Modulated Wave Trains in Lattice Differential Systems

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Abstract

The existence of weak sinks in mixed parabolic-lattice systems on the real line is established for systems that incorporate discrete coupling on an underlying lattice in addition to continuous diffusion. Sinks can be thought of as interfaces that separate two spatially periodic structures with different wave numbers: the corresponding modulated wave train is time periodic in the frame that moves with the speed of the interface. In this paper, the existence of weak sinks is proved that connect wave trains with almost identical wave number. The main difficulty is the global coupling between points on the underlying lattice, since its presence turns the equation solved by sinks into an ill-posed functional differential equation of mixed type.

Key words: mixed type functional differential equation, modulated waves, travelling waves, lattice differential equation, finite dimensional reduction, global center manifold, advanced and retarded arguments.

1 Introduction

In this paper we consider the partially discrete reaction-diffusion system

$$\partial_t y(x,t) = \gamma \partial_{xx} y(x,t) + \sum_{j=0}^{N} A_j y(x+n_j,t) + g\big(y(x,t)\big), \tag{1.1}$$

which mixes a continuous Laplacian with a discrete Laplacian on a one-dimensional lattice. In particular, we take $x \in \mathbb{R}$, $y(x,t) \in \mathbb{R}^n$ and $A_j \in \mathbb{R}^{n \times n}$. We require $\gamma > 0$, but allow the shifts $n_j \in \mathbb{Z}$ to be both positive and negative. We are especially interested in wave train solutions to (1.1). Such solutions can be written in the form

$$y(x,t) = u(\omega t - kx) \tag{1.2}$$

for some 2π-periodic function u. Here ω stands for the temporal frequency of the wave train, while k denotes the spatial wave number. In general, these solutions will persist as the wave number k is

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varied, giving rise to a one-parameter family of wave train solutions to (1.1) that we will write in the form

$$y(x,t) = u(\omega_{\mathrm{nl}}(k)t - kx; k). \tag{1.3}$$

The function ω_{nl} is referred to as the nonlinear dispersion relation.

Let us consider two nearby wave numbers k_- and k_+. The main goal of this paper is to construct solutions to (1.1) that are periodic in time when viewed in an appropriate co-moving coordinate frame and that 'connect' the wave train $u(\omega_{\mathrm{nl}}(k_-)t - k_-x; k_-)$ at $x \approx -\infty$ to the wave train $u(\omega_{\mathrm{nl}}(k_+)t - k_+x; k_+)$ at $x \approx \infty$. In particular, we will look for solutions of the form $y(x,t) = u_*(x - c_*t, \omega_*t)$, that behave as

$$u_*(x - c_*t, \omega_*t) \to u\big(\omega_{\mathrm{nl}}(k_\pm)t - k_\pm x - \vartheta_\pm; k_\pm\big) \tag{1.4}$$

as $x \to \pm\infty$, while satisfying the periodicity condition $u_*(\xi,\tau) = u_*(\xi, \tau + 2\pi)$.

The existence of modulated waves that satisfy these properties has already been established for the reaction-diffusion system (1.1) without the shifted arguments, i.e., in the situation where $A_j = 0$ for $0 \le j \le N$. This was achieved in [17] by using the spatial-dynamical approach developed by Kirchgässner [38] and applying a center manifold result due to Mielke [49]. In order to outline this construction, let us introduce new functions v and w that take values in an appropriate space of 2π-periodic functions and write $v(\xi)(\tau) = u_*(\xi,\tau)$ and $w(\xi)(\tau) = \partial_\xi u_*(\xi,\tau)$. In the absence of the shifted terms in (1.1), the search for modulated waves leads to the equation

$$\begin{array}{lcl} \partial_\xi v(\xi)(\cdot) & = & w(\xi)(\cdot), \\ -\gamma\partial_\xi w(\xi)(\cdot) & = & c_* w(\xi)(\cdot) - \omega_*\partial_\tau v(\xi)(\cdot) + g\big(v(\xi)(\cdot)\big). \end{array} \tag{1.5}$$

After fixing a wave number k_0, we write $c_* = \omega_{\mathrm{nl}}'(k_0)$ and consider temporal frequencies

$$\omega_* = \omega_{\mathrm{nl}}(k_0) - k_0 c_* + \overline{\omega}, \tag{1.6}$$

in which $\overline{\omega}$ shares the sign of $\omega_{\mathrm{nl}}''(k_0)$ and has small absolute value. For any such value of $\overline{\omega}$, there exist two wave numbers $k_\pm = k_\pm(\overline{\omega})$ such that the choices $v_\pm(\xi)(\tau) = u(\tau - k_\pm\xi; k_\pm)$ lead to solutions of (1.5). These wave numbers are distinct if $\overline{\omega} \neq 0$ and satisfy the limits $k_\pm(\overline{\omega}) \to k_0$ as $\overline{\omega} \to 0$. Under some generic assumptions, the results in [49] can be applied to construct, for $\overline{\omega}$ as above, a two dimensional center manifold $\mathcal{M}$ that captures all solutions to (1.5) that remain orbitally close to the periodic function (v_0, w_0). Here $v_0(\xi)(\tau) = u(-k_0\xi + \tau; k_0)$ and $w_0(\xi)(\tau) = -k_0 u'(-k_0\xi + \tau; k_0)$. Of course, this manifold will contain the two solutions $(v_\pm, D_\xi v_\pm)$ mentioned above.

The crucial observation that allows the dynamics on this center manifold to be explicitly analyzed, is that the change of variables $\sigma = \tau - k_0\xi$ turns the periodic solution (v_0, w_0) and its spatial translates into a ring of equilibria for the transformed version of (1.5). Exploiting this change of variables allows one to derive a two dimensional ODE that describes the dynamical behaviour on $\mathcal{M}$. The desired modulated wave can subsequently be read off from this ODE.

The results in [49] however cease to apply in the presence of the shifted terms. This situation was partially remedied in [33], where center manifolds were constructed for differential equations involving such shifted terms. Unfortunately, these manifolds are as of yet unable to capture solutions that merely remain orbitally close to a prescribed periodic solution. In addition, the results in [33] do not cover differential equations posed on general Hilbert spaces. For these reasons, we deviate from the approach in [17] and choose to work directly with the transformed coordinates (ξ, σ) in this paper. This choice ensures that the variational equation that is encountered has autonomous instead of periodic coefficients. After constructing an appropriate global center manifold, the analysis in [17] can be carried over to the current setting.

The primary motivation for the inclusion of the shifted terms in (1.1) comes from the wish to understand and classify the structure of solutions to so-called lattice differential equations (LDEs).

Such equations are infinite systems of ordinary differential equations indexed by points on a spatial lattice. A typical example is given by the discrete Nagumo equation

$$\dot{u}_i(t) = \alpha[u_{i-1}(t) + u_{i+1}(t) - 2u(t)] - (u(t)^2 - 1)(u(t) - \rho), \tag{1.7}$$

for some $-1 < \rho < 1$, which arises when one discretizes the scalar reaction diffusion equation

$$y_t = \Delta y - (y^2 - 1)(y - \rho) \tag{1.8}$$

on a one-dimensional lattice with spacing $h = \alpha^{-1/2}$. In the literature, the discrete Nagumo equation has served as a prototype system for investigating the properties of LDEs. We mention here the work of Mallet-Paret [44, 45], who analyzed travelling wave solutions to this equation connecting the two equilibria $\overline{u} = \pm 1$ and found that in general there exist nontrivial intervals of the detuning parameter ρ for which the wave speed satisfies $c = 0$ and hence the waves fail to propagate.

It turns out that propagation failure is one of many features that distinguish LDEs from their continuous counterparts. The relatively rich structure of LDEs, combined with the ability to incorporate nonlocal interactions into a model, have presented a strong motivation for the study of such systems. At present, models involving LDEs can be found in many scientific disciplines, including chemical reaction theory [21, 41], image processing and pattern recognition [14], material science [2, 8] and biology [3, 36]. Apart from these modelling considerations, LDEs also arise when one studies numerical methods to solve PDEs and analyzes the effects of the employed spatial discretization [5, 6, 7, 19].

Since travelling waves provide a convenient starting point in the analysis of LDEs, they have attracted considerable interest during the past two decades. Early papers on the subject by Chi, Bell and Hassard [10] and by Keener [37] were followed by many others which developed the basic theory; see, for example, [9, 13, 27, 34, 35, 42, 44, 45, 46, 60, 62, 63]. To appreciate the difficulties that arise, let us plug the travelling wave Ansatz $u_i = \phi(i - ct)$ into the LDE (1.7). We find that the profile ϕ must satisfy the following scalar functional differential equation of mixed type (MFDE),

$$-c\phi'(\xi) = \alpha[\phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)] - \big(\phi(\xi)^2 - 1\big)\big(\phi(\xi) - \rho\big). \tag{1.9}$$

Although ϕ is $\mathbb{R}$-valued, the relevant state space associated to (1.9) is necessarily infinite dimensional. A typical choice is given by $X_{\mathrm{tw}} = C([-1, 1], \mathbb{R})$, although the Hilbert space $L^2([-1, 1], \mathbb{R})$ has also been used in the literature. The linearization of (1.9) around a wave profile ϕ will in general be ill-posed [28], which prevents the use of the semigroup techniques developed for retarded differential equations [15]. For this reason, one needs to resort to Fredholm techniques and exponential dichotomies when analyzing linear MFDEs. These tools were developed for use in an MFDE setting by Mallet Paret [43], Verduyn Lunel [46] and Härterich et al. [28]. In [32, 33] these results were used in a nonlinear setting to construct center manifolds around equilibria and periodic solutions to MFDEs posed on the finite dimensional spaces $\mathbb{C}^n$ and $\mathbb{R}^n$. For a detailed discussion of these issues we refer to [30].

This paper is intended primarily as a step towards understanding modulated wave solutions to LDEs, which are a logical next stop on the way upwards from travelling waves in the chain of complexity. As one would expect, such a step is accompanied by an increase in complexity of the equations that need to be analyzed. Indeed, upon introducing the new functions $v(\xi)(\sigma) = u_*(\xi, \sigma + k_0\xi)$ and $w(\xi)(\sigma) = D_1 u_*(\xi, \sigma + k_0\xi)$, the system to solve becomes

$$\begin{array}{rcl} \partial_\xi v(\xi)(\cdot) & = & w(\xi)(\cdot) + k_0 \partial_\sigma v(\xi)(\cdot), \\ -\gamma \partial_\xi w(\xi)(\cdot) & = & k_0 \partial_\sigma w(\xi)(\cdot) + c_* w(\xi)(\cdot) - \omega_* \partial_\sigma v(\xi)(\cdot) \\ & & + \sum_{j=0}^{N} A_j v(\xi + n_j)(\cdot - n_j k_0) + g\big(v(\xi)(\cdot)\big). \end{array} \tag{1.10}$$

This equation now contains shifts with respect to both ξ and σ. There exists a ring of equilibrium solutions

$$\big(v_\vartheta(\xi)(\cdot), w_\vartheta(\xi)(\cdot)\big) = \big(u(\vartheta + \cdot\,; k_0), -k_0 u'(\vartheta + \cdot\,; k_0)\big), \tag{1.11}$$

parametrized by $\vartheta \in \mathbb{R}$. In order to construct modulated waves, we will need to consider the state space $X_{\mathrm{mw}} = C([n_{\min}, n_{\max}], H)$. Here $n_{\min} = \min\{n_j\}$, $n_{\max} = \max\{n_j\}$ and H stands for a Hilbert space that encodes pairs of 2π-periodic functions, which will be fixed in the sequel.

The fact that we are forced to use X_{mw} instead of X_{tw} presents two immediate complications. To understand the first of these, let us consider any $\phi \in C(\mathbb{R}, \mathbb{R})$ that solves (1.9). The differential equation then immediately implies that $\phi' \in C(\mathbb{R}, \mathbb{R})$ and allows bounds on ϕ to be turned into bounds on ϕ'. The presence of the derivatives with respect to σ in (1.10) however prevents the use of such direct bootstrapping methods. As a consequence, special care needs to be taken when constructing solution operators to linear inhomogeneous systems and we will need additional smoothness on the part of the nonlinearity g.

The second complication caused by the use of X_{mw} is that it is no longer easy to construct and analyze characteristic functions. Indeed, substituting $\phi(\xi) = 1 + \beta e^{z\xi}$ into (1.9) and keeping only terms that are linear in β, we find the characteristic equation $\Delta_{\mathrm{ch}}(z) = 0$, in which

$$\Delta_{\mathrm{ch}}(z) = -cz - \alpha[e^{z} + e^{-z} - 2] + 2(1-\rho). \tag{1.12}$$

By now, a lot is known about the root distribution and asymptotic behaviour of such quasi-polynomials [4, 15]. A very useful feature is that in vertical strips in the complex plane, the uniform estimates $\Delta_{\mathrm{ch}}(z) = -czI + O(1)$ hold as $\operatorname{Im} z \to \pm\infty$. This allows us to obtain uniform bounds on $\Delta_{\mathrm{ch}}(z)^{-1}$ for large $|\operatorname{Im} z|$, which considerably eases the construction of Greens functions to solve linear MFDEs [43]. However, upon plugging the analogous Ansatz into (1.10), we shall see in the sequel that we need to look for pairs (z, v) for which $\mathcal{L}_{\mathrm{ch}}(z)v = 0$. Here v is a 2π-periodic function and the characteristic operator $\mathcal{L}_{\mathrm{ch}}(z)$ for $z \in \mathbb{C}$ is given by

$$\begin{array}{rcl} \mathcal{L}_{\mathrm{ch}}(z)v & = & [zc_* + \gamma\nu^2 - (\omega_* + k_0 c_* + 2\gamma k_0 z)D + \gamma k_0^2 D^2]v \\ & & + \sum_{j=0}^{N} A_j e^{z n_j} v(\cdot - n_j k_0) + Dg\big(u(\cdot\,; k_0)\big)v. \end{array} \tag{1.13}$$

Let us note that for $\gamma = 0$, this expression reduces to

$$\mathcal{L}_{\mathrm{ch}}(z)v = [zc_* - (\omega_* + k_0 c_*)D]v + \sum_{j=0}^{N} A_j e^{z n_j} v(\cdot - n_j k_0) + Dg\big(u(\cdot\,; k_0)\big)v. \tag{1.14}$$

Consider any two integers $\ell \in \mathbb{Z}$ and $\Delta k \in \mathbb{Z}$. Under the transformation $\widetilde{v} = \exp[i\Delta k \cdot]v$ and $\widetilde{z} = z + ik_0\Delta k + 2\pi i\ell$, we have

$$\exp[-i\Delta k\cdot]\mathcal{L}_{\mathrm{ch}}(\widetilde{z})\widetilde{v} = \mathcal{L}_{\mathrm{ch}}(z)v + i\big(2\pi c_*\ell - \omega_*\Delta k\big)v. \tag{1.15}$$

Let us suppose that $\mathcal{L}_{\mathrm{ch}}(z_0)$ admits an eigenvalue λ with $\operatorname{Re}\lambda = \operatorname{Re} z_0$. If ω_* and πc_* are not rationally related, then in view of (1.15) we can have no hope of obtaining a uniform bound on $\mathcal{L}_{\mathrm{ch}}(z)^{-1}$ on the vertical line $\operatorname{Re} z = \operatorname{Re} z_0$. In fact, this is precisely the reason that we have to exclude $\gamma = 0$ in (1.1). We emphasize here that this issue is unrelated to the usual artificial ambiguity that surrounds Floquet exponents.

We remark that in [1, 18, 31] an additional second order dispersion term of order γ was also added to LDEs to allow for the numerical computation of travelling waves. The feasibility of this approach was established in [31], where conditions were given under which the limit $\gamma \to 0$ yields solutions to the original LDE. The present set up can hence be seen as a step towards obtaining information on (1.1) with $\gamma = 0$, by analyzing the path $\gamma \to 0$. Let us emphasize here that there are also physical reasons to introduce a second order term in (1.1). Indeed, such a term arises naturally if we consider systems that have local as well as nonlocal interactions. In addition, it allows continuation techniques to be used to study the effect of moving from a continuous to a discrete model. An example of such an approach can be found in [20].

Aside from the technical issues connected to the state space X_{mw}, the most important challenge that needs to be overcome in the present work is the construction of a global center manifold for functional differential equations of mixed type. To set the scene, let us first sketch such a construction for the planar ODE

$$y'(\xi) = f\big(y(\xi)\big), \tag{1.16}$$

in which $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function. For any $\vartheta \in \mathbb{R}$, we write $\rho(\vartheta) \in SO_2(\mathbb{R}^2)$ for the linear operator that rotates vectors $v \in \mathbb{R}^2$ around the origin by the angle ϑ. Let us suppose that f is invariant under these transformations, i.e.,

$$\rho(-\vartheta) f\big(\rho(\vartheta)v\big) = f(v) \tag{1.17}$$

for all $v \in \mathbb{R}^2$ and $\vartheta \in \mathbb{R}$. In addition, let us suppose that (1.16) admits an equilibrium solution $\overline{y} \neq 0$, which in view of (1.17) implies the existence of an entire circle of equilibria $\{\rho(\vartheta)\overline{y}\}_{\vartheta\in\mathbb{R}}$.

A simple differentiation yields the identity $Df(\overline{y})D\rho(0)\overline{y} = 0$, showing that the matrix $Df(\overline{y})$ admits zero as an eigenvector. If the algebraic multiplicity of this zero eigenvector is two, then one may hope to find solutions to (1.17) that remain close to the circle $\{\rho(\vartheta)\overline{y}\}_{\vartheta\in\mathbb{R}}$. To capture such solutions, let us write

$$y(\xi) = \rho(\theta(\xi))[\overline{y} + u(\xi)], \tag{1.18}$$

in which $\theta : \mathbb{R} \to \mathbb{R}$ is a smooth function that is chosen in such a way that $|y(\xi) - \rho(\theta(\xi))\overline{y}|$ is minimized for every $\xi \in \mathbb{R}$. This choice yields the normalization condition

$$\langle D\rho(0)\overline{y}, u(\xi)\rangle = 0 \tag{1.19}$$

for every $\xi \in \mathbb{R}$, where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^2$. Differentiating (1.18), plugging the result into (1.16) and using (1.17) yields

$$\theta'(\xi)D\rho(0)[\overline{y} + u(\xi)] + u'(\xi) = f\big(\overline{y} + u(\xi)\big). \tag{1.20}$$

Using this expression to differentiate (1.19), we find

$$\theta'(\xi) = [\langle D\rho(0)\overline{y}, D\rho(0)\overline{y}\rangle + \langle D\rho(0)\overline{y}, D\rho(0)u(\xi)\rangle]^{-1}\langle D\rho(0)\overline{y}, f\big(\overline{y} + u(\xi)\big)\rangle. \tag{1.21}$$

We hence see that the function θ can be completely eliminated from (1.20), yielding an autonomous differential equation for u that can be analyzed using standard center manifold theory. This approach was initiated by Krupa in [39]. Related developments for compact symmetry groups can be found in [11, 24, 51], while [23, 25, 48, 54, 55] contain results that apply in the presence of non-compact symmetries.

Let us now return to the setting of MFDEs and consider the prototype system

$$y'(\xi) = f\big(y(\xi), y(\xi - 1), y(\xi + 1)\big), \tag{1.22}$$

in which y takes values in a Hilbert space H that contains 2π-periodic functions. There are two main problems that preclude the use of the approach sketched above in this setting. The first issue concerns the operator $D\rho(0)$. In our MFDE setting, the symmetry acts as $[\rho(\vartheta)y(\xi)](\sigma) = y(\xi)(\vartheta + \sigma)$, which implies that $D\rho(0)y(\xi) = \partial_\sigma y(\xi)$. The nonlinear term $\theta'(\xi)D\rho(0)u(\xi)$ in (1.20) hence loses an order of smoothness compared to u, which in our setting would prevent the use of the standard Lyapunov-Perron fixed point method to construct the center manifold. The second problem is caused by the advanced and retarded terms present in (1.22), which break the mechanism by which the variable θ was previously eliminated from (1.20). Indeed, the equivalent of (1.20) in the MFDE setting will depend not only on $\theta'(\xi)$, but also on the quantities $\theta(\xi \pm 1)$.

The first of these problems can be resolved by writing

$$y(\xi) = \rho(\theta(\xi))\overline{y} + u(\xi) \tag{1.23}$$

and arguing much as above to obtain the coupled system

$$\begin{aligned} \theta'(\xi) &= [\langle D\rho(0)\overline{y}, D\rho(0)\overline{y}\rangle + \langle D\rho(\theta(\xi))\overline{y}, u(\xi)\rangle]^{-1}\langle D\rho(\theta(\xi))\overline{y}, f(\theta_\xi, u_\xi)\rangle, && (1.24a) \\ u'(\xi) &= -\theta'(\xi)D\rho(\theta(\xi))\overline{y} + f(\theta_\xi, u_\xi), && (1.24b) \end{aligned}$$

in which

$$f(\theta_\xi, u_\xi) = f\big(\rho(\theta(\xi))\overline{y} + u(\xi), \rho(\theta(\xi-1))\overline{y} + u(\xi-1), \rho(\theta(\xi+1))\overline{y} + u(\xi+1)\big). \quad (1.25)$$

Notice that the problematic term involving the operator $D\rho(0)$ applied to u has indeed disappeared. The linearization of (1.24b) with respect to u however continues to include a dependence on θ. The main technical tool that is developed in this paper is the construction of a center manifold that contains all solutions (θ, u) with sufficiently small u to a system of the form (1.24). The variable θ does not have to be bounded, although we remark that θ' will automatically be small whenever u is small.

We remark here that our constructions are closely related to singular perturbation theory. Indeed, the classical Fenichel theorems established in [22] consider a fast-slow ODE of the form

$$\begin{aligned} \theta' &= \epsilon g_{\mathrm{s}}(\theta, u, \epsilon), \\ u' &= g_{\mathrm{f}}(\theta, u, \epsilon), \end{aligned} \quad (1.26)$$

that for $\epsilon = 0$ admits a manifold of equilibria given by $g_{\mathrm{f}}\big(\vartheta, \widetilde{u}(\vartheta), 0\big) = 0$ for some smooth $\widetilde{u}$. One of the principal questions addressed by the theory is the persistence of the invariant manifold $\mathcal{M}_0 = \{(\vartheta, \widetilde{u}(\vartheta))\}$ as ϵ becomes nonzero. Fenichel's first theorem [22] states that whenever (1.26) is normally hyperbolic, the invariant set $\mathcal{M}_0$ can be extended smoothly to a center manifold $\mathcal{M}_\epsilon = \{(\vartheta, \widetilde{u}_\epsilon(\vartheta))\}$, defined for small $\epsilon > 0$, that captures all solutions to (1.26) that remain in the neighbourhood of $\mathcal{M}_0$. This condition of normal hyperbolicity requires the eigenvalues of the linearizations $D_2 g_{\mathrm{f}}(\vartheta, \widetilde{u}(\vartheta), 0)$ to be uniformly bounded away from the imaginary axis.

By now a rich literature has developed concerning the existence and persistence of center manifolds for rather general normally-hyperbolic invariant sets [16, 29, 40, 47, 52, 59]. For our purposes however, normal hyperbolicity is too restrictive, as the dynamical behaviour that we will be interested in is generated precisely by the extra center directions available for u in (1.24b). An important result in this respect was obtained by Chow, Liu and Yi [12], who constructed center manifolds for a general class of invariant sets without requiring normal hyperbolicity to hold.

Unfortunately, most of the center manifold results mentioned above were obtained using the so-called Hadamard graph transform technique [26], which is very geometrical in nature. Indeed, as a crucial part of these constructions, the vector field must be modified to ensure that it points outwards at the boundary of a neighbourhood of the invariant set under consideration. However, in the infinite dimensional setup that we will use to analyze MFDEs, the vector field does not even map into the underlying state space. In addition, we mention that the construction in [12] utilizes finite dimensional results such as Whitney's embedding theorem [57] and Nash's embedding theorem [50].

For these reasons, we prefer to construct center manifolds using Lyapunov-Perron type techniques, which are far more analytical and thus easier to generalize to our infinite dimensional setting. The work of Sakamoto [53] is very interesting in this respect, as it proves Fenichel's first theorem using a fixed point argument on appropriate weighted function spaces. However, his construction breaks down as soon as normal hyperbolicity is lost. The crucial ingredient that we will use to generalize Sakamoto's approach was inspired by the work of Yi [61], where exponential dichotomies are used to construct integral manifolds for nonautonomous ODE versions of (1.24), again under a normal hyperbolicity condition. Instead of attempting to construct a center manifold by solving a single fixed point problem, we will use a two-stepped approach and set up two distinct fixed point equations. Taken together, the results will yield the desired invariant manifold for (1.24). More specifically, we will assume that the center manifold can be written in the form $\mathcal{M} = \{(\vartheta, \widetilde{u}(\vartheta, b))\}$ for some prescribed $\widetilde{u}$, in which $\vartheta \in \mathbb{R}$ and $b \in [-\epsilon, \epsilon]$. Using this form, we will show that the dynamics for the pair (ϑ, b) are uniquely fixed, which allows us to check if $\widetilde{u}$ indeed yields a valid invariant manifold. This latter criterion can be reformulated as a fixed point problem for $\widetilde{u}$.

