h} + \left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{expl}}(U) \right\|_{\ell^\infty_h} + \left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^\infty_h} \leq K. \tag{D.52}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\begin{aligned} \left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^2_h} &\leq Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}] \leq K\overline{S}_{\mathrm{sh};\mathrm{full}}(U), \\ \left\| \mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^\infty_h} &\leq Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}] \leq K\overline{S}_{\mathrm{sh};\mathrm{full}}(U). \end{aligned} \tag{D.53}$$

Proof. Recalling (3.30), the bounds follow by inspection. □

Lemma D.16. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bound*

$$\begin{aligned} \left\| \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh}}[V] \right\|_{\ell^2_h} &\leq Kh\Big[\|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ V\|_{\ell^2_h} \Big] \\ &\quad + Kh \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \Big[\|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \Big] \\ &\leq K\overline{\mathcal{E}}_{\mathrm{sm};U}(V) \end{aligned} \tag{D.54}$$

for any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. Recalling (3.14), we make the decomposition

$$\mathcal{Y}^{+}_{2b;\mathrm{lin};U}[V] = \widetilde{M}_{U;I}[V] + \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};a}[V] \tag{D.55}$$

by writing

$$\begin{aligned} \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};a}[V] &= 2h[6\gamma_U^{-8} - 5\gamma_U^{-6}][\partial^{(2)}U]\partial^+[\partial^{(2)}U]\partial^0 V \\ &\quad + 4h[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^{(2)}U]\partial^+[\partial^{(2)}U]\partial^0 V \\ &\quad + 2h\gamma_U^{-6}\partial^0 U\Big[\partial^+[\partial^{(2)}U]\partial^{(2)}V + [\partial^{(2)}U]\partial^+[\partial^{(2)}V]\Big] \\ &\quad + 4h\gamma_U^{-6}\partial^0 U\Big[\partial^+[\partial^{(2)}U]T^+[\partial^{(2)}V] + [\partial^{(2)}U]\partial^+[\partial^{(2)}V]\Big]. \end{aligned} \tag{D.56}$$

Introducing the function

$$\mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};b}[V] = -\gamma_U^{-2}\Big[\partial^+\big[M_U[V]\big] - M^{+}_{U;\mathrm{apx}}[V]\Big] - \frac{1}{2}hc_*\gamma_U^{-3}\partial^+[\partial^{(2)}V] \tag{D.57}$$

and recalling (3.13), we see that

$$\begin{aligned} \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{expl}}[V] + \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};b}[V] &= \gamma_U^{-2}M^{+}_{U;\mathrm{apx}}[V] + 2\gamma_U^{-4}\partial^0 U[\partial^{(2)}U]M_U[V] - \widetilde{M}_{U;III}[V] \\ &\quad + c_*\Big[3\gamma_U^{-5}\partial^0 U[\partial^{(2)}U]\partial^0 V + \gamma_U^{-3}[\partial^{(2)}V]\Big] \\ &= \widetilde{M}_{U;I}[V] + \widetilde{M}_{U;II}[V] + \widetilde{M}_{U;III}[V] \\ &\quad - \widetilde{M}_{U;III}[V] - \widetilde{M}_{U;II}[V] \\ &= \mathcal{Y}^{+}_{2b;\mathrm{lin};U} - \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};a}[V]. \end{aligned} \tag{D.58}$$

In particular, we obtain

$$\mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh}}[V] = \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{sh};b}[V]. \tag{D.59}$$

In view of Proposition C.1, the desired bound now follows by inspection. □

Proof of Propositions 4.1-4.6. The statements can be readily verified by inspecting the results in §D.1-§D.6. □

E Reductions for $\mathcal{G}$

Our goal here is to construct the functions $\mathcal{G}_{\mathrm{apx;sh};b}$, $\mathcal{G}_{\mathrm{apx;rem};b}$, $\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}$ and $\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b}$ and demonstrate that they satisfy the corresponding bounds in Propositions 5.2-5.3. We proceed in a relatively direct fashion, treating each of the components in (3.42) separately and subsequently combining the results.

E.1 Simplifications for $\mathcal{G}_A$

We recall the definition

$$\mathcal{G}_{A;\mathrm{apx};II}(U) = \Big[1 - \mathcal{Y}_{1;\mathrm{apx;expl}}(U)T^{-}\big[\mathcal{X}_{A;\mathrm{apx;expl}}(U)\big]\Big]\mathcal{Y}_{2;\mathrm{apx;expl}}(U). \tag{E.1}$$

Substituting the relevant expressions from §D we find

$$\mathcal{G}_{A;\mathrm{apx};II}(U) = \Big[1 - \partial^0 U T^{-}\big[\partial^0 U\big]\Big]c_*\gamma_U^{-1}\partial^0 U. \tag{E.2}$$

We now make the decomposition

$$\mathcal{G}_{A;\mathrm{apx};II}(U) = \mathcal{G}_{A;\mathrm{apx};III}(U) + \mathcal{G}_{A;\mathrm{apx;sh};b}(U), \tag{E.3}$$

by introducing

$$\begin{aligned} \mathcal{G}_{A;\mathrm{apx};III}(U) &= c_*\Big[1 - (\partial^0 U)^2\Big]\gamma_U^{-1}\partial^0 U \\ &= c_*\gamma_U\partial^0 U, \end{aligned} \tag{E.4}$$

together with

$$\mathcal{G}_{A;\mathrm{apx;sh};b}(U) = -c_* h\partial^0 U\partial^{-}\big[\partial^0 U\big]\gamma_U^{-1}\partial^0 U. \tag{E.5}$$

We also recall the definition

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;II}[V] = & -\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]T^{-1}\big[\mathcal{X}_{A;\mathrm{apx;expl}}(U)\big]\mathcal{Y}_{2;\mathrm{apx;expl}}(U) \\ & -\mathcal{Y}_{1;\mathrm{apx;expl}}(U)T^{-1}\big[\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{Y}_{2;\mathrm{apx;expl}}(U) \\ & +\Big[1 - \mathcal{Y}_{1;\mathrm{apx;expl}}(U)T^{-1}\big[\mathcal{X}_{A;\mathrm{apx;expl}}(U)\big]\Big]\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]. \end{aligned} \tag{E.6}$$

Substituting the relevant expressions from §D, we find

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;II}[V] = & -\partial^0 V T^{-1}[\partial^0 U]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & -\partial^0 U T^{-1}[\partial^0 V]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & +\Big[1 - \partial^0 U T^{-1}\big[\partial^0 U\big]\Big]\big(\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\big). \end{aligned} \tag{E.7}$$

