Analysis Of Newton's Method to Compute Travelling Wave Solutions to Lattice Differential Equations.

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Lattice differential equations

System of ODE's, indexed by a lattice Λ,

$$\dot{x}_\eta = F_\eta(\{x_\lambda\}_{\lambda \in \Lambda}), \qquad \eta \in \Lambda. \tag{1}$$

- Lattice Λ often infinite, leading to infinite dimensional systems.
- Nonlinearities F_η reflect geometry of the lattice.
- Often only short range interactions.

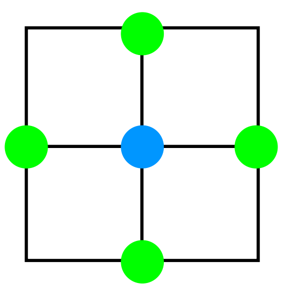
Example

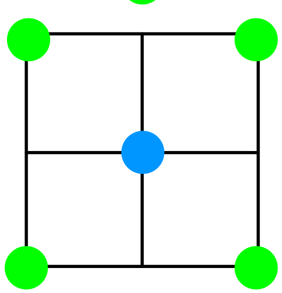
Typical example of LDE on the integer lattice $\Lambda = \mathbb{Z}^2$,

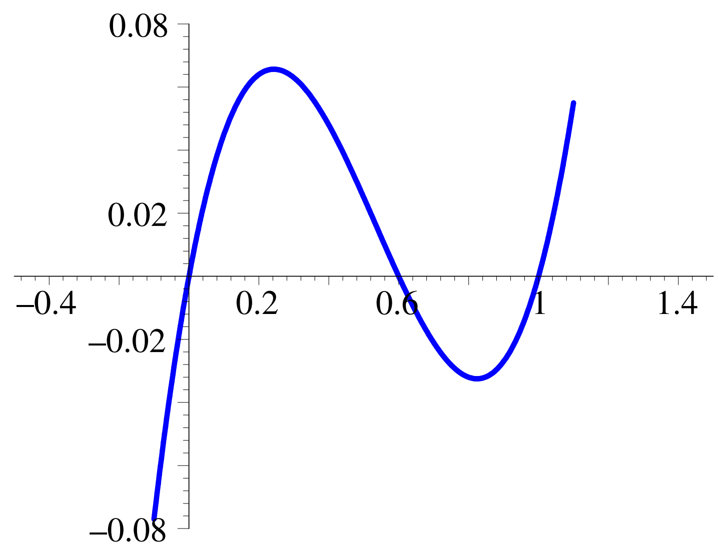
$$\dot{u}_{i,j} \quad = \quad \alpha L_D u_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2, \tag{2}$$

L_D is a discrete Laplacian, which could be given by

$$\begin{aligned} L_D u_{i,j} &= (\Delta^+ u)_{i,j} \equiv u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \qquad \textit{or} \\ L_D u_{i,j} &= (\Delta^\times u)_{i,j} \equiv u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j}. \end{aligned} \tag{3}$$


Discrete Laplacian Δ^+.


Discrete Laplacian $\Delta^\times$.



Bistable nonlinearity, typically

$$f_{\text{cub}}(u) = u(u-a)(u-1). \tag{4}$$

Example Continued

The system (2), i.e.

$$\dot{u}_{i,j} \quad = \quad \alpha(\Delta^{+}u)_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2, \tag{5}$$

with $\alpha = h^{-2}$, arises from discretization of the reaction diffusion equation on $\mathbb{R}^2$,

$$\dot{u} = \Delta u - f(u), \tag{6}$$

to a rectangular lattice with spacing h.

- Large values of α correspond with the continuous limit $h \to 0$.
- One can also study (5) with small α and $\alpha < 0$.
- Away from the continuous limit, (5) has a much richer structure that (6).

Models

Models leading to LDES can be found in

- Chemical reaction theory
- Image processing and pattern recognition
- Biology
- Material science (Crystals)

The numerical and experimental work of Leon Chua and Martin Hasler is a strong motivation for the study of LDEs.

They are developing algorithms based on LDEs which identify various prescribed patterns, for example edges, or corners, in a digitized image.

Cellular Neural Networks

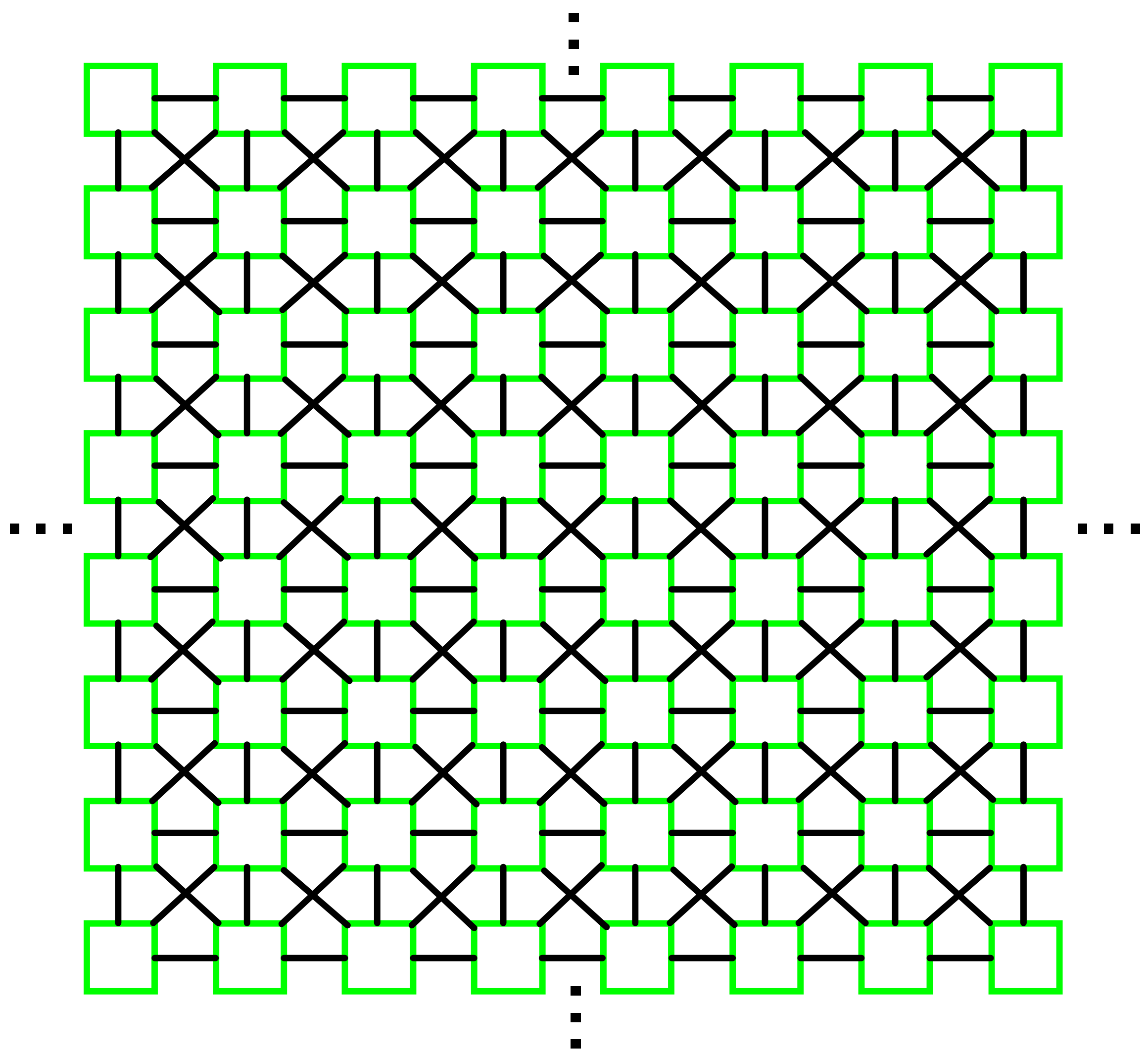


Figure 1: Already in 1988 Leon O. Chua and Lin Yang developed the concept of Cellular Neural Networks: large neural nets with local interactions.

CNN Automata

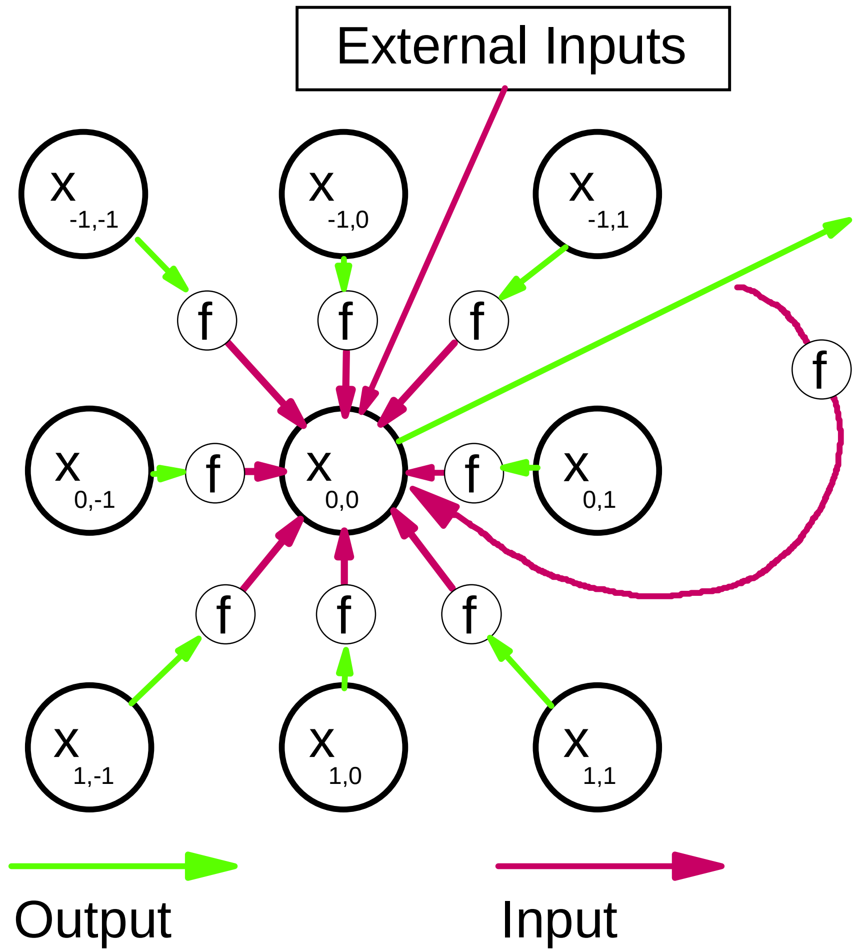
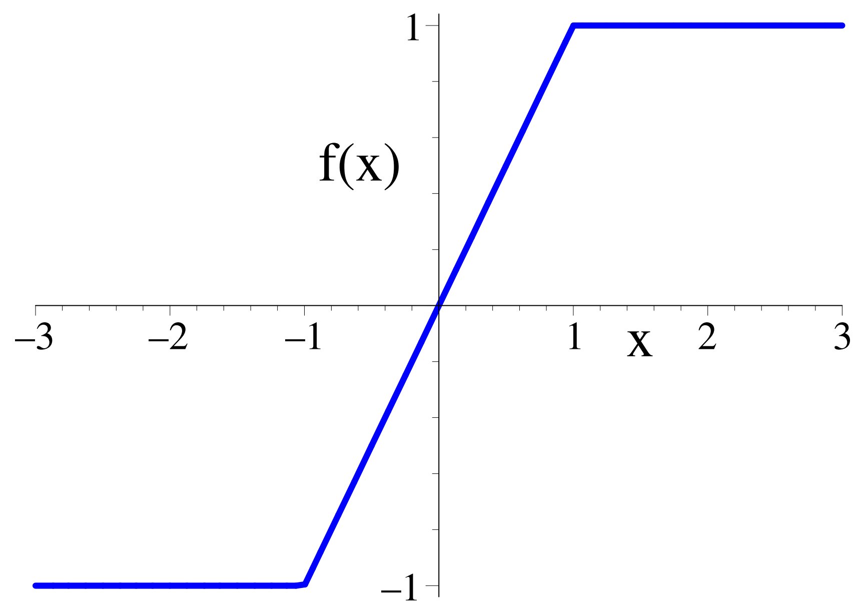