Our constructions here depend heavily on the center manifold theory for equilibrium solutions to MFDEs that was developed in [32]. In particular, in Section 7 we will rely on [32, Theorem 2.2] to analyze the dynamical behaviour of the pair (ϑ, b), which is given by an MFDE. In addition, the constructions in [32, Section 5] will be needed to solve (1.24a) and find θ whenever u is prescribed.

In Section 2 we formulate our main result that establishes the existence of modulated wave solutions to (1.1). After dealing with some preliminary issues in Section 3, we derive the relevant version of the coupled system (1.24) in Section 4. In Section 5 we consider the linearization of (1.24b) for constant functions θ and use Laplace and Fourier transforms to construct a solution operator. This forms the basis for Section 6, where we again consider the linearization of (1.24b) but now allow θ to vary. Finally, in Section 7 we construct the desired center manifold using the approach sketched above.

2 Main Results

We recall our main partially discrete reaction-diffusion system

$$\partial_t y(x,t) = \gamma \partial_{xx} y(x,t) + \sum_{j=0}^{N} A_j y(x+n_j, t) + g\big(y(x,t)\big), \tag{2.1}$$

in which $\gamma > 0$, $x \in \mathbb{R}$, $n_j \in \mathbb{Z}$, $y(x,t) \in \mathbb{R}^n$ and $A_j \in \mathbb{R}^{n \times n}$. We will make the following assumption on the nonlinearity g.

(Hg) The nonlinearity $g : \mathbb{R}^n \to \mathbb{R}^n$ is C^r-smooth for some integer $r \geq 5$.

Let us first focus our attention on wave train solutions to (2.1), i.e., solutions that can be written in the form $y(x,t) = u(\omega t - kx)$ for some wave number k, frequency ω and 2π-periodic profile $u \in C(\mathbb{R}, \mathbb{R}^n)$. To facilitate the search for these solutions, we introduce the family of Hilbert spaces

$$H^s = \{ v \in L^2_{\mathrm{per}}([0,2\pi], \mathbb{R}^n) \mid D^\ell v \in L^2_{\mathrm{per}}([0,2\pi], \mathbb{R}^n) \text{ for } 1 \leq \ell \leq s \}, \tag{2.2}$$

parametrized by integers $s \geq 0$. We also introduce the shift operator $T_\vartheta : H^s \to H^s$ for any $\vartheta \in \mathbb{R}$, which acts as

$$[T_\vartheta v](\sigma) = v(\vartheta + \sigma) \tag{2.3}$$

after extending v in a periodic fashion. We will consider the functional $\mathcal{F} : H^2 \times \mathbb{R} \times \mathbb{R} \to H^0$ that is given by

$$\mathcal{F}(u, \omega, k)(\zeta) = -\gamma k^2 u''(\zeta) + \omega u'(\zeta) - \sum_{j=0}^{N} A_j u(\zeta - n_j k) - g\big(u(\zeta)\big) \tag{2.4}$$

and look for $u \in H^2$ that have $\mathcal{F}(u, \omega, k) = 0$. Here ζ should be interpreted as the wave variable associated to (2.1), i.e., $\zeta = \omega t - kx$. Let us write $\mathcal{L}(u, \omega, k) : H^2 \to H^0$ for the operator associated to the linearization of (2.4), which is given by

$$[\mathcal{L}(u, \omega, k) v](\zeta) = -\gamma k^2 v''(\zeta) + \omega v'(\zeta) - \sum_{j=0}^{N} A_j v(\zeta - n_j k) - Dg\big(u(\zeta)\big) v(\zeta). \tag{2.5}$$

We are interested in situations where (2.4) admits a nondegenerate solution, as is made precise in the following assumption.

(H$\mathcal{F}$) There exist quantities $k_0 \neq 0$, $\omega_0 \in \mathbb{R}$ and $u_0 \in H^2$ such that $\mathcal{F}(u_0, \omega_0, k_0) = 0$. In addition, the linear operator $\mathcal{L}(u_0, \omega_0, k_0) : H^2 \to H^0$ admits a simple eigenvalue $\lambda = 0$.

Note that $\mathcal{L}(u_0, \omega_0, k_0)u_0' = 0$, hence (H$\mathcal{F}$) implies that $u_0' \notin \mathcal{R}\big(\mathcal{L}(u_0, \omega_0, k_0)\big)$. This observation can be used to establish the existence of a family of wave train solutions, as our first result shows. We remark here that all the lemmas stated in this section will be proved in Section 3.

Lemma 2.1. *Consider the equation (2.4) and suppose that the assumptions (Hg) and (H$\mathcal{F}$) are satisfied. Then there exist an open set $V \subset \mathbb{R}$ that has $k_0 \in V$ and $0 \notin V$, together with two functions $\omega_{\mathrm{nl}} : V \to \mathbb{R}$ and $u : V \to H^{r+2}$, such that for all $k \in V$ we have $\mathcal{F}(u(k), \omega_{\mathrm{nl}}(k), k) = 0$. The function ω_{nl} is C^r-smooth, while for every integer $0 \leq \ell \leq k$, the map u is C^ℓ smooth when viewed as a map into $H^{r+2-\ell}$. The pairs (ω, u) thus obtained are locally unique up to translations. In particular, if $\mathcal{F}(u, \omega, k) = 0$ for some k near k_0, ω near $\omega_{\mathrm{nl}}(k)$ and u orbitally close to $u(k)$, then $\omega = \omega_{\mathrm{nl}}(k)$ and $u = T_\vartheta u(k)$ for some $\vartheta \in \mathbb{R}$.*

The result above allows us to define the so-called group velocity that is given by $c_g(k) = \omega_{\mathrm{nl}}'(k)$, together with the phase velocity $c_p(k) = \omega_{\mathrm{nl}}(k)/k$. We will use the shorthands $c_g = c_g(k_0)$ and $c_p = c_p(k_0)$. In addition, throughout the sequel we will often use the notation $u(\zeta; k) = u(k)(\zeta)$.

It is essential to consider the linear stability of the wave train $u(k_0)$. Let us therefore look for solutions to (2.1) that have the special Floquet form

$$y(x,t) = u(\zeta; k_0) + e^{\lambda t} e^{-\nu \zeta / k_0} w(\zeta), \tag{2.6}$$

in which ζ is now given by $\zeta = \omega_0 t - k_0 x$, while w is a 2π-periodic function. Recalling that $u(k_0)$ satisfies (2.4) and ignoring higher order correction terms, we find that w must satisfy the following linear equation,

$$\begin{array}{rcl} \gamma k_0^2 w''(\zeta) & = & [\omega_0 + 2\gamma k_0 \nu] w'(\zeta) + [\lambda - \nu c_p - \gamma \nu^2] w(\zeta) \\ & & - \sum_{j=0}^N A_j e^{n_j \nu} w(\zeta - n_j k_0) - Dg\big(u(\zeta; k_0)\big) w(\zeta). \end{array} \tag{2.7}$$

This can be interpreted as an eigenvalue problem involving the linear operator $\mathcal{L}_{\mathrm{st}}(\nu) : H^2 \to H^0$, that for any $\nu \in \mathbb{C}$ is defined by

$$\mathcal{L}_{\mathrm{st}}(\nu) w = [\nu c_p + \gamma \nu^2 - (\omega_0 + 2\gamma k_0 \nu) D + \gamma k_0^2 D^2] w + \sum_{j=0}^N A_j e^{\nu n_j} T_{-n_j k_0} w + Dg\big(u(\cdot; k_0)\big) w. \tag{2.8}$$

Lemma 2.2. *For each $\nu \in \mathbb{C}$, the linear operator $\mathcal{L}_{\mathrm{st}}(\nu) : H^2 \to H^0$ has discrete spectrum. In addition, let $\lambda \in \mathbb{C}$ be a simple eigenvalue of $\mathcal{L}_{\mathrm{st}}(\nu_*)$ for some $\nu_* \in \mathbb{C}$. Then there exists an analytic map $\lambda_* : \nu \mapsto \lambda_*(\nu) \in \mathbb{C}$, defined for ν sufficiently close to ν_*, such that $\lambda_*(\nu)$ is a simple eigenvalue for $\mathcal{L}_{\mathrm{st}}(\nu)$. In addition, if $\mathcal{L}_{\mathrm{st}}(\nu)$ has an eigenvalue $\widetilde{\lambda}$ that is sufficiently close to λ for some ν sufficiently close to ν_*, then we must have $\widetilde{\lambda} = \lambda_*(\nu)$.*

Observe that whenever (H$\mathcal{F}$) is satisfied, the operator $\mathcal{L}_{\mathrm{st}}(0)$ admits a simple eigenvalue $\lambda = 0$. We may therefore introduce the so-called linear dispersion function $\lambda_{\mathrm{lin}}(\nu) = \lambda_*(\nu)$, which tracks this eigenvalue in the spirit of Lemma 2.2 as ν is varied. For our purposes in this paper, we will need to assume that both the nonlinear and the linear dispersion functions are non-degenerate in the following sense.

(HD) The linear dispersion satisfies $\lambda_{\mathrm{lin}}''(0) \neq 0$, while the nonlinear dispersion satisfies $\omega_{\mathrm{nl}}''(k_0) \neq 0$.

As in [17], we will need to exclude Floquet solutions for which the phase speed of the modulation is equal to the group velocity c_g. Turning back to (2.6), this means that we need to exclude solutions of the special form

$$y(x,t) = u(\zeta; k_0) + e^{i(\omega t - \nu x)} w(\zeta) \tag{2.9}$$

that have $\omega/\nu = c_g$. Our next condition guarantees that this is the case.

(HL) For every $z \in i\mathbb{R} \setminus \{0\}$ and every $\lambda \in \operatorname{pointspec} \mathcal{L}_{\rm st}(z)$, we have $\lambda \neq (c_p - c_g) z$.

In this paper, we will be interested in solutions to our main equation (2.1) that can be said to connect two wave train solutions with nearby wave numbers. In particular, we fix a speed c_* and a temporal frequency ω_* and introduce the new variables $\xi = x - c_* t$ and $\tau = \omega_* t$. We seek solutions of the form $y(x,t) = u_*(\xi, \tau)$, that have $u_*(\xi, \tau + 2\pi) = u_*(\xi, \tau)$ for all ξ and τ. After fixing two asymptotic wave numbers k_- and k_+, we require the limits

$$\begin{array}{l} \Big\| u_*(\xi, \cdot) - u\big(\frac{\omega(k_\pm) - k_\pm c_*}{\omega_*} \cdot - k_\pm \xi + \vartheta_\pm ; k_\pm \big) \Big\|_{H^2} \\ + \Big\| \partial_\xi u_*(\xi, \cdot) + k_\pm u'\big(\frac{\omega(k_\pm) - k_\pm c_*}{\omega_*} \cdot - k_\pm \xi + \vartheta_\pm ; k_\pm \big) \Big\|_{H^1} \qquad \to 0 \end{array} \tag{2.10}$$

to hold as $\xi \to \pm\infty$, for some pair of asymptotic phases $\vartheta_\pm \in \mathbb{R}$. Roughly speaking, this means that on bounded time intervals we have $y(x,t) \to u\big(\omega(k_\pm)t - k_\pm x + \vartheta_\pm ; k_\pm\big)$ as $x \to \pm\infty$. We recall here that the parameters $k_\pm$ and (c_*, ω_*) cannot be chosen arbitrarily. Indeed, after fixing the asymptotic wave numbers $k_\pm$, the speed c_* and frequency ω_* are fixed by the Rankine-Hugoniot conditions

$$\begin{array}{lllll} c_* & = & c_*(k_-, k_+) & = & \frac{\omega_{\rm nl}(k_+) - \omega_{\rm nl}(k_-)}{k_+ - k_-}, \\ \omega_* & = & \omega_*(k_-, k_+) & = & \frac{k_+ \omega_{\rm nl}(k_-) - k_- \omega_{\rm nl}(k_+)}{k_+ - k_-}, \end{array} \tag{2.11}$$

which follow directly from our requirement of periodicity in τ. Note that $c_* \to c_g$ and $\omega_* \to k_0(c_p - c_g)$ as $k_\pm \to k_0$.

We remark that up to this point, our setup does not differ significantly from the approach in [17]. Before we can state our main theorem however, we will need to introduce two conditions that are specifically related to the discrete nature of (2.1). As a preparation, we note here that throughout this paper we will overload the notation H^s and use it to refer to both the Hilbert spaces H^s introduced previously as well as their complex-valued counterparts $H^s_{\mathbb{C}}$ that are given by

$$H^s_{\mathbb{C}} = \{ v \in L^2_{\rm per}([0, 2\pi], \mathbb{C}^n) \mid D^\ell v \in L^2_{\rm per}([0, 2\pi], \mathbb{C}^n) \text{ for } 1 \le \ell \le s \}. \tag{2.12}$$

The details should be clear from the context.

We are now ready to introduce the linear operator $\mathcal{T}(z) : H^2 \times H^1 \to H^1 \times H^0$ for $z \in \mathbb{C}$, that in matrix form is given by

$$\mathcal{T}(z) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(z + \frac{1}{\gamma} c_g - k_0 D) & 1 \end{pmatrix} \begin{pmatrix} -\gamma z + \gamma k_0 D & \gamma \\ \mathcal{L}_{\rm ch}(z) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.13}$$

in which the characteristic operator $\mathcal{L}_{\rm ch}(z) : H^2 \to H^0$ acts as

$$\mathcal{L}_{\rm ch}(z) v \quad = \quad [-\mathcal{L}_{\rm st}(z) + z(c_p - c_g)] v. \tag{2.14}$$

We hence see that the operator $\mathcal{T}(z)$ is closely related to $\mathcal{L}_{\rm st}(z)$. It arises when studying the variational equation that u_* must satisfy with $c_* = c_g$. After writing $\mathbf{u}_0 = \big(u(k_0), -k_0 u'(k_0)\big)$, it is not hard to see that $\mathcal{T}(0)\mathbf{u}_0' = 0$. We will need to make the following two assumptions concerning $\mathcal{T}$.

(H$\mathcal{T}$1) We have $\langle \mathbf{u}_0', \mathcal{T}'(0)\mathbf{u}_0' \rangle \neq 0$ and $\langle \mathbf{u}_0', \mathcal{T}(i\kappa)\mathbf{u}_0' \rangle \neq 0$ for $\kappa \in \mathbb{R} \setminus \{0\}$, where the inner product is the one on $H^1 \times H^0$.

(H$\mathcal{T}$2) Let $\mathbf{u}_1 \in H^2 \times H^1$ be the uniquely defined function that has $\langle \mathbf{u}_0', \mathbf{u}_1 \rangle_{H^1 \times H^0} = 0$ and $\mathcal{T}(0)\mathbf{u}_1 = -\mathcal{T}'(0)\mathbf{u}_0'$. In addition, define

$$\begin{array}{lll} \Delta(z) & = & -\gamma z \, \|\mathbf{u}_1\|^2_{H^1 \times H^0} \, \langle \mathbf{u}_0', \mathcal{T}(z)\mathbf{u}_0' \rangle - \langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0' \rangle \langle \mathbf{u}_0', \mathcal{T}(z)\mathbf{u}_0' \rangle \\ & & + \langle \mathbf{u}_0', \mathcal{T}'(0)\mathbf{u}_0' \rangle \langle \mathbf{u}_1, \mathcal{T}(z)\mathbf{u}_0' \rangle, \end{array} \tag{2.15}$$

in which the inner products are the ones on $H^1 \times H^0$. Then we have $\Delta(i\kappa) \neq 0$ for all $\kappa \in \mathbb{R} \setminus \{0\}$ and $\Delta''(0) \neq 0$.

The choice $c_* = c_g$ is crucial to ensure that the function $\mathbf{u}_1$ mentioned in (H$\mathcal{T}$2) exists, as will become clear in Section 3. Roughly speaking, (H$\mathcal{T}$1) ensures that (1.24a) can be solved uniquely for prescribed u, while (H$\mathcal{T}$2) is needed to ensure well-posedness of the flow on the two dimensional center manifold.

The following result hints at the intricate relation between (H$\mathcal{T}$1) and the geometry of the lattice. We emphasize here that the criteria are far from being sharp, but already exhibit a large class of systems for which (H$\mathcal{T}$1) is satisfied.

Lemma 2.3. *Consider a scalar version of (2.1) that can be written in the form*

$$\partial_t y(x,t) = \gamma \partial_{xx} y(x,t) + \sum_{j=0}^{N} [A_j^+ y(x+j,t) + A_j^- y(x-j,t)] + g\big(y(x,t)\big), \tag{2.16}$$

in which $A_j^\pm \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that (Hg), (H$\mathcal{F}$) and (HD) are satisfied. Then there exists a constant γ_* that depends only on the set $\{A_j^\pm\}_{j=1}^N$, such that (H$\mathcal{T}$1) is satisfied if $|\gamma| \geq \gamma_*$ and $A_j^+ = A_j^-$ for $1 \leq j \leq N$. If on the other hand $A_j^+ = -A_j^-$ for $1 \leq j \leq N$, then (H$\mathcal{T}$1) is satisfied for every $\gamma > 0$.

The next result concerns the validity of (H$\mathcal{T}$1) and (H$\mathcal{T}$2) for the scalar system

$$\partial_t y(x,t) = \gamma \partial_{xx} y(x,t) + \alpha^{-2}[y(x-1,t) + y(x+1,t) - 2y(x,t)] + g\big(y(x,t)\big), \tag{2.17}$$

in which α should be seen as a small parameter. Such a system arises when studying the PDE

$$u_t = \gamma \alpha^2 u_{xx} + u_{xx} + g(u) \tag{2.18}$$

and replacing the second Laplacian by its discrete counterpart, posed on a lattice with internode distance α. The relation between y and u in this context is given by $y(x,t) = u(\alpha x, t)$. Under the assumption that the PDE (2.18) with $\alpha = 0$ admits wave train solutions that survive under the discretization (2.17), the result shows that (H$\mathcal{T}$1) and (H$\mathcal{T}$2) are satisfied for small α. We emphasize that in this situation the non-local terms are large relative to γ and the nonlinearity f, showing that our results do not merely cover small discrete perturbations to reaction-diffusion systems.

Lemma 2.4. *Consider the lattice equation (2.17) and the associated PDE (2.18), in which the nonlinearity g is at least C^5-smooth. Assume that for some $\widetilde{k}_0 \neq 0$, the PDE (2.18) with $\alpha = 0$ admits for $\widetilde{k}$ near $\widetilde{k}_0$ the real valued wave solutions $u = \widetilde{u}(\widetilde{\omega}(\widetilde{k})t - \widetilde{k}x; \widetilde{k})$. Assume furthermore that for sufficiently small $\alpha > 0$, the lattice equation (2.17) satisfies the assumptions (Hg), (H$\mathcal{F}$) and (HD) with $k_0 = \alpha \widetilde{k}_0$ and write $u(\widetilde{k}, \alpha)$ and $\omega(\widetilde{k}, \alpha)$ for the branches introduced in Lemma 2.1 with $k = \alpha \widetilde{k}$. Assume finally that the following properties hold.*

- *(i) The limit $u(\widetilde{k}_0, \alpha) \to \widetilde{u}(\widetilde{k}_0) \in H^2$ holds as $\alpha \to 0$.*
- *(ii) The wave trains for (2.18) have been normalized in such a way that $\langle \widetilde{u}_1, \widetilde{u}'(\widetilde{k}_0) \rangle_{H^1} = 0$, in which $\widetilde{u}_1 = -D_{\widetilde{k}} \widetilde{u}(\widetilde{k}_0)$.*
- *(iii) The limit $-D_{\widetilde{k}} u(\widetilde{k}_0, \alpha) \to \widetilde{u}_1 \in H^1$ holds as $\alpha \to 0$, where $\widetilde{u}_1$ is as introduced in (ii).*
- *(iv) We have the inequality $\langle \widetilde{u}_1', \widetilde{u}''(\widetilde{k}_0) \rangle_{H^0} \neq 0$, in which $\widetilde{u}_1$ is as introduced in (ii).*

Then (H$\mathcal{T}$2) is satisfied for all sufficiently small $\alpha > 0$. In addition, if $|\gamma| \geq 2\widetilde{k}_0^2$, then also (H$\mathcal{T}$1) is satisfied for all sufficiently small $\alpha > 0$.

Condition (iv) in the result above is a technical condition that considerably eases the relevant computations. If it fails however, it can be replaced by a more involved condition involving higher order derivatives.

We are now ready to state the main result of this paper, which transfers the result in [17] to the lattice setting.

Theorem 2.5. *Consider the nonlinear system (2.1) and suppose that (Hg), (H$\mathcal{F}$), (HD), (HL), (H$\mathcal{T}$1) and (H$\mathcal{T}$2) are all satisfied. Then for every wave number $k_1 < k_0$ that is sufficiently close to k_0, there exist a complementary wave number $k_2 > k_0$ such that $c_g = c_*(k_1, k_2)$. In addition, there exists a bounded function $u_* = u_*(k_1, k_2) \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1)$ such that the following two properties are satisfied.*

(i) Recalling the quantities c_* and ω_* defined in (2.11) and writing $y(x,t) = u_*(x - c_*t, \omega_*t)$, the function y satisfies (2.1).

(ii) There exist constants $\vartheta_\pm \in \mathbb{R}$ such that u_* satisfies the asymptotics (2.10), with $k_- = k_1$ and $k_+ = k_2$ if $\omega''_{\mathrm{nl}}(k_0)\lambda''_{\mathrm{lin}}(0) < 0$, and with $k_- = k_2$ and $k_+ = k_1$ if $\omega''_{\mathrm{nl}}(k_0)\lambda''_{\mathrm{lin}}(0) > 0$.

Furthermore, there exists a constant $\epsilon > 0$ such that the following holds true. Let $u \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1)$ satisfy (i) and (ii) with u_* replaced by u. Suppose furthermore that for some $\theta \in C^1(\mathbb{R}, \mathbb{R})$ with $\sup_{\xi\in\mathbb{R}} |\theta'(\xi)| < \infty$, the function u can be written in the form

$$\begin{array}{lcl} u(\xi,\tau) & = & u\big(\theta(\xi) + \tau - k_0\xi; k_0\big) + v_1(\xi,\tau), \\ \partial_\xi u(\xi,\tau) & = & -k_0 u'\big(\theta(\xi) + \tau - k_0\xi; k_0\big) + v_2(\xi,\tau), \end{array} \tag{2.19}$$

in which the pair (v_1, v_2) satisfies the orthogonality condition

$$\big\langle u'\big(\theta(\xi) - k_0\xi + \cdot\,; k_0\big), v_1(\xi,\cdot)\big\rangle_{H^1} - k_0\big\langle u''\big(\theta(\xi) - k_0\xi + \cdot\,; k_0\big), v_2(\xi,\cdot)\big\rangle_{H^0} = 0 \tag{2.20}$$

for all $\xi \in \mathbb{R}$. Finally, suppose that the pair (v_1, v_2) is small in the sense that

$$\sup_{\xi\in\mathbb{R}} \|v_1(\xi,\cdot)\|_{H^2} + \sup_{\xi\in\mathbb{R}} \|v_2(\xi,\cdot)\|_{H^1} \quad < \quad \epsilon. \tag{2.21}$$

Then u is a translate of u_*, i.e., for some $\vartheta \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}$ we have

$$u(\xi,\tau) = u_*(\xi_0 + \xi, \vartheta + \tau) \tag{2.22}$$

for all $\xi \in \mathbb{R}$ and $\tau \in [0, 2\pi]$, after periodically extending u_* in the second variable.

3 Notation and Proofs of Lemmas 2.1 through 2.4

In this section we introduce the notation we will use throughout this paper. In addition, we investigate the operators $\mathcal{F}$ and $\mathcal{T}(z)$ introduced in (2.4) and (2.13) and establish Lemmas 2.1 through 2.4.

We start by defining, for any Hilbert space H, the family of Banach spaces

$$BC_\eta(\mathbb{R}, H) = \{x \in C(\mathbb{R}, H) \mid \|x\|_\eta := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}\,|x(\xi)|_H < \infty\}, \tag{3.1}$$

parametrized by $\eta \in \mathbb{R}$. We also recall the Fourier transform $\widehat{f}(\eta)$ and the inverse Fourier transform $\check{f}(\xi)$ of a function $f \in L^2(\mathbb{R}, H)$, that are given by

$$\widehat{f}(\eta) = \textstyle\int_{-\infty}^{\infty} e^{-i\eta\xi} f(\xi)d\xi, \qquad \check{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi} f(\eta)d\eta. \tag{3.2}$$

We remark here that the integrals above are well-defined only if $f \in L^1(\mathbb{R}, H)$. If this is not the case, the integrals have to be understood as integrals in the Fourier sense, i.e., the functions

$$h_n(k) = \int_{-n}^{n} e^{-ik\xi} f(\xi)d\xi \tag{3.3}$$

satisfy $h_n \to \hat{f}$ in $L^2(\mathbb{R}, H)$ and in addition there is a subsequence $\{n'\}$ such that $h_{n'}(k) \to \widehat{f}(k)$ almost everywhere. We recall that the Fourier transform takes convolutions into products, i.e., $\widehat{(f * g)}(\eta) = \widehat{f}(\eta)\widehat{g}(\eta)$ for almost every η.