We now make the decomposition

$$\mathcal{G}_{A;\mathrm{lin};U;II}[V] = \mathcal{G}_{A;\mathrm{lin};U;III}[V] + \mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] \tag{E.8}$$

by introducing

$$\begin{aligned}
\mathcal{G}_{A;\mathrm{lin};U;III}[V] &= -c_*\partial^0 V[\partial^0 U]\gamma_U^{-1}\partial^0 U - c_*\partial^0 U[\partial^0 V]\gamma_U^{-1}\partial^0 U \\
&\quad +\Big[1-\partial^0 U\partial^0 U\Big]\big(\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\big) \\
&= c_*\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V + M_U[V],
\end{aligned} \tag{E.9}$$

together with

$$\begin{aligned}
\mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] &= -c_* h\partial^0 V\partial^-\Big[\partial^0 U\Big]\gamma_U^{-1}\partial^0 U \\
&\quad -c_* h\partial^0 U\partial^-\Big[\partial^0 V\Big]\gamma_U^{-1}\partial^0 U \\
&\quad -h\partial^0 U\partial^-\big[\partial^0 U\big]\big(\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\big).
\end{aligned} \tag{E.10}$$

We summarize our results by writing

$$\begin{aligned}
\mathcal{G}_{A;\mathrm{apx}}(U) &= \mathcal{G}_{A;\mathrm{apx};III}(U) &&= c_*\gamma_U\partial^0 U, \\
\mathcal{G}_{A;\mathrm{lin};U}[V] &= \mathcal{G}_{A;\mathrm{lin};U;III}[V] &&= c_*\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V + M_U[V]
\end{aligned} \tag{E.11}$$

and obtaining the following bound.

Lemma E.1. *Assume that (Hg) is satisfied and pick $0<\kappa<\frac{1}{12}$. Then there exists a constant $K>0$ so that the following properties hold true.*

(i) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell_h^2$, we have the identities

$$\begin{aligned}
\mathcal{G}_{A;\mathrm{apx};II}(U) &= \mathcal{G}_{A;\mathrm{apx}}(U) + \mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U), \\
\mathcal{G}_{A;\mathrm{lin};U;II}[V] &= \mathcal{G}_{A;\mathrm{lin};U}[V] + \mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V].
\end{aligned} \tag{E.12}$$

(ii) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell_h^2$ we have the bounds

$$\begin{aligned}
\|\mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U)\|_{\ell_h^2} &\le Kh &&= KS_{\mathrm{sh};\mathrm{full}}(U), \\
\|\mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V]\|_{\ell_h^2} &\le Kh\,\|V\|_{\ell_h^{2;2}} &&\le K\mathcal{E}_{\mathrm{sh};U}(V).
\end{aligned} \tag{E.13}$$

Proof. In view of Proposition C.1, the bounds follow by inspection. □

E.2 Simplifications for $\mathcal{G}_B$

We recall the definition

$$\mathcal{G}_{B;\mathrm{apx};II}(U) = \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond-;+}_{\mathrm{apx}}(U). \tag{E.14}$$

Substituting the relevant expressions from §D we find

$$\mathcal{G}_{B;\mathrm{apx};II}(U) = c_*\partial^0 U\textstyle\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^{(2)}U. \tag{E.15}$$

In view of Lemma B.5, we introduce the expressions

$$\begin{aligned}
\mathcal{G}_{B;\mathrm{apx};III}(U) &= c_*\partial^0 U(1-\gamma_U), \\
\mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U) &= \mathcal{G}_{B;\mathrm{apx};II}(U) - \mathcal{G}_{B;\mathrm{apx};III}(U).
\end{aligned} \tag{E.16}$$

We also recall the definition

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond-;+}_{\mathrm{apx}}(U)\\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond-;+}_{\mathrm{apx}}(U)\\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{D}^{\diamond-;+}_{\mathrm{apx}}(U)\\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond-;+}_{\mathrm{lin};U}[V].\end{array} \tag{E.17}$$

Substituting the relevant expressions from §D, we find

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & \partial^0 V\sum_{-;h}\big[c_*\gamma_U^{-1}\partial^0 U\big]\partial^{(2)}U\\ & & +\partial^0 U\sum_{-;h}\big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\big]\partial^{(2)}U\\ & & -3\partial^0 U\sum_{-;h}\big[c_*\gamma_U^{-1}\partial^0 U\big]\big[\partial^0 U\partial^0 V\big]\gamma_U^{-2}\partial^{(2)}U\\ & & +\partial^0 U\sum_{-;h}\big[c_*\gamma_U^{-1}\partial^0 U\big]\big(3\gamma_U^{-2}\partial^0 U[\partial^{(2)}U]\partial^0 V+\partial^{(2)}V\big).\end{array} \tag{E.18}$$

A little algebra yields

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & c_*\partial^0 V\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^{(2)}U\\ & & +\partial^0 U\sum_{-;h}\big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\big]\partial^{(2)}U\\ & & +c_*\partial^0 U\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^{(2)}V.\end{array} \tag{E.19}$$

In view of Lemma's B.4 and B.5, we introduce the expressions

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;III}[V] & = & c_*\partial^0 V(1-\gamma_U)\\ & & +\partial^0 U\sum_{-;h}\big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\big]\partial^{(2)}U\\ & & +c_*\partial^0 U\gamma_U^{-1}[\partial^0 U]\partial^0 V-c_*\partial^0 U\sum_{-;h}\Big[\gamma_U^{-3}[\partial^{(2)}U]\partial^0 V\Big],\\ \mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V] & = & \mathcal{G}_{B;\mathrm{lin};U;II}[V]-\mathcal{G}_{B;\mathrm{lin};U;III}[V].\end{array} \tag{E.20}$$

After a short computation, we find

$$\mathcal{G}_{B;\mathrm{lin};U;III}[V] = c_*\partial^0 V(1+\gamma_U^{-1}-2\gamma_U)+\partial^0 U\sum_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V]. \tag{E.21}$$

We summarize our results by writing

$$\mathcal{G}_{B;\mathrm{apx}}(U) = \mathcal{G}_{B;\mathrm{apx};III}(U) \qquad \mathcal{G}_{B;\mathrm{lin};U}[V] = \mathcal{G}_{B;\mathrm{lin};U;III}[V] \tag{E.22}$$

and obtaining the following bounds.