Figure 2: Overview of inputs and outputs for the cell at $(0,0)$.

State equation

$$C\dot{x}_{i,j}(t) = -\frac{1}{R_x}x_{i,j}(t) + \sum_{(k,l)\in N} A_{k,l} f(x_{i+k,j+l}) + I_{\text{ext}} \quad (7)$$

Here N denotes the 3×3 neighbourhood $\{(i,j) \mid -1 \le i \le 1,\ -1 \le j \le 1\}$.



$f(x) = \frac{1}{2}(|x+1| - |x-1|)$.

CNN Pattern Recognition

- One CNN Cell represents one pixel.
- Original state and I_{ext} correspond with input picture.
- Input picture is greyscale with values in range $[-1, 1]$.
- Neural Network should converge to equilibrium state $x(\infty)$.
- Output should be black and white, i.e. $f(x(\infty)) \in \{-1, 1\}$. This is equivalent to $|x(\infty)| \geq 1$.

Theorem 1. *Suppose that $A_{0,0} > R_x^{-1}$. Then for inputs corresponding to greyscale images, the limits*

$$\lim_{t \to \infty} x_{i,j}(t) = x_{i,j}(\infty) \tag{8}$$

exist and satisfy $|x_{i,j}(\infty)| \geq 1$.

This theorem guarantees that the final output $f(x_{i,j}(\infty))$ is a black and white image.

CNN Pattern Recognition - Line Detection

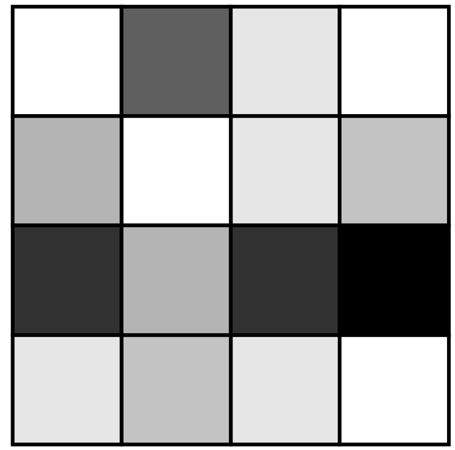
The coupling constants $A_{i,j}$ should be chosen according to the task at hand.

0.0	0.0	0.0
1.0	2.0	1.0
0.0	0.0	0.0

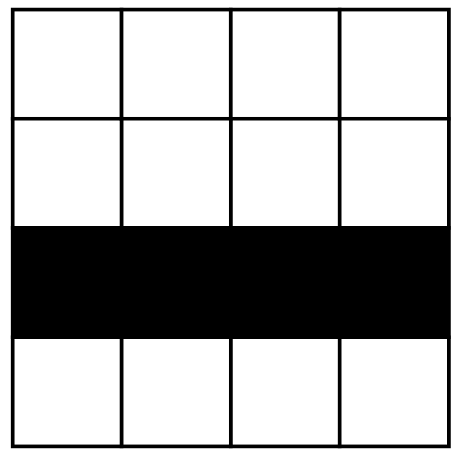
Horizontal line detector template.

0.0	1.0	0.0
0.0	2.0	0.0
0.0	1.0	0.0

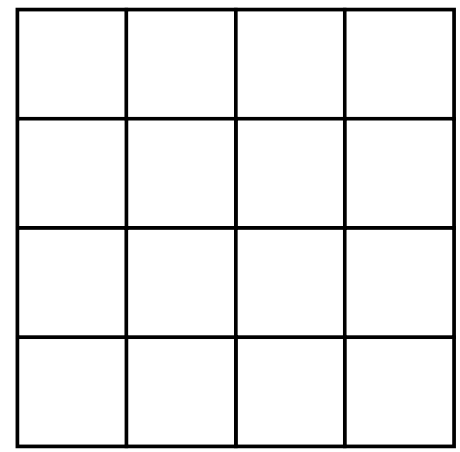
Vertical line detector template.



Original greyscale image.



Horizontal line after.



Vertical line after.

CNN Noise Reduction

Goal is to eliminate random noise applied to image.

0.0	1.0	0.0
1.0	2.0	1.0
0.0	1.0	0.0

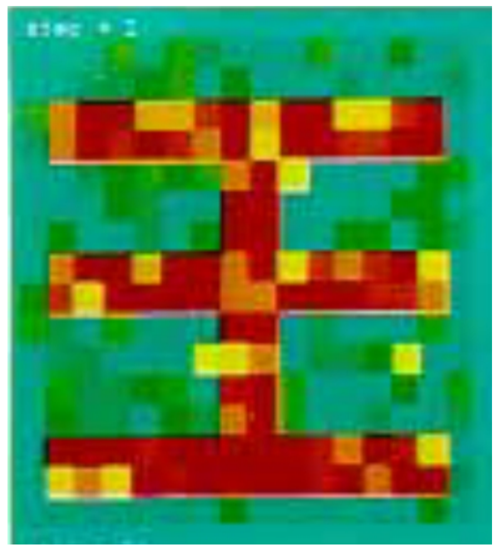
Noise reduction template A.

0.0	1.0	0.0
1.0	4.0	1.0
0.0	1.0	0.0

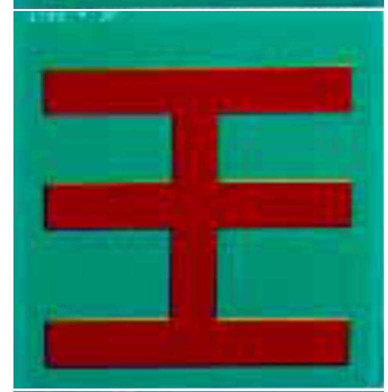
Noise reduction template B.

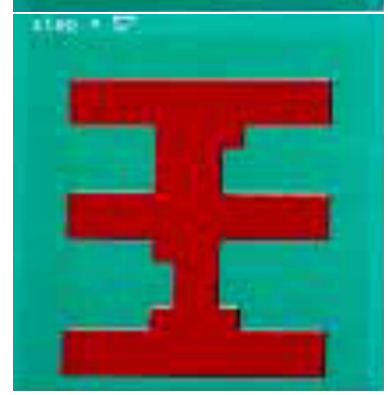
0.5	1.0	0.5
1.0	4.0	1.0
0.5	1.0	0.5

Noise reduction template C.



Original Image.





CNN Edge Recognition

Goal is to extract edges from an image.

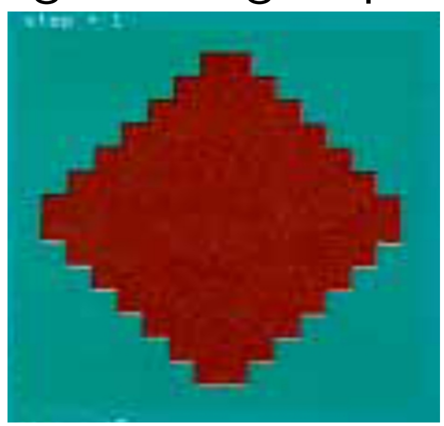
0.0	-1.0	0.0
-1.0	4.0	-1.0
0.0	-1.0	0.0

Edge recognition template A.

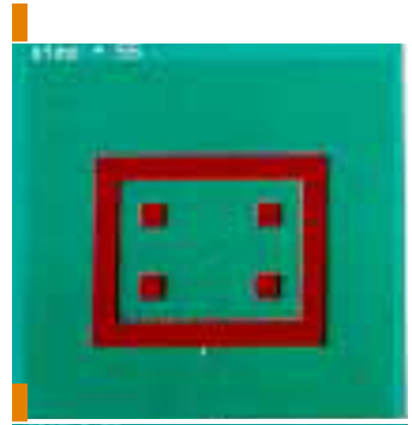
-0.25	-0.25	-0.25
-0.25	2.0	-0.25
-0.25	-0.25	-0.25

Edge recognition template B.

Original Image Square.



Original Image Diamond.



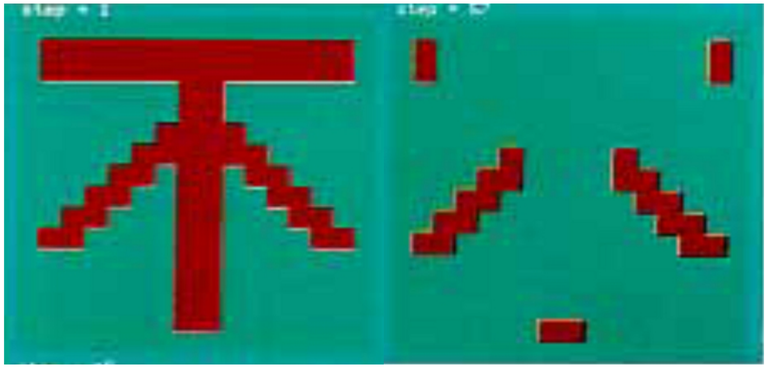
A.