Now suppose that $f : \mathbb{R} \to H$ satisfies $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$. Then for any z with $\operatorname{Re} z > -a$, define the Laplace transform

$$\widetilde{f}_+(z) = \int_0^\infty e^{-z\xi} f(\xi) d\xi. \tag{3.4}$$

Similarly, if $f(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then for any z with $\operatorname{Re} z < b$, define

$$\widetilde{f}_-(z) = \int_0^\infty e^{z\xi} f(-\xi) d\xi. \tag{3.5}$$

The inverse transformation is described in the next result, which can be found in the standard Laplace transform literature [58, 7.3-5].

Lemma 3.1. *Let $f : \mathbb{R} \to H$ satisfy a growth condition $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ and suppose that f is of bounded variation on bounded intervals. Then for any $\gamma > -a$ and $\xi > 0$ we have the inversion formula*

$$\frac{f(\xi+) + f(\xi-)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz, \tag{3.6}$$

whereas for $\xi = 0$ we have

$$\frac{f(0+)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz. \tag{3.7}$$

After these definitions, we are ready to proceed to the study of the operator $\mathcal{F}$ defined in (2.4). We prove Lemma 2.1 by repeatedly appealing to the implicit function theorem.

Proof of Lemma 2.1. Notice first that $\mathcal{F} : H^2 \times \mathbb{R} \times \mathbb{R} \to H^0$ does not depend C^r-smoothly on k, due to the presence of this variable in the shifts of the argument of u. This precludes a direct application of the implicit function theorem. Instead, we will consider $\mathcal{F}$ as an operator from $H^{r+1} \times \mathbb{R} \times \mathbb{R}$ into H^{r-1}. In this setting, it is not hard to see that $\mathcal{F}$ is C^2-smooth with respect to the variable k and C^1-smooth with respect to the first variable u. Observe also that $D_1\mathcal{F}(u, \omega, k) = \mathcal{L}(u, \omega, k)$, in which the latter map is viewed as an element in the space $\mathcal{L}(H^{r+1}, H^{r-1})$. The differentiation map $v \mapsto v''$ from H^{r+1} into H^{r-1} is Fredholm with index zero, while the single differentiation $v \mapsto v'$ from H^{r+1} into H^{r-1} and the inclusion $H^{r+1} \subset H^{r-1}$ are compact. In particular, this implies that also $\mathcal{L}(u_0, \omega_0, k_0)$ is Fredholm with index zero. Due to (H$\mathcal{F}$), we know that zero is a simple eigenvalue for $\mathcal{L}(u_0, \omega_0, k_0)$, which means that $u_0' \notin \mathcal{R}(\mathcal{L}(u_0, \omega_0, k_0))$. This implies that the linear map $D_{1,2}\mathcal{F}(u_0, \omega_0, k_0) : H^{r+1} \times \mathbb{R} \to H^{r-1}$ has full range. Using the implicit function theorem, we thus obtain a C^1-smooth branch $u : V \to H^{r+1}$ and $\omega : V \to \mathbb{R}$ of solutions to the equation $\mathcal{F}(u, \omega, k) = 0$, which is locally unique up to translations in u.

Using the identity (2.4) one may easily establish that $k \mapsto u(k) \in H^{r+2}$ is C^0-smooth. In addition, the continuous derivatives $D_k u : V \to H^{r+1}$ and $D_k\omega : V \to \mathbb{R}$ satisfy the equation

$$\mathcal{F}^1(D_k u, D_k \omega, k) := \mathcal{L}(u(k), \omega(k), k) D_k u - 2\gamma k u''(k) + u'(k) D_k \omega + \sum_{j=0}^{N} A_j n_j T_{-n_j k} u'(k) = 0, \tag{3.8}$$

in which $\mathcal{F}^1$ is a map from $H^{r+1} \times \mathbb{R} \times \mathbb{R}$ into H^{r-1}. Note that $\mathcal{F}^1$ is now not C^1-smooth in the third variable, due to the presence of the term involving $Dg(u(\cdot; k)) D_k u$. However, $\mathcal{F}^1$ recovers its C^1-smoothness when considered as a map from $H^r \times \mathbb{R} \times \mathbb{R}$ into H^{r-2}. Arguing as above, one may establish that $D_{1,2}\mathcal{F}^1(k_0)$ has full range in H^{r-2} and hence apply the implicit function theorem to show that $k \mapsto D_k u \in H^r$ and $k \mapsto D_k \omega \in \mathbb{R}$ are C^1-smooth. One may now complete the proof by repeating this argument a sufficient number of times. ∎

It is now time to investigate the relation between $\mathcal{T}(z)$ and $\mathcal{L}_{\rm st}$. It will be convenient to explicitly include the dependence on ω into the definitions of $\mathcal{T}$, $\mathcal{L}_{\rm ch}$ and $\mathcal{L}_{\rm st}$. To this end, let us write

$$\begin{array}{lcl} \mathcal{L}_{\rm st}(\omega,\nu)w & = & [\nu c_p + \gamma\nu^2 - (\omega + 2\gamma k_0 \nu)D + \gamma k_0^2 D^2]w \\ & & + \sum_{j=0}^{N} A_j e^{\nu n_j} T_{-n_j k_0} w + Dg\big(u(\cdot;k_0)\big)w, \\ \mathcal{L}_{\rm ch}(\omega,\nu)w & = & [-\mathcal{L}_{\rm st}(\omega,\nu) + \nu(c_p - c_g)]w \end{array} \tag{3.9}$$

and define $\mathcal{T}_\omega(z)$ according to (2.13) with $\mathcal{L}_{\rm ch} = \mathcal{L}_{\rm ch}(\omega,\cdot)$. The following result extends Lemma 2.2 and will be used to bound the length of Jordan chains associated to $\mathcal{T}_\omega(z)$ for ω near ω_0 and z near zero. We remark that Jordan chains in this nonlinear setting are defined as in [15, Section IV.4].

Lemma 3.2. *For each $\omega \in \mathbb{R}$ and $\nu \in \mathbb{C}$, the linear operator $\mathcal{L}_{\rm st}(\omega,\nu) : H^2 \to H^0$ has discrete spectrum. In addition, let $\lambda \in \mathbb{C}$ be a simple eigenvalue of $\mathcal{L}_{\rm st}(\omega_*,\nu_*)$ for some pair $\omega_* \in \mathbb{R}$ and $\nu_* \in \mathbb{C}$. Then there exist a neighbourhood $V \subset \mathbb{R}\times\mathbb{C}$ with $(\omega_*,\nu_*) \in V$, a C^∞-smooth map $\lambda_* : V \to \mathbb{C}$ together with a C^∞-smooth map $w_{\rm st} : V \to H^2$, such that for all pairs $(\omega,\nu) \in V$ the following identity holds,*

$$\mathcal{L}_{\rm st}(\omega,\nu)w_{\rm st}(\omega,\nu) = \lambda_*(\omega,\nu)w_{\rm st}(\omega,\nu). \tag{3.10}$$

The map $\nu \mapsto \lambda_*(\omega,\nu)$ is analytic where it is defined, while for every $(\omega,\nu) \in V$, the eigenvalue $\lambda_*(\omega,\nu)$ is simple. In addition, if $\mathcal{L}_{\rm st}(\omega,\nu)$ has an eigenvalue $\widetilde{\lambda}$ that is sufficiently close to λ for some pair $(\omega,\nu) \in V$, then we must have $\widetilde{\lambda} = \lambda_*(\omega,\nu)$. Finally, for any $(\omega,\nu) \in V$, the length of a maximal Jordan chain for $\mathcal{T}_\omega(z)$ around $z = \nu$ is equal to the algebraic multiplicity of $z = \nu$ as a root of the function $z \mapsto \lambda_*(\omega,z) - z(c_p - c_g)$.

Proof. As in the proof of Lemma 2.1, one may argue that the linear map $\mathcal{L}_{\rm st}(\omega,\nu) : H^2 \to H^0$ is Fredholm with index zero. Due to the compact embedding $H^2 \hookrightarrow H^0$, the resolvent $[\lambda - \mathcal{L}_{\rm st}(\omega,\nu)]^{-1}$ is compact when viewed as a map from $H^0 \to H^0$. This in turn implies that the spectrum of $\mathcal{L}_{\rm st}(\omega,\nu)$ is discrete. Whenever λ is a simple eigenvalue for $\mathcal{L}_{\rm st}(\omega_*,\nu_*)$ for some pair $(\omega_*,\nu_*) \in \mathbb{R}\times\mathbb{C}$, the existence of the smooth branches λ_* and $w_{\rm st}$ follows from a standard application of the implicit function theorem.

To establish the final claim involving the Jordan chains of $\mathcal{T}_\omega$, we observe that there is a one to one correspondence between the Jordan chains of $\mathcal{T}_\omega$ and those of $\mathcal{L}_{\rm ch}(\omega,\cdot)$. To see this, suppose that $\mathcal{T}_\omega$ admits a Jordan chain of length ℓ at $z = z_0$. Referring back to (2.13), this means that there exist $v_1^i \in H^2$ and $v_2^i \in H^1$ for integers $0 \le i \le \ell - 1$, such that

$$\big|(z - k_0 D)[v_1^0 + \ldots + (z-z_0)^{\ell-1}v_1^{\ell-1}] - [v_2^0 + \ldots + (z-z_0)^{\ell-1}v_2^{\ell-1}]\big|_{H^1} = O(|z-z_0|^\ell) \tag{3.11}$$

as $z \to z_0$. Plugging this into (2.13), we see that we must also have

$$\big|\mathcal{L}_{\rm ch}(\omega,z)[v_1^0 + \ldots + (z-z_0)^{\ell-1}v_1^{\ell-1}]\big|_{H^0} = O(|z-z_0|^\ell) \tag{3.12}$$

as $z \to z_0$, implying that also $\mathcal{L}_{\rm ch}(\omega,\cdot)$ has a Jordan chain of length ℓ at $z = z_0$. Conversely, any such Jordan chain for $\mathcal{L}_{\rm ch}(\omega,\cdot)$ can be expanded into a Jordan chain for $\mathcal{T}_\omega$, since one can always choose v_2^i in such a way that (3.11) is satisfied.

It therefore suffices to consider the Jordan chains for $\mathcal{L}_{\rm ch}(\omega,\cdot)$. We recall that

$$\mathcal{L}_{\rm ch}(\omega,z) = -\mathcal{L}_{\rm st}(\omega,z) + z(c_p - c_g). \tag{3.13}$$

For convenience, let us also write

$$\mu(\omega,z) = \lambda_*(\omega,z) - z(c_p - c_g). \tag{3.14}$$

Repeated differentiation of the identity (3.10) and substitution of (3.13) shows that for any integer $\ell \geq 0$ and all $(\omega, z) \in V$, we have

$$-\sum_{i=0}^{\ell} \binom{\ell}{i} D_2^i \mathcal{L}_{\rm ch}(\omega, z) D_2^{\ell-i} w_{\rm st}(\omega, z) = \sum_{i=0}^{\ell} \binom{\ell}{i} D_2^i \mu(\omega, z) D_2^{\ell-i} w_{\rm st}(\omega, z). \tag{3.15}$$

We now fix a pair $(\omega, \nu) \in V$ and let m denote the algebraic multiplicity of the root $z = \nu$ of $\mu(\omega, z) = 0$. If $m = 0$, then we have $\operatorname{Ker} \mathcal{L}_{\rm ch}(\omega, \nu) = \{0\}$ and hence we are done. Let us therefore assume that $m > 0$ and hence $\lambda(\omega, \nu) = \nu(c_p - c_g)$. Observe that the right hand side of (3.15) with $z = \nu$ vanishes for $0 \leq \ell \leq m-1$, showing that $\mathcal{L}_{\rm ch}(\omega, z)$ indeed admits a Jordan chain of length m for z near ν. However, inserting $\ell = m$ leads to

$$-\sum_{i=0}^{m} \binom{m}{i} D_2^i \mathcal{L}_{\rm ch}(\omega, \nu) D_2^{m-i} w_{\rm st}(\omega, \nu) = \kappa w_{\rm st}(\omega, \nu) \tag{3.16}$$

for some nonzero κ. Since $\lambda_*(\omega, \nu)$ is a simple eigenvalue, there is no $v \in H^2$ with

$$\mathcal{L}_{\rm ch}(\omega, \nu) v = [\mathcal{L}_{\rm st}(\omega, \nu) - \lambda_*(\omega, \nu)] v = w_{\rm st}(\omega, \nu). \tag{3.17}$$

This shows that the Jordan chain of length m constructed above cannot be extended to a Jordan chain of length $m+1$. Let us now consider any other Jordan chain $\{w_{\rm st}(\omega, \nu), \widetilde{w}_1, \ldots, \widetilde{w}_{m-1}\}$ of length m. For every $1 \leq i \leq m-1$ we must then have

$$\widetilde{w}_i = D_2^i w_{\rm st}(\omega, \nu) + \sum_{j=0}^{i-1} \alpha_i D_2^j w_{\rm st}(\omega, \nu), \tag{3.18}$$

for suitable coefficients $\alpha_i \in \mathbb{C}$. It is now not hard to verify that in fact no Jordan chain with length $m+1$ can exist for $\mathcal{L}_{\rm ch}(\omega, z)$ around $z = \nu$, which completes the proof. □

With Lemma 3.2 at hand, we are ready to explicitly construct a maximal Jordan chain for $\mathcal{L}_{\rm ch}(z) = \mathcal{L}_{\rm ch}(\omega_0, z)$ around $z = 0$. An easy computation shows that $\mathcal{L}_{\rm ch}(0) u'(k_0) = 0$. To proceed, we compute

$$-\mathcal{L}_{\rm ch}'(0) u'(k_0) = -2\gamma k_0 u''(k_0) + c_g u'(k_0) + \sum_{j=0}^{N} n_j A_j T_{-n_j k_0} u'(k_0). \tag{3.19}$$

On the other hand, using (3.8) and recalling $c_g = D_k \omega_{\rm nl}(k_0)$, we obtain

$$\mathcal{L}_{\rm ch}(0) D_k u(k_0) = 2\gamma k_0 u''(k_0) - c_g u'(k_0) - \sum_{j=0}^{N} n_j A_j T_{-n_j k_0} u'(k_0). \tag{3.20}$$

We thus find the Jordan chain $\{u'(k_0), -D_k u(k_0)\}$ for $\mathcal{L}_{\rm ch}(z)$ around $z = 0$. The corresponding Jordan chain for $\mathcal{T}(z)$ is now given by $\{\mathbf{u}_0', \mathbf{u}_1\}$, with

$$\begin{array}{rcl} \mathbf{u}_0' & = & \big(u'(k_0), -k_0 u''(k_0)\big), \\ \mathbf{u}_1 & = & \big(-D_k u(k_0), u'(k_0) + k_0 D_k u'(k_0)\big). \end{array} \tag{3.21}$$

Using Lemma 3.2, we may conclude $\lambda_{\rm lin}'(0) = c_p - c_g$. In addition, using this lemma in combination with (HD), we find that the Jordan chain $\{\mathbf{u}_0', \mathbf{u}_1\}$ thus constructed cannot be extended further. Notice however that the branch $u(k)$ can be modified by picking $\alpha \in \mathbb{R}$ and writing $\widetilde{u}(k) = T_{\alpha k} u(k)$. This yields $D_k \widetilde{u}(k_0) = \alpha u'(k_0) + D_k u(k_0)$. By choosing α appropriately, we can therefore ensure that the orthogonality condition $\langle \mathbf{u}_0', \mathbf{u}_1 \rangle_{H^1 \times H^0} = 0$ holds.

For future reference, we shall also derive an expression for $\frac{1}{2}\mathcal{T}''(0)\mathbf{u}_0' + \mathcal{T}'(0)\mathbf{u}_1$. We proceed by differentiating the identity

$$[z(c_p - c_g) - \mathcal{L}_{\mathrm{ch}}(z)]w_{\mathrm{st}}(\omega_0, z) = \lambda_{\mathrm{lin}}(z) w_{\mathrm{st}}(\omega_0, z). \tag{3.22}$$

Using $\lambda_{\mathrm{lin}}'(0) = c_p - c_g$, a single differentiation of (3.22) yields $-\mathcal{L}_{\mathrm{ch}}(0)w_{\mathrm{st}}(\omega_0, 0) = \mathcal{L}_{\mathrm{ch}}(0)D_2 w_{\mathrm{st}}(\omega_0, 0)$, which implies that for some $\alpha \in \mathbb{C}$ we have $D_2 w_{\mathrm{st}}(\omega_0, 0) = -D_k u(k_0) + \alpha u'(k_0)$. A further differentiation yields

$$\lambda_{\mathrm{lin}}''(0)u'(k_0) = -\mathcal{L}_{\mathrm{ch}}''(0)u'(k_0) + 2\mathcal{L}_{\mathrm{ch}}'(0)D_k u(k_0) - 2\alpha\mathcal{L}_{\mathrm{ch}}'(0)u'(k_0) - \mathcal{L}_{\mathrm{ch}}(0)D_2^2 w_{\mathrm{st}}(\omega_0, 0). \tag{3.23}$$

After a short calculation one may now verify that

$$\begin{array}{rcl} \frac{1}{2}\mathcal{T}''(0)\mathbf{u}_0' + \mathcal{T}'(0)\mathbf{u}_1 & = & -\frac{1}{2}\lambda_{\mathrm{lin}}''(0)\big(0, u'(k_0)\big) - \mathcal{T}(0)\big(\alpha D_k u(k_0), \alpha k_0 D_k u'(k_0)\big) \\ & & +\mathcal{T}(0)\big(0, D_k u(k_0)\big) - \frac{1}{2}\mathcal{T}(0)\big(D_2^2 w_{\mathrm{st}}(\omega_0, 0), -k_0 D_2^2 w_{\mathrm{st}}'(\omega_0, 0)\big). \end{array} \tag{3.24}$$

We conclude this section by establishing Lemmas 2.3 and 2.4, which give conditions under which (H$\mathcal{T}$1) and (H$\mathcal{T}$2) are satisfied.

Proof of Lemma 2.3. We may use $\mathcal{T}(0)\mathbf{u}_0' = 0$ to calculate

$$\begin{array}{rcl} \langle \mathbf{u}_0', \mathcal{T}(z)\mathbf{u}_0' \rangle_{H^1 \times H^0} & = & \langle \mathbf{u}_0', [\mathcal{T}(z) - \mathcal{T}(0)]\mathbf{u}_0' \rangle_{H^1 \times H^0} \\ & = & -\gamma z \langle \mathbf{u}_0', \mathbf{u}_0' \rangle_{H^1 \times H^0} + k_0 \sum_{j=0}^N A_j^+ (e^{jz} - 1)\langle u''(k_0), T_{-jk_0} u'(k_0)\rangle_{H^0} \\ & & +k_0 \sum_{j=0}^N A_j^- (e^{-jz} - 1)\langle u''(k_0), T_{jk_0} u'(k_0)\rangle_{H^0}. \end{array} \tag{3.25}$$

Since $u(k_0)$ is real-valued, we may compute

$$\begin{array}{rcl} \langle u''(k_0), T_{jk_0} u'(k_0)\rangle_{H^0} & = & -\langle u'(k_0), T_{jk_0} u''(k_0)\rangle_{H^0} = -\langle T_{-jk_0} u'(k_0), u''(k_0)\rangle_{H^0} \\ & = & -\langle u''(k_0), T_{-jk_0} u'(k_0)\rangle_{H^0}. \end{array} \tag{3.26}$$

In the situation that $A_j := A_j^+ = A_j^-$ for $1 \le j \le N$, we thus find

$$\begin{array}{rcl} \langle \mathbf{u}_0', \mathcal{T}(i\kappa)\mathbf{u}_0' \rangle_{H^1 \times H^0} & = & -\gamma i\kappa \langle \mathbf{u}_0', \mathbf{u}_0' \rangle_{H^1 \times H^0} \\ & & -ik_0 \sum_{j=1}^N 2A_j \langle u''(k_0), T_{jk_0} u'(k_0)\rangle_{H^0} \sin j\kappa. \end{array} \tag{3.27}$$

Since

$$\begin{array}{rcl} 2\,|k_0 \langle u''(k_0), T_{jk_0} u'(k_0)\rangle_{H^0}| & \le & 2\,|k_0|\, \|u''(k_0)\|_{H^0}\, \|u'(k_0)\|_{H^0} \\ & \le & |k_0|^2\, \|u''(k_0)\|_{H^0}^2 + \|u'(k_0)\|_{H^0}^2 \\ & \le & \langle \mathbf{u}_0', \mathbf{u}_0' \rangle_{H^1 \times H^0}, \end{array} \tag{3.28}$$

it is not hard to see that (H$\mathcal{T}$1) is automatically satisfied if $|\gamma| \ge \sum_{j=1}^N j\,|A_j|$.