Lemma E.2. *Assume that (Hg) is satisfied and pick $0<\kappa<\frac{1}{12}$. Then there exists a constant $K>0$ so that the following properties hold true.*

(i) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell^2_h$, we have the identities

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{apx};II}(U) & = & \mathcal{G}_{B;\mathrm{apx}}(U)+\mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U),\\ \mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & \mathcal{G}_{B;\mathrm{lin};U}[V]+\mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{E.23}$$

(ii) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell^2_h$ we have the bounds

$$\begin{array}{rcll}\|\mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U)\|_{\ell^2_h} & \leq & Kh & = KS_{\mathrm{sh;full}}(U),\\ \|\mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V]\|_{\ell^2_h} & \leq & Kh\big[\,\|\partial^+V\|_{\ell^2_h}+\|\partial^+\partial^+V\|_{\ell^2_h}\,\big] & \leq K\mathcal{E}_{\mathrm{sh};U}(V).\end{array} \tag{E.24}$$

Proof. The estimates follow from Lemma's B.4 and B.5. □

E.3 Simplifications for $\mathcal{G}_C$ and $\mathcal{G}_D$

We recall the definition

$$\mathcal{G}_{\#;\mathrm{apx};II}(U) \quad = \quad \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \tag{E.25}$$

for $\# \in \{C, D\}$. Inspecting (D.25), we see that

$$\mathcal{G}_{C;\mathrm{apx};II}(U) = -\mathcal{G}_{D;\mathrm{apx};II}(U). \tag{E.26}$$

We also recall the definition

$$\begin{array}{rcl}\mathcal{G}_{\#;\mathrm{lin};U;II}[V] & = & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{D}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{D}^{\diamond_0;+}_{\mathrm{lin};U}[V]\Big]\end{array} \tag{E.27}$$

for $\# \in \{C, D\}$. Using (D.25) once more, we hence see

$$\mathcal{G}_{C;\mathrm{lin};U;II}[V] = -\mathcal{G}_{D;\mathrm{lin};U;II}[V]. \tag{E.28}$$

E.4 Final decomposition

Recalling the definitions (5.1), we observe that

$$\mathcal{G}_{A;\mathrm{apx}}(U) + \mathcal{G}_{B;\mathrm{apx}}(U) = c_*\gamma_U\partial^0 U + c_*\partial^0 U(1-\gamma_U) = c_*\partial^0 U = \mathcal{G}_{\mathrm{apx}}(U), \tag{E.29}$$

together with

$$\begin{array}{rcl}\mathcal{G}_{A;\mathrm{lin};U}[V] + \mathcal{G}_{B;\mathrm{lin};U}[V] & = & c_*\gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V + M_U[V] \\ & & \quad +c_*\partial^0 V(1+\gamma_U^{-1} - 2\gamma_U) + \partial^0 U\sum\nolimits_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V] \\ & = & c_*\partial^0 V + M_U[V] + \partial^0 U\sum\nolimits_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V] \\ & = & \mathcal{G}_{\mathrm{lin};U}[V].\end{array} \tag{E.30}$$

Proof of Propositions 5.2-5.3 for $\mathcal{G}$. Upon writing

$$\begin{array}{rcl}\mathcal{G}_{\mathrm{apx};\mathrm{sh};b}(U) & = & \mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U), \\ \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};b}[V] & = & \mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V],\end{array} \tag{E.31}$$

together with $\mathcal{G}_{\mathrm{apx};\mathrm{rem};b} = \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};b} = 0$, the statements follow from Lemma's E.1 and E.2. □

F Reductions for $\mathcal{G}^+$

Our goal here is to construct the functions $\mathcal{G}^+_{\mathrm{apx};\mathrm{sh};b}$, $\mathcal{G}^+_{\mathrm{apx};\mathrm{rem};b}$, $\mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b}$ and $\mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b}$ and demonstrate that they satisfy the corresponding bounds in Propositions 5.2-5.3. As in the previous section, we treat each of the components in (3.46) and (3.47) separately and subsequently combine the results.

F.1 Simplifications for $\mathcal{G}^+_{A'b}$

We recall the definition

$$\mathcal{G}^+_{A'b;\mathrm{apx};II}(U) = \Big[1 - \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\Big]\mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U). \tag{F.1}$$

Substituting the relevant expressions from §D, we find

$$\begin{aligned} \mathcal{G}^+_{A'b;\mathrm{apx};II}(U) &= \gamma_U^2\Big[\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U + c_*\gamma_U^{-3}\partial^{(2)}U\Big] \\ &= \gamma_U^2\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-2}\partial^+\partial^{(2)}U + c_*\gamma_U^{-1}\partial^{(2)}U. \end{aligned} \tag{F.2}$$

We now make the decomposition

$$\mathcal{G}^+_{A'b;\mathrm{apx};II}(U) = \mathcal{G}^+_{A'b;\mathrm{apx};III}(U) + \mathcal{G}^+_{A'b;\mathrm{apx};\mathrm{sh};b}(U) \tag{F.3}$$

by introducing

$$\mathcal{G}^+_{A'b;\mathrm{apx};III}(U) = \gamma_U^2\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-2}\partial^+\partial^{(2)}U + c_*\gamma_U^{-1}S^+[\partial^{(2)}U], \tag{F.4}$$

together with

$$\mathcal{G}^+_{A'b;\mathrm{apx};\mathrm{sh};b}(U) = -\frac{1}{2}c_*h\gamma_U^{-1}\partial^+[\partial^{(2)}U]. \tag{F.5}$$

We also recall the definition

$$\begin{aligned} \mathcal{G}^+_{A'b;\mathrm{lin};U;II}[V] = &-\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U) \\ &-\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U) \\ &\big[1 - \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V]. \end{aligned} \tag{F.6}$$

Substituting the relevant expressions from §D, we find

$$\begin{aligned} \mathcal{G}^+_{A'b;\mathrm{lin};U;II}[V] &= -2\partial^0U\partial^0V\Big[\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U + c_*\gamma_U^{-3}\partial^{(2)}U\Big] \\ &\quad +\gamma_U^2\Big[\gamma_U^{-2}\partial^+\big[M_U[V]\big] + 2\gamma_U^{-4}\partial^0U[\partial^{(2)}U]M_U[V] - \widetilde{M}_{U;III}[V]\Big] \\ &\quad +\gamma_U^2c_*\Big[3\gamma_U^{-5}\partial^0U[\partial^{(2)}U]\partial^0V + \gamma_U^{-3}S^+[\partial^{(2)}V]\Big] \\ &= c_*\gamma_U^{-3}\partial^0U[\partial^{(2)}U]\partial^0V + c_*\gamma_U^{-1}S^+[\partial^{(2)}V] \\ &\quad +\partial^+\big[M_U[V]\big] + 2\gamma_U^{-2}\partial^0U[\partial^{(2)}U]M_U[V] - \gamma_U^2\widetilde{M}_{U;III}[V] \\ &\quad -2\partial^0U\Big[\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U\Big]\partial^0V. \end{aligned} \tag{F.7}$$

We conclude by writing

$$\begin{aligned} \mathcal{G}^+_{A'b;\mathrm{apx}}(U) &= \mathcal{G}^+_{A'b;\mathrm{apx};III}(U) \\ &= \gamma_U^2\mathcal{E}^+_{\mathrm{tw};\mathrm{apx}}(U) - \gamma_U^{-2}\partial^+\partial^{(2)}U + c_*\gamma_U^{-1}S^+[\partial^{(2)}U], \\ \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] &= \mathcal{G}^+_{A'b;\mathrm{lin};U;II}[V] \end{aligned} \tag{F.8}$$

and obtaining the following bound.