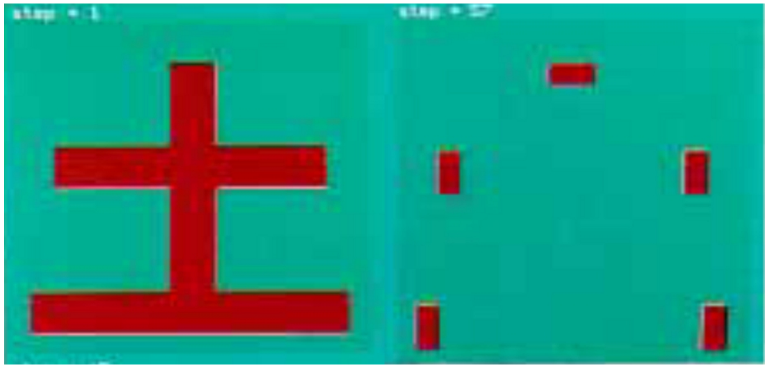
B.

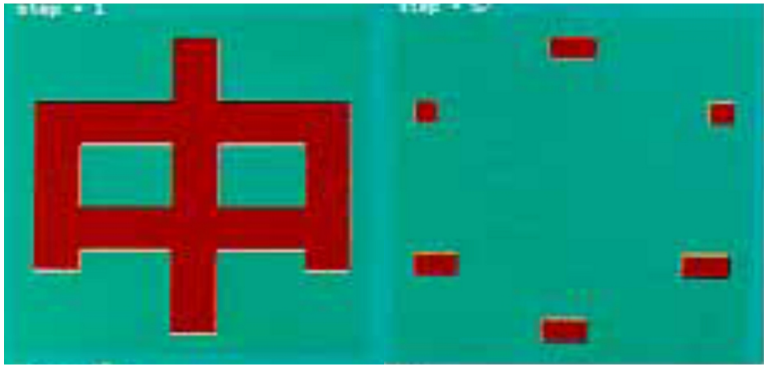
B.

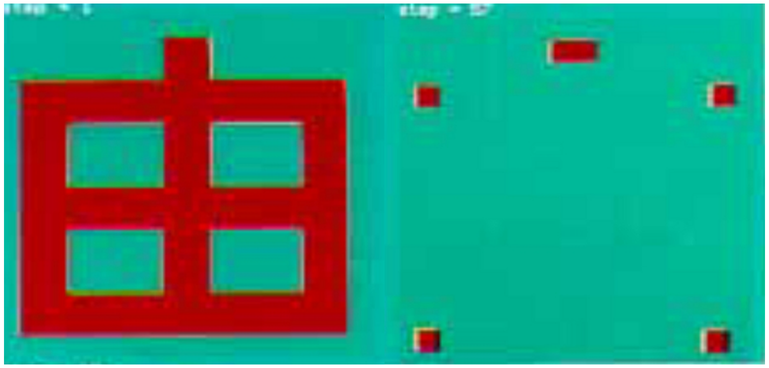
CNN Corner Recognition

Template the same as for edge recognition; Inputs I_{ext} get extra biasterm.









CNN Circuits

- Cellular Neural Networks can be implemented as electronic circuits.
- Couplings $A_{k,l}$ can be set by changing impedances of circuit elements.
- Very fast parallel processing possible.

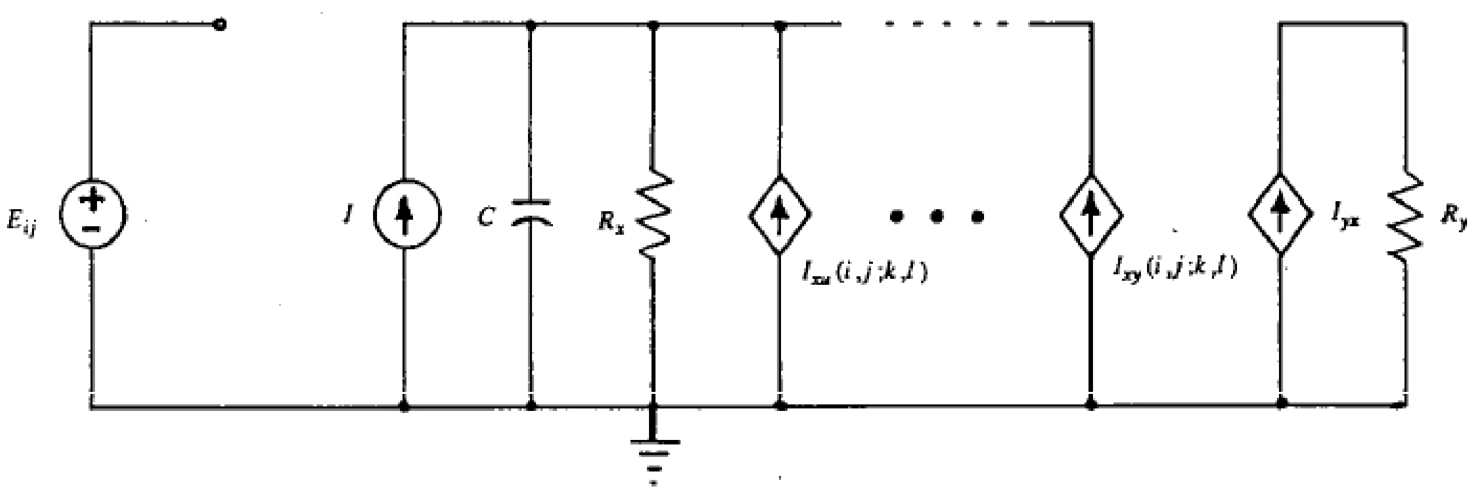


Figure 3: Circuit

CNN Final Example

Large scale edge recognition using CNN's is possible.

Understanding behaviour of LDE's

- Equilibrium Solutions.
- Transient behaviour.

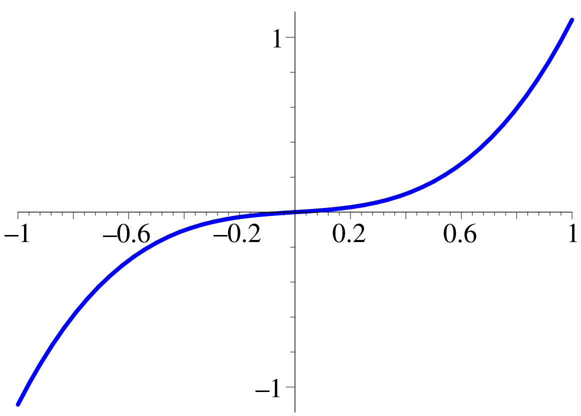
Equilibrium Solutions

Mallet-Paret has studied the equilibrium solutions of the system

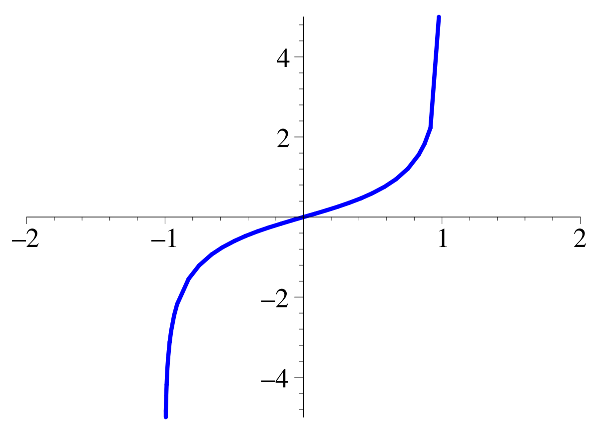
$$\dot{u}_{i,j} = -\beta^{+}(\Delta^{+}u)_{i,j} - \beta^{\times}(\Delta^{\times}u)_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2. \tag{9}$$

The nonlinearity f is assumed to be an odd function, one of either

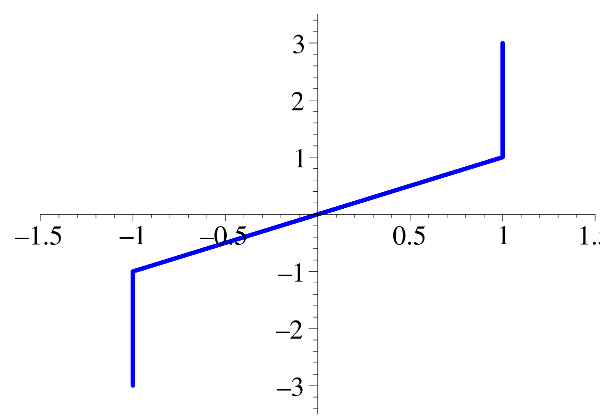
$$\begin{array}{lcll}
f_{\mathrm{cub},0}(z) & = & \gamma z + z^3 & \gamma > 0 \\
f_{\log}(z) & = & (\gamma - 2)z + \log((1+z)/(1-z)) & \gamma > 0 \\
f_{\mathrm{cart}}(z) & = & \gamma z, \quad -1 < z < 1, \quad (-\infty, -1], \quad z = -1, \quad [1, \infty), \quad z = 1 & \gamma > 0
\end{array} \tag{10}$$



Cubic $f_{\mathrm{cub},0}$



Logarithmic $f_{\log}$



Cartoon f_{cart}

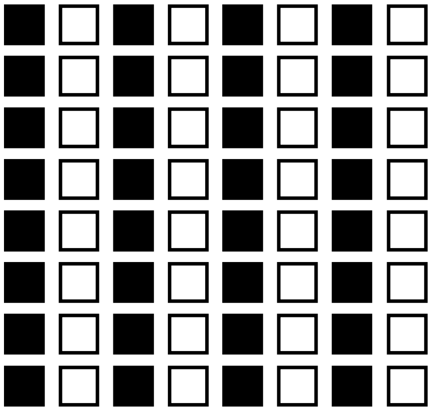
Equilibrium Solutions Continued

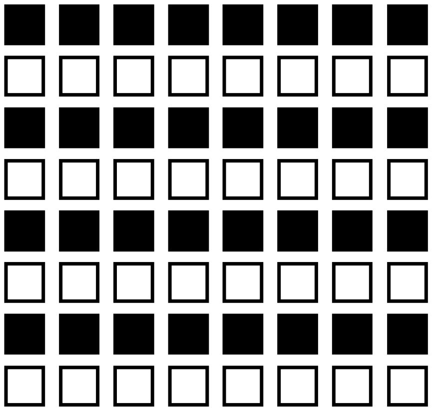
- Pattern of vertical stripes given by $u_{i,j} = (-1)^i k$ for some $k \in \mathbb{R}$.
- Pattern of horizontal stripes given by $u_{i,j} = (-1)^j k$ for some $k \in \mathbb{R}$.
  These two patterns are solutions when