If on the other hand $A_j := A_j^+ = -A_j^-$ for $1 \le j \le N$, then we may compute

$$\begin{array}{rcl} \langle \mathbf{u}_0', \mathcal{T}(i\kappa)\mathbf{u}_0' \rangle_{H^1 \times H^0} & = & -\gamma i\kappa \langle \mathbf{u}_0', \mathbf{u}_0' \rangle_{H^1 \times H^0} \\ & & -k_0 \sum_{j=1}^N 2A_j \langle u''(k_0), T_{jk_0} u'(k_0)\rangle_{H^0} (\cos j\kappa - 1) \end{array} \tag{3.29}$$

and we see that (H$\mathcal{T}$1) is satisfied for any $\gamma > 0$. □

Proof of Lemma 2.4. Let us first recall the definitions

$$\begin{array}{rcl} \mathbf{u}_0'(\alpha) & = & \big(u'(\widetilde{k}_0, \alpha), -\alpha\widetilde{k}_0 u''(\widetilde{k}_0, \alpha)\big), \\ \mathbf{u}_1(\alpha) & = & \big(u_1(\alpha), u'(\widetilde{k}_0, \alpha) - \alpha\widetilde{k}_0 u_1'(\alpha)\big), \end{array} \tag{3.30}$$

in which $u_1(\alpha) = -\alpha^{-1} D_{\widetilde{k}} u(\widetilde{k}_0, \alpha) + \beta u'(\widetilde{k}_0, \alpha)$, where β is chosen in such a way that

$$\langle \mathbf{u}_1(\alpha), \mathbf{u}_0'(\alpha) \rangle_{H^1 \times H^0} = 0. \tag{3.31}$$

Using (ii) and (iii), we find $\alpha u_1(\alpha) \to \widetilde{u}_1$ as $\alpha \to 0$. Throughout the rest of this proof we will write $\widetilde{u}_0 = \widetilde{u}(\widetilde{k}_0)$ and $k = \widetilde{k}_0$. We compute

$$\mathcal{T}(z, \alpha)\mathbf{u}_0'(\alpha) = -\gamma z \mathbf{u}_0'(\alpha) - \alpha^{-2}\big(0, [(e^z - 1) T_{-\alpha k} u'(k, \alpha) + (e^{-z} - 1) T_{\alpha k} u'(k, \alpha)]\big). \tag{3.32}$$

Furthermore, we find

$$D_1 \mathcal{T}(0, \alpha)\mathbf{u}_0'(\alpha) = -\gamma \mathbf{u}_0'(\alpha) - \alpha^{-2}\big(0, [T_{-\alpha k} u'(k, \alpha) - T_{\alpha k} u'(k, \alpha)]\big). \tag{3.33}$$

Let us write

$$\begin{array}{rcl}
\Gamma(\alpha) & = & \langle \mathbf{u}_1(\alpha), \mathbf{u}_1(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u'(k, \alpha), T_{\alpha k} u'(k, \alpha) \rangle_{H^0} \\
 & = & \langle \mathbf{u}_1(\alpha), \mathbf{u}_1(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u'(k, \alpha), T_{-\alpha k} u'(k, \alpha) \rangle_{H^0}, \\
\Gamma_0^+(\alpha) & = & \langle \mathbf{u}_0'(\alpha), \mathbf{u}_0'(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u''(k, \alpha), T_{\alpha k} u'(k, \alpha) \rangle_{H^0}, \\
\Gamma_0^-(\alpha) & = & \langle \mathbf{u}_0'(\alpha), \mathbf{u}_0'(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u''(k, \alpha), T_{-\alpha k} u'(k, \alpha) \rangle_{H^0}, \\
\Gamma_1^+(\alpha) & = & \langle \mathbf{u}_1(\alpha), \mathbf{u}_1(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u_1'(\alpha), T_{\alpha k} u'(k, \alpha) \rangle_{H^0}, \\
\Gamma_1^-(\alpha) & = & \langle \mathbf{u}_1(\alpha), \mathbf{u}_1(\alpha) \rangle^{-1}_{H^1 \times H^0} \langle u_1'(\alpha), T_{-\alpha k} u'(k, \alpha) \rangle_{H^0}.
\end{array} \tag{3.34}$$

As before, the calculation (3.26) implies that $\Gamma_0^+(\alpha) = -\Gamma_0^-(\alpha)$. For later use, note that in the limit $\alpha \to 0$ we have

$$\begin{array}{lcl}
\alpha^{-2}\Gamma(\alpha) & \to & \langle \widetilde{u}_1, \widetilde{u}_1 \rangle^{-1}_{H^1} \langle \widetilde{u}_0, \widetilde{u}_0 \rangle_{H^0}, \\
\alpha^{-1}[\Gamma_0^+(\alpha) - \Gamma_0^-(\alpha)] & \to & 2k \langle \widetilde{u}_0', \widetilde{u}_0' \rangle^{-1}_{H^1} \langle \widetilde{u}_0'', \widetilde{u}_0'' \rangle_{H^0}, \\
\alpha^{-2}[\Gamma_1^+(\alpha) - \Gamma_1^-(\alpha)] & \to & 2k \langle \widetilde{u}_1, \widetilde{u}_1 \rangle^{-1}_{H^1} \langle \widetilde{u}_1', \widetilde{u}_0'' \rangle_{H^0}, \\
\alpha^{-1}[\Gamma_1^+(\alpha) + \Gamma_1^-(\alpha)] & \to & 2 \langle \widetilde{u}_1, \widetilde{u}_1 \rangle^{-1}_{H^1} \langle \widetilde{u}_1', \widetilde{u}_0' \rangle_{H^0}.
\end{array} \tag{3.35}$$

Let us now drop the dependence on α and compute

$$\begin{array}{rcl}
\langle \mathbf{u}_0', \mathbf{u}_0' \rangle^{-1} \langle \mathbf{u}_0', \mathcal{T}'(0)\mathbf{u}_0' \rangle & = & -\gamma + \alpha^{-1} k [\Gamma_0^- - \Gamma_0^+], \\
\langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{-1} \langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0' \rangle & = & \alpha^{-1} k [\Gamma_1^- - \Gamma_1^+], \\
\langle \mathbf{u}_0', \mathbf{u}_0' \rangle^{-1} \langle \mathbf{u}_0', \mathcal{T}(z)\mathbf{u}_0' \rangle & = & -\gamma z + \alpha^{-1} k [e^z \Gamma_0^- + e^{-z} \Gamma_0^+], \\
\langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{-1} \langle \mathbf{u}_1, \mathcal{T}(z)\mathbf{u}_0' \rangle & = & -\alpha^{-2}(e^z + e^{-z} - 2)\Gamma + \alpha^{-1} k [(e^z - 1)\Gamma_1^- + (e^{-z} - 1)\Gamma_1^+],
\end{array} \tag{3.36}$$

in which the inner products are those on $H^1 \times H^0$. We may now calculate

$$\begin{array}{rcl}
\Delta(z) & = & \gamma^2 z^2 - \gamma \alpha^{-1} k z (e^z \Gamma_0^- + e^{-z} \Gamma_0^+ + \Gamma_1^+ - \Gamma_1^-) \\
 & & + \frac{1}{2} \alpha^{-2} k^2 [\Gamma_0^- - \Gamma_0^+][\Gamma_1^+ + \Gamma_1^-][e^z + e^{-z} - 2] \\
 & & + \gamma \alpha^{-2} (e^z + e^{-z} - 2)\Gamma - \gamma \alpha^{-1} k [(e^z - 1)\Gamma_1^- + (e^{-z} - 1)\Gamma_1^+] \\
 & & - \alpha^{-3} k (\Gamma_0^- - \Gamma_0^+)(e^z + e^{-z} - 2).
\end{array} \tag{3.37}$$

This allows us to write

$$\begin{array}{rcl}
\operatorname{Im} \Delta(i\kappa) & = & -\gamma \alpha^{-1} k \kappa (\Gamma_1^+ - \Gamma_1^-) \\
 & & + \gamma \alpha^{-1} k (\Gamma_1^+ - \Gamma_1^-) \sin \kappa,
\end{array} \tag{3.38}$$

which in view of (3.35) and (iv) shows that $\operatorname{Im} \Delta(i\kappa) \neq 0$ for $\kappa \neq 0$ and sufficiently small α. We also compute

$$\begin{array}{rcl}
\Delta''(0) & = & 2\gamma^2 - 2\gamma k \alpha^{-1} (\Gamma_0^- - \Gamma_0^+) \\
 & & + \alpha^{-2} (\Gamma_0^- - \Gamma_0^+)(\Gamma_1^+ + \Gamma_1^-) \\
 & & - 2\gamma \alpha^{-2} \Gamma - \gamma \alpha^{-1} k (\Gamma_1^+ + \Gamma_1^-) \\
 & & - 2\alpha^{-3} k (\Gamma_0^- - \Gamma_0^+).
\end{array} \tag{3.39}$$

In view of the scalings (3.35), we find that in the limit $\alpha \to 0$ one has

$$\alpha^2 \Delta''(0) \to -4k^2 \langle \widetilde{u}_0', \widetilde{u}_0' \rangle_{H^1}^{-1} \langle \widetilde{u}_0'', \widetilde{u}_0'' \rangle_{H^0}, \tag{3.40}$$

which implies that $\Delta''(0) \neq 0$ whenever α is sufficiently small. This establishes (H$\mathcal{T}$2). To see that (H$\mathcal{T}$1) holds for small α, we write

$$\langle \mathbf{u}_0', \mathbf{u}_0' \rangle_{H^1 \times H^0}^{-1} \langle \mathbf{u}_0', \mathcal{T}(i\kappa) \mathbf{u}_0' \rangle_{H^1 \times H^0} = -\gamma i \kappa - \alpha^{-1} ik(\Gamma_0^+ - \Gamma_0^-) \sin \kappa, \tag{3.41}$$

recall that $|\gamma| \geq 2k^2$ and use (3.35). □

4 Coordinate System near Wave Trains

In this section we study solutions to (2.1) that can be written in the form $y(x,t) = u_*(x - c_* t, \omega_* t)$, in which u_* is 2π-periodic in the second variable. We derive a differential equation for u_* and transform this equation in such a way that a center manifold reduction can be applied. This reduction allows us to capture on a finite dimensional manifold all solutions u_* that remain orbitally close to a wave train solution $y(x,t) = u(\omega_{\rm nl}(k)t - kx; k)$. Our main result Theorem 2.5 can subsequently be read off from a two dimensional ODE that encodes all the relevant dynamics.

We start by substituting the Ansatz $y(x,t) = u_*(x - c_* t, \omega_* t)$ into (2.1) and find

$$-c_* \partial_\xi u(\xi, \tau) + \omega_* \partial_\tau u(\xi, \tau) = \gamma \partial_{\xi\xi} u_*(\xi, \tau) + \sum_{j=0}^{N} A_j u_*(\xi + n_j, \tau) + g\big(u_*(\xi, \tau)\big). \tag{4.1}$$

In order to recover the solution $y(x,t) = u(\omega_0 t - k_0 x; k_0)$, it suffices to choose $c_* = c_g$, $\omega_* = \omega_0 - k_0 c_g$ and write $u_*(\xi, \tau) = u(\tau - k_0 \xi; k_0)$. Since we need to consider functions $u_*(\xi, \tau)$ that remain close to translates of $u(k_0)$, we introduce the new variables $\theta \in C(\mathbb{R}, \mathbb{R})$, $v_1 \in C(\mathbb{R}, H^2)$ and $v_2 \in C(\mathbb{R}, H^1)$ and write

$$\begin{array}{rcl} u_*(\xi, \tau) & = & u\big(\theta(\xi) + \tau - k_0 \xi; k_0\big) + v_1(\xi, \tau), \\ \partial_\xi u_*(\xi, \tau) & = & -k_0 u'\big(\theta(\xi) + \tau - k_0 \xi; k_0\big) + v_2(\xi, \tau). \end{array} \tag{4.2}$$

We note that θ', v_1 and v_2 should be thought of as small functions, but θ itself need to be bounded. To prevent ambiguity, we supplement (4.1) with the pointwise orthogonality condition

$$\big\langle u'\big(\theta(\xi) - k_0 \xi + \cdot\,; k_0\big), v_1(\xi, \cdot) \big\rangle_{H^1} + \big\langle -k_0 u''\big(\theta(\xi) - k_0 \xi + \cdot\,; k_0\big), v_2(\xi, \cdot) \big\rangle_{H^0} = 0. \tag{4.3}$$

Let us now vary ω_* slightly by writing $\omega_* = \omega_0 - k_0 c_g(k_0) + \overline{\omega}$, while keeping $c_* = c_g$ fixed. Since $\omega_{\rm nl}'(k_0) = c_g$, it is not hard to see that for any sufficiently small $\overline{\omega}$ that has $\operatorname{sign} \overline{\omega} = \operatorname{sign} \omega_{\rm nl}''(k_0)$, there exist two wave numbers $k_1 < k_0 < k_2$ such that

$$\omega_{\rm nl}(k_i) - \omega_0 - (k_i - k_0) c_g = \overline{\omega} \tag{4.4}$$

for $i = 1, 2$. Upon writing $u_*^i(\xi, \tau) = u(\tau - k_i \xi; k_i)$ and $y^i(x,t) = u_*^i(x - c_* t, \omega_* t)$, it is easily verified that

$$y^i(x,t) = u_*^i\big(x - c_g t, \omega_{\rm nl}(k_i) t - k_i c_g t\big) = u\big(\omega_{\rm nl}(k_i) t - k_i x; k_i\big), \tag{4.5}$$

from which we see that y^i satisfies (2.1) and u_*^i satisfies (4.1). Now, as long as k_i is sufficiently close to k_0, it is possible to apply a suitable shift to $u(k_i)$ to arrange for the identity

$$\big\langle u'(k_0), u(k_i) - u(k_0) \big\rangle_{H^1} + \big\langle k_0 u''(k_0), k_i u'(k_i) - k_0 u'(k_0) \big\rangle_{H^0} = 0. \tag{4.6}$$

This ensures that u_*^i can be written in the form (4.2) in such a way that (4.3) holds. In particular, writing $\theta(\xi) = (k_0 - k_i)\xi$ and $\mathbf{u}_*^i = (u_*^i, \partial_\xi u_*^i)$, we find

$$\mathbf{u}_*^i(\xi,\tau) = T_{\theta(\xi)}\mathbf{u}_0(\tau - k_0\xi) - (k_i - k_0)T_{\theta(\xi)}\mathbf{u}_1(\tau - k_0\xi) + O(|k_i - k_0|)^2. \tag{4.7}$$

The modulating waves that we are interested in will connect appropriate translates of $\mathbf{u}_*^1$ and $\mathbf{u}_*^2$.

Let us now move on and derive a differential equation for the pair $v = (v_1, v_2)$. To represent this equation, we introduce for any $\theta \in C(\mathbb{R},\mathbb{R})$ the notation

$$\mathrm{ev}_\xi\theta = \big(\theta(\xi), \theta(\xi + n_0), \ldots, \theta(\xi + n_N)\big) \in \mathbb{R}^{N+2}. \tag{4.8}$$

Plugging (4.2) into (4.1), we find

$$\begin{array}{lcl}
-\gamma\partial_\xi v_1(\xi,\tau) & = & -\gamma v_2(\xi,\tau) + \gamma\theta'(\xi)u'\big(\theta(\xi) + \tau - k_0\xi; k_0\big), \\
-\gamma\partial_\xi v_2(\xi,\tau) & = & -\gamma k_0\theta'(\xi)u''\big(\theta(\xi) + \tau - k_0\xi; k_0\big) + c_g v_2(\xi,\tau) - \overline{\omega}u'\big(\theta(\xi) + \tau - k_0\xi; k_0\big) \\
 & & -[\omega_0 - k_0 c_g + \overline{\omega}]\partial_\tau v_1(\xi,\tau) \\
 & & + \sum_{j=0}^N A_j v_1(\xi + n_j, \tau) + Dg\big(u(\theta(\xi) + \tau - k_0\xi; k_0)\big)v_1(\xi,\tau) \\
 & & + \sum_{j=0}^N A_j u'\big(\theta(\xi) + \tau - k_0\xi - n_j k_0; k_0\big)\big(\theta(\xi + n_j) - \theta(\xi)\big) \\
 & & + g_{\mathrm{nl}}\big(\theta(\xi), \tau - k_0\xi, v_1(\xi,\tau)\big) + h_{\mathrm{nl}}(\mathrm{ev}_\xi\theta, \tau - k_0\xi).
\end{array} \tag{4.9}$$

Here we have introduced the new nonlinearity $g_{\mathrm{nl}} : \mathbb{R} \times [0, 2\pi] \times \mathbb{R}^n \to \mathbb{R}^n$ that is given by

$$g_{\mathrm{nl}}(\vartheta, \sigma, v) \;=\; g\big(u(\vartheta + \sigma; k_0) + v\big) - g\big(u(\vartheta + \sigma; k_0)\big) - Dg\big(u(\vartheta + \sigma; k_0)\big)v. \tag{4.10}$$

In addition, the second new nonlinearity $h_{\mathrm{nl}} : \mathbb{R}^{N+2} \times [0, 2\pi] \to \mathbb{R}^n$ can, after a slight abuse of notation, be written as

$$\begin{array}{lcl}
h_{\mathrm{nl}}(\mathrm{ev}_\xi\theta, \sigma) & = & \sum_{j=0}^N A_j[u\big(\theta(\xi + n_j) + \sigma - n_j k_0; k_0\big) - u\big(\theta(\xi) + \sigma - n_j k_0; k_0\big)] \\
 & & - \sum_{j=0}^N A_j u'\big(\theta(\xi) + \sigma - n_j k_0; k_0\big)\big(\theta(\xi + n_j) - \theta(\xi)\big).
\end{array} \tag{4.11}$$

In order to transform (4.9) into an autonomous differential equation, we introduce the new variable $\sigma = \tau - k_0\xi$ and consider the function $\widetilde{v}(\xi,\sigma) = v(\xi,\tau)$. Upon dropping the tilde, the new function v must satisfy

$$\begin{array}{lcl}
-\gamma\partial_\xi v_1(\xi,\sigma) & = & -\gamma k_0\partial_\sigma v_1(\xi,\sigma) - \gamma v_2(\xi,\sigma) + \gamma\theta'(\xi)u'\big(\theta(\xi) + \sigma; k_0\big), \\
-\gamma\partial_\xi v_2(\xi,\sigma) & = & -\gamma k_0\partial_\sigma v_2(\xi,\sigma) - \gamma k_0\theta'(\xi)u''\big(\theta(\xi) + \sigma; k_0\big) + c_g v_2(\xi,\sigma) - \overline{\omega}u'\big(\theta(\xi) + \sigma; k_0\big) \\
 & & -[\omega_0 - k_0 c_g + \overline{\omega}]\partial_\sigma v_1(\xi,\sigma) \\
 & & + \sum_{j=0}^N A_j v_1(\xi + n_j, \sigma - n_j k_0) + Dg\big(u(\theta(\xi) + \sigma; k_0)\big)v_1(\xi,\sigma) \\
 & & + \sum_{j=0}^N A_j u'\big(\theta(\xi) + \sigma - n_j k_0; k_0\big)\big(\theta(\xi + n_j) - \theta(\xi)\big) \\
 & & + g_{\mathrm{nl}}\big(\theta(\xi), \sigma, v_1(\xi,\sigma)\big) + h_{\mathrm{nl}}\big(\mathrm{ev}_\xi\theta, \sigma\big).
\end{array} \tag{4.12}$$

The next two results shows that the functions g_{nl} and h_{nl} have vanishing linear parts.

Lemma 4.1. *For any $\delta > 0$, there exists a constant C_δ, which behaves as $C_\delta = O(1)$ as $\delta \to 0$, such that the following two properties are satisfied.*

(i) For any $v \in \mathbb{R}^n$ with $|v| < \delta$, any $\sigma \in [0, 2\pi]$ and any $\vartheta \in \mathbb{R}$, we have

$$|g_{\mathrm{nl}}(\vartheta, \sigma, v)| < C_\delta\delta^2. \tag{4.13}$$

(ii) For any pair $v_1, v_2 \in \mathbb{R}^n$ with $|v_i| < \delta$ for $i = 1, 2$, the following bound holds for arbitrary $\sigma \in [0, 2\pi]$ and arbitrary pairs $\vartheta_1, \vartheta_2 \in \mathbb{R}$,

$$|g_{\mathrm{nl}}(\vartheta_1, \sigma, v_1) - g_{\mathrm{nl}}(\vartheta_2, \sigma, v_2)| < C_\delta\delta\,|v_1 - v_2| + C_\delta\delta^2\,|\vartheta_1 - \vartheta_2|\,. \tag{4.14}$$

Proof. Observe first that (i) follows directly from (ii), since $g_{\rm nl}(\vartheta, \sigma, 0) = 0$. To establish (ii), let us write $\Delta g = |g_{\rm nl}(\vartheta_1, \sigma, v) - g_{\rm nl}(\vartheta_2, \sigma, v)|$ for some $v \in \mathbb{R}^n$ with $|v| < \delta$ and use the shorthand $u_0 = u(k_0)$ to estimate

$$\begin{array}{lcl} \Delta g & = & \big|[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)] \int_0^1 Dg(u_0(\vartheta_2 + \sigma) + v + s[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)])ds \\ & & -[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)] \int_0^1 Dg(u_0(\vartheta_2 + \sigma) + s[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)])ds \\ & & -[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)] \int_0^1 D^2 g(u_0(\vartheta_2 + \sigma) + s[u_0(\vartheta_1 + \sigma) - u_0(\vartheta_2 + \sigma)])v ds\big| \\ & \leq & C_\delta \, |v|^2 \, |\vartheta_1 - \vartheta_2| \, , \end{array} \tag{4.15}$$

with $C_\delta = O(1)$ as $\delta \to 0$. The inequality follows from the fact that g is at least C^3-smooth and that u_0 is bounded. Furthermore, it is not hard to see that for arbitrary $\vartheta \in \mathbb{R}$ and $\sigma \in [0, 2\pi]$, one may write $\Delta g = |g_{\rm nl}(\vartheta, \sigma, v_1) - g_{\rm nl}(\vartheta, \sigma, v_2)|$ and compute

$$\begin{array}{lcl} \Delta g & \leq & |g(u_0(\vartheta + \sigma) + v_1) - g(u_0(\vartheta + \sigma) + v_2) - Dg(u_0(\vartheta + \sigma) + v_2)(v_1 - v_2)| \\ & & + |Dg(u_0(\vartheta + \sigma) + v_2) - Dg(u_0(\vartheta + \sigma))| \, |v_1 - v_2| \\ & \leq & C_\delta \delta \, |v_1 - v_2| \, , \end{array} \tag{4.16}$$

again with $C_\delta = O(1)$ as $\delta \to 0$. This completes the proof. □

Before we state the analogous result for $h_{\rm nl}$, we introduce, for any $\theta \in C(\mathbb{R}, \mathbb{R})$, the notation

$$\mathrm{cev}_\xi \theta = \big(\theta(\xi + n_0) - \theta(\xi), \ldots, \theta(\xi + n_N) - \theta(\xi)\big) \in \mathbb{R}^{N+1}. \tag{4.17}$$

Lemma 4.2. *For any $\delta > 0$, there exists a constant C_δ, which behaves as $C_\delta = O(1)$ as $\delta \to 0$, such that the following two properties are satisfied.*

(i) Suppose that for some $\theta \in C(\mathbb{R}, \mathbb{R})$ and $\xi \in \mathbb{R}$ we have $|\mathrm{cev}_\xi \theta| < \delta$. Then the following inequality holds for any $\sigma \in [0, 2\pi]$,

$$|h_{\rm nl}(\mathrm{ev}_\xi \theta, \sigma)| < C_\delta \delta^2. \tag{4.18}$$

(ii) Consider any pair $\theta^1, \theta^2 \in C(\mathbb{R}, \mathbb{R})$ and any $\xi \in \mathbb{R}$ such that $\left|\mathrm{cev}_\xi \theta^i\right| < \delta$ for $i = 1, 2$. Then the following bound holds for arbitrary $\sigma \in [0, 2\pi]$,

$$\left|h_{\rm nl}(\mathrm{ev}_\xi \theta^1, \sigma) - h_{\rm nl}(\mathrm{ev}_\xi \theta^2, \sigma)\right| < C_\delta \delta \left|\mathrm{ev}_\xi \theta^1 - \mathrm{ev}_\xi \theta^2\right|. \tag{4.19}$$

Proof. In view of the fact that $h_{\rm nl}(\mathrm{ev}_\xi \theta, \sigma) = 0$ for any constant function θ and any $\sigma \in [0, 2\pi]$, it again suffices to prove (ii). Let us consider the function $q : \mathbb{R}^3 \to \mathbb{R}^n$ given by

$$q(\vartheta, \vartheta_\Delta, \sigma) = u\big(\vartheta + \vartheta_\Delta + \sigma; k_0\big) - u\big(\vartheta + \sigma; k_0\big) - u'\big(\vartheta + \sigma; k_0\big)\vartheta_\Delta. \tag{4.20}$$

Let us suppose that $\left|\vartheta_\Delta^i\right| < \delta$ for $i = 1, 2$. We may then mimic the computations in (4.15) and (4.16) to obtain the estimate

$$\left|q(\vartheta^1, \vartheta^1_\Delta, \sigma) - q(\vartheta^2, \vartheta^2_\Delta, \sigma)\right| \leq C_\delta \delta^2 \left|\vartheta^1 - \vartheta^2\right| + C_\delta \delta \left|\vartheta^1_\Delta - \vartheta^2_\Delta\right|, \tag{4.21}$$

again with $C_\delta = O(1)$ as $\delta \to 0$. In view of the definition (4.11), this computation suffices to establish (ii). □

We now proceed to rewrite (4.12) in a more compact fashion. To this end, let us introduce for $v = (v_1, v_2) \in L^1_{\rm loc}(\mathbb{R}, H^2 \times H^1)$ the notation

$$\mathrm{vev}_\xi(v_1, v_2) = \big(v_1(\xi), v_1(\xi + n_0), \ldots, v_1(\xi + n_N), v_2(\xi)\big) \in (H^2)^{N+2} \times H^1. \tag{4.22}$$

We also introduce, for $\vartheta \in \mathbb{R}$ and $\overline{\omega} \in \mathbb{R}$, the operator $L(\vartheta, \overline{\omega}) : (H^2)^{N+2} \times H^1 \to H^1 \times H^0$ that acts as

$$\begin{array}{rcl} L(\vartheta, \overline{\omega})\mathrm{vev}_\xi v & = & -\gamma k_0 \partial_\sigma v(\xi) + \big(-\gamma v_2(\xi), c_g v_2(\xi)\big) - \big(0, [\omega_0 - k_0 c_g + \overline{\omega}]\partial_\sigma v_1(\xi)\big) \\ & & + \big(0, Dg(u(\vartheta + \cdot; k_0))v_1(\xi)\big) + \big(0, \sum_{j=0}^N A_j T_{-n_j k_0} v_1(\xi + n_j)\big), \end{array} \tag{4.23}$$

in which we have again abused notation slightly. In addition, we define the linear operator $M : \mathbb{R}^{N+1} \to H^1 \times H^0$ via

$$M \mathrm{cev}_\xi \theta = \Big(0, \sum_{j=0}^N \big(\theta(\xi + n_j) - \theta(\xi)\big) A_j T_{-n_j k_0} u'(k_0)\Big). \tag{4.24}$$

Finally, we introduce the operators $G : \mathbb{R} \times H^2 \times H^1 \to H^1 \times H^0$ and $H : \mathbb{R}^{N+2} \to H^1 \times H^0$ via

$$\begin{array}{rcl} G(\vartheta, v)(\sigma) & = & \Big(0, g_{\mathrm{nl}}\big(\vartheta, \sigma, v_1(\sigma)\big)\Big), \\ H(\mathrm{ev}_\xi \theta)(\sigma) & = & \Big(0, h_{\mathrm{nl}}\big(\mathrm{ev}_\xi \theta, \sigma\big)\Big). \end{array} \tag{4.25}$$

The differential equation (4.12) can now be simplified to

$$\begin{array}{rcl} -\gamma \partial_\xi v(\xi) & = & L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v + T_{\theta(\xi)} M \mathrm{cev}_\xi \theta + \gamma \theta'(\xi) T_{\theta(\xi)} \mathbf{u}_0' - \overline{\omega} T_{\theta(\xi)}\big(0, u'(k_0)\big) \\ & & + G\big(\theta(\xi), v(\xi)\big) + H(\mathrm{ev}_\xi \theta), \end{array} \tag{4.26}$$

which we view as an equation on the space $H^1 \times H^0$.