Lemma F.1. *Assume that (Hg) is satisfied and pick* $0 < \kappa < \frac{1}{12}$*. Then there exists a constant* $K > 0$ *so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$, we have the identity

$$\mathcal{G}^+_{A'b;\mathrm{apx};II}(U) \quad = \quad \mathcal{G}^+_{A'b;\mathrm{apx}}(U) + \mathcal{G}^+_{A'b;\mathrm{apx;sh};b}(U). \tag{F.9}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\left\| \mathcal{G}^+_{A'b;\mathrm{apx;sh};b}(U) \right\|_{\ell_h^2} \quad \leq \quad Kh \, \|\partial^+\partial^+\partial^+ U\|_{\ell_h^2} \quad \leq \quad K\overline{S}_{\mathrm{sh;full}}(U). \tag{F.10}$$

Proof. The results follow by inspection. □

F.2 Simplifications for $\mathcal{G}^+_{A'c}$

We recall the definition

$$\mathcal{G}^+_{A'c;\mathrm{apx};II}(U) \quad = \quad -\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)\mathcal{X}_{A;\mathrm{apx;expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx;expl}}(U)\big]. \tag{F.11}$$

Substituting the relevant expressions from §D, we find

$$\mathcal{G}^+_{A'c;\mathrm{apx};II}(U) \quad = \quad -\partial^{(2)}U[\partial^0 U]T^+\big[c_*\gamma_U^{-1}\partial^0 U\big]. \tag{F.12}$$

We now make the decomposition

$$\mathcal{G}_{A'c;\mathrm{apx};II}(U) = \mathcal{G}_{A'c;\mathrm{apx};III}(U) + \mathcal{G}_{A'c;\mathrm{apx;sh};b}(U) \tag{F.13}$$

by introducing

$$\begin{aligned} \mathcal{G}^+_{A'c;\mathrm{apx};III}(U) \quad &= \quad -c_*[\partial^{(2)}U]\partial^0 U\big[\gamma_U^{-1}\partial^0 U\big] \\ &= \quad -c_*\gamma_U^{-1}(1-\gamma_U^2)\partial^{(2)}U, \end{aligned} \tag{F.14}$$

together with

$$\mathcal{G}^+_{A'c;\mathrm{apx;sh};b}(U) \quad = \quad -h[\partial^{(2)}U]\partial^0 U\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big]. \tag{F.15}$$

In addition, we make the splitting

$$\mathcal{G}_{A'c;\mathrm{apx};III}(U) = \mathcal{G}_{A'c;\mathrm{apx};IV}(U) + \mathcal{G}_{A'c;\mathrm{apx;sh};c}(U) \tag{F.16}$$

by writing

$$\mathcal{G}^+_{A'c;\mathrm{apx};IV}(U) \quad = \quad -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^{(2)}U], \tag{F.17}$$

together with

$$\mathcal{G}^+_{A'c;\mathrm{apx;sh};c}(U) \quad = \quad \tfrac{1}{2}hc_*\gamma_U^{-1}(1-\gamma_U^2)\partial^+[\partial^{(2)}U]. \tag{F.18}$$

We also recall the definition

$$\begin{aligned} \mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] \quad = \quad & -\mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{X}_{A;\mathrm{apx;expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx;expl}}(U)\big] \\ & -\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]T^+\big[\mathcal{Y}_{2;\mathrm{apx;expl}}(U)\big] \\ & -\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)\mathcal{X}_{A;\mathrm{apx;expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]\big]. \end{aligned} \tag{F.19}$$

Substituting the relevant expressions from §D, we find

$$\begin{aligned} \mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] \quad = \quad & -S^+[\partial^{(2)}V]\partial^0 UT^+\Big[c_*\gamma_U^{-1}\partial^0 U\Big] - \partial^{(2)}U[\partial^0 V]T^+\Big[c_*\gamma_U^{-1}\partial^0 U\Big] \\ & -\partial^{(2)}U[\partial^0 U]T^+\Big[\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\Big]. \end{aligned} \tag{F.20}$$

We now make the decomposition

$$\mathcal{G}^{+}_{A'c;\mathrm{lin};U;II}[V] = \mathcal{G}^{+}_{A'c;\mathrm{lin};U;III}[V] + \mathcal{G}^{+}_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V] \tag{F.21}$$

by introducing

$$\begin{array}{rcl} \mathcal{G}^{+}_{A'c;\mathrm{lin};U;III}[V] & = & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^{(2)}V] - c_*\gamma_U^{-3}(1+\gamma_U^2)[\partial^{(2)}U]\partial^0 U\partial^0 V \\ & & -[\partial^{(2)}U]\partial^0 U\Big[\gamma_U^{-2}M_U[V]\Big], \end{array} \tag{F.22}$$

together with

$$\begin{array}{rcl} \mathcal{G}^{+}_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V] & = & -hS^+[\partial^{(2)}V]\partial^0 U\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big] \\ & & -h[\partial^{(2)}U]\partial^0 V\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big] \\ & & -h[\partial^{(2)}U]\partial^0 U\partial^+\Big[\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\Big]. \end{array} \tag{F.23}$$

We summarize our results by writing

$$\mathcal{G}^{+}_{A'c;\mathrm{apx}}(U) \;=\; \mathcal{G}^{+}_{A'c;\mathrm{apx};IV}(U) \qquad \mathcal{G}^{+}_{A'c;\mathrm{lin};U}[V] \;=\; \mathcal{G}^{+}_{A'c;\mathrm{lin};U;III}[V] \tag{F.24}$$

and obtaining the following bounds.