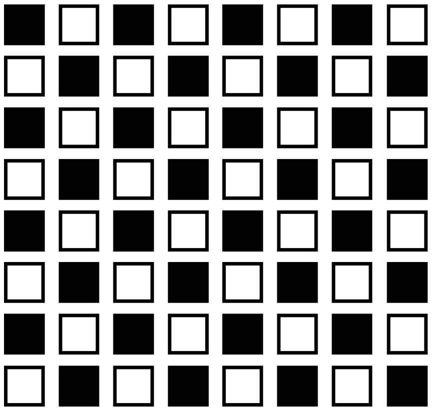
$$0 = (4\beta^+ + 8\beta^\times)k - f(k). \tag{11}$$

- Checkerboard pattern $u_{i,j} = (-1)^{i+j} k$ for some $k \in \mathbb{R}$.

$$0 = 8\beta^+ k - f(k). \tag{12}$$







Bifurcation Analysis

The equilibrium conditions take the form

$$0 = \lambda k - f(k). \tag{13}$$

For $\lambda = f'(0) = \gamma$ the solution $k = 0$ bifurcates. Writing

$$f(k) = \gamma k + \frac{1}{6} f'''(0) k^3 + O(k^5), \tag{14}$$

we have the solutions for λ nearby γ, with $\lambda > \gamma$

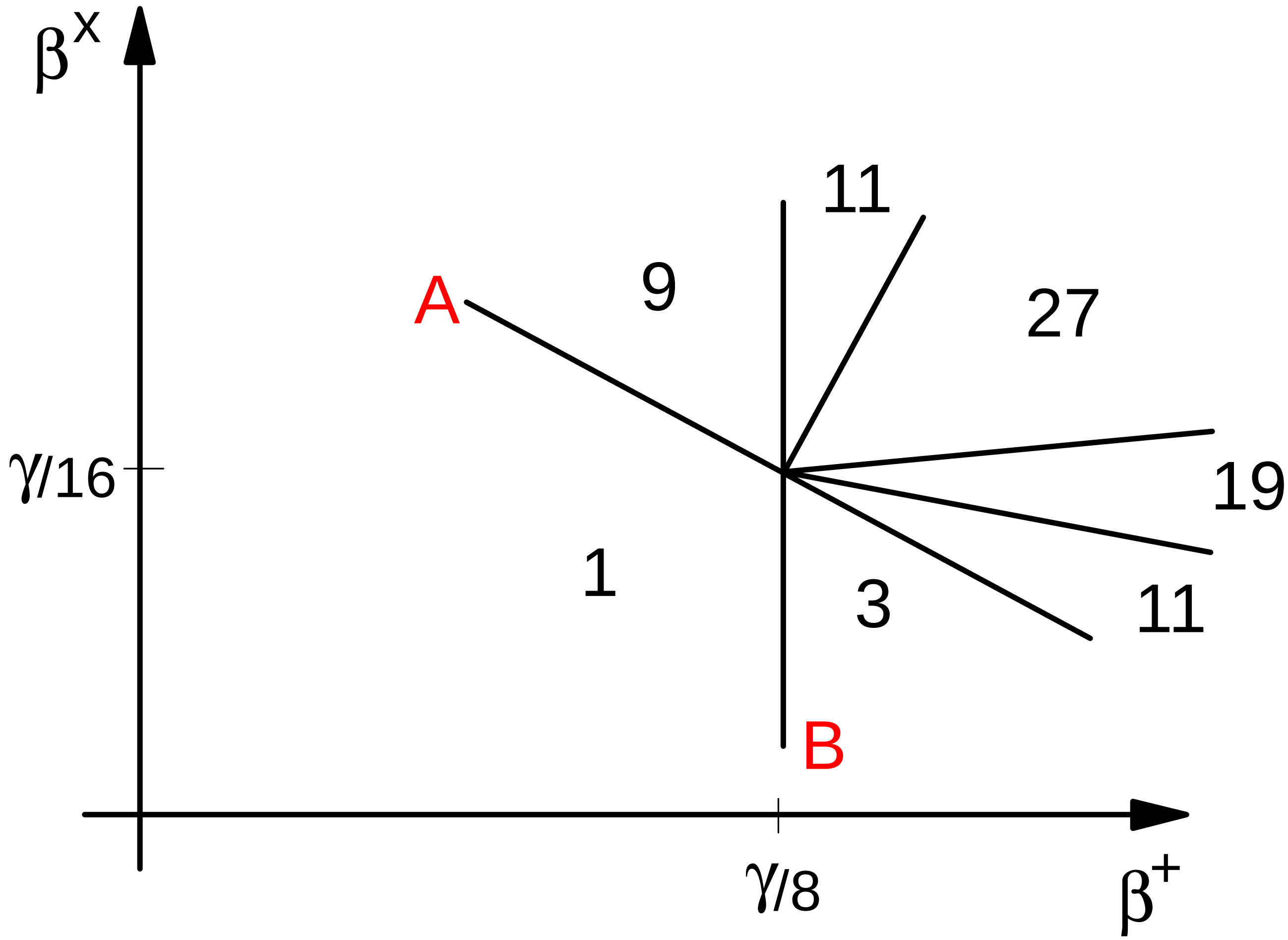
$$k \sim \pm \sqrt{6(\lambda - \gamma)/f'''(0)}. \tag{15}$$

One can extend this analysis to all solutions with spatial period two, which can be written as

$$u_{i,j} = (-1)^i v + (-1)^j w + (-1)^{i+j} x + y, \tag{16}$$

for real v, w, x, y.

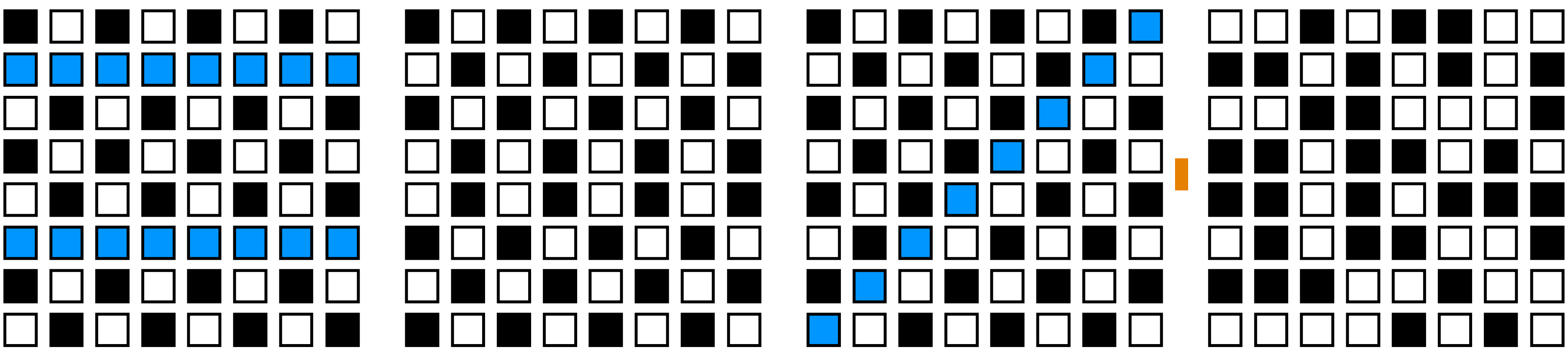
Bifurcation Diagram



Bifurcation diagram for equilibrium solutions to (9).
Line A: $4\beta^{+} + 8\beta^{\times} = \gamma$. Bifurcation line for stripes.
Line B: $8\beta^{+} = \gamma$. Bifurcation line for checkerboard.

Mosaic Solutions

- One can obtain global results when one considers (9) with the cartoon nonlinearity f_4.
- The parameter space $\{(\beta^+, \beta^\times)\}$ can be divided into finitely many regions.
- Each region admits a set of asymptotically stable equilibrium solutions u for which $u_{i,j} \in \{-1, 0, 1\}$.
- Some have patterns; others are spatially chaotic.



Travelling Wave Solutions

Consider the LDE (2),

$$\dot{u}_{i,j} = \alpha L_D u_{i,j} - f_{\text{cub}}(u_{i,j}, a), \quad (i,j) \in \mathbb{Z}^2, \tag{17}$$

with the cubic nonlinearity $f_{\text{cub}}(u, a) = u(u-1)(u-a)$.
One often is interested in travelling wave solutions and makes the ansatz

$$u_{i,j}(t) = \phi(ik_1 + jk_2 - ct). \tag{18}$$

Substitution into (2) with $L_D = \Delta^+$ yields

$$-c\phi'(\xi) = \alpha(\phi(\xi + k_1) + \phi(\xi - k_1) + \phi(\xi + k_2) + \phi(\xi - k_2) - 4\phi(\xi)) - f_{\text{cub}}(\phi(\xi), a).$$

This is a mixed-type functional differential equation, also called a differential-difference equation (DDE).

One often imposes the limits

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \qquad \lim_{\xi \to \infty} \phi(\xi) = 1. \tag{19}$$

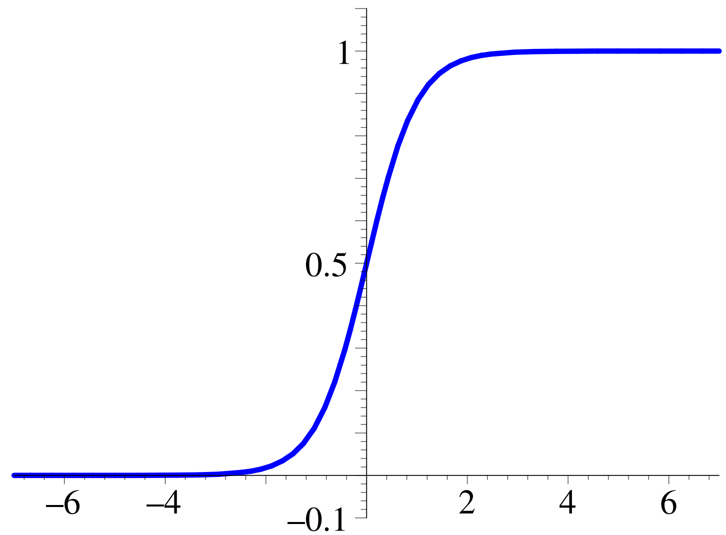
Travelling Wave Solutions - Existence

Theorem 1 (Mallet-Paret). *The differential difference equation*

$$-c\phi'(\xi) = \alpha(\phi(\xi + k_1) + \phi(\xi - k_1) + \phi(\xi + k_2) + \phi(\xi - k_2) - 4\phi(\xi)) - f_{\text{cub}}(\phi(\xi), a),$$

together with the side conditions

$$\begin{aligned} \lim_{\xi \to -\infty} \phi(\xi) &= 0, \\ \lim_{\xi \to \infty} \phi(\xi) &= 1, \\ \phi(0) &= a, \end{aligned} \qquad (20)$$



has a unique solution (ϕ, c) whenever $c \neq 0$. Moreover, this solution depends C^1-smoothly on a when $c(a) \neq 0$.