In the comoving variables (ξ, σ), the orthogonality condition (4.3) becomes

$$\langle T_{\theta(\xi)} \mathbf{u}_0', v(\xi) \rangle_{H^1 \times H^0} = 0. \tag{4.27}$$

Differentiation of this identity yields

$$\theta'(\xi) \langle T_{\theta(\xi)} \mathbf{u}_0'', v(\xi) \rangle_{H^1 \times H^0} = -\langle T_{\theta(\xi)} \mathbf{u}_0', \partial_\xi D_1 v(\xi) \rangle_{H^1 \times H^0}. \tag{4.28}$$

Substitution of (4.26) yields

$$\begin{array}{rcl} \gamma \theta'(\xi) \langle T_{\theta(\xi)} \mathbf{u}_0'', v(\xi) \rangle & = & \langle T_{\theta(\xi)} \mathbf{u}_0', L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v \rangle + \gamma \theta'(\xi) \langle T_{\theta(\xi)} \mathbf{u}_0', T_{\theta(\xi)} \mathbf{u}_0' \rangle \\ & & + \langle \mathbf{u}_0', M \mathrm{cev}_\xi \theta \rangle + \langle T_{\theta(\xi)} \mathbf{u}_0', G\big(\theta(\xi), v(\xi)\big) \rangle \\ & & + \langle T_{\theta(\xi)} \mathbf{u}_0', H(\mathrm{ev}_\xi \theta) \rangle, \end{array} \tag{4.29}$$

in which the inner products are those on $H^1 \times H^0$. In order to simplify this expression, we introduce the projections $P_\vartheta : H^1 \times H^0 \to H^3 \times H^2$ and the associated operators $Q_\vartheta : H^1 \times H^0 \to \mathbb{R}$ that are given by

$$\begin{array}{rcl} Q_\vartheta v & = & \|\mathbf{u}_0'\|_{H^1 \times H^0}^{-2} \langle T_\vartheta \mathbf{u}_0', v \rangle_{H^1 \times H^0}, \\ P_\vartheta v & = & \mathbf{u}_0' Q_\vartheta v. \end{array} \tag{4.30}$$

Throughout this paper, we will often use the shorthands $P = P_0$ and $Q = Q_0$. In addition, we introduce the functions $V : \mathbb{R} \times H^1 \times H^0 \to \mathbb{R}$ and $\mathcal{S} : \mathbb{R}^{N+2} \times (H^2)^{N+1} \times H^1 \times \mathbb{R} \to \mathbb{R}$ that act as

$$\begin{array}{rcl} V(\vartheta, v) & = & [1 - \|\mathbf{u}_0'\|_{H^1 \times H^0}^{-2} \langle T_\vartheta \mathbf{u}_0'', v \rangle_{H^1 \times H^0}]^{-1} - 1, \\ \mathcal{S}(\mathrm{ev}_\xi \theta, \mathrm{vev}_\xi v, \overline{\omega}) & = & Q_{\theta(\xi)} G\big(\theta(\xi), v(\xi)\big) + Q_{\theta(\xi)} H(\mathrm{ev}_\xi \theta) \\ & & + V\big(\theta(\xi), v(\xi)\big)\big[Q M \mathrm{cev}_\xi \theta + Q_{\theta(\xi)} L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v \\ & & + Q_{\theta(\xi)} G\big(\theta(\xi), v(\xi)\big) + Q_{\theta(\xi)} H(\mathrm{ev}_\xi \theta)\big]. \end{array} \tag{4.31}$$

We may now rearrange (4.29) to yield

$$-\gamma \theta'(\xi) \;=\; Q M \mathrm{cev}_\xi \theta + Q_{\theta(\xi)} L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v + \mathcal{S}(\mathrm{ev}_\xi \theta, \mathrm{vev}_\xi v, \overline{\omega}). \tag{4.32}$$

Notice that this is a functional differential equation of mixed type. The following result shows that the linear part of this equation can be solved uniquely up to an initial condition, if the function space is chosen appropriately.

Lemma 4.3. *Consider the inhomogeneous functional differential equation of mixed type*

$$-\gamma\theta'(\xi) = QM\mathrm{cev}_\xi\theta + f(\xi), \tag{4.33}$$

with $\gamma > 0$ and suppose that (H$\mathcal{T}$1) is satisfied. Then there exist two constants $0 < \eta_{\min} < \eta_{\max}$, together with linear mappings $\mathcal{K}_\eta : BC_\eta(\mathbb{R},\mathbb{R}) \to BC^1_\eta(\mathbb{R},\mathbb{R})$, defined and uniformly bounded for $\eta \in [\eta_{\min}, \eta_{\max}]$, such that for any $f \in BC_\eta(\mathbb{R},\mathbb{R})$, the function $\theta = \mathcal{K}_\eta f$ satisfies (4.33) and has $\theta(0) = 0$. Moreover, if $f \in BC_{\eta_0}(\mathbb{R},\mathbb{R}) \cap BC_{\eta_1}(\mathbb{R},\mathbb{R})$ for some pair $\eta_0, \eta_1 \in [\eta_{\min}, \eta_{\max}]$, then $\mathcal{K}_{\eta_0} f = \mathcal{K}_{\eta_1} f$.

Proof. The characteristic equation $\Delta(z)$ associated to (4.33) is given by

$$\Delta(z) \quad = \quad -\gamma z + k_0 \langle \mathbf{u}_0', \mathbf{u}_0' \rangle^{-1}_{H^1\times H^0} \sum_{j=0}^N \langle u''(k_0), A_j T_{-n_j k_0} u'(k_0)\rangle_{H^0}(e^{n_j z} - 1). \tag{4.34}$$

On the other hand, we may compute

$$\begin{array}{rcl} \langle \mathbf{u}_0', \mathcal{T}(z)\mathbf{u}_0'\rangle_{H^1\times H^0} & = & \langle \mathbf{u}_0', [\mathcal{T}(z) - \mathcal{T}(0)]\mathbf{u}_0'\rangle_{H^1\times H^0} \\ & = & -\gamma z \langle \mathbf{u}_0', \mathbf{u}_0'\rangle_{H^1\times H^0} + k_0 \sum_{j=0}^N \langle u''(k_0), A_j T_{-n_j k_0} u'(k_0)\rangle_{H^0}(e^{n_j z} - 1). \end{array} \tag{4.35}$$

In view of (H$\mathcal{T}$1), we see that $\Delta(i\kappa) \neq 0$ for $\kappa \in \mathbb{R} \setminus \{0\}$, while $\Delta(0) = 0$ and $\Delta'(0) \neq 0$. The generalized eigenspace associated to all eigenvalues on the imaginary axis is thus one dimensional and spanned by $\theta = \mathbf{1}$. The operators $\mathcal{K}_\eta$ can now be constructed as in [32, Section 5]. □

In order to ensure that the operators $\mathcal{K}_\eta$ constructed above can be used to solve (4.32), we need to add cut-offs to the nonlinear functions V, G and H. Let us therefore consider a C^∞-smooth function $\chi : [0, \infty) \to \mathbb{R}$ that has $\chi(\zeta) = 1$ for $0 \le \zeta \le 1$ and $\chi(\zeta) = 0$ for $\zeta \ge 2$. For any $\delta > 0$, we write χ_δ for the function $\chi_\delta(\zeta) = \chi(\zeta/\delta)$. We are now in a position to define, for small quantities $\delta_v > 0$ and $\delta_\theta > 0$,

$$\begin{array}{rcl} G^c(\vartheta, v) & = & \chi_{\delta_v}(|v_1|_{H^2}) G(\vartheta, v), \\ H^c(\mathrm{ev}_\xi\theta) & = & \chi_{\delta_\theta}(|\mathrm{cev}_\xi\theta|) H(\mathrm{ev}_\xi\theta), \\ V^c(\mathrm{ev}_\xi\theta, v) & = & \chi_{\delta_\theta}(|\mathrm{cev}_\xi\theta|)\chi_{\delta_v}(|v|_{H^1\times H^0}) V(\theta(\xi), v). \end{array} \tag{4.36}$$

The use of these cut-offs allow us to turn the local estimates obtained in Lemmas 4.1 and 4.2 into global estimates.

Lemma 4.4. *For arbitrary $\vartheta \in \mathbb{R}$, $v \in H^2 \times H^1$, $\theta \in C(\mathbb{R},\mathbb{R})$ and $w \in H^1 \times H^0$, the functions G^c, H^c and V^c defined in (4.36) can be bounded as follows,*

$$\begin{array}{rcl} |G^c(\vartheta, v)|_{H^3\times H^2} & \le & C_{\delta_v}\delta_v^2, \\ |H^c(\mathrm{ev}_\xi\theta)|_{H^3\times H^2} & \le & C_{\delta_\theta}\delta_\theta^2, \\ |V^c(\mathrm{ev}_\xi\theta, w)| & \le & C_{\delta_v}\delta_v, \end{array} \tag{4.37}$$

in which $C_\delta = O(1)$ as $\delta \to 0$. In addition, for arbitrary $\vartheta^1, \vartheta^2 \in \mathbb{R}$, $v^1, v^2 \in H^2 \times H^1$, $\theta^1, \theta^2 \in C(\mathbb{R},\mathbb{R})$ and $w^1, w^2 \in H^1 \times H^0$, the following Lipschitz estimates hold,

$$\begin{array}{rcl} \left|G^c(\vartheta^1, v^1) - G^c(\vartheta^2, v^2)\right|_{H^3\times H^2} & \le & C_{\delta_v}\delta_v^2 \left|\vartheta^1 - \vartheta^2\right| + C_{\delta_v}\delta_v \left\|v_1^1 - v_1^2\right\|_{H^2}, \\ \left|H^c(\mathrm{ev}_\xi\theta^1) - H^c(\mathrm{ev}_\xi\theta^2)\right|_{H^3\times H^2} & \le & C_{\delta_\theta}\delta_\theta \left|\mathrm{ev}_\xi\theta^1 - \mathrm{ev}_\xi\theta^2\right|, \\ \left|V^c(\mathrm{ev}_\xi\theta^1, w^1) - V^c(\mathrm{ev}_\xi\theta^2, w^2)\right| & \le & C_{\delta_v}\delta_v(1 + \delta_\theta^{-1}) \left|\mathrm{ev}_\xi\theta^1 - \mathrm{ev}_\xi\theta^2\right| + C_{\delta_v} \left\|w^1 - w^2\right\|_{H^1\times H^0}, \end{array} \tag{4.38}$$

again with $C_\delta = O(1)$ as $\delta \to 0$.

Proof. Let us first note that in view of the fact that g is C^5-smooth, the computation in the proof of Lemma 4.1 can be extended to the operator $\widetilde{g}_{\mathrm{nl}} : \mathbb{R} \times [0, 2\pi] \times (\mathbb{R}^n)^3 \to \mathbb{R}^n$, that is defined in such a way that

$$\frac{d^2}{d\sigma^2}[\sigma \mapsto g_{\mathrm{nl}}(\vartheta, \sigma, v(\sigma))] = \widetilde{g}_{\mathrm{nl}}(\vartheta, \sigma, v(\sigma), v'(\sigma), v''(\sigma)). \tag{4.39}$$

The bounds (4.37) now follow immediately from this observation together with Lemma 4.2. To see the bound involving G^c in (4.38), suppose without loss of generality that $\left|v_1^1\right|_{H^1} \le 2\delta$. We compute

$$\begin{array}{rcl} \left|G^c(\vartheta^1, v^1) - G^c(\vartheta^2, v^2)\right|_{H^3 \times H^2} & \le & \left|\chi_{\delta_v}(\left|v_1^1\right|_{H^2}) - \chi_{\delta_v}(\left|v_1^2\right|_{H^2})\right| \left|G(\vartheta^1, v^1)\right|_{H^3 \times H^2} \\ & & + \chi_{\delta_v}(\left|v_1^2\right|_{H^2}) \left|G(\vartheta^1, v^1) - G(\vartheta^2, v^2)\right|_{H^3 \times H^2} \\ & \le & C\delta_v^{-1} \left|v_1^1 - v_1^2\right|_{H^2} \delta_v^2 \\ & & + C\delta_v \left|v_1^1 - v_1^2\right|_{H^2} + C\delta_v^2 \left|\vartheta_1 - \vartheta_2\right|, \end{array} \tag{4.40}$$

in which the constant C depends on δ_v in the correct fashion. The remaining estimates in (4.38) can be proven in a similar fashion. □

After applying these cut-offs to the definition of $\mathcal{S}$, we obtain the new operator $\mathcal{S}^c : \mathbb{R}^{N+2} \times (H^2)^{N+2} \times H^1 \times \mathbb{R} \to \mathbb{R}$ that is given by

$$\begin{array}{rcl} \mathcal{S}^c(\mathrm{ev}_\xi\theta, \mathrm{vev}_\xi v, \overline{\omega}) & = & Q_{\theta(\xi)} G^c\big(\theta(\xi), v(\xi)\big) + Q_{\theta(\xi)} H^c(\mathrm{ev}_\xi\theta) \\ & & + V^c\big(\mathrm{ev}_\xi\theta, v(\xi)\big)\big[QM\mathrm{cev}_\xi\theta + Q_{\theta(\xi)} L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v \\ & & + Q_{\theta(\xi)} G^c\big(\theta(\xi), v(\xi)\big) + Q_{\theta(\xi)} H^c(\mathrm{ev}_\xi\theta)\big]. \end{array} \tag{4.41}$$

Let us recall the interval $[\eta_{\min}, \eta_{\max}]$ appearing in the statement of Lemma 4.3 and choose a constant $\eta \in [\eta_{\min}, \eta_{\max}]$. The bounds obtained in Lemma 4.4 now imply that for any $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ and $\theta \in C(\mathbb{R}, \mathbb{R})$, the function $\xi \mapsto \mathcal{S}^c(\mathrm{ev}_\xi\theta, \mathrm{vev}_\xi v, \overline{\omega})$ belongs to $BC_\eta(\mathbb{R}, \mathbb{R})$. This allows us to define the operators

$$\begin{array}{rcl} \Theta_{\mathrm{L}} & : & C(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \to \mathcal{L}\big(BC_\eta(\mathbb{R}, H^2 \times H^1), BC_\eta(\mathbb{R}, \mathbb{R})\big), \\ \Theta_{\mathrm{NL}} & : & C(\mathbb{R}, \mathbb{R}) \times BC_\eta(\mathbb{R}, H^2 \times H^1) \times \mathbb{R} \to BC_\eta(\mathbb{R}, \mathbb{R}), \end{array} \tag{4.42}$$

that act as

$$\begin{array}{rcl} \Theta_{\mathrm{L}}(\theta, \overline{\omega})v & = & \mathcal{K}_\eta Q_{\theta(\xi')} L(\theta(\xi'), \overline{\omega})\mathrm{vev}_{\xi'} v, \\ \Theta_{\mathrm{NL}}(\theta, v, \overline{\omega}) & = & \mathcal{K}_\eta \mathcal{S}^c(\mathrm{ev}_{\xi'}\theta, \mathrm{vev}_{\xi'} v, \overline{\omega}), \end{array} \tag{4.43}$$

where $\mathcal{K}_\eta$ acts with respect to the variable ξ'.

Upon using $\theta = \Theta_{\mathrm{L}}(\theta, \overline{\omega})v + \Theta_{\mathrm{NL}}(\theta, v, \overline{\omega})$ to rewrite the terms $\theta'(\xi)$ and $\mathrm{cev}_\xi\theta$ appearing in (4.26) and replacing G and H by their cut-off counterparts G^c and H^c, we arrive at the equation

$$-\gamma\partial_\xi v(\xi) = [I - P_{\theta(\xi)}]L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v + [I - P_{\theta(\xi)}]T_{\theta(\xi)} M\mathrm{cev}_\xi \Theta_{\mathrm{L}}(\theta, \overline{\omega})v + \mathcal{R}^c(\theta, v, \overline{\omega})(\xi). \tag{4.44}$$

Here we have introduced the nonlinearity

$$\mathcal{R}^c : C(\mathbb{R}, \mathbb{R}) \times BC_\eta(\mathbb{R}, H^2 \times H^1) \times \mathbb{R} \to BC_\eta(\mathbb{R}, H^3 \times H^2) \tag{4.45}$$

that is given by

$$\begin{array}{rcl} \mathcal{R}^c(\theta, v, \overline{\omega})(\xi) & = & [I - P_{\theta(\xi)}]G^c\big(\theta(\xi), v(\xi)\big) + [I - P_{\theta(\xi)}]H^c(\mathrm{ev}_\xi\theta) \\ & & + [I - P_{\theta(\xi)}]T_{\theta(\xi)} M\mathrm{cev}_\xi \Theta_{\mathrm{NL}}(\theta, v, \overline{\omega}) - \overline{\omega} T_{\theta(\xi)}\big(0, u'(k_0)\big) \\ & & - P_{\theta(\xi)} V^c\big(\mathrm{ev}_\xi\theta, v(\xi)\big)\big[T_{\theta(\xi)} M\mathrm{cev}_\xi\theta + L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v \\ & & + G^c\big(\theta(\xi), v(\xi)\big) + H^c(\mathrm{ev}_\xi\theta)\big]. \end{array} \tag{4.46}$$

To summarize, we have now arrived at the system

$$\begin{array}{rcl} -\gamma\theta'(\xi) & = & QM\mathrm{cev}_\xi\theta + Q_{\theta(\xi)}L(\theta(\xi),\overline{\omega})\mathrm{vev}_\xi v + \mathcal{S}^c(\mathrm{ev}_\xi\theta, \mathrm{vev}_\xi v, \overline{\omega}), \\ -\gamma\partial_\xi v(\xi) & = & [I - P_{\theta(\xi)}]L(\theta(\xi),\overline{\omega})\mathrm{vev}_\xi v + [I - P_{\theta(\xi)}]T_{\theta(\xi)}M\mathrm{cev}_\xi\Theta_{\mathrm{L}}(\theta,\overline{\omega})v \\ & & +\mathcal{R}^c(\theta, v, \overline{\omega})(\xi). \end{array} \tag{4.47}$$

We are now ready to state our center manifold result that captures all sufficiently small solutions to our transformed system (4.47). The proof of this result can be found in Section 7.

Theorem 4.5. *Consider the system (4.47) with $\delta_v = \delta_\theta^{7/4}$ and suppose that (Hg), (H$\mathcal{F}$), (HD), (HL), (H$\mathcal{T}$1) and (H$\mathcal{T}$2) are all satisfied. For any sufficiently small δ_θ, there exist constants $\delta > 0$, $\epsilon > 0$ and $\eta > 0$, together with an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and a function $h : \mathbb{R}^2 \times \Omega \to H^2 \times H^1$, such that the following properties are satisfied.*

(i) The function h is C^{r-3} smooth, in which we have recalled the integer r appearing in (Hg). In addition, we have $h(\kappa, \vartheta, \overline{\omega}) = O(|\kappa|^2 + |\overline{\omega}|)$ as $\kappa \to 0$ and $\overline{\omega} \to 0$.

(ii) Let $(\theta, v) \in BC_\eta(\mathbb{R},\mathbb{R}) \times BC_\eta(\mathbb{R}, H^2 \times H^1)$ be any pair of functions that satisfies (4.47). Suppose furthermore that $\|v(\xi)\|_{H^2\times H^1} < \delta$ for all $\xi \in \mathbb{R}$. Then upon writing

$$\kappa(\xi) = \langle \mathbf{u}_1, \mathbf{u}_1\rangle^{-1}_{H^1\times H^0}\langle T_{\theta(\xi)}\mathbf{u}_1, v(\xi)\rangle_{H^1\times H^0}, \tag{4.48}$$

we have

$$v(\xi) = \kappa(\xi)T_{\theta(\xi)}\mathbf{u}_1 + h\big(\kappa(\xi), \theta(\xi), \overline{\omega}\big) \tag{4.49}$$

for all $\xi \in \mathbb{R}$. In addition, the pair (θ, κ) satisfies the ODE

$$\begin{array}{rcl} \theta'(\xi) & = & f_1(\kappa(\xi), \overline{\omega}), \\ \kappa'(\xi) & = & 2\lambda''_{\mathrm{lin}}(0)^{-1}\overline{\omega} + f_2(\kappa(\xi), \overline{\omega}), \end{array} \tag{4.50}$$

in which f_1 and f_2 are both C^{r-3}-smooth, while f_2 satisfies the bound

$$f_2(\kappa, \overline{\omega}) = O\big((|\overline{\omega}| + |\kappa|)^2\big) \tag{4.51}$$

as $\kappa, \overline{\omega} \to 0$.

(iii) Consider any pair (θ, κ) that satisfies the ODE (4.50) for some $\overline{\omega} \in \Omega$ and in addition has $|\kappa(\xi)| < \epsilon$ for all $\xi \in \mathbb{R}$. Then upon writing

$$v(\xi) = \kappa(\xi)T_{\theta(\xi)}\mathbf{u}_1 + h\big(\kappa(\xi), \theta(\xi), \overline{\omega}\big), \tag{4.52}$$

the pair (θ, v) satisfies (4.47).

With this center manifold result in hand, we are able to provide the proof of our main theorem.

Proof of Theorem 2.5. We consider the terminology of Theorem 4.5. Let us first consider any pair (θ, v) that solves (4.47), with $|v(\xi)|_{H^2\times H^1} < \delta_v$ for all $\xi \in \mathbb{R}$. For any $f \in BC_\eta(\mathbb{R},\mathbb{R})$, notice that $\mathrm{cev}_\xi\mathcal{K}f = \mathrm{cev}_0\mathcal{K}T^{(1)}_\xi f$, in which $[T^{(1)}_\xi f](\xi') = f(\xi + \xi')$. In view of the fact that $\delta_v = o(\delta_\theta)$, the equation for θ in (4.47) now automatically implies that $|\mathrm{cev}_\xi\theta| < \delta_\theta$ for all $\xi \in \mathbb{R}$, provided δ_θ is chosen to be sufficiently small. Using (4.2), such a pair (θ, v) hence leads to a solution of (4.1).

Let us now consider all the equilibria $\overline{\kappa}$ of the differential equation

$$\kappa'(\xi) = 2\lambda''_{\mathrm{lin}}(0)^{-1}\overline{\omega} + f_2(\kappa(\xi), \overline{\omega}) \tag{4.53}$$

that have $|\overline{\kappa}| < \epsilon$. In view of (4.4) and the subsequent discussion, (4.53) admits at least two equilibria, namely $\overline{\kappa}^i = -(k_i - k_0)$ for $i = 1, 2$. On the other hand, any sufficiently small equilibrium $\overline{\kappa}$ for

(4.53) leads to a wave train solution of (2.1). The local uniqueness of the branches $u(k)$ and $\omega(k)$ as established in Lemma 2.1 now guarantees that there are no additional equilibria between $\overline{\kappa}^2$ and $\overline{\kappa}^1$ if ϵ is chosen to be sufficiently small. Looking back at (4.53), we see that

$$\operatorname{sign}\big(\kappa'(0)\big) = \operatorname{sign}(\overline{\omega})\operatorname{sign}\big(\lambda''_{\mathrm{lin}}(0)\big) = \operatorname{sign}\big(\omega''_{\mathrm{nl}}(k_0)\big)\operatorname{sign}\big(\lambda''_{\mathrm{lin}}(0)\big). \tag{4.54}$$

If $\operatorname{sign}\big(\kappa'(0)\big) > 0$, we find a solution for κ that has $\kappa(-\infty) = \overline{\kappa}^2$ and $\kappa(+\infty) = \overline{\kappa}^1$. Conversely, if $\operatorname{sign}\big(\kappa'(0)\big) < 0$, we find a solution for κ that has $\kappa(-\infty) = \overline{\kappa}^1$ and $\kappa(+\infty) = \overline{\kappa}^2$. Lifting these heteroclinic connections for κ back to solutions u_* of (4.1), we find that (i) and (ii) are satisfied. The uniqueness claims follow directly from the fact that the heteroclinic solutions to (4.53) obtained above are unique up to translations. □

5 Linearization near Wave Trains

In this section, we will construct a solution operator for the linear localized system

$$-\gamma\partial_\xi v(\xi) = L(\vartheta, \overline{\omega})\mathrm{vev}_\xi v + f(\xi), \tag{5.1}$$

in which we take $v \in L^1_{\mathrm{loc}}(\mathbb{R}, H^2 \times H^1)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}, H^1 \times H^0)$ for the moment, with $\gamma > 0$. We emphasize that $\vartheta \in \mathbb{R}$ is fixed, which means that (5.1) is an autonomous differential equation for v. Our study of (5.1) will serve as a stepping stone towards solving the linear part of (4.47), which will be discussed in Section 6. We remark that we choose to include the dependence on $\overline{\omega}$ in the linear system (5.1) for reasons that should become clear in Section 7.