Lemma F.2. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the identities

$$\begin{array}{rcl} \mathcal{G}^{+}_{A'c;\mathrm{apx};II}(U) & = & \mathcal{G}^{+}_{A'c;\mathrm{apx}}(U) + \mathcal{G}^{+}_{A'c;\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}^{+}_{A'c;\mathrm{apx};\mathrm{sh};c}(U), \\ \mathcal{G}^{+}_{A'c;\mathrm{lin};U;II}[V] & = & \mathcal{G}^{+}_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^{+}_{A;\mathrm{lin};U;\mathrm{sh};b}[V]. \end{array} \tag{F.25}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rclcl} \left\|\mathcal{G}^{+}_{A'c;\mathrm{apx};\mathrm{sh};b}(U)\right\|_{\ell^2_h} & \leq & Kh & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^{+}_{A'c;\mathrm{apx};\mathrm{sh};c}(U)\right\|_{\ell^2_h} & \leq & Kh\left\|\partial^+\partial^{(2)}U\right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{sh;full}}(U). \end{array} \tag{F.26}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{rcl} \left\|\mathcal{G}^{+}_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V]\right\|_{\ell^2_h} & \leq & Kh\left\|V\right\|_{\ell^{2;3}_h} + Kh\left\|\partial^+\partial^+\partial^+ U\right\|_{\ell^\infty_h}\left\|\partial^+ V\right\|_{\ell^2_h} \\ & \leq & Kh\overline{\mathcal{E}}_{\mathrm{sh};U}[V]. \end{array} \tag{F.27}$$

Proof. Recalling Proposition C.1, the bounds follow by inspection. □

F.3 Simplifications for $\mathcal{G}^{+}_{B'}$

We recall the definition

$$\mathcal{G}^{+}_{B';\mathrm{apx};II}(U) \;=\; \mathcal{Y}^{+}_{1;\mathrm{apx;expl}}(U)T^+\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx;expl}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx;expl}}(U)\Big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U) \tag{F.28}$$

Substituting the relevant expressions from §D, we find

$$\begin{array}{rcl} \mathcal{G}^{+}_{B';\mathrm{apx};II}(U) & = & [\partial^{(2)}U]T^+\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[\gamma_U^3[\gamma_U^{-3}\partial^{(2)}U]\Big] \\ & = & c_*[\partial^{(2)}U]T^+\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^{(2)}U. \end{array} \tag{F.29}$$

In view of Lemma B.5, we introduce the expressions

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{apx};III}(U) & = & c_*[\partial^{(2)}U]T^{+}(1-\gamma_U), \\ \mathcal{G}^{+}_{B';\mathrm{apx};\mathrm{sh};b}(U) & = & \mathcal{G}^{+}_{B';\mathrm{apx};II}(U) - \mathcal{G}^{+}_{B';\mathrm{apx};III}(U). \end{array} \tag{F.30}$$

In addition, we make the splitting

$$\mathcal{G}^{+}_{B';\mathrm{apx};III}(U) = \mathcal{G}^{+}_{B';\mathrm{apx};IV}(U) + \mathcal{G}^{+}_{B';\mathrm{apx};\mathrm{sh};c}(U) \tag{F.31}$$

by writing

$$\mathcal{G}^{+}_{B';\mathrm{apx};IV}(U) \quad = \quad c_* S^{+}[\partial^{(2)}U](1-\gamma_U), \tag{F.32}$$

together with

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{apx};\mathrm{sh};c}(U) & = & -\frac{1}{2}c_* h\partial^{+}[\partial^{(2)}U]T^{+}(1-\gamma_U) \\ & & \quad +c_* h S^{+}[\partial^{(2)}U]\partial^{+}(1-\gamma_U). \end{array} \tag{F.33}$$

We also recall the definition

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] & = & \mathcal{Y}^{+}_{1;\mathrm{lin};U;\mathrm{expl}}[V]T^{+}\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\mathcal{Y}^{+}_{1;\mathrm{apx};\mathrm{expl}}(U)T^{+}\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\mathcal{Y}^{+}_{1;\mathrm{apx};\mathrm{expl}}(U)T^{+}\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{D}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\mathcal{Y}^{+}_{1;\mathrm{apx};\mathrm{expl}}(U)T^{+}\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{D}^{\diamond_-;+}_{\mathrm{lin};U}[V]. \end{array} \tag{F.34}$$

Substituting the relevant expressions from §D, we find

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] & = & S^{+}[\partial^{(2)}V]T^{+}\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[\gamma_U^3[\gamma_U^{-3}\partial^{(2)}U]\Big] \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\Big[\gamma_U^3[\gamma_U^{-3}\partial^{(2)}U]\Big] \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[(-3)\gamma_U\partial^0 U\partial^0 V[\gamma_U^{-3}\partial^{(2)}U]\Big] \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[\gamma_U^3[3\gamma_U^{-5}\partial^0 U\partial^{(2)}U\partial^0 V+\gamma_U^{-3}\partial^{(2)}V]\Big]. \end{array} \tag{F.35}$$

A little algebra yields

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] & = & S^{+}[\partial^{(2)}V]T^{+}\sum_{-;h}c_*\gamma_U^{-1}[\partial^0 U]\partial^{(2)}U \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\partial^{(2)}U \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U[\partial^{(2)}V]. \end{array} \tag{F.36}$$

In view of Lemma's B.5 and B.4, we introduce the expressions

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V] & = & c_* S^{+}[\partial^{(2)}V]T^{+}(1-\gamma_U) \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\partial^{(2)}U \\ & & \quad +c_*[\partial^{(2)}U]T^{+}\Big[\gamma_U^{-1}[\partial^0 U]\partial^0 V-\sum_{-;h}\gamma_U^{-3}[\partial^{(2)}U]\partial^0 V\Big], \\ \mathcal{G}^{+}_{B';\mathrm{lin};U;\mathrm{sh};b}[V] & = & \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V]-\mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V]. \end{array} \tag{F.37}$$

A short computation yields

$$\begin{array}{lcl} \mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V] & = & c_* S^{+}[\partial^{(2)}V]T^{+}(1-\gamma_U)+c_*[\partial^{(2)}U]T^{+}\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\ & & \quad +[\partial^{(2)}U]T^{+}\sum_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V]. \end{array} \tag{F.38}$$