Notice that the DDE is translation invariant. The normalization $\phi(0) = a$ picks out a unique translate.

Travelling Wave Solutions - Stability

Theorem 2. *Suppose $p_{i,j}(t) = \phi(ik_1 + jk_2 - ct)$ is a travelling wave solution to the LDE*

$$\dot{u}_{i,j} \quad = \quad \alpha L_D u_{i,j} - f_{\text{cub}}(u_{i,j}, a), \quad (i,j) \in \mathbb{Z}^2. \tag{21}$$

Then p is asymptotically stable, i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that for any solution x to the LDE (21) satisfying

$$\|x(t_0) - p(t_0)\| < \delta \tag{22}$$

for some t_0, we have

$$\|x(t) - p(t)\| < \epsilon \tag{23}$$

for all $t > t_0$. Furthermore, there exists t^* such that

$$\lim_{t \to \infty} \|x(t) - p(t + t^*)\| = 0. \tag{24}$$

Travelling Wave Solutions - Spatial anisotropy

A feature which distinguishes LDEs from PDE's is spatial anisotropy. Substitution of the travelling wave ansatz

$$u(x,t) = \phi(k \cdot x - ct) \tag{25}$$

into the continuous reaction diffusion equation

$$\dot{u} = \Delta u - f_{\mathrm{cub}}(u,a), \tag{26}$$

leads to

$$-c\phi'(\xi) = \phi''(\xi) - f_{\mathrm{cub}}(\phi(\xi),a), \tag{27}$$

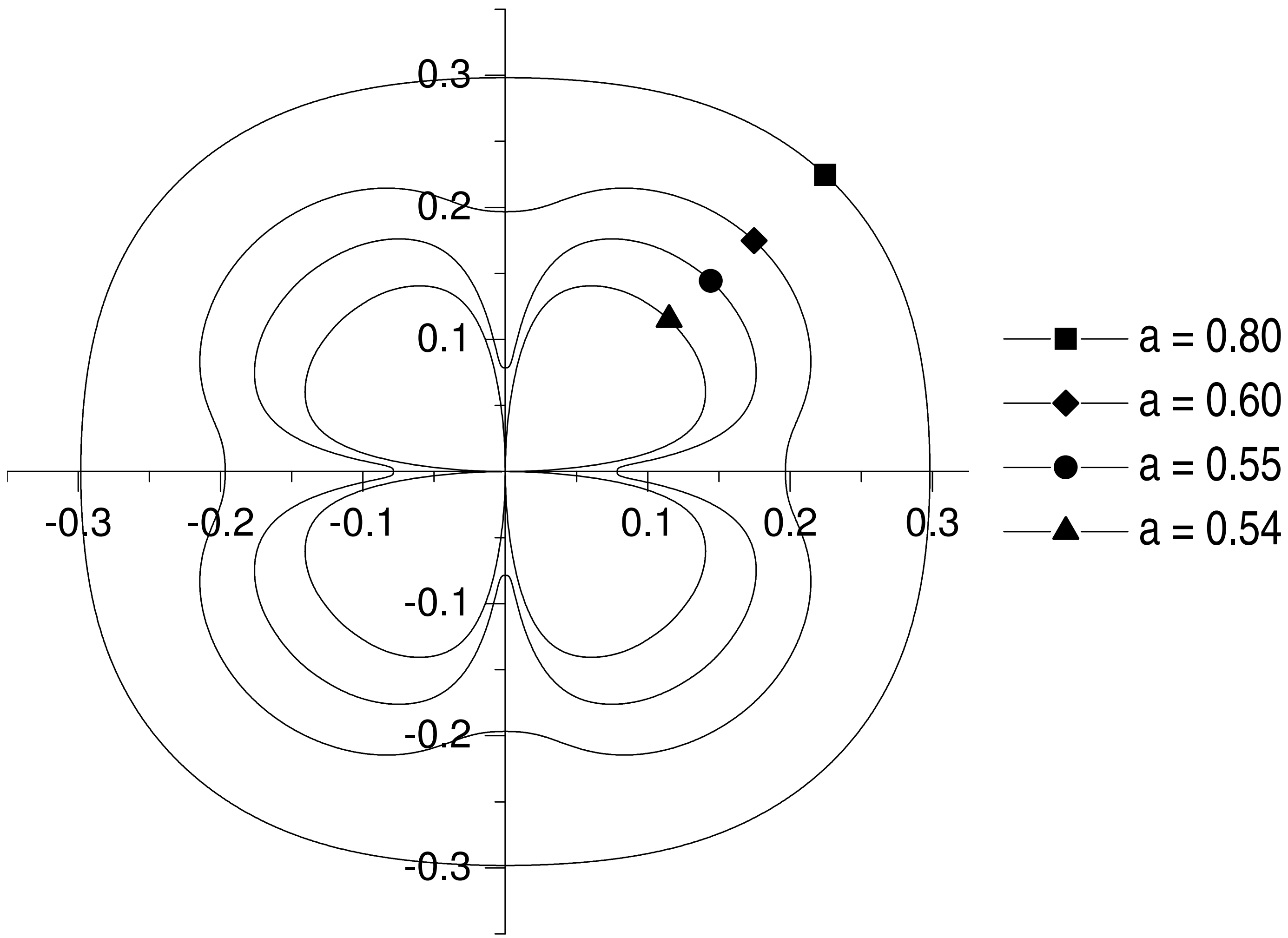
which is independent of k. Compare to

$$-c\phi'(\xi) = \alpha(\phi(\xi+k_1) + \phi(\xi-k_1) + \phi(\xi+k_2) + \phi(\xi-k_2) - 4\phi(\xi)) - f_{\mathrm{cub}}(\phi(\xi),a),$$

which depends on k.

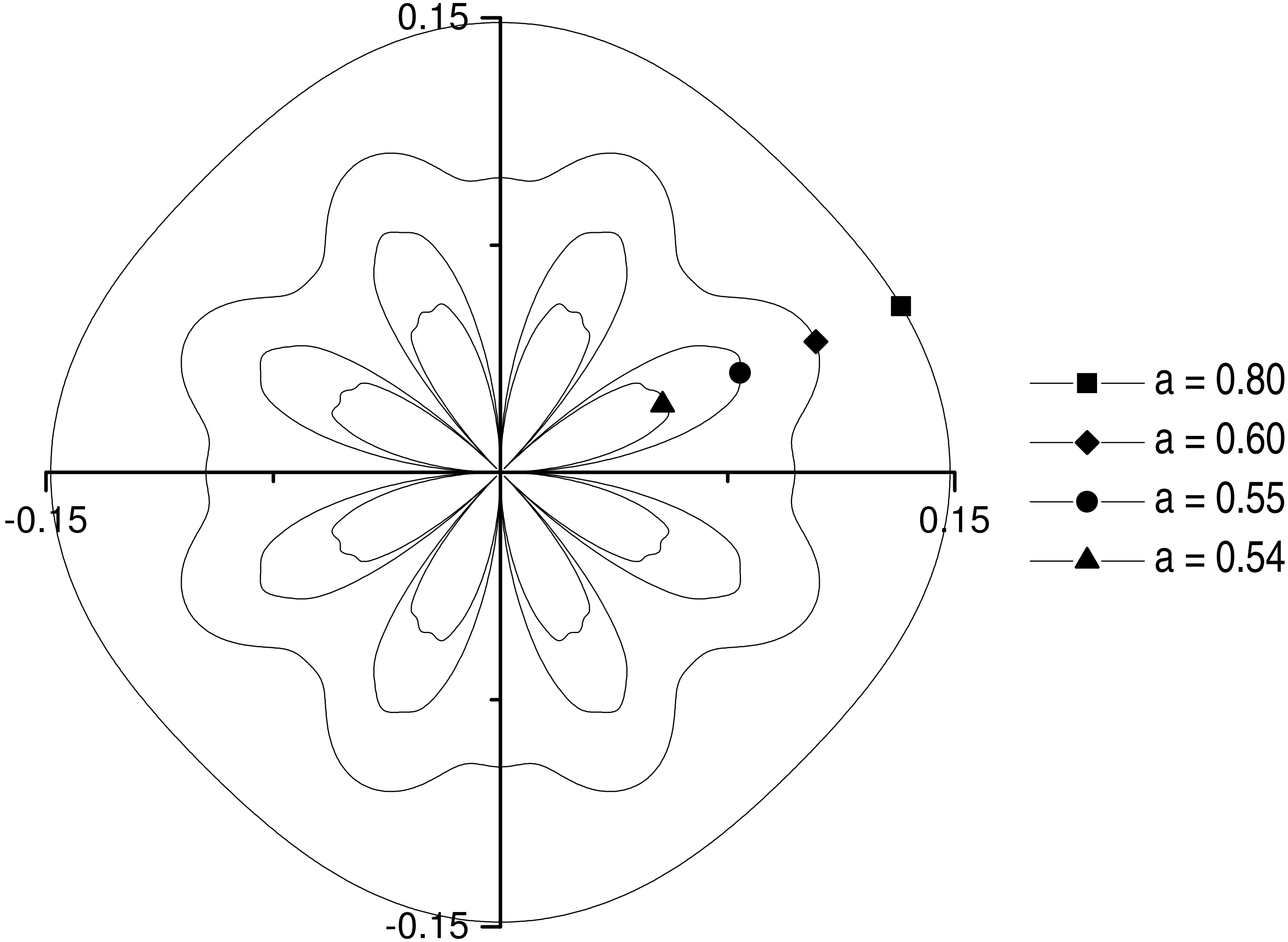
Spatial anisotropy

The lattice anisotropy can be illustrated by taking $k = (\cos\theta, \sin\theta)$ and studying the $c(\theta)$ relation. Example LDE: $\dot{u}_{i,j} = (\Delta^+ u)_{i,j} - 10 f_{\text{cub}}(u_{i,j}, a)$.



Spatial anisotropy Continued

Another $c(\theta)$ plot for $\dot{u}_{i,j} = \frac{1}{4}\big((\Delta^{+}u)_{i,j} + (\Delta^{\times}u)_{i,j}\big) - f_{\text{cub}}(u_{i,j}, a)$.