In order to state our results, we will need to introduce the following family of function spaces,

$$BY_\eta(\mathbb{R}, H) = \{v \in L^1_{\mathrm{loc}}(\mathbb{R}, H) \mid \|v\|_{BY_\eta} := e^{-\eta|\xi|} \sup_{\xi\in\mathbb{R}} \big[\int_{\xi-1}^{\xi+1} |v(\zeta)|^2_H \, d\zeta\big]^{1/2} < \infty\}, \tag{5.2}$$

in which H is a Hilbert space and $\eta \in \mathbb{R}$. We also need to introduce the point evaluation operator

$$\mathrm{pev}_\xi v = v(\xi), \tag{5.3}$$

for $\xi \in \mathbb{R}$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}, H)$, together with the projections $\pi_1 : H^2 \times H^1 \to H^2$ and $\pi_2 : H^2 \times H^1 \to H^1$ that are given by

$$v = \big(\pi_1 v, \pi_2 v\big) \tag{5.4}$$

for any $v \in H^2 \times H^1$.

Our main result shows how (5.1) can be solved for inhomogeneities $f \in BY_\eta(\mathbb{R}, H^3 \times H^2)$. Due to the fact the linear operator $L(\vartheta, \overline{\omega})$ contains both hyperbolic and elliptic terms, the solution v will not in general gain any regularity with respect to f. This is the reason that we cannot restrict ourselves to $f \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ if we wish to find solutions $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$.

Proposition 5.1. *Consider the linear system (5.1) and suppose that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Then there exist constants $0 < \eta_{\min} < \eta_{\max}$ and an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$, together with maps*

$$\mathcal{K}^{\mathrm{lc}}_\eta : \mathbb{R} \times \Omega \to \mathcal{L}\big(BY_\eta(\mathbb{R}, H^3 \times H^2), BY_\eta(\mathbb{R}, H^3 \times H^2) \cap BC_\eta(\mathbb{R}, H^2 \times H^1)\big), \tag{5.5}$$

defined for $\eta \in [\eta_{\min}, \eta_{\max}]$, such that the following properties are satisfied.

(i) For any $\eta \in [\eta_{\min}, \eta_{\max}]$, $f \in BY_\eta(\mathbb{R}, H^3 \times H^2)$, $\vartheta \in \mathbb{R}$ and $\overline{\omega} \in \Omega$, the function $v = \mathcal{K}^{\mathrm{lc}}_\eta(\vartheta, \overline{\omega})f$ solves (5.1) and in addition has

$$\langle T_\vartheta \mathbf{u}'_0, v(0)\rangle_{H^1\times H^0} = 0, \qquad \langle T_\vartheta \mathbf{u}_1, v(0)\rangle_{H^1\times H^0} = 0. \tag{5.6}$$

(ii) Suppose that for some $\eta \in [\eta_{\min}, \eta_{\max}]$, $\vartheta \in \mathbb{R}$ and $\overline{\omega} \in \Omega$ there exists a function $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ that solves (5.1) with $f = 0$. Then there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$v(\xi) = \alpha_1 T_\vartheta \mathbf{u}_0' + \alpha_2 T_\vartheta[\mathbf{u}_1 + \xi \mathbf{u}_0'] - \overline{\omega}\mathrm{pev}_\xi \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})\big(0, \pi_1[\alpha_1 T_\vartheta \mathbf{u}_0'' + \alpha_2 T_\vartheta[\mathbf{u}_1' + \xi' \mathbf{u}_0'']]\big). \quad (5.7)$$

(iii) There exists a constant $C > 0$ such that $\big\|\mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})\big\| < C$ for all $\eta \in [\eta_{\min}, \eta_{\max}]$, $\vartheta \in \mathbb{R}$ and $\overline{\omega} \in \Omega$.

(iv) Recall the integer r defined in (Hg). For every $\eta \in [\eta_{\min}, \eta_{\max}]$, the map $(\vartheta, \overline{\omega}) \to \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})$ is C^{r-3}-smooth.

(v) For every $\eta \in [\eta_{\min}, \eta_{\max}]$, $\overline{\omega} \in \Omega$ and any pair $\vartheta_1, \vartheta_2 \in \mathbb{R}$, we have the identity

$$\mathcal{K}_\eta^{\mathrm{lc}}(\vartheta_1, \overline{\omega}) = T_{\vartheta_1 - \vartheta_2} \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta_2, \overline{\omega}) T_{\vartheta_2 - \vartheta_1}. \quad (5.8)$$

(vi) Consider a pair $\eta_0, \eta_1 \in [\eta_{\min}, \eta_{\max}]$ and consider

$$f \in BY_{\eta_0}(\mathbb{R}, H^3 \times H^2) \cap BY_{\eta_1}(\mathbb{R}, H^3 \times H^2). \quad (5.9)$$

Then for any $\overline{\omega} \in \Omega$ and $\vartheta \in \mathbb{R}$, we have

$$\mathcal{K}_{\eta_0}^{\mathrm{lc}}(\vartheta, \overline{\omega}) f = \mathcal{K}_{\eta_1}^{\mathrm{lc}}(\vartheta, \overline{\omega}) f. \quad (5.10)$$

In view of (vi) in the result above, we will often write $\mathcal{K}^{\mathrm{lc}}(\vartheta, \overline{\omega}) = \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})$ whenever the exact choice of η is irrelevant. Until stated otherwise, we will take $\vartheta = 0$ in (5.1), employ the shorthands $L(\overline{\omega}) = L(0, \overline{\omega})$ and $B = Dg\big(u(\cdot\,; k_0)\big)$ and concentrate on constructing the operators $\mathcal{K}^{\mathrm{lc}}(0, \overline{\omega})$ first. For convenience, we introduce the notation

$$\mathbb{C}_{\eta_-, \eta_+} = \{z \in \mathbb{C} \mid \eta_- \le \mathrm{Re}\, z \le \eta_+\}. \quad (5.11)$$

Let us write $\widehat{v} : \mathbb{R} \to H^2 \times H^1$ for the Fourier transform (3.2) of v with respect to ξ, together with $\widehat{f} : \mathbb{R} \to H^1 \times H^0$ for the Fourier transform of f. Taking the Fourier transform of (5.1) posed on the space $H^1 \times H^0$, we find

$$-i\eta\gamma\widehat{v}(\eta) = \begin{pmatrix} -\gamma k_0 \partial_\sigma & -\gamma \\ -(\omega_0 - k_0 c_g + \overline{\omega})\partial_\sigma + \sum_{j=0}^N A_j e^{i\eta n_j} T_{-n_j k_0} + B & -\gamma k_0 \partial_\sigma + c_g \end{pmatrix} \widehat{v}(\eta) + \widehat{f}(\eta), \quad (5.12)$$

which can be rewritten as

$$\mathcal{T}_\omega(i\eta)\widehat{v}(\eta) = \widehat{f}(\eta), \quad (5.13)$$

with $\omega = \omega_0 + \overline{\omega}$.

In view of the equivalence (2.13), we will proceed towards solving (5.12) by studying the behaviour of the related operator $\mathcal{L}_{\mathrm{ch}}$. As in the proof of Lemma 2.1, we may argue that $\mathcal{L}_{\mathrm{ch}}(\omega, i\eta) : H^2 \to H^0$ is Fredholm with index zero, which means that a bounded inverse $\mathcal{L}_{\mathrm{ch}}(\omega, i\eta)^{-1} : H^0 \to H^2$ exists if and only if $\mathcal{L}_{\mathrm{ch}}(\omega, i\eta)$ is injective. This latter condition can be related to the point spectrum of $\mathcal{L}_{\mathrm{st}}(\omega, i\eta)$, via the relation

$$\mathcal{L}_{\mathrm{ch}}(\omega, i\eta) = -\mathcal{L}_{\mathrm{st}}(\omega, i\eta) + i\eta(c_p - c_g). \quad (5.14)$$

However, for the purposes of this section we will need to obtain bounds on the inverse $\mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1}$ that are uniform for z in vertical strips in the complex plane. Indeed, in the sequel this will allow us to apply the inverse Fourier transform to solve (5.1) on exponentially weighted function spaces. Let us therefore fix two reals $\eta_- \leq \eta_+$ and set out to explicitly construct $\mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1}$ for $z \in \mathbb{C}_{\eta_-,\eta_+}$, wherever this is defined.

We start by introducing, for $s \in \{0, 1, 2\}$, the sequence spaces

$$\ell_2^s \quad = \quad \Big\{ v = \{v_k\}_{k\in\mathbb{Z}} \mid v_k \in \mathbb{C}^n \text{ and } \|v\|_{\ell_2^s}^2 := \textstyle\sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |v_k|^2 < \infty \Big\}. \tag{5.15}$$

Let us recall that any $v \in H^s$ can be represented as

$$v(\sigma) = \sum_{k\in\mathbb{Z}} v_k e^{ik\sigma} \tag{5.16}$$

for some sequence $\{v_k\}_{k\in\mathbb{Z}} \in \ell_2^s$. Throughout this section, we will use the same symbol for a function $v \in H^s$ and its sequence representation $v \in \ell_2^s$ given by (5.16). Note that if (Hg) and (H$\mathcal{F}$) are satisfied, then we may write

$$B(\sigma) = \sum_{\ell\in\mathbb{Z}} B_k e^{i\ell\sigma}, \tag{5.17}$$

with coefficients that satisfy the estimate $\|B\|_\ell \leq C/(1+|\ell|)^3$. On the level of sequence spaces, the operator B becomes a convolution mapping, acting as

$$[Bv]_k = \sum_{\ell\in\mathbb{Z}} B_\ell v_{k-\ell}. \tag{5.18}$$

Lemma 5.2. *Suppose that (Hg) and (H$\mathcal{F}$) are satisfied. Then the linear map B given by (5.18) satisfies $B \in \mathcal{L}(\ell_2^0)$ and $B \in \mathcal{L}(\ell_2^1)$.*

Proof. This follows directly from the fact that for some $C > 0$ and all $\sigma \in [0, 2\pi]$ we have $|B(\sigma)| < C$ and $|B'(\sigma)| < C$. □

Using the sequence space representation, the identity $\mathcal{L}_{\mathrm{ch}}(\omega, z)v = w$ becomes

$$\gamma k_0^2 k^2 v_k + (\omega + 2\gamma k_0 z) ikv_k - (\gamma z^2 + c_g z)v_k - \sum_{j=0}^{N} A_j e^{zn_j} e^{-ikn_j k_0} v_k = \sum_{\ell\in\mathbb{Z}} B_l v_{k-\ell} + w_k. \tag{5.19}$$

Let us introduce the notation

$$\begin{aligned} \Delta(\overline{\omega}, z, k) &= -\gamma z^2 + (2i\gamma k_0 k - c_g)z + \gamma k_0^2 k^2 + (\omega_0 + \overline{\omega})ik - \textstyle\sum_{j=0}^{N} A_j e^{zn_j} e^{-ikn_j k_0} \\ &= -\gamma(z - ik_0 k)^2 - c_g(z - ik_0 k) + (\overline{\omega} + (c_p - c_g)k_0)ik - \textstyle\sum_{j=0}^{N} A_j e^{(z-ik_0 k)n_j}, \end{aligned} \tag{5.20}$$

using which (5.19) becomes

$$\Delta(\overline{\omega}, z, k)v_k = [Bv]_k + w_k. \tag{5.21}$$

Throughout this section, we will need to use the following assumption.

(ha) There exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$, $\overline{\omega} \in \mathbb{R}$ and $z \in \mathbb{C}_{\eta_-,\eta_+}$, the matrix $\Delta(\overline{\omega}, z, k)$ is invertible and satisfies the uniform bound

$$\left|\Delta(\overline{\omega}, z, k)^{-1}\right| \leq C. \tag{5.22}$$

We emphasize here that we can always arrange for (ha) to hold true. Indeed, by choosing a constant $\kappa \in \mathbb{R}$ that shares the sign of γ and has $|\kappa|$ sufficiently large, we may ensure that for all $\overline{\omega} \in \mathbb{R}$, $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $k \in \mathbb{Z}$ we have

$$\big|\kappa + \mathrm{Re}\,[-\gamma z^2 - c_g z]\big| > 2 \max\big(1, \sum_{j=0}^{N} |A_j|\,|e^{z n_j}|\big). \tag{5.23}$$

We can now modify (5.1) by adding an extra matrix $A_{N+1} = \kappa I$ with associated shift $r_{N+1} = 0$ and replacing $B(\sigma)$ by $B(\sigma) - \kappa I$. This modification obviously leaves (5.12) invariant, but ensures that (ha) is satisfied.

If (ha) holds, then for any $\overline{\omega} \in \mathbb{R}$ and $z \in \mathbb{C}_{\eta_-,\eta_+}$, we may introduce the multiplication operator $\Delta^{\mathrm{inv}}(\overline{\omega}, z) : \ell^0_2 \to \ell^0_2$ that acts as

$$[\Delta^{\mathrm{inv}}(\overline{\omega}, z)v]_k = \Delta(\overline{\omega}, z, k)^{-1} v_k. \tag{5.24}$$

The following two results show that this operator is compact and uniformly bounded for $z \in \mathbb{C}_{\eta_-,\eta_+}$ and small $\overline{\omega}$.

Lemma 5.3. *Fix two constants $\eta_- \le \eta_+$ and suppose that (ha) is satisfied. Then for any $\overline{\omega} \in \mathbb{R}$ and $z \in \mathbb{C}_{\eta_-,\eta_+}$, the operator $\Delta^{\mathrm{inv}}(\overline{\omega}, z) : \ell^0_2 \to \ell^0_2$ is compact.*

Proof. We proceed much as in [49]. To see that $\Delta^{\mathrm{inv}}(\overline{\omega}, z)$ is a compact operator, consider any bounded sequence $\{w^n\}_{n\in\mathbb{N}} \subset \ell^0_2$. Write $v^n = \Delta^{\mathrm{inv}}(\overline{\omega}, z)w^n$ and use a diagonal argument to pass to a subsequence for which each component v^n_k converges as $n \to \infty$. For any $K > 0$ we find

$$\|v^n - v^m\|^2_2 \le \sum_{|k|<K} |v^n_k - v^m_k|^2 + (1 + K)^{-1} \sum_{|k|\ge K} (1 + |k|)\,|v^n_k - v^m_k|^2\,. \tag{5.25}$$

For fixed $z \in \mathbb{C}$, notice that $\Delta(\overline{\omega}, z, k) = O(k^2)$ as $|k| \to \infty$. This means that the second sum on the right hand side of (5.25) can be bounded independently on K, n and m. For any $\epsilon > 0$, we can choose $K > 0$ sufficiently large to ensure that the norm of the second term in (5.25) is bounded by $\frac{\epsilon}{2}$. By restricting to sufficiently large n and m the norm of the first term in (5.25) can also be bounded by $\frac{\epsilon}{2}$, showing that $\{v^n\}$ is a Cauchy-sequence. This completes the proof. □

Lemma 5.4. *Fix two constants $\eta_- \le \eta_+$ and suppose that (ha) is satisfied. Then there exist a small open subset $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and a constant $C > 0$, such that for any $\overline{\omega} \in \Omega$ and any $z \in \mathbb{C}_{\eta_-,\eta_+}$, we have the bounds*

$$\big\|\Delta^{\mathrm{inv}}(\overline{\omega}, z)\big\|_{\mathcal{L}(\ell^0_2,\ell^1_2)} \le C, \tag{5.26a}$$

$$\big\|z\Delta^{\mathrm{inv}}(\overline{\omega}, z)\big\|_{\mathcal{L}(\ell^0_2,\ell^0_2)} \le C. \tag{5.26b}$$

Proof. Note first that (ha) implies the uniform bound

$$\big\|\Delta^{\mathrm{inv}}(\overline{\omega}, z)\big\|_{\mathcal{L}(\ell^0_2,\ell^0_2)} \le C_1. \tag{5.27}$$

We will assume that $\Omega \subset \mathbb{R}$ is sufficiently small to ensure that for all $\overline{\omega} \in \Omega$, the quantity $p(\overline{\omega}) := \overline{\omega} + (c_p - c_g)k_0$ satisfies $|p(\overline{\omega})| > \overline{p}$ for some $\overline{p} > 0$. As a final preparation, we consider the function $q : z \mapsto \gamma z^2 + c_g z$. Note that there exist two constants $C_2 > 0$ and $C_3 > 0$ such that

$$|\mathrm{Re}\, q(z)| + C_2 \ge C_3\, |\mathrm{Im}\, q(z)|^2 \tag{5.28}$$

holds for all $z \in \mathbb{C}_{\eta_-,\eta_+}$.

To see (5.26a), observe that $v = \Delta^{\mathrm{inv}}(\overline{\omega}, z)w$ implies that

$$\big[ip(\overline{\omega}) - \frac{q(z - ik_0k)}{k}\big]kv_k = w_k + \sum_{j=0}^{N} A_j e^{(z-ik_0k)n_j} v_k. \tag{5.29}$$

We fix a constant K such that $K \geq C_2/\overline{p}$ and $K \geq 6/(\overline{p}C_3)$. Consider any pair (z, k) with $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $|k| > K$. We claim that

$$\left| ip(\overline{\omega}) - \frac{q(z)}{k} \right| \geq \frac{1}{2}\overline{p}. \tag{5.30}$$

Indeed, if this inequality is violated, we must have

$$\left| \mathrm{Im}\, \frac{q(z)}{k} \right| \geq \frac{1}{2}\overline{p}, \tag{5.31}$$

after which (5.28) allows us to obtain the contradiction

$$\left| \frac{\mathrm{Re}\, q(z)}{k} \right| \geq C_3 \, |k| \, \frac{1}{4}\overline{p}^2 - \frac{C_2}{|k|} \geq \frac{1}{2}\overline{p}. \tag{5.32}$$

In view of the uniform bound (5.27), the identity (5.29) now implies that $\|\{kv_k\}\|_{\ell^0_2} \leq C_4 \|w\|_{\ell^0_2}$ for some $C_4 > 0$ and hence (5.26a) follows.

To see (5.26b), observe that $v = \Delta^{\mathrm{inv}}(\overline{\omega}, z)w$ yields

$$(z - ik_0k)[-c_g - \gamma(z - ik_0k)]v_k = -ip(\overline{\omega})kv_k + w_k + \sum_{j=0}^{N} A_j e^{(z-ik_0k)n_j} v_k. \tag{5.33}$$

We hence obtain

$$|(z - ik_0k)v_k| \leq \frac{|c_g| + 1}{\gamma} |v_k| + |p(\overline{\omega})kv_k| + |w_k| + \left| \sum_{j=0}^{N} A_j e^{(z-ik_0k)n_j} v_k \right|, \tag{5.34}$$

in which we used (5.33) whenever $|z - ik_0k| > \frac{|c_g|+1}{\gamma}$. Using (5.26a) we hence find $\|\{zv_k\}\|_{\ell^0_2} \leq C_6 \|w\|_{\ell^0_2}$, as desired. □

Before we proceed, let us note that if $[I - \Delta^{\mathrm{inv}}(\overline{\omega}, z)B]v = 0$ for some $v \in \ell^0_2$, then Lemma 5.4 ensures that $v \in \ell^1_2$ and the identity (5.20) immediately implies that also $v \in \ell^2_2$. Now, for any $v \in \ell^2_2$, we may write

$$\begin{aligned} [I - \Delta^{\mathrm{inv}}(\overline{\omega}, z)B]v &= \Delta^{\mathrm{inv}}(\overline{\omega}, z)[\mathcal{L}_{\mathrm{ch}}(\omega_0 + \overline{\omega}, z)]v \\ &= \Delta^{\mathrm{inv}}(\overline{\omega}, z)[-\mathcal{L}_{\mathrm{st}}(\omega_0 + \overline{\omega}, z) + z(c_p - c_g)]v. \end{aligned} \tag{5.35}$$

By the Fredholm alternative, we thus find that $[I - \Delta^{\mathrm{inv}}(\overline{\omega}, z)] : \ell^0_2 \to \ell^0_2$ is invertible if and only if

$$z(c_p - c_g) \notin \mathrm{pointspec}\, \mathcal{L}_{\mathrm{st}}(\omega_0 + \overline{\omega}, z). \tag{5.36}$$

For later use, we state this as the following assumption.

(hb) For every $\overline{\omega} \in \Omega$, $z \in \mathbb{C}_{\eta_-,\eta_+}$ and all $\lambda \in \mathrm{pointspec}\, \mathcal{L}_{\mathrm{st}}(\omega_0 + \overline{\omega}, z)$, we have $\lambda \neq (c_p - c_g)z$.

Lemma 5.5. *Assume that (Hg) and (H$\mathcal{F}$) are satisfied. Fix a pair $\eta_- \le \eta_+$ together with an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and assume that (ha) and (hb) are satisfied. Then $\mathcal{L}_{\mathrm{ch}}(\omega, z) : H^2 \to H^0$ is invertible for all $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $\omega \in \omega_0 + \Omega$. In addition, there exists a constant C such that for all $\omega \in \omega_0 + \Omega$ and $z \in \mathbb{C}_{\eta_-,\eta_+}$, we have*

$$\left\| \mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1} \right\|_{\mathcal{L}(H^s, H^{s+1})} \le C \tag{5.37}$$

for $s \in \{0, 1, 2\}$, together with

$$\left\| z\mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1} \right\|_{\mathcal{L}(H^s, H^s)} \le C. \tag{5.38}$$

Finally, if $s \in \{1, 2\}$ we also have

$$\left\| z^2\mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1} \right\|_{\mathcal{L}(H^s, H^{s-1})} \le C. \tag{5.39}$$

Proof. In view of the criterion (5.36), we see that $[I - \Delta^{\mathrm{inv}}(\overline{\omega}, z)B] : \ell_2^0 \to \ell_2^0$ is invertible for $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $\overline{\omega} \in \Omega$. Writing $\omega = \omega_0 + \overline{\omega}$, we hence find

$$\mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1} = [I - \Delta^{\mathrm{inv}}(\overline{\omega}, z)B]^{-1}\Delta^{\mathrm{inv}}(\overline{\omega}, z). \tag{5.40}$$

Due to (5.26b), we find that for $|\mathrm{Im}\, z|$ sufficiently large we have

$$\left\| \Delta^{\mathrm{inv}}(\overline{\omega}, z) \right\|_{\mathcal{L}(\ell_2^0, \ell_2^0)} \|B\|_{\mathcal{L}(\ell_2^0, \ell_2^0)} \le \frac{1}{2}, \tag{5.41}$$

which leads to the uniform bound

$$\left\| \mathcal{L}_{\mathrm{ch}}(\omega, z)^{-1} \right\|_{\mathcal{L}(H^0, H^0)} \le C_1. \tag{5.42}$$

Consider any $w \in \ell_2$ and suppose that $\mathcal{L}_{\mathrm{ch}}(\omega, z)v = w$. We then obtain

$$v = \Delta^{\mathrm{inv}}(\overline{\omega}, z)(w + Bv) \tag{5.43}$$

and after applying (5.26a) together with (5.42) the uniform bound (5.37) with $s = 0$ follows.

Notice that for any $v \in H^2$ that has $\mathcal{L}_{\mathrm{ch}}(\omega, z)v = w$ with $w \in H^1$, we may compute

$$\mathcal{L}_{\mathrm{ch}}(\omega, z)v' = w' + B'(\cdot)v, \tag{5.44}$$

which in view of the boundedness of B' implies that in fact $v \in H^3$ and establishes (5.37) with $s = 1$. In addition, if also $w \in H^2$, we obtain

$$\mathcal{L}_{\mathrm{ch}}(\omega, z)v'' = w'' + 2B'(\cdot)v' + B''(\cdot)v. \tag{5.45}$$

The assumption (Hg) implies that also B'' is bounded, which establishes (5.37) with $s = 2$.

The uniform bounds (5.38) for $s \in \{0, 1, 2\}$ follow in a similar fashion as above, by using (5.43) in combination with (5.26b). Finally, to see the bound (5.39) for $s \in \{1, 2\}$, suppose that $\mathcal{L}_{\mathrm{ch}}(\omega, z)v = w$ for some $w \in H^s$, write

$$-\gamma z^2 v = \gamma k_0^2 v'' - (\omega + 2\gamma k_0 z)v' + c_g z v + \sum_{j=0}^{N} A_j e^{z n_j} T_{-n_j k_0} v + Bv + w \tag{5.46}$$

and use (5.37) and (5.38) to estimate the H^{s-1}-norm of the right hand side. □

These bounds on $\mathcal{L}_{\mathrm{ch}}(\omega, z)$ can now easily be turned into bounds on $\mathcal{T}_\omega(z)$, using the representation (2.13).