We now make the decomposition

$$\mathcal{G}^+_{B';\mathrm{lin};U;III}[V] = \mathcal{G}^+_{B';\mathrm{lin};U;IV}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V] \tag{F.39}$$

by writing

$$\begin{array}{lcl} \mathcal{G}^+_{B';\mathrm{lin};U;IV}[V] & = & c_* S^+[\partial^{(2)}V](1-\gamma_U) + c_*[\partial^{(2)}U]\gamma_U^{-1}[\partial^0 U]\partial^0 V \\ & & +S^+[\partial^{(2)}U]T^+ \sum_{-;h} \gamma_U^{-2}[\partial^{(2)}U]M_U[V], \end{array} \tag{F.40}$$

together with

$$\begin{array}{lcl} \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V] & = & c_* h S^+[\partial^{(2)}V]\partial^+[1-\gamma_U] + hc_*[\partial^{(2)}U]\partial^+\big[\gamma_U^{-1}\partial^0 U \partial^0 V\big] \\ & & -\frac{1}{2}h\partial^+[\partial^{(2)}U]T^+ \sum_{-;h} \gamma_U^{-2}[\partial^{(2)}U]M_U[V]. \end{array} \tag{F.41}$$

We summarize our results by writing

$$\mathcal{G}^+_{B';\mathrm{apx}}(U) \;=\; \mathcal{G}^+_{B';\mathrm{apx};IV}(U) \qquad\qquad \mathcal{G}^+_{B';\mathrm{lin};U}[V] \;=\; \mathcal{G}^+_{B';\mathrm{lin};U;IV}[V] \tag{F.42}$$

and obtaining the following bounds.

Lemma F.3. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(a) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the identities

$$\begin{array}{lcl} \mathcal{G}^+_{B';\mathrm{apx};II}(U) & = & \mathcal{G}^+_{B';\mathrm{apx}}(U) + \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U), \\ \mathcal{G}^+_{B';\mathrm{lin};U;II}[V] & = & \mathcal{G}^+_{B';\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V]. \end{array} \tag{F.43}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{lclcl} \left\|\mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};b}(U)\right\|_{\ell^2_h} & \leq & Kh & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U)\right\|_{\ell^2_h} & \leq & Kh\left\|\partial^+\partial^{(2)}U\right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{sh;full}}(U). \end{array} \tag{F.44}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{lcl} \left\|\mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};b}[V]\right\|_{\ell^2_h} & \leq & Kh\big[\,\|\partial^+V\|_{\ell^2_h} + \|\partial^+\partial^+V\|_{\ell^2_h}\,\big] \\ & \leq & Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\|\mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V]\right\|_{\ell^2_h} & \leq & Kh\,\|V\|_{\ell^{2;2}_h} + Kh\,\|\partial^+\partial^+\partial^+U\|_{\ell^2_h}\,\|V\|_{\ell^{2;2}_h} \\ & \leq & Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V). \end{array} \tag{F.45}$$

Proof. Recalling Lemma's B.4 and B.5, the bounds in (ii) and the first bound in (iii) follow by inspection. The final bound in (iii) follows from Proposition C.1. □

F.4 Final decomposition

Arguing as in §E.3 we see that

$$\mathcal{G}^+_{C;\mathrm{apx};II}(U) \;=\; -\mathcal{G}^+_{D;\mathrm{apx};II}(U), \qquad\qquad \mathcal{G}^+_{C;\mathrm{lin};U;II}[V] \;=\; -\mathcal{G}^+_{D;\mathrm{lin};U;II}[V], \tag{F.46}$$

so these can be neglected. Leaving the remaining component $\mathcal{G}^+_{A'a}$ intact, we recall (3.46), (3.48) and (4.39) to write

$$\begin{array}{rcl} \mathcal{G}^+_{A'a;\mathrm{apx}}(U) & = & \mathcal{G}^+_{A'a;\mathrm{apx};II}(U) \;\; = \;\; \gamma_U^{-2}\partial^+\partial^{(2)}U, \\ \mathcal{G}^+_{A'a;\mathrm{lin};U}[V] & = & \mathcal{G}^+_{A'a;\mathrm{lin};U;II}[V] \;\; = \;\; \gamma_U^2\widetilde{M}_{U;III}[V] - 2\gamma_U^{-4}\partial^0 U[\partial^+\partial^{(2)}U]\partial^0 V. \end{array} \tag{F.47}$$

This allows us to define the total

$$\mathcal{G}^+_{\mathrm{apx};III}(U) \;\; = \;\; \mathcal{G}^+_{A'a;\mathrm{apx}}(U) + \mathcal{G}^+_{A'b;\mathrm{apx}}(U) + \mathcal{G}^+_{A'c;\mathrm{apx}}(U) + \mathcal{G}^+_{B';\mathrm{apx}}(U). \tag{F.48}$$

Substituting the relevant expressions from §F.1-F.3 we obtain

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx};III}(U) & = & \gamma_U^{-2}\partial^+\partial^{(2)}U \\ & & \quad \gamma_U^2\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \gamma_U^{-2}\partial^+\partial^{(2)}U + c_*\gamma_U^{-1}S^+[\partial^{(2)}U] \\ & & \quad -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^{(2)}U] \\ & & \quad c_*S^+[\partial^{(2)}U](1-\gamma_U) \\ & = & c_*S^+[\partial^{(2)}U] + \gamma_U^2\mathcal{E}^+_{\mathrm{tw;apx}}(U). \end{array} \tag{F.49}$$

In order to suppress the final term, we introduce the expressions

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx;sh};b;i}(U) & = & \gamma_U^2\big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\big], \\ \mathcal{G}^+_{\mathrm{apx;rem};b;i}(U) & = & \gamma_U^2\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]. \end{array} \tag{F.50}$$

Moving on to the linear approximants, we define the function

$$\mathcal{G}^+_{\mathrm{lin};U;III}[V] \;\; = \;\; \mathcal{G}^+_{A'a;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U}[V]. \tag{F.51}$$

As a first step towards evaluating this expression, we substitute the relevant identities from §F.1-F.3 to compute

$$\begin{array}{rcl} \mathcal{G}^+_{A'a;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] & = & \gamma_U^2\widetilde{M}_{U;III}[V] - 2\gamma_U^{-4}\partial^0 U[\partial^+\partial^{(2)}U]\partial^0 V \\ & & \quad +c_*\gamma_U^{-3}\partial^0 U[\partial^{(2)}U]\partial^0 V + c_*\gamma_U^{-1}S^+[\partial^{(2)}V] \\ & & \quad +\partial^+\big[M_U[V]\big] + 2\gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V] - \gamma_U^2\widetilde{M}_{U;III}[V] \\ & & \quad -2\partial^0 U\Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \gamma_U^{-4}\partial^+\partial^{(2)}U\Big]\partial^0 V \\ & = & c_*\gamma_U^{-3}\partial^0 U[\partial^{(2)}U]\partial^0 V + c_*\gamma_U^{-1}S^+[\partial^{(2)}V] \\ & & \quad +\partial^+\big[M_U[V]\big] + 2\gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V] \\ & & \quad -2\partial^0 U\Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U)\Big]\partial^0 V. \end{array} \tag{F.52}$$