Travelling Wave Solutions - Propagation Failure

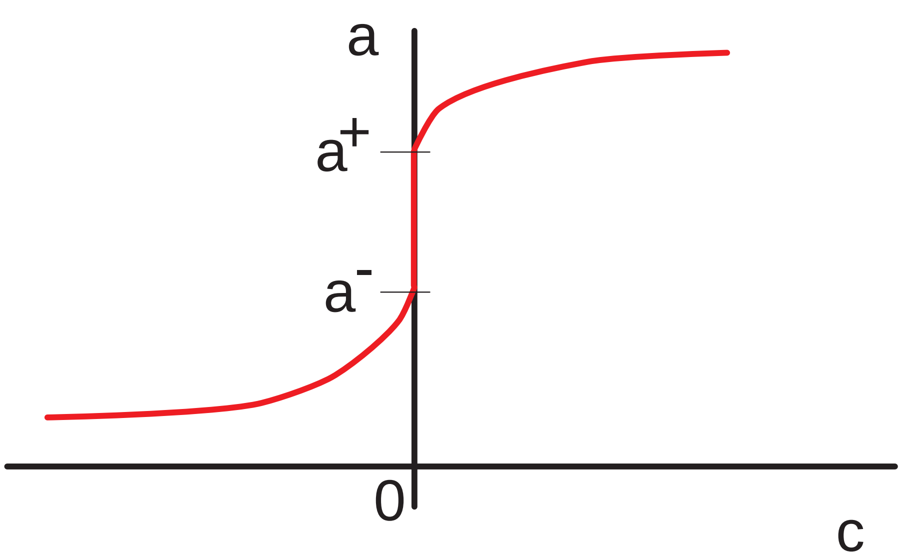
Another feature which distinguishes LDEs from PDEs is propagation failure.

Theorem 3 (Mallet-Paret). *The differential difference equation*

$$-c\phi'(\xi) = \alpha(\phi(\xi + k_1) + \phi(\xi - k_1) + \phi(\xi + k_2) + \phi(\xi - k_2) - 4\phi(\xi)) - f_{cub}(\phi(\xi), a)$$

generally admits a nontrivial interval $[a_-, a_+]$ *for which the wavespeed* c *vanishes, i.e.*

$$c(a) = 0, \quad a \in [a_-, a_+]. \tag{28}$$



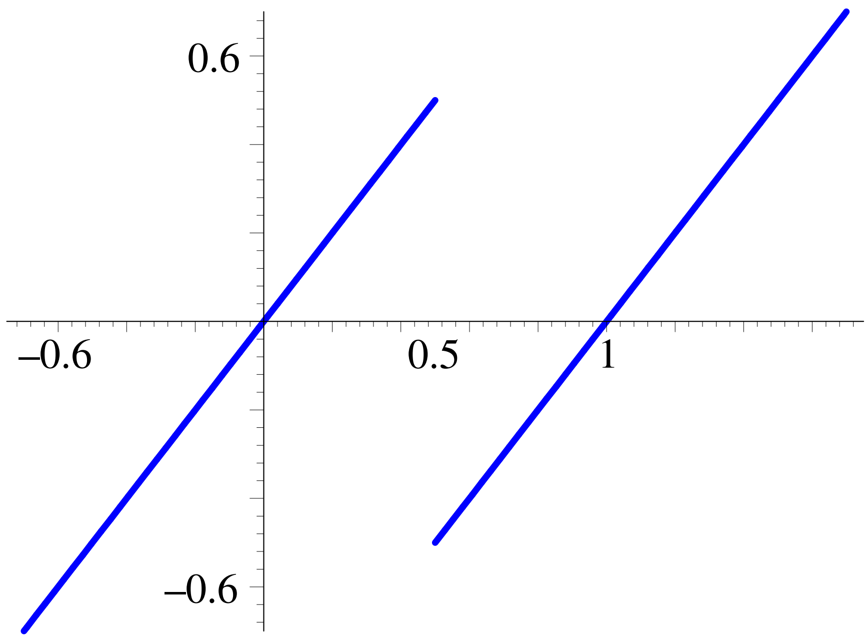
Propagation failure and Spatial Anisotropy

Theorem 4. *Consider the travelling wave equation*

$$-c\phi'(\xi) = \alpha\Big(\phi(\xi + k_1) + \phi(\xi - k_1) + \phi(\xi + k_2) + \phi(\xi - k_2) - 4\phi(\xi)\Big) - \widetilde{f}(\phi(\xi), a),$$

with $k = (\cos\theta, \sin\theta)$*. Write* $a^+(\theta)$ *for the critical value of* a *at which propagation failure sets in. Then* $a^+(\theta)$ *is continuous whenever* $\tan\theta$ *is irrational and discontinuous whenever* $\tan\theta$ *is rational or infinite.*

An analogous result has recently been obtained for the cubic f_{cub}.



The idealized nonlinearity

$$\widetilde{f}(x, a) = x - \mathrm{Heaviside}(x - a). \quad (29)$$

Our contribution

Analysis of numerical method to solve class of DDEs including

$$-\gamma\phi''(\xi) - c\phi'(\xi) = \epsilon \sum_{j=1}^{N} (\phi(\xi + r_j) - \phi(\xi)) - f_{\mathrm{cub}}(\phi(\xi), a) \tag{30}$$

for $\gamma > 0$ and $\epsilon > 0$, under the conditions

$$\begin{array}{lcl} \lim_{\xi\to-\infty} \phi(\xi) & = & 0, \\ \lim_{\xi\to\infty} \phi(\xi) & = & 1, \\ \phi(0) & = & a. \end{array} \tag{31}$$

- The extra second order term required for computational purposes.
- Physically, this term arises if we incorporate local as well as nonlocal effects into the model.
- It allows us to perform continuation between discrete and continuous Laplacian.

A connecting solution to the DDE (30) is a pair $(\phi, c) \in W^{2,\infty} \times \mathbb{R}$ which satisfies the DDE (30) and the conditions (31).

Newton iteration

Write $\overline{\phi}(\xi) = \big(\phi(\xi + r_1), \dots, \phi(\xi + r_N)\big)$ and split the DDE as

$$-\gamma\phi''(\xi) - c\phi'(\xi) = F(\phi(\xi)) + G(\overline{\phi}(\xi))$$

Solutions correspond to zeroes of $\mathcal{G} : W^{2,\infty} \times \mathbb{R} \to L^\infty$, defined by

$$\mathcal{G}(\phi, c)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F(\phi(\xi)) - G(\overline{\phi}(\xi)). \quad (32)$$

We seek zeroes of the map $\mathcal{G}$ via Newton iteration. Normally, this would involve the iteration step

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{G}(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n), \quad (33)$$

where $D_{1,2}\mathcal{G}$ is the Frechet derivative of $\mathcal{G}$, given by

$$[D_{1,2}\mathcal{G}(\phi, c)](\psi, b)(\xi) = -\gamma\psi''(\xi) - c\psi'(\xi) - D_1F(\phi)\psi(\xi) - D_1G(\overline{\phi})\overline{\psi(\xi)} - b\phi'(\xi).$$

The iteration step involves solving a linear DDE and thus is hard.

Variant of Newton Iteration

Goal is to relax dependence on shifted arguments. Introduce the operator

$$\mathcal{F}^{\mu}(\phi, c)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F(\phi(\xi)) - \mu G(\overline{\phi}(\xi)), \tag{34}$$

where $\mu \in [0, 1]$ is a relaxation parameter.

The numerical method uses the iteration step

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{F}^{\mu}(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n). \tag{35}$$

- This is an ODE for $\mu = 0$ of the form $\gamma\phi''_{n+1} + c_n\phi'_{n+1} = H(\phi_{n+1}, c_{n+1}, \xi)$.
- Can use standard boundary solver (like COLMOD) to solve at each iteration step.
- Second order term ensures that solutions remain smooth, even when $c \to 0$.
- Essential in light of propagation failure!

Convergence of The Method

Theorem 5. *Let* (ϕ, c) *be a connecting solution of the DDE*

$$-\gamma\phi''(\xi) - c\phi'(\xi) = F(\phi(\xi)) + G(\overline{\phi}(\xi)).$$

Then the Newton iteration given by

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n),$$

is well-defined and converges to the solution (ϕ, c) *for all initial values* (ϕ_0, c_0) *which are sufficiently close to the solution* (ϕ, c) *and for all* μ *sufficiently close to* 1.

- Does the DDE with the second order term have a solution?
- How does the second order term affect the solution?
- How do we get an appropriate initial value (ϕ_0, c_0)?
- Can we take $\mu = 0$?

Main result

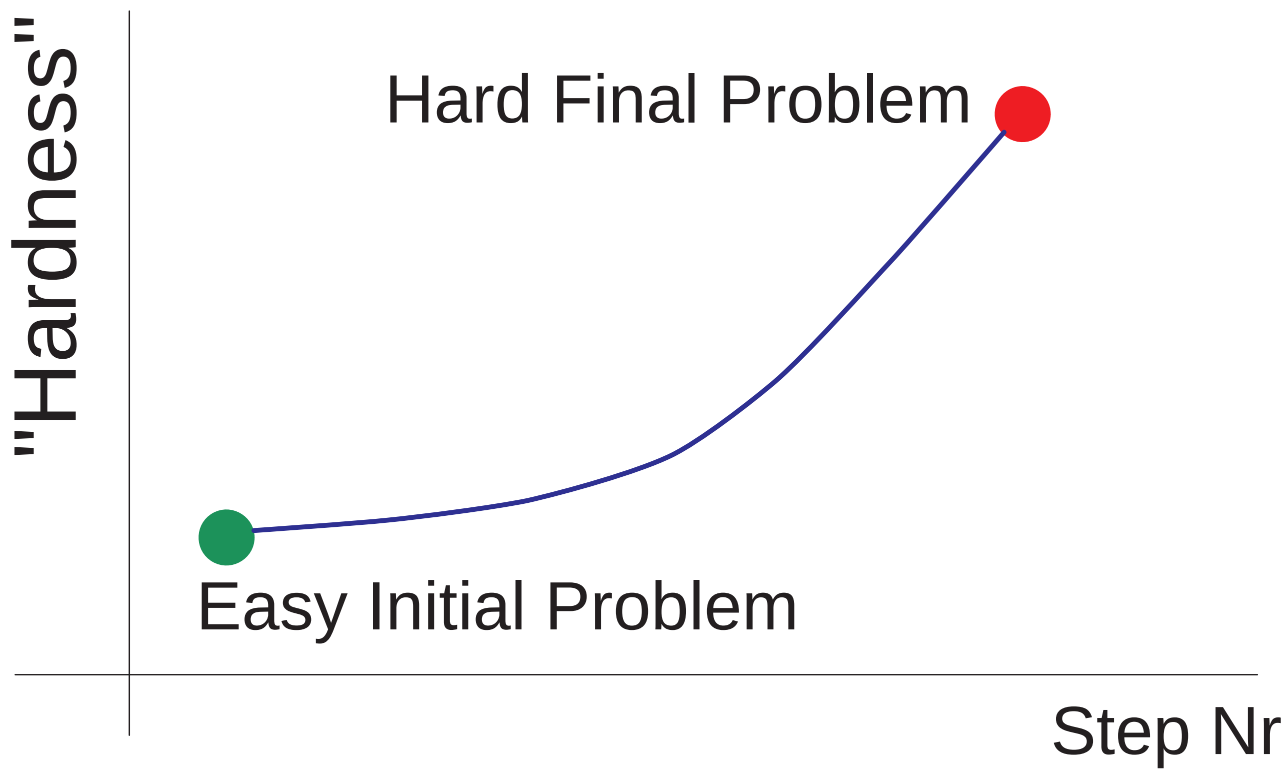
Theorem 6. *The differential difference equation*