Corollary 5.6. *Assume that (Hg) and (H$\mathcal{F}$) are satisfied. Fix a pair $\eta_- \leq \eta_+$ together with an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and assume that (ha) and (hb) are satisfied. Then $\mathcal{T}_\omega(z) : H^2 \times H^1 \to H^1 \times H^0$ is invertible for all $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $\omega \in \omega_0 + \Omega$. In addition, there exists a constant C such that for $s \in \{0, 1, 2\}$ we have*

$$\left\|\mathcal{T}_\omega(z)^{-1}\right\|_{\mathcal{L}(H^{s+1}\times H^s, H^{s+1}\times H^s)} \quad \leq \quad C \tag{5.47}$$

for all $z \in \mathbb{C}_{\eta_-,\eta_+}$ and $\omega \in \omega_0 + \Omega$.

Let us recall for $p \in \{2, \infty\}$, $\eta \in \mathbb{R}$ and a Hilbert space H, the function spaces

$$\begin{array}{lcl} L^p_\eta(\mathbb{R}, H) & = & \left\{v \in L^1_{\mathrm{loc}}(\mathbb{R}, H) \mid e^{-\eta \cdot} v(\cdot) \in L^p(\mathbb{R}, H)\right\}, \\ W^{1,p}_\eta(\mathbb{R}, H) & = & \left\{v \in L^1_{\mathrm{loc}}(\mathbb{R}, H) \mid e^{-\eta \cdot} v(\cdot) \in W^{1,p}(\mathbb{R}, H)\right\}, \end{array} \tag{5.48}$$

with norms given by $\|v\|_{L^p_\eta} = \|e^{-\eta \cdot} v(\cdot)\|_{L^p}$ and similarly $\|v\|_{W^{1,p}_\eta} = \|e^{-\eta \cdot} v(\cdot)\|_{W^{1,p}}$. With the uniform estimates on $\mathcal{T}_\omega(z)$ obtained in Corollary 5.6 in hand, we are ready to solve (5.1) for inhomogeneities $f \in L^2_\eta(\mathbb{R}, H^3 \times H^2)$.

Lemma 5.7. *Consider the linear system (5.1) with $\gamma > 0$ and suppose that (Hg) and (H$\mathcal{F}$) are satisfied. Fix a pair $\eta_- \leq \eta_+$ together with an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and assume that (ha) and (hb) are satisfied. Then for every $\overline{\omega} \in \Omega$ and $\eta \in [\eta_-, \eta_+]$ there exists a bounded operator*

$$\Lambda^{\mathrm{inv}}_\eta(\overline{\omega}) : L^2_\eta(\mathbb{R}, H^3 \times H^2) \to L^2_\eta(\mathbb{R}, H^3 \times H^2) \cap W^{1,2}_\eta(\mathbb{R}, H^2 \times H^1) \tag{5.49}$$

so that the function $\Lambda^{\mathrm{inv}}_\eta(\overline{\omega})f$ solves (5.1) for any $f \in L^2_\eta(\mathbb{R}, H^3 \times H^2)$. The norm of $\Lambda^{\mathrm{inv}}_\eta(\overline{\omega})$ can be bounded uniformly for $\overline{\omega} \in \Omega$ and $\eta \in [\eta_-, \eta_+]$. Finally, we have the explicit representation formula

$$[\Lambda^{\mathrm{inv}}_\eta(\overline{\omega})f](\xi) = \frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty} e^{z\xi}\mathcal{T}_{\omega_0+\overline{\omega}}(z)^{-1}[\widetilde{f}_+(z) + \widetilde{f}_-(z)]dz. \tag{5.50}$$

Proof. The result follows in a standard fashion by applying an exponential shift to (5.1) and using the uniform bounds on $\mathcal{T}_\omega(z)^{-1}$ obtained in Corollary 5.6 to solve (5.12), which represents (5.1) in Fourier space. Similar computations can be found in [32, Section 3]. □

In order to turn the L^2-estimates obtained above into L^∞-estimates, we need to exploit the property that the effect of any compactly supported inhomogeneity f on the solution of (5.1) decays exponentially. To make this precise, we introduce for any Hilbert space H the new function space

$$L^{2,\infty}_\eta(\mathbb{R}, H) = \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, H) \mid \|x\|_{L^{2,\infty}_\eta} := \sup_{\xi\in\mathbb{R}} e^{-\eta\xi}\left[\int_{\xi-1}^{\xi+1} |x(\zeta)|^2_H \, d\zeta\right]^{1/2} < \infty\right\}. \tag{5.51}$$

Lemma 5.8. *Consider any $\eta \in \mathbb{R}$. Consider the linear system (5.1) with $\gamma > 0$ and assume that (Hg) and (H$\mathcal{F}$) are satisfied. Fix a constant $\epsilon > 0$, write $\eta_\pm = \eta \pm \epsilon$, choose an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and assume that (ha) and (hb) are satisfied. Assume furthermore that for all $\overline{\omega} \in \Omega$ we have*

$$\Lambda^{\mathrm{inv}}_{\eta+\epsilon}(\overline{\omega})g = \Lambda^{\mathrm{inv}}_{\eta-\epsilon}(\overline{\omega})g \tag{5.52}$$

for all $g \in L^2_{\eta+\epsilon}(\mathbb{R}, H^3 \times H^2) \cap L^2_{\eta-\epsilon}(\mathbb{R}, H^3 \times H^2)$. Choose an $\overline{\omega} \in \Omega$ together with a function

$$f \in L^{2,\infty}_\eta(\mathbb{R}, H^3 \times H^2) \cap L^2_{\eta+\epsilon}(\mathbb{R}, H^3 \times H^2) \tag{5.53}$$

and write $x = \Lambda^{\mathrm{inv}}_{\eta+\epsilon}(\overline{\omega})f$. Then we have

$$x \in C(\mathbb{R}, H^2 \times H^1) \cap L^\infty_\eta(\mathbb{R}, H^2 \times H^1) \cap L^{2,\infty}_\eta(\mathbb{R}, H^3 \times H^2). \tag{5.54}$$

In addition, for some constant $C > 0$ that does not depend on f, we have the bound

$$\|x\|_{L^\infty_\eta(\mathbb{R},H^2\times H^1)} + \|x\|_{L^{2,\infty}_\eta(\mathbb{R},H^3\times H^2)} \leq C\,\|f\|_{L^{2,\infty}_\eta}. \tag{5.55}$$

The analogous results hold for $f \in L^{2,\infty}_\eta(\mathbb{R}, H^3 \times H^2) \cap L^2_{\eta-\epsilon}(\mathbb{R}, H^3 \times H^2)$.

Proof. Our arguments here are an adaptation of those presented by Mielke in [49] for elliptic PDEs. Without loss of generality, we will assume that $\eta = 0$ and consider $f \in L^{2,\infty}(\mathbb{R}, H^3 \times H^2)$ for which also $f \in L^2_\epsilon(\mathbb{R}, H^3 \times H^2)$. For any $n \in \mathbb{Z}$, let χ_n denote the indicator function for the interval $[n, n+1]$. Writing $f_n = \chi_n f$, we see that

$$f_n \in L^2_\epsilon(\mathbb{R}, H^3 \times H^2) \cap L^2_{-\epsilon}(\mathbb{R}, H^3 \times H^2), \tag{5.56}$$

with $\sum_{n\in\mathbb{Z}} f_n \to f$ in $L^2_\epsilon(\mathbb{R}, H^3 \times H^2)$. We can hence define $x_n = \Lambda^{\mathrm{inv}}_\epsilon(\overline{\omega}) f_n = \Lambda^{\mathrm{inv}}_{-\epsilon}(\overline{\omega}) f_n$ and observe that

$$x_n \in W^{1,2}_\epsilon(\mathbb{R}, H^2 \times H^1) \cap W^{1,2}_{-\epsilon}(\mathbb{R}, H^2 \times H^1), \tag{5.57}$$

again with $\sum_{n\in\mathbb{Z}} x_n \to x$ in $W^{1,2}_\epsilon(\mathbb{R}, H^2 \times H^1)$. We can exploit the fact that $T^{(1)}_n$ and $\Lambda^{\mathrm{inv}}_{\pm\epsilon}(\overline{\omega})$ commute to compute

$$\begin{array}{rcl} \|x_n\|_{W^{1,2}([m,m+1],H^2\times H^1)} & = & \big[\int_m^{m+1} |x_n(\xi)|^2_{H^2\times H^1} + |\partial_\xi x_n(\xi)|^2_{H^2\times H^1}\, d\xi\big]^{1/2} \\ & = & \big[\int_{m-n}^{m-n+1} |x_n(\xi+n)|^2_{H^2\times H^1} + |\partial_\xi x_n(\xi+n)|^2_{H^2\times H^1}\, d\xi\big]^{1/2} \\ & \le & e^{-\epsilon(m-n)}\big[\int_{m-n}^{m-n+1} \big(e^{\epsilon\xi}\, |x_n(\xi+n)|_{H^2\times H^1}\big)^2 \\ & & +\big(e^{\epsilon\xi}\, |\partial_\xi x_n(\xi+n)|_{H^2\times H^1}\big)^2 d\xi\big]^{1/2} \\ & \le & C_1 e^{-\epsilon(m-n)} \left\| T^{(1)}_n x_n \right\|_{W^{1,2}_{-\epsilon}(\mathbb{R},H^2\times H^1)} \\ & \le & C_2 e^{-\epsilon(m-n)} \left\| T^{(1)}_n f_n \right\|_{L^2_{-\epsilon}(\mathbb{R},H^3\times H^2)} \\ & \le & C_2 e^{-\epsilon(m-n)} e^{\epsilon} \left\| T^{(1)}_n f_n \right\|_{L^2(\mathbb{R},H^3\times H^2)} \\ & \le & C_2 e^{-\epsilon(m-n)} e^{\epsilon}\, \|f\|_{L^{2,\infty}(\mathbb{R},H^3\times H^2)}, \end{array} \tag{5.58}$$

for some constants $C_1 > 0$ and $C_2 > 0$. In a similar fashion, we obtain for some $C_3 > 0$,

$$\|x_n\|_{W^{1,2}([m,m+1],H^2\times H^1)} \le C_3 e^{\epsilon(m-n)} e^{\epsilon}\, \|f\|_{L^{2,\infty}(\mathbb{R},H^3\times H^2)}. \tag{5.59}$$

Summing these identities over $n \in \mathbb{Z}$, we obtain

$$\begin{array}{rcl} \|x\|_{W^{1,2}([m,m+1],H^2\times H^1)} & \le & e^{\epsilon}(C_2 + C_3)\, \|f\|_{L^{2,\infty}(\mathbb{R},H^3\times H^2)} \\ & & \quad \big[\sum_{n\ge m} e^{\epsilon(m-n)} + \sum_{n<m} e^{\epsilon(n-m)}\big] \\ & \le & C_4\, \|f\|_{L^{2,\infty}(\mathbb{R},H^3\times H^2)}. \end{array} \tag{5.60}$$

Observe that this bound no longer depends on m. By a Sobolev embedding, we thus obtain that $x \in BC_0(\mathbb{R}, H^2 \times H^1)$. Moreover, this bound also implies that $\partial_{\sigma\xi} x \in L^{2,\infty}(\mathbb{R}, H^1 \times H^0)$. Using the differential equation (5.1), we find that $\partial_{\sigma\sigma} x \in L^{2,\infty}(\mathbb{R}, H^1 \times H^0)$ and hence $x \in L^{2,\infty}(\mathbb{R}, H^3 \times H^2)$. The bound (5.55) now follows from (5.60) together with (5.1). □

We now set out to find all $w \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ that satisfy (5.1) with $f = 0$. As a first step towards this goal, we will derive a representation formula for solutions to (5.1) with f in appropriate exponentially weighted function spaces. In particular, we will use the family of spaces

$$BX_{\mu,\nu}(\mathbb{R}, H) = \big\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, H) \mid \|x\|_{BX_{\mu,\nu}} := \sup_{\xi<0} e^{-\mu\xi}\, |x(\xi)|_H + \sup_{\xi\ge 0} e^{-\nu\xi}\, |x(\xi)|_H < \infty\big\}, \tag{5.61}$$

defined for any Hilbert space H and parametrized by $\mu, \nu \in \mathbb{R}$.

Lemma 5.9. *Consider the linear system (5.1) with $\gamma > 0$. Fix a pair of constants $\eta_- < \eta_+$ and an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$. Assume that for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z \in \{\eta_-, \eta_+\}$ and all $\omega \in \omega_0 + \Omega$, we have*

$$(c_p - c_g) z \notin \mathrm{pointspec}\, \mathcal{L}_{\mathrm{st}}(\omega, z). \tag{5.62}$$

Consider any pair $\mu, \nu \in \mathbb{R}$ for which $\eta_- < \mu < \nu < \eta_+$ and fix $f \in BX_{\mu,\nu}(\mathbb{R}, H^1 \times H^0)$. Then if $v \in C(\mathbb{R}, H^2 \times H^1) \cap BX_{\mu,\nu}(\mathbb{R}, H^2 \times H^1)$ is a solution to (5.1) for this choice of f, the following identity must hold,

$$\begin{array}{rcl} v(\xi) & = & \frac{1}{2\pi i}\int_{\eta_+ - i\infty}^{\eta_+ + i\infty} e^{\xi z}\big(K(\xi, z, v) + \mathcal{T}_\omega(z)^{-1}\widetilde{f}_+(z)\big)dz \\ & & +\frac{1}{2\pi i}\int_{\eta_- + i\infty}^{\eta_- - i\infty} e^{\xi z}\big(K(\xi, z, v) - \mathcal{T}_\omega(z)^{-1}\widetilde{f}_-(z)\big)dz. \end{array} \tag{5.63}$$

Here $\omega = \omega_0 + \overline{\omega}$ and $K : \mathbb{R} \times \mathbb{C} \times C(\mathbb{R}, H^2 \times H^1) \to H^2 \times H^1$ is given by

$$K(\xi, z, v) = \int_\xi^0 e^{-z\sigma} v(\sigma)d\sigma + \mathcal{T}_\omega(z)^{-1}\big[-\gamma v(0) + \big(0, \sum_{j=0}^N A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma}\pi_1 v(\sigma)d\sigma\big)\big]. \tag{5.64}$$

Proof. The differential equation (5.1) implies that $\partial_\xi v \in BX_{\mu,\nu}(\mathbb{R}, H^1 \times H^0)$. This ensures that the Laplace transform $\widetilde{\partial_\xi v}_+(z)$ is well-defined for $\mathrm{Re}\, z > \nu$. This allows us to take the Laplace transform of the entire system (5.1), which yields

$$\mathcal{T}_\omega(z)\widetilde{v}_+(z) = \widetilde{f}_+(z) - \gamma v(0) + \big(0, \sum_{j=0}^N A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma}\pi_1 v(\sigma)d\sigma\big). \tag{5.65}$$

Since $\mathcal{T}_\omega(z)$ is invertible for every z with $\mathrm{Re}\, z = \eta_+$, we find that for such z we must have

$$\widetilde{v}_+(z) = \mathcal{T}_\omega(z)^{-1}[\widetilde{f}_+(z) - \gamma v(0) + \big(0, \sum_{j=0}^N A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma}\pi_1 v(\sigma)d\sigma\big)]. \tag{5.66}$$

Similar arguments can be used to obtain an expression for $\widetilde{x}_-(z)$. We may now use the inverse Laplace transform as described in Lemma 3.1 to obtain (5.63). A detailed derivation can be found in [32, Proposition 5.1]. □

The operator K appearing in (5.63) can be linked to a spectral projection operator that is closely connected to $\mathcal{T}$. In order to make this precise, we write $r_{\min} = \min\{0, n_0, \ldots, n_N\}$, $r_{\max} = \max\{0, n_0, \ldots, n_N\}$ and introduce the state space

$$X = C([r_{\min}, r_{\max}], H^1 \times H^0). \tag{5.67}$$

In addition, we fix a small $\overline{\omega}$, write $\omega = \omega_0 + \overline{\omega}$ and introduce the closed and densely defined operator $A : \mathcal{D}(A) \subset X \to X$ given by

$$\begin{array}{rcl} \mathcal{D}(A) & = & \big\{\phi \in X \cap C^1([r_{\min}, r_{\max}], H^1 \times H^0) \mid \phi(0) \in H^2 \times H^1 \text{ and } -\gamma\partial_\xi\phi(0) = L(\overline{\omega})\phi\big\}, \\ A\phi & = & \partial_\xi\phi. \end{array} \tag{5.68}$$

Lemma 5.10. *The operator A has only point spectrum, with*

$$\sigma(A) = \sigma_p(A) = \big\{z \in \mathbb{C} \mid \mathcal{T}_\omega(z)v = 0 \text{ for some } v \in H^2 \times H^1\big\}. \tag{5.69}$$

For $z \in \rho(A)$, the resolvent of A is given by

$$(zI - A)^{-1}\psi = e^{\cdot z}K(\cdot, z, \psi), \tag{5.70}$$

in which $K : [r_{\min}, r_{\max}] \times \mathbb{C} \times X \to H^1 \times H^0$ is given by

$$K(\tau, z, \psi) = \int_\tau^0 e^{-z\sigma}\psi(\sigma)d\sigma + \mathcal{T}_\omega(z)^{-1}\big[-\gamma\psi(0) + \big(0, \sum_{j=0}^N A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma}\pi_1\psi(\sigma)d\sigma\big)\big]. \tag{5.71}$$

Proof. Fix $\psi \in X$ and consider the equation $(zI - A)\phi = \psi$ for $\phi \in \mathcal{D}(A)$, which is equivalent to the system

$$\begin{array}{rcl} \partial_\xi \phi & = & z\phi - \psi \\ -\gamma \partial_\xi \phi(0) & = & L(\overline{\omega})\phi. \end{array} \tag{5.72}$$

Suppose that $\mathcal{T}_\omega(z) : H^2 \times H^1 \to H^1 \times H^0$ is invertible. Solving the first equation yields

$$\phi(\tau) = e^{z\tau}\phi(0) + e^{z\tau} \int_\tau^0 e^{-z\sigma}\psi(\sigma)d\sigma. \tag{5.73}$$

It is not hard to verify that choosing

$$\phi(0) = \mathcal{T}_\omega(z)^{-1}\big[- \gamma\psi(0) + \big(0, \sum_{j=0}^{N} A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma}\pi_1\psi(\sigma)d\sigma\big)\big] \tag{5.74}$$

ensures that also the second line of (5.72) is satisfied. On the other hand, consider any $z \in \mathbb{C}$ and $v \in H^2 \times H^1$ such that $\mathcal{T}_\omega(z)v = 0$. Inspecting the function $\phi(\tau) = e^{z\tau}v$, we find that $\phi \in \mathcal{D}(A)$ with $A\phi = z\phi$, showing that $z \in \sigma_p(A)$ and completing the proof. ◻

The representation (5.70) implies that we need to study the map $z \mapsto \mathcal{T}_\omega(z)^{-1}$ in order to understand the behaviour of the resolvent $z \mapsto (zI - A)^{-1}$. In particular, we will need to determine the order of the poles.

Lemma 5.11. *Suppose that $z = \lambda$ is an isolated singularity for the map $z \mapsto \mathcal{T}_\omega(z)^{-1}$. Suppose furthermore that every $v \in \operatorname{Ker} \mathcal{T}_\omega(\lambda)$ has finite Jordan rank and let k be the maximum of such Jordan ranks. Then $z \mapsto \mathcal{T}_\omega(z)^{-1}$ has a pole of order k at $z = \lambda$.*

Proof. Recall first that $\mathcal{T}_\omega(\lambda) : H^2 \times H^1 \to H^1 \times H^0$ is Fredholm with index zero, which allows us to define the integer $n = \dim \operatorname{Ker} \mathcal{T}_\omega(\lambda)$. As customary in the matrix-valued case, we may construct a canonical Jordan basis by repeatedly choosing a kernel element $v \in \operatorname{Ker} \mathcal{T}_\omega(\lambda)$ that has maximal Jordan rank in the subspace of $\operatorname{Ker} \mathcal{T}_\omega(\lambda)$ that has not been spanned by previously chosen elements. In this fashion we find quantities $v^i_{j,\ell} \in H^2 \times H^1$, in which the rank j ranges from 1 to k, the index ℓ ranges from 1 to m_j and i ranges from 0 to $j - 1$. Here m_j denotes the number of Jordan chains with rank j that were constructed. For each fixed j and ℓ the quantities $v^i_{j,\ell}$ form a Jordan chain of length j, which means that

$$\mathcal{T}_\omega(z)[v^0_{j,\ell} + (z - \lambda)v^1_{j,\ell} + \ldots + (z - \lambda)^{j-1}v^{j-1}_{j,\ell}] = O((z - \lambda)^j) \tag{5.75}$$

as $z \to \lambda$. By nature of the construction these Jordan chains cannot be extended, which means that

$$\beta_{j,\ell} := \frac{1}{j!}\mathcal{T}^{(j)}_\omega(\lambda)v^0_{j,\ell} + \ldots + \mathcal{T}^{(1)}_\omega(\lambda)v^{j-1}_{j,\ell} \notin \mathcal{R}(\mathcal{T}_\omega(\lambda)). \tag{5.76}$$

In addition, these n vectors $\beta_{j,\ell}$ are linearly independent over $\mathcal{R}(\mathcal{T}_\omega(\lambda))$, which allows us to choose n quantities $\alpha_{j,\ell} \in \mathcal{R}(\mathcal{T}_\omega(\lambda))^\perp$ that satisfy the orthogonality relations

$$\langle \alpha_{j,\ell}, \beta_{j',\ell'} \rangle_{H^1 \times H^0} = \delta_{jj'}\delta_{\ell\ell'}. \tag{5.77}$$

To show that $z \mapsto \mathcal{T}_\omega(z)^{-1}$ has a pole of order k, we will construct a holomorphic function $z \mapsto H(z) \in \mathcal{L}(H^1 \times H^0, H^2 \times H^1)$ that is analytic in a neighbourhood of $z = \lambda$ and satisfies

$$\mathcal{T}_\omega(z)H(z) = (z - \lambda)^k I. \tag{5.78}$$

In fact, one easily sees that it suffices to find operators $H_i \in \mathcal{L}(H^1 \times H^0, H^2 \times H^1)$ for $0 \le i \le k$ such that

$$\mathcal{T}_\omega(z)[H_0 + (z-\lambda)H_1 + \ldots + (z-\lambda)^k H_k] = (z-\lambda)^k I + O((z-\lambda)^{k+1}). \tag{5.79}$$

A short computation shows that this can be done by writing

$$H_i = \sum_{j=k-i}^{k} \sum_{l=1}^{m_j} v_{j,\ell}^{j+(i-k)} \langle \alpha_{j,\ell}, \cdot \rangle_{H^1 \times H^0} \tag{5.80}$$

for $0 \le i < k$ and choosing H_k in such a way that

$$\mathcal{T}_\omega(\lambda) H_k = I - \sum_{j,\ell} \beta_{j,\ell} \langle \alpha_{j,\ell}, \cdot \rangle_{H^1 \times H^0}. \tag{5.81}$$

This is always possible, since $\mathcal{R}(\mathcal{T}_\omega(\lambda))^\perp = \mathrm{span}\{\alpha_{j,\ell}\}$, while (5.77) implies that for all $v \in H^1 \times H^0$

$$\big\langle \alpha_{j,\ell}, v - \sum_{j',\ell'} \beta_{j',\ell'} \langle \alpha_{j',\ell'}, v \rangle_{H^1 \times H^0} \big\rangle_{H^1 \times H^0} = 0. \tag{5.82}$$

□

The next result shows how Jordan chains for the operator $zI - A$ can be constructed from Jordan chains for $\mathcal{T}_\omega(z)$. We introduce a new operator $\widehat{A}$ on the larger space $\widehat{X} = H^1 \times H^0 \times X$, given by

$$\begin{array}{lcl} \mathcal{D}(\widehat{A}) & = & \Big\{ (v,\phi) \in \widehat{X} \mid \partial_\xi \phi \in X, \phi(0) \in H^2 \times H^1, v = -\gamma\phi(0) \Big\}, \\ \widehat{A}(v,\phi) & = & (L(\overline{\omega})\phi, \partial_\xi \phi). \end{array} \tag{5.83}$$

Let us write $j : X \to \widehat{X}$ for the continuous embedding $\phi \mapsto (-\gamma\phi(0), \phi)$. As in [32], one may argue that the part of $\widehat{A}$ in jX is equivalent to A and that the closure of $\mathcal{D}(\widehat{A})$ is given by jX. Hence the spectral analysis of A and $\widehat{A}$ is one and the same. We have the following equivalence.