In a similar fashion, we find

$$\begin{array}{rcl} \mathcal{G}^+_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U}[V] & = & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^{(2)}V] \\ & & \quad -c_*\gamma_U^{-3}(1+\gamma_U^2)[\partial^{(2)}U]\partial^0 U\partial^0 V \\ & & \quad -[\partial^{(2)}U]\partial^0 U\Big[\gamma_U^{-2}M_U[V]\Big] \\ & & \quad +c_* S^+[\partial^{(2)}V](1-\gamma_U) + c_*[\partial^{(2)}U]\gamma_U^{-1}[\partial^0 U]\partial^0 V \\ & & \quad +S^+[\partial^{(2)}U]T^+\sum_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V] \\ & = & c_*S^+[\partial^{(2)}V] - c_*\gamma_U^{-1}S^+[\partial^{(2)}V] - c_*\gamma_U^{-3}\partial^0 U[\partial^{(2)}U]\partial^0 V \\ & & \quad -\gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V] \\ & & \quad +S^+[\partial^{(2)}U]T^+\sum_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V]. \end{array} \tag{F.53}$$

In particular, we see that

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx;lin};U;III}[V] & = & c_*S^+[\partial^{(2)}V] + \partial^+\big[M_U[V]\big] + \gamma_U^{-2}\partial^0 U[\partial^{(2)}U]M_U[V] \\ & & \quad +S^+[\partial^{(2)}U]T^+\sum_{-;h}\gamma_U^{-2}[\partial^{(2)}U]M_U[V] \\ & & \quad -2\partial^0 U[\mathcal{E}^+_{\mathrm{tw;apx}}(U)]\partial^0 V. \end{array} \tag{F.54}$$

Comparing this expression with (5.4), we set out to suppress the final term by introducing the functions

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b;i}[V] & = & -2\partial^0 U\big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\big]\partial^0 V, \\ \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b;i}[V] & = & -2\partial^0 U\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\partial^0 V. \end{array} \tag{F.55}$$

Lemma F.4. *Assume that (Hg) is satisfied, pick $0 < \kappa < \frac{1}{12}$ and recall the definitions (5.4). There exists a constant $K > 0$ so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the identities

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx};III}(U) & = & \mathcal{G}^+_{\mathrm{apx}}(U) + \mathcal{G}^+_{\mathrm{apx;sh};b;i}(U) + \mathcal{G}^+_{\mathrm{apx;rem};b;i}(U), \\ \mathcal{G}^+_{\mathrm{lin};U;III}[V] & = & \mathcal{G}^+_{\mathrm{lin};U}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b;i}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b;i}[V]. \end{array} \tag{F.56}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rcccl} \left\|\mathcal{G}^+_{\mathrm{apx;sh};b;i}(U)\right\|_{\ell^2_h} & \leq & Kh & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^+_{\mathrm{apx;rem};b;i}(U)\right\|_{\ell^2_h} & \leq & K\left\|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{rem;full}}(U). \end{array} \tag{F.57}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{rcl} \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};b;i}[V]\right\|_{\ell^2_h} & \leq & Kh\left\|\partial^+ V\right\|_{\ell^2_h} \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\|\mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};b;i}[V]\right\|_{\ell^2_h} & \leq & K\left\|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\right\|_{\ell^\infty_h}\left\|\partial^+ V\right\|_{\ell^2_h} \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{F.58}$$

Proof. Recalling (C.5), the bounds follow by inspection. □

Proof of Proposition 5.2-5.3 for $\mathcal{G}^+$. Upon introducing the full remainder functions

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{apx;rem:}b}(U) & = & \mathcal{G}^+_{\mathrm{apx;rem;}b;i}(U), \\ \mathcal{G}^+_{\mathrm{lin;}U\mathrm{;rem;}b}[V] & = & \mathcal{G}^+_{\mathrm{lin;}U\mathrm{;rem;}b;i}[V], \end{array} \tag{F.59}$$

together with their counterparts

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{apx;sh;}b}(U) & = & \mathcal{G}^+_{A'b\mathrm{;apx;sh;}b}(U) + \mathcal{G}^+_{A'c\mathrm{;apx;sh;}b}(U) + \mathcal{G}^+_{A'c\mathrm{;apx;sh;}c}(U) \\ & & \quad + \mathcal{G}^+_{B'\mathrm{;apx;sh;}b}(U) + \mathcal{G}^+_{B'\mathrm{;apx;sh;}c}(U) + \mathcal{G}^+_{\mathrm{apx;sh;}b;i}(U), \\ \mathcal{G}^+_{\mathrm{lin;}U\mathrm{;sh;}b}[V] & = & \mathcal{G}^+_{A'c\mathrm{;lin;}U\mathrm{;sh;}b}[V] \\ & & \quad + \mathcal{G}^+_{B'\mathrm{;lin;}U\mathrm{;sh;}b}[V] + \mathcal{G}^+_{B'\mathrm{;lin;}U\mathrm{;sh;}c}[V] + \mathcal{G}^+_{\mathrm{lin;}U\mathrm{;sh;}b;i}[V], \end{array} \tag{F.60}$$

the desired estimates follow directly from Lemma's F.1-F.4. □

References

[1] I. Barashenkov, O. Oxtoby and D. Pelinovsky (2005), Translationally Invariant Discrete Kinks from One-Dimensional Maps. *Phys. Rev. E* **72**, 035602.

[2] S. Benzoni-Gavage, P. Huot and F. Rousset (2003), Nonlinear Stability of Semidiscrete Shock Waves. *SIAM J. Math. Anal.* **35**, 639–707.

[3] H. Berestycki, F. Hamel and H. Matano (2009), Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.* **62**(6), 729–788.

[4] A. Bonnet and F. Hamel (1999), Existence of nonplanar Solutions of a Simple Model of Premixed Bunsen Flames. *SIAM J. Math. Anal.* **31**, 80–118.

[5] M. Brucal-Hallare and E. S. Van Vleck (2011), Traveling Wavefronts in an Antidiffusion Lattice Nagumo Model. *SIAM J. Appl. Dyn. Syst.* **10**, 921–959.

[6] X. Chen, J. S. Guo and C. C. Wu (2008), Traveling Waves in Discrete Periodic Media for Bistable Dynamics. *Arch. Ration. Mech. Anal.* **189**, 189–236.

[7] S. N. Chow, J. Mallet-Paret and W. Shen (1998), Traveling Waves in Lattice Dynamical Systems. *J. Diff. Eq.* **149**, 248–291.