$$-\gamma\phi''(\xi) - c\phi'(\xi) = \epsilon \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\text{cub}}(\phi(\xi), a),$$

with $\gamma > 0$ *and* $\epsilon > 0$ *has a unique connecting solution* $\big(\phi(a), c(a)\big) \in W^{2,\infty} \times \mathbb{R}$ *for all* $0 < a < 1$*. Moreover, this connecting solution* $\big(\phi(a), c(a)\big)$ *depends* C^1*-smoothly on the detuning parameter* a*.*

Continuation

- In general it is hard to find an appropriate initial solution (ϕ_0, c_0).
- The continuity in parameter space established in the previous theorem allows us to use continuation.
- Progessively advance from easy problems to hard problems, using solution of a problem as initial condition for next problem.



Main Results continued

Theorem 7. *Let* (ϕ_n, c_n) *be a sequence of connecting solutions to the DDEs*

$$-\gamma_n \phi''(\xi) - c\phi'(\xi) = \epsilon \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a),$$

with $\gamma_n \to 0$*. Then, after passing to a subsequence, the pointwise limits*

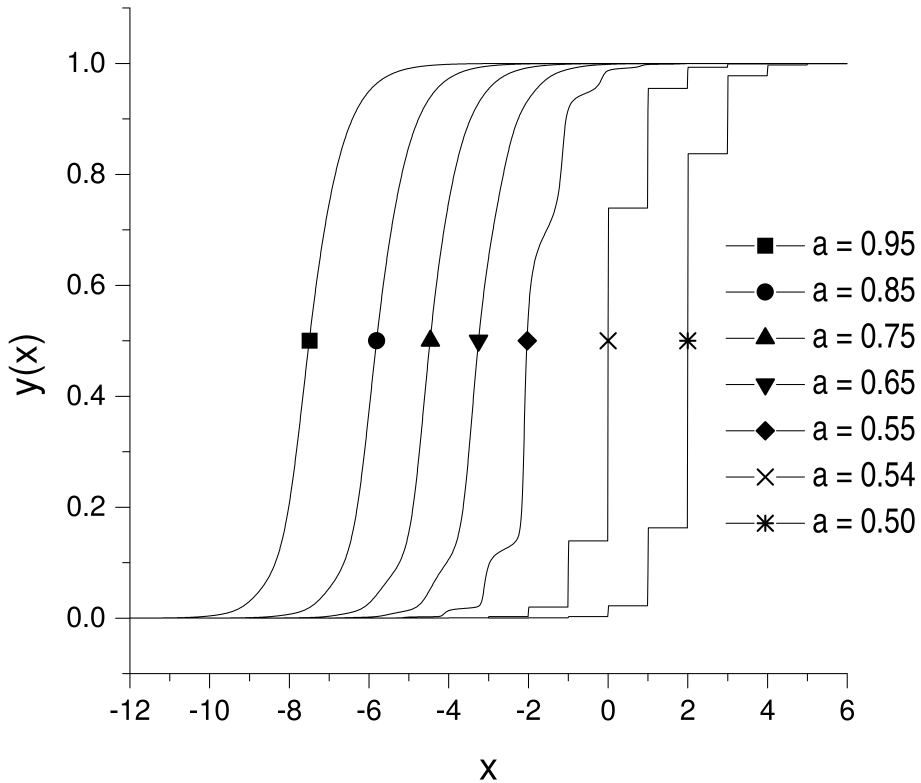
$$\begin{array}{rcl} \phi_0(\xi) & = & \lim_{n\to\infty} \phi_n(\xi), \\ c_0 & = & \lim_{n\to\infty} c_n \end{array} \tag{36}$$

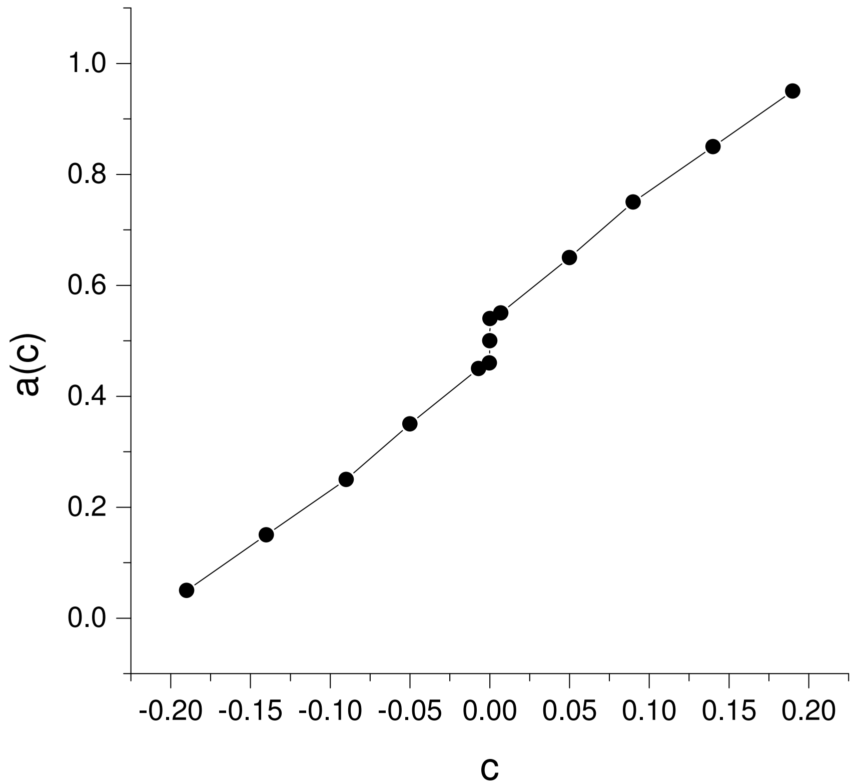
both exist and (ϕ_0, c_0) *is a connecting solution to the limiting DDE*

$$-c\phi'(\xi) = \epsilon \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a).$$

We can thus hope to uncover the rich behaviour at $\gamma = 0$ by choosing γ small enough.