[8] S. V. Dmitriev, P. G. Kevrekidis and N. Yoshikawa (2005), Discrete Klein–Gordon Models with Static Kinks Free of the Peierls–Nabarro Potential. *J. Phys. A.* **38**, 7617–7627.

[9] A. Doelman, B. Sandstede, A. Scheel and G. Schneider (2009), The Dynamics of Modulated Wave Trains. *Memoirs of the American Mathematical Society* **199**.

[10] C. E. Elmer (2006), Finding Stationary Fronts for a Discrete Nagumo and Wave Equation; Construction. *Physica D* **218**, 11–23.

[11] C. E. Elmer and E. S. Van Vleck (2003), Anisotropy, Propagation Failure, and Wave Speedup in Traveling Waves of Discretizations of a Nagumo PDE. *J. Comput. Phys.* **185**(2), 562–582.

[12] C. E. Elmer and E. S. Van Vleck (2003), Existence of Monotone Traveling Fronts for BDF Discretizations of Bistable Reaction-Diffusion Equations. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10**(1-3), 389–402. Second International Conference on Dynamics of Continuous, Discrete and Impulsive Systems (London, ON, 2001).

[13] C. E. Elmer and E. S. Van Vleck (2005), Dynamics of Monotone Travelling Fronts for Discretizations of Nagumo PDEs. *Nonlinearity* **18**, 1605–1628.

[14] P. C. Fife and J. B. McLeod (1977), The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65**(4), 335–361.

[15] G. Friesecke and R. L. Pego (2004), Solitary waves on Fermi-Pasta-Ulam lattices. III. Howland-type Floquet theory. *Nonlinearity* **17**(1), 207–227.

[16] M. Haragus and A. Scheel (2006), Corner Defects in Almost Planar Interface Propagation. In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, Vol. 23. Elsevier, pp. 283–329.

[17] J. Härterich, B. Sandstede and A. Scheel (2002), Exponential Dichotomies for Linear Non-Autonomous Functional Differential Equations of Mixed Type. *Indiana Univ. Math. J.* **51**, 1081–1109.

[18] A. Hoffman and J. Mallet-Paret (2010), Universality of Crystallographic Pinning. *J. Dyn. Diff. Eq.* **22**, 79–119.

[19] W. Huang, H. J. Hupkes, G. Lozada-Cruz and E. S. Van Vleck, Propagation failure for traveling waves of reaction-diffusion equations under moving mesh discretization. *In preparation.*

[20] W. Huang and R. D. Russell (2011), Adaptive Mesh Movement in 1D. In: *Adaptive Moving Mesh Methods.* Springer, pp. 27–135.

[21] H. J. Hupkes, L. Morelli, W. M. Schouten-Straatman and E. S. Van Vleck (2020), Traveling Waves and Pattern Formation for Spatially Discrete Bistable Reaction-Diffusion Equations. In: *Springer Proceedings in Mathematics & Statistics - Difference Equations and Discrete Dynamical Systems with Applications*, Vol. 312. pp. 55–112.

[22] H. J. Hupkes, D. Pelinovsky and B. Sandstede (2011), Propagation failure in the discrete Nagumo equation. *Proc. Amer. Math. Soc.* **139**(10), 3537–3551.

[23] H. J. Hupkes and B. Sandstede (2010), Travelling Pulse Solutions for the Discrete FitzHugh-Nagumo System. *SIAM J. Appl. Dyn. Sys.* **9**, 827–882.

[24] H. J. Hupkes and B. Sandstede (2013), Stability of Pulse Solutions for the Discrete FitzHugh-Nagumo System. *Transactions of the AMS* **365**, 251–301.

[25] H. J. Hupkes and E. S. Van Vleck (2016), Travelling Waves for Complete Discretizations of Reaction Diffusion Systems. *J. Dyn. Diff. Eqns* **28**(3), 955–1006.

[26] H. J. Hupkes and E. S. Van Vleck (2021), Travelling Waves for Adaptive Grid Discretizations of Reaction Diffusion Systems I: Well-posedness. *in press J. Dyn. Diff. Eqn.*

[27] H. J. Hupkes and E. S. Van Vleck (2021), Travelling Waves for Adaptive Grid Discretizations of Reaction Diffusion Systems II: Linear theory. *in press J. Dyn. Diff. Eqn.*

[28] J. P. Keener (1987), Propagation and its Failure in Coupled Systems of Discrete Excitable Cells. *SIAM J. Appl. Math.* **47**, 556–572.

[29] J. Mallet-Paret (1999), The Fredholm Alternative for Functional Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **11**, 1–48.

[30] J. Mallet-Paret (1999), The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems. *J. Dyn. Diff. Eq.* **11**, 49–128.

[31] J. Mallet-Paret and S. M. Verduyn-Lunel, Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differential Equations. *J. Diff. Eq., to appear.*

[32] J. Nolen, J.-M. Roquejoffre, L. Ryzhik and A. Zlatoš (2012), Existence and Non-Existence of Fisher-KPP Transition Fronts. *Arch. Ration. Mech. Anal.* **203**(1), 217–246.

[33] V. Roussier (2004), Stability of radially symmetric travelling waves in reaction–diffusion equations. In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, Vol. 21. Elsevier, pp. 341–379.

[34] A. Rustichini (1989), Functional Differential Equations of Mixed Type: the Linear Autonomous Case. *J. Dyn. Diff. Eq.* **11**, 121–143.

[35] A. Rustichini (1989), Hopf Bifurcation for Functional-Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **11**, 145–177.

[36] W. Schouten and H. J. Hupkes (2019), Nonlinear Stability of Pulse Solutions for the Discrete FitzHugh-Nagumo equation with Infinite-Range Interactions. *Discrete Contin. Dyn. Syst.* **39**(9), 5017–5083.

[37] W. Schouten-Straatman and H. J. Hupkes (2019), Travelling waves for spatially discrete systems of FitzHugh-Nagumo type with periodic coefficients. *SIAM J. Math. Anal.* **51**(4), 3492–3532.

[38] H.-O. Walther (2010), Linearized stability for semiflows generated by a class of neutral equations, with applications to state-dependent delays. *Journal of Dynamics and Differential Equations* **22**(3), 439–462.

[39] H. F. Weinberger (1979), Genetic wave propagation, convex sets, and semi-infinite programming. In: *Constructive approaches to mathematical models (Proc. Conf. in honor of R. J. Duffin, Pittsburgh, Pa., 1978).* New York: Academic Press, pp. 293–317.

[40] B. Zinner (1992), Stability of Traveling Wavefronts for the Discrete Nagumo Equation. *J. Diff. Eq.* **96**, 1–27.