Propagation failure

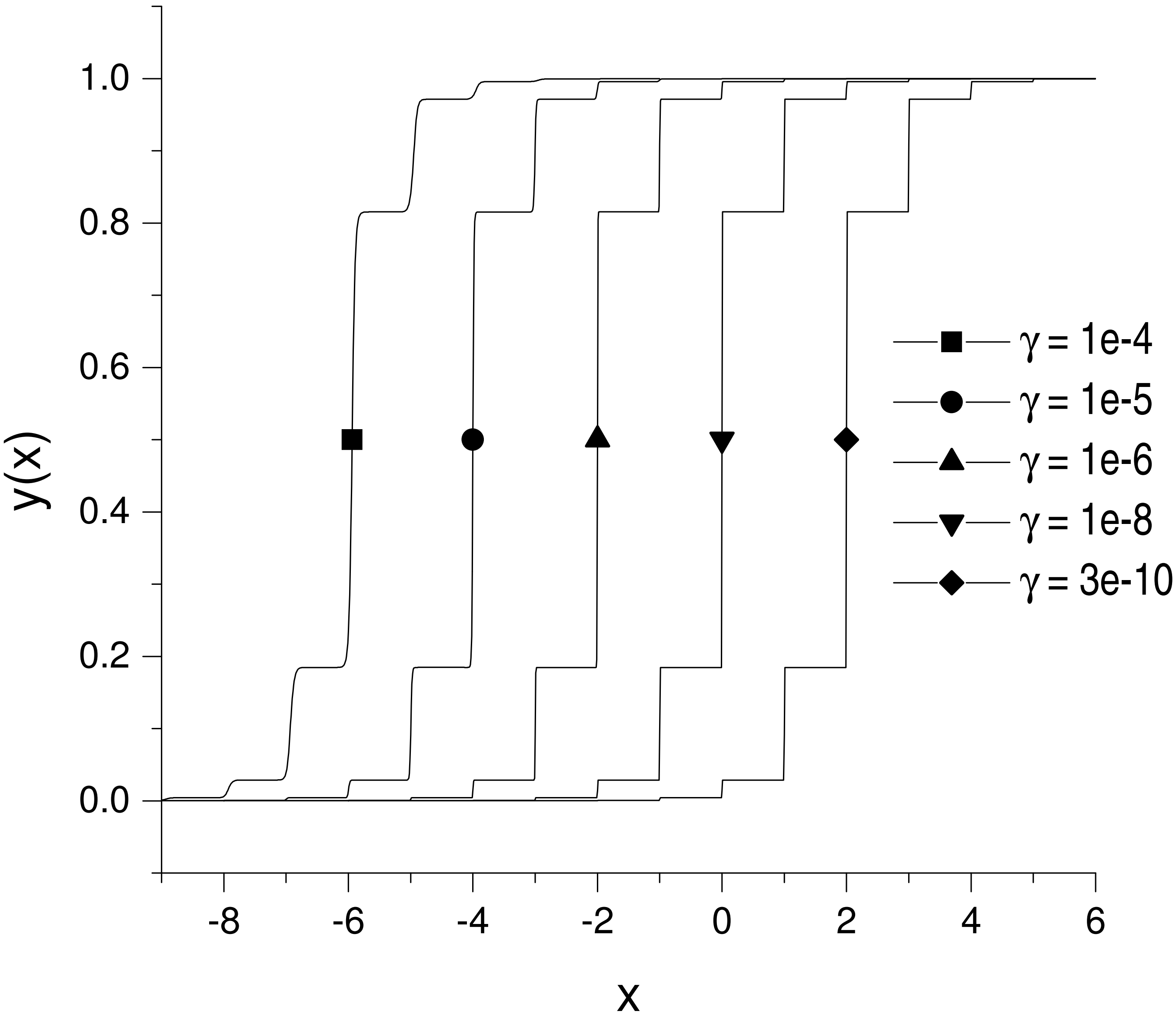




Solutions to the test problem

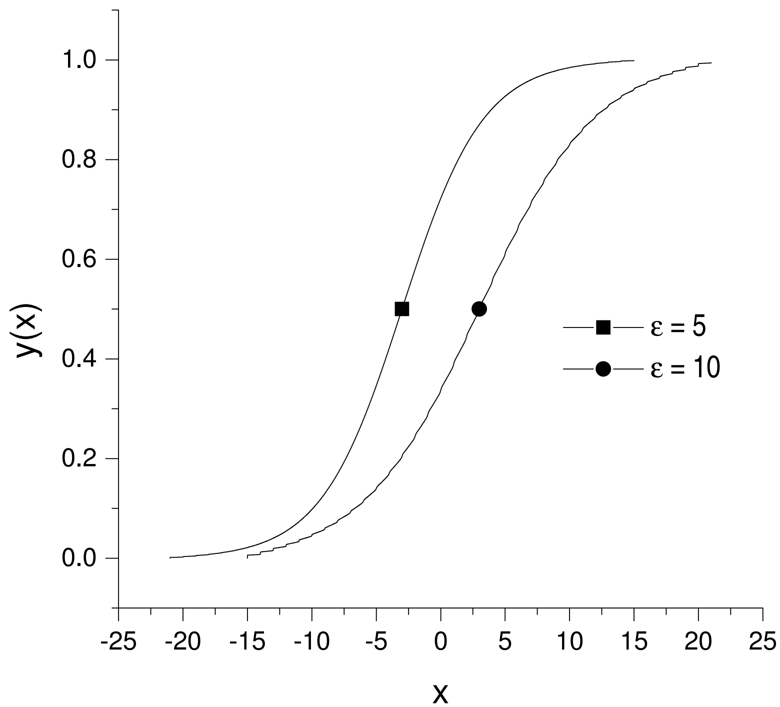
$$-10^{-8}\phi''(\xi) - c\phi'(\xi) = 0.1\big(\phi(\xi+1) - \phi(\xi-1) - 2\phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a).$$

Limit $\gamma \to 0$ in critical case $a = 0.5$



Large delay term

- When ϵ increases, delay term becomes dominant $\Rightarrow$ difficult to converge.
- However, high ϵ corresponds to the PDE limit and is thus uninteresting.



Solutions to the test problem

$$-10^{-4}\phi''(\xi) - c\phi'(\xi) = \epsilon\big(\phi(\xi+1) - \phi(\xi-1) - 2\phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), 0.50).$$

Higher Dimensional Systems

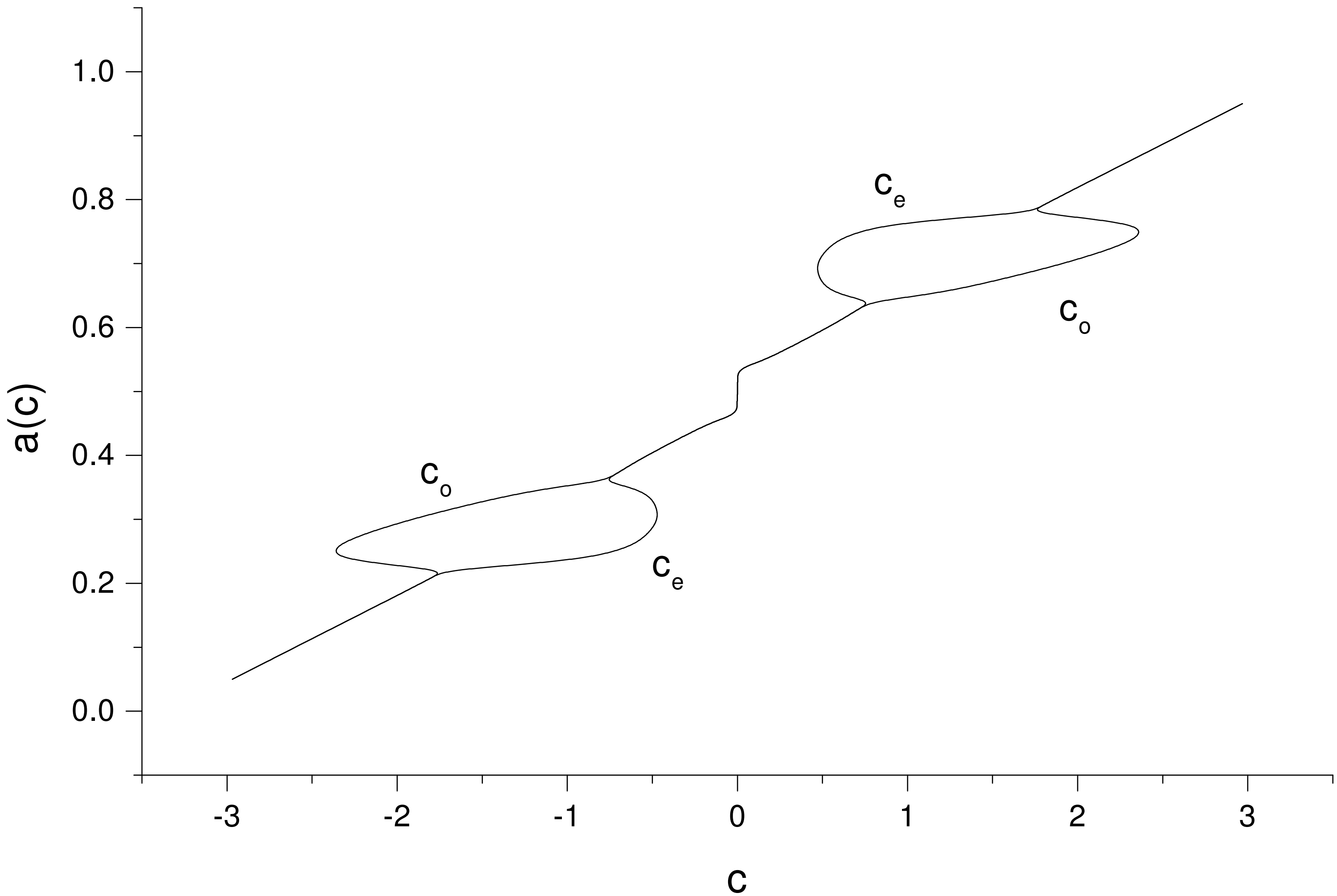
- Our complete analysis has been for one dimensional systems.
- One can also study higher dimensional DDEs.
- These have a richer structure!

For example, consider the system

$$\begin{cases} -10^{-5}\phi_e''(\xi) - c_e\phi_e'(\xi) = 1.6\big(\phi_o(\xi) - 2\phi_e(\xi) + \phi_o(\xi-2)\big) - 15 f_{\text{cub}}(\phi_e(\xi), a) \\ -10^{-5}\phi_o''(\xi) - c_o\phi_o'(\xi) = 1.6\big(\phi_e(\xi+2) - 2\phi_o(\xi)) + \phi_e(\xi)\big) - 15 f_{\text{cub}}(\phi_o(\xi), a). \end{cases} \tag{37}$$

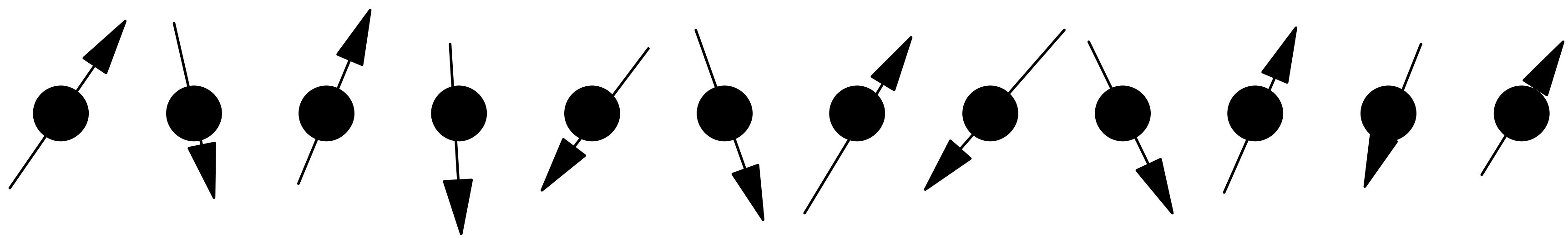
The solutions were normalized to have $\phi_e(0) = a$ and $\phi_o(-1) = a$. If we choose $c_e = c_o$ and $\phi_o(\xi) = \phi_e(\xi+1)$, the system (37) reduces to a one dimensional problem which has a unique solution.

Period Two Bifurcation - Solution Is No Longer Unique



Ising Spin Model

Application of higher dimensional systems.



Atoms arranged on 1d lattice, each atom has spin vector (s_1, s_2, s_3). Very important model in solid state physics.

In magnetic field governed by dipole neighbour-neighbour interactions.

$$
\begin{aligned}
\dot{s}_1(x,t) &= \lambda s_2(x,t), \\
\dot{s}_2(x,t) &= -\lambda x_1(x,t) + s_3(x,t)\big(s_1(x-1,t) + s_1(x+1,t)\big), \\
\dot{s}_3(x,t) &= -s_2(x,t)\big(s_1(x-1,t) + s_1(x+1,t)\big),
\end{aligned}
$$

with normalization $s_1^2 + s_2^2 + s_3^2 = 1$.

Ising Spin Model continued

Using travelling wave ansatz, we get

$$\begin{aligned} cs_1'(\xi) &= -\lambda s_2(\xi), \\ cs_2'(\xi) &= \lambda s_1(\xi) - \sqrt{1 - s_1^2(\xi) - s_2^2(\xi)}\big(s_1(\xi-1) + s_1(\xi+1)\big). \end{aligned} \quad (38)$$

- Our method cannot solve this equation as yet.
- Presence of periodic solutions complicates matters.
- Want to adapt method to handle this case.

Possible future research

- Study high dimensional bifurcations in greater detail. Attempts to find period 4 bifurcation in 4d systems has failed.
- Generalize results to higher dimensions.
- Include periodic solutions.

The End