Lemma 5.12. *Consider the holomorphic functions $E : \mathbb{C} \to \mathcal{L}\big(H^2 \times H^1 \times X, \mathcal{D}(\widehat{A})\big)$ and $F : \mathbb{C} \to \mathcal{L}(\widehat{X}, \widehat{X})$ given by*

$$\begin{array}{lcl} E(z)(v,\psi)(\tau) & = & (-\gamma v, e^{z\tau} v + e^{z\tau} \int_\tau^0 e^{-z\sigma} \psi(\sigma) d\sigma), \\ F(z)(v,\psi)(\tau) & = & \Big(v + \big(0, \sum_{j=0}^N A_j e^{z n_j} T_{-n_j k_0} \int_{n_j}^0 e^{-z\sigma} \pi_1 \psi(\sigma) d\sigma\big), \psi(\tau) \Big), \end{array} \tag{5.84}$$

in which $\mathcal{D}(\widehat{A})$ is considered as a Banach space with the graph norm. Then $E(z)$ and $F(z)$ are bijective for every $z \in \mathbb{C}$ and we have the identity

$$\begin{pmatrix} \mathcal{T}_\omega(z) & 0 \\ 0 & I \end{pmatrix} = F(z)(zI - \widehat{A})E(z). \tag{5.85}$$

Proof. The bijectivity of F is immediate. To show that also E is invertible, write $E_2(z)$ for the X-component of $E(z)$, and observe that

$$\psi(\tau) = zE_2(z)(v,\psi)(\tau) - \partial_\xi E_2(z)(v,\psi)(\tau), \tag{5.86}$$

which means E has a left inverse. Using partial integration, we may compute

$$E_2(z)\big(\psi(0), (zI - \partial_\xi)\psi\big)(\tau) = e^{z\tau}\psi(0) + e^{z\tau} \int_\tau^0 e^{-z\sigma}(z\psi(\sigma) - \partial_\xi \psi(\sigma)) d\sigma = \psi(\tau), \tag{5.87}$$

which shows that E has a right inverse. A simple calculation now shows

$$(zI - \widehat{A})E(z)(v, \psi) = \Big(\mathcal{T}_\omega(z)v - \big(0, \sum_{j=0}^{N} A_j e^{zn_j} T_{-n_j k_0} \int_{n_j}^{0} e^{-z\sigma} \pi_1 \psi(\sigma) d\sigma \big), \psi \Big), \tag{5.88}$$

from which the identity (5.85) follows immediately. □

Lemma 5.13. *For any pair $\mu \le \nu$, set $\Sigma = \Sigma_{\mu,\nu} = \{ z \in \sigma(A) \mid \mu \le \operatorname{Re} z \le \nu \}$. Suppose that Σ is a finite set and that for each $z \in \Sigma$ the Jordan rank of $\mathcal{T}_\omega(z)$ is finite. Then each $z \in \Sigma$ is a pole of $(zI - A)^{-1}$ of finite order. In addition, we have the decomposition*

$$X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma, \tag{5.89}$$

in which $\mathcal{M}_\Sigma$ is the generalized eigenspace corresponding to the eigenvalues in Σ and $\mathcal{R}_\Sigma$ is the null space of the associated spectral projection $Q_\Sigma : X \to X$.

Consider any pair of constants $\eta_\pm$ with $\eta_- < \mu$ and $\eta_+ > \nu$, for which $\Sigma_{\eta_-,\eta_+} = \Sigma$. In addition, consider any $\phi \in X$ that has $\phi(0) \in H^2 \times H^1$ and $\partial_\xi \pi_1 \phi \in L^1([r_{\min}, r_{\max}], H^0)$. Then we have the identity

$$(Q_\Sigma \phi)(\tau) = \frac{1}{2\pi i} \int_{\eta_+ - i\infty}^{\eta_+ + i\infty} e^{z\tau} K(\tau, z, \phi) dz + \frac{1}{2\pi i} \int_{\eta_- + i\infty}^{\eta_- - i\infty} e^{z\tau} K(\tau, z, \phi) dz, \tag{5.90}$$

in which K is given by (5.71).

Proof. Lemma 5.11 together with the representation (5.70) implies that each $\lambda \in \Sigma$ is a pole of finite order for $z \mapsto (zI - A)^{-1}$. The spectral splitting (5.89) follows directly from [15, Theorem IV.2.5]. Using a Dunford integral to represent the spectral projection Q_Σ, it suffices to show that

$$\int_{\eta_- + i\kappa}^{\eta_+ + i\kappa} e^{z\tau} \mathcal{T}_\omega(z)^{-1} \big[-\gamma \phi(0) + \big(0, \sum_{j=0}^{N} A_j e^{zn_j} T_{-n_j k_0} \int_{n_j}^{0} e^{-z\sigma} \pi_1 \phi(\sigma) d\sigma \big) \big] dz \to 0 \tag{5.91}$$

as $\kappa \to \pm\infty$ in order to establish (5.90). Since $z \mapsto \mathcal{T}_\omega(z)^{-1}$ does not decay as $\operatorname{Im} z \to \pm\infty$ this is not immediately clear. An integration by part yields

$$\int_{n_j}^{0} e^{-z\sigma} \pi_1 \phi(\sigma) d\sigma = z^{-1} \big[e^{-zn_j} \pi_1 \phi(r_j) - \pi_1 \phi(0) + \int_{n_j}^{0} e^{-z\sigma} \partial_\xi \pi_1 \phi(\sigma) d\sigma \big]. \tag{5.92}$$

In view of the assumption on $\pi_1 \phi$, it thus only remains to show that

$$\int_{\eta_- + i\kappa}^{\eta_+ + i\kappa} e^{z\tau} \mathcal{T}_\omega(z)^{-1} \phi(0) dz \to 0 \tag{5.93}$$

as $\kappa \to \pm\infty$. To see this, we write

$$-\gamma \phi(0) \;\; = \;\; z^{-1}[-\gamma z \phi(0) - \mathcal{T}_\omega(z)\phi(0)] + z^{-1}[\mathcal{T}_\omega(z)\phi(0)] \tag{5.94}$$

and use the fact that $\mathcal{T}_\omega(z) + \gamma z I$ remains bounded as $\operatorname{Im} z \to \infty$. □

The relation thus obtained between the representation (5.63) and the spectral projection (5.90) allows us to lift the inverses Λ^{inv} obtained in Lemma 5.7 to inverses defined on $L^{2,\infty}(\mathbb{R}, H^3 \times H^2)$. As a preparation, we introduce for any function Hilbert space H and any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}, H)$ the notation $\Phi_+ f \in L^1_{\mathrm{loc}}(\mathbb{R}, H)$ to denote the function has $[\Phi_+ f](\xi) = f(\xi)$ for $\xi \ge 0$ and $[\Phi_+ f](\xi) = 0$ for $\xi < 0$. In addition, we write $\Phi_- f = f - \Phi_+ f$.

Lemma 5.14. *Consider any $\eta \in \mathbb{R}$. Consider the linear system (5.1) with $\gamma > 0$ and assume that (Hg) and (H$\mathcal{F}$) are satisfied. Fix a constant $\epsilon > 0$, write $\eta_{\pm} = \eta \pm 2\epsilon$, choose an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ and assume that (ha) and (hb) are satisfied. Fix any $\overline{\omega} \in \Omega$. Consider any*

$$f \in L^{2,\infty}_{\eta}(\mathbb{R}, H^3 \times H^2) \tag{5.95}$$

and write

$$v = \Lambda^{\mathrm{inv}}_{\eta+\epsilon}(\overline{\omega})\Phi_+ f + \Lambda^{\mathrm{inv}}_{\eta-\epsilon}(\overline{\omega})\Phi_- f. \tag{5.96}$$

Then v solves the linear system (5.1). In addition, we have

$$v \in L^{2,\infty}_{\eta}(\mathbb{R}, H^3 \times H^2) \cap L^{\infty}_{\eta}(\mathbb{R}, H^2 \times H^1) \cap W^{1,\infty}_{\eta}(\mathbb{R}, H^1 \times H^0) \tag{5.97}$$

and there exists a constant $C > 0$ that does not depend on f and $\overline{\omega}$, such that

$$\|v\|_{L^{2,\infty}_{\eta}(\mathbb{R},H^3\times H^2)} + \|v\|_{L^{\infty}_{\eta}(\mathbb{R},H^2\times H^1)} + \|v\|_{W^{1,\infty}_{\eta}(\mathbb{R},H^1\times H^0)} \le C\,\|f\|_{L^{2,\infty}_{\eta}(\mathbb{R},H^3\times H^2)}\,. \tag{5.98}$$

Proof. Notice first that the assumptions of Lemma 5.8 are satisfied. Indeed, for any function $g \in L^2_{\eta+\epsilon}(\mathbb{R}, H^3 \times H^2) \cap L^2_{\eta-\epsilon}(\mathbb{R}, H^3 \times H^2)$, write

$$w = \Lambda^{\mathrm{inv}}_{\eta+\epsilon}(\overline{\omega})g - \Lambda^{\mathrm{inv}}_{\eta-\epsilon}(\overline{\omega})g \tag{5.99}$$

and notice that $w \in C(\mathbb{R}, H^2 \times H^1) \cap BX_{\eta-\epsilon,\eta+\epsilon}(\mathbb{R}, H^2 \times H^1)$. In addition, w satisfies (5.1) with $f = 0$. Write $\psi = w_{|[r_{\min},r_{\max}]}$. Choosing $\mu = \eta - \epsilon$, $\nu = \eta + \epsilon$ and comparing (5.63) with (5.90), we find that $Q_{\Sigma_{\mu,\nu}}\psi = \psi$. However, our condition on ϵ implies that $Q_{\Sigma_{\mu,\nu}} = 0$. After repeating this argument for shifted versions of g we may conclude $w = 0$. The claims now follow directly from Lemma 5.8. □

Let us now introduce for $\eta \in \mathbb{R}$ and $\overline{\omega} \in \mathbb{R}$ the set

$$\mathcal{N}^{\mathrm{lc}}_{\eta}(\overline{\omega}) = \{v \in BC_{\eta}(\mathbb{R}, H^2 \times H^1) \mid v \text{ satisfies (5.1) with } f = 0\}. \tag{5.100}$$

Arguing similarly as in the proof of Lemma 5.14, (5.63) in combination with (5.90) allows us to obtain a characterization of $\mathcal{N}^{\mathrm{lc}}_{\eta}(\overline{\omega})$.

Lemma 5.15. *Consider the linear system (5.1) with $\gamma > 0$ and assume that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Then there exist a small open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$, together with two constants $0 < \eta^*_{\min} < \eta^*_{\max}$, such that $\mathcal{T}_{\omega}(z) : H^2 \times H^1 \to H^1 \times H^0$ is invertible for all $\omega \in \omega_0 + \Omega$ and all $z \in \mathbb{C}$ that have $\eta^*_{\min} \le |\mathrm{Re}\, z| \le \eta^*_{\max}$. In addition, for each $\overline{\omega} \in \Omega$ and $\eta \in (\eta^*_{\min}, \eta^*_{\max})$, the set $\mathcal{N}^{\mathrm{lc}}_{\eta}(\overline{\omega})$ is two dimensional.*

Proof. The uniform bound (5.26b) implies that there exist a small $\eta^*_{\max} > 0$, large $\kappa > 0$ and small neighbourhood $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ such that $\mathcal{T}_{\omega}(z)$ is invertible for all $\omega \in \omega_0 + \Omega$ and $z \in \mathbb{C}_{-\eta^*_{\max},\eta^*_{\max}}$ that have $|z| > \kappa$. Let us recall the function $\mu^*(\omega, z) = \lambda_{\mathrm{lin}}(\omega, z) - z(c_p - c_g)$ introduced in Lemma 3.2. Since $z \mapsto \mu^*(\omega, z)$ is analytic, its set of roots is discrete. This implies that there exists a small $\delta > 0$ such that $\mathcal{T}_{\omega_0}(z)$ is invertible for all $z \in \mathbb{C}$ with $0 < |z| < \delta$. After possibly further decreasing $\eta^*_{\max}$, we hence see that $\mathcal{T}_{\omega_0}(z)$ is invertible for all $z \in \mathbb{C}$ with $0 < |\mathrm{Re}\, z| \le \eta_{\max}$. As ω is varied, the double root of $\mu^*(\omega_0, \cdot)$ at $z = 0$ can be split into two components $z_1(\omega)$ and $z_2(\omega)$ that depend continuously on ω. For any choice of $\eta^*_{\min}$ that satisfies $0 < \eta^*_{\min} < \eta^*_{\max}$, we may therefore choose Ω sufficiently small to ensure that $\mathcal{T}_{\omega}(z)$ is invertible for all $\omega \in \omega_0 + \Omega$ and $z \in \mathbb{C}$ with $\eta^*_{\min} \le |\mathrm{Re}\, z| \le \eta^*_{\max}$. In addition, Lemma 3.2 ensures that $\dim \mathrm{Ker}\, \mathcal{T}_{\omega}(z_i(\omega)) = 1$ for $i = 1, 2$ and $\omega \in \omega_0 + \Omega$, while the total number of elements in the combined Jordan chains associated to $\mathcal{T}_{\omega}(z_i(\omega))$ remains equal to two. Let us now consider $w \in \mathcal{N}^{\mathrm{lc}}_{\eta}(\overline{\omega})$. Combining the representation formula (5.63) with the spectral projection (5.90) and writing $\psi = w_{|[r_{\min},r_{\max}]}$ shows that $\psi = Q_{\Sigma_{-\eta,\eta}}\psi$. □

From now on, we fix two constants $\eta_{\min}$ and $\eta_{\max}$ in such a way that $\eta^*_{\min} < \eta_{\min} < \eta_{\max} < \eta^*_{\max}$. We also fix an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ that is sufficiently small to ensure that the results developed previously in this section are all applicable. We are now ready to introduce for $\eta \in [\eta_{\min}, \eta_{\max}]$ and $\overline{\omega} \in \Omega$ the pseudo-inverses

$$\widetilde{\mathcal{K}}_\eta(\overline{\omega}) : BY_\eta(\mathbb{R}, H^3 \times H^2) \to BC_\eta(\mathbb{R}, H^2 \times H^1) \cap BY_\eta(\mathbb{R}, H^3 \times H^2), \tag{5.101}$$

that are given by

$$\widetilde{\mathcal{K}}_\eta(\overline{\omega}) f = \Lambda^{\text{inv}}_{\eta+\epsilon}(\overline{\omega})\Phi_+ f + \Lambda^{\text{inv}}_{-\eta-\epsilon}(\overline{\omega})\Phi_- f, \tag{5.102}$$

for sufficiently small $\epsilon > 0$. By construction we have that $x = \widetilde{\mathcal{K}}_\eta(\overline{\omega}) f$ solves (5.1), but the normalization conditions (5.6) may still be violated.

To repair this, observe that for all $\eta \in [\eta_{\min}, \eta_{\max}]$, we have

$$\mathcal{N}^{\text{lc}}_\eta(0) = \text{span}\{\mathbf{u}_0', \xi \mathbf{u}_0' + \mathbf{u}_1\}. \tag{5.103}$$

Let us introduce the projection $\Pi : H^2 \times H^1 \to \mathbb{R}^2$ that is given by

$$\Pi v = \big(\|\mathbf{u}_0'\|^{-2}_{H^1 \times H^0} \langle \mathbf{u}_0', v \rangle_{H^1 \times H^0}, \|\mathbf{u}_1\|^{-2}_{H^1 \times H^0} \langle \mathbf{u}_1, v \rangle_{H^1 \times H^0} \big). \tag{5.104}$$

Our goal here is to construct, for $\overline{\omega} \in \Omega$, extension operators $E^{\text{lc}}(\overline{\omega}) : \mathbb{R}^2 \to BC_{\eta_{\min}}(\mathbb{R}, H^3 \times H^2)$, such that for any $a \in \mathbb{R}^2$ we have $E^{\text{lc}}(\overline{\omega}) a \in \mathcal{N}^{\text{lc}}_{\eta_{\min}}(\overline{\omega})$ and $\Pi \text{pev}_0 E^{\text{lc}}(\overline{\omega}) a = a$. It is not hard to see that

$$[E^{\text{lc}}(0) a](\xi) = (a_1 + a_2 \xi) \mathbf{u}_0' + a_2 \mathbf{u}_1 \tag{5.105}$$

satisfies these conditions. Observe in addition that for any $a \in \mathbb{R}^2$ we have

$$[-\gamma \partial_\xi - L(\overline{\omega})] E^{\text{lc}}(0) a = \big(0, \overline{\omega} \partial_\sigma \pi_1 E^{\text{lc}}(0) a\big). \tag{5.106}$$

After possibly decreasing the size of Ω, we may introduce the operators $E^{\text{lc}}(\overline{\omega})$ by way of

$$E^{\text{lc}}(\overline{\omega}) a = [E^{\text{lc}}(0) - \widetilde{\mathcal{K}}(\overline{\omega})\big(0, \overline{\omega} \partial_\sigma \pi_1 E^{\text{lc}}(0)\big)][I - \Pi \text{pev}_0 \widetilde{\mathcal{K}}(\overline{\omega})\big(0, \overline{\omega} \partial_\sigma \pi_1 E^{\text{lc}}(0)\big)]^{-1} a. \tag{5.107}$$

By construction, for any $a \in \mathbb{R}^2$ the function $E^{\text{lc}}(\overline{\omega}) a$ satisfies (5.1) with $f = 0$, while also $\Pi \text{pev}_0 E^{\text{lc}}(\overline{\omega}) a = a$. The fact that $E^{\text{lc}}(\overline{\omega})$ maps into $BC_{\eta_{\min}}(\mathbb{R}, H^3 \times H^2)$ follows from the smoothness of functions in $\text{Ker}\, \mathcal{T}_\omega(z)$.

With these operators $E^{\text{lc}}(\overline{\omega})$ in hand, we may introduce the new pseudo inverses

$$\mathcal{K}_\eta(\overline{\omega}) : BY_\eta(\mathbb{R}, H^3 \times H^2) \to BC_\eta(\mathbb{R}, H^2 \times H^1) \cap BY_\eta(\mathbb{R}, H^3 \times H^2) \tag{5.108}$$

that are given by

$$\mathcal{K}_\eta(\overline{\omega}) f = \widetilde{\mathcal{K}}_\eta(\overline{\omega}) f - E^{\text{lc}}(\overline{\omega}) \Pi \text{pev}_0 \widetilde{\mathcal{K}}_\eta(\overline{\omega}) f, \tag{5.109}$$

which now do satisfy the normalization conditions (5.6). Finally, we can include the dependence on ϑ by writing

$$\mathcal{K}^{\text{lc}}_\eta(\vartheta, \overline{\omega}) = T_\vartheta \mathcal{K}_\eta(\overline{\omega}) T_{-\vartheta}. \tag{5.110}$$

With this definition we have gathered all the ingredients necessary to establish Proposition 5.1.

Proof of Proposition 5.1. Items (i), (ii) and (vi) follow easily from the discussion above. Item (iii) follows from the definition of $\mathcal{K}_\eta^{\mathrm{lc}}$ together with Lemma 5.7 and the bound (5.55). Item (v) follows directly from (5.110). It remains to establish (iv), which concerns the smoothness of the map $(\vartheta, \overline{\omega}) \mapsto \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})$. Let us pick any $\vartheta \in \mathbb{R}$ and $\overline{\omega} \in \Omega$ and write $v^1 = \mathcal{K}^{\mathrm{lc}}(\vartheta, \overline{\omega})f$ and $v^2 = \mathcal{K}^{\mathrm{lc}}(0,0)f$. Upon defining $w = v^1 - v^2$, it is not hard to see that w satisfies the differential equation

$$-\gamma \partial_\xi w(\xi) = L(0,0)\mathrm{vev}_\xi w + \big(0, [Dg\big(u(\vartheta + \cdot\,; k_0)\big) - Dg\big(u(\cdot\,; k_0)\big) - \overline{\omega}\partial_\sigma]\pi_1 v^1(\xi)\big) \tag{5.111}$$

and hence

$$\begin{array}{rcl} w & = & E^{\mathrm{lc}}(0)[\Pi - \Pi T_{-\vartheta}]\mathrm{pev}_0 \mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})f \\ & & + \mathcal{K}_\eta^{\mathrm{lc}}(0,0)\big(0, [Dg\big(u(\vartheta + \cdot\,; k_0)\big) - Dg\big(u(\cdot\,; k_0)\big) - \overline{\omega}\partial_\sigma]\pi_1\big)\mathcal{K}_\eta^{\mathrm{lc}}(\vartheta, \overline{\omega})f. \end{array} \tag{5.112}$$

To help interpret the first term, we note that for $v \in H^2 \times H^1$ we have

$$-[\Pi - \Pi T_{-\vartheta}]v = \big(\, \|\mathbf{u}_0'\|_{H^1 \times H^0}^{-2} \, \langle T_\vartheta \mathbf{u}_0' - \mathbf{u}_0', v \rangle_{H^1 \times H^0}, \|\mathbf{u}_1\|_{H^1 \times H^0}^{-2} \, \langle T_\vartheta \mathbf{u}_1 - \mathbf{u}_1, v\rangle_{H^1 \times H^0}\big). \tag{5.113}$$

The smoothness of $(\vartheta, \overline{\omega}) \mapsto \mathcal{K}^{\mathrm{lc}}(\vartheta, \overline{\omega})$ can now be established using the smoothness of g and the fact that $u(k_0) \in H^{r+2}$, much along the lines of [53, Lemma 2.5]. We lose three orders of smoothness since we need to get estimates on terms of the form

$$\big|[Dg\big(u(\vartheta + \cdot\,; k_0)\big) - Dg\big(u(\cdot\,; k_0)\big)]\pi_1 v^1(\xi)\big|_{H^2}\,, \tag{5.114}$$

which leads to expressions involving $D^3 g$. □

6 Slowly Varying Coefficients

We are now ready to consider the linear system

$$-\gamma \partial_\xi v(\xi) = [I - P_{\theta(\xi)}]L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v + [I - P_{\theta(\xi)}]T_{\theta(\xi)} M \mathrm{cev}_\xi \Theta_{\mathrm{L}}(\theta, \overline{\omega})v + f(\xi), \tag{6.1}$$

which is nonlocal on account of the term involving Θ_{L}. Recall the interval $[\eta_{\min}, \eta_{\max}]$ appearing in Lemma 4.3 and consider any $\theta \in C(\mathbb{R}, \mathbb{R})$. For any $\eta \in [\eta_{\min}, \eta_{\max}]$ and $\overline{\omega} \in \mathbb{R}$ we may then associate to (6.1) the linear operator

$$\Lambda(\theta, \overline{\omega}) : BC_\eta(\mathbb{R}, H^2 \times H^1) \cap BC_\eta^1(\mathbb{R}, H^1 \times H^0) \to BC_\eta(\mathbb{R}, H^1 \times H^0) \tag{6.2}$$

that is given by

$$[\Lambda(\theta, \overline{\omega})v](\xi) = -\gamma \partial_\xi v(\xi) - [I - P_{\theta(\xi)}]L(\theta(\xi), \overline{\omega})\mathrm{vev}_\xi v - [I - P_{\theta(\xi)}]T_{\theta(\xi)} M \mathrm{cev}_\xi \Theta_{\mathrm{L}}(\theta, \overline{\omega})v. \tag{6.3}$$

The main result that we set out to prove in this section, shows that (6.1) can be solved if $\|\theta'\|_\infty$ is sufficiently small. However, the solution operator will be constructed in such a way that it can be defined for all continuous functions θ.

Proposition 6.1. *Consider the linear system (6.1) and suppose that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Then there exist constants $0 < \eta_{\min} < \eta_{\max}$ and an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$, together with maps*

$$\mathcal{K}_\eta^{\mathrm{gb}} : C(\mathbb{R}, \mathbb{R}) \times \Omega \to \mathcal{L}\big(BY_\eta(\mathbb{R}, H^3 \times H^2), BC_\eta(\mathbb{R}, H^2 \times H^1) \cap BY_\eta(\mathbb{R}, H^3 \times H^2)\big), \tag{6.4}$$

defined for $\eta \in [\eta_{\min}, \eta_{\max}]$, such that the following properties are satisfied.

(i) There exists $\epsilon > 0$, such that if $\theta \in C^1(\mathbb{R}, \mathbb{R})$ and $|\theta'(\xi)| < \epsilon$ for all $\xi \in \mathbb{R}$, then $v = \mathcal{K}_\eta^{\mathrm{gb}}(\theta, \overline{\omega})f$ satisfies $\Lambda(\theta, \overline{\omega})v = f$ for any $\eta \in [\eta_{\min}, \eta_{\max}]$, $\overline{\omega} \in \Omega$ and $f \in BY_\eta(\mathbb{R}, H^3 \times H^2)$.

(ii) We have $\Pi T_{-\theta(0)}\mathrm{pev}_0\mathcal{K}^{\mathrm{gb}}_\eta(\theta,\overline{\omega})f = 0$ *for all* $\eta \in [\eta_{\min},\eta_{\max}]$, $\theta \in C(\mathbb{R},\mathbb{R})$, $\overline{\omega} \in \Omega$ *and* $f \in BY_\eta(\mathbb{R}, H^3 \times H^2)$.

(iii) The norm $\|\mathcal{K}^{\mathrm{gb}}_\eta(\theta,\overline{\omega})\|$ *can be bounded independently of* $\eta \in [\eta_{\min},\eta_{\max}]$, $\theta \in C(\mathbb{R},\mathbb{R})$ *and* $\overline{\omega} \in \Omega$.

(iv) There exists a constant $C > 0$ *such that for any three* $\eta_1,\eta_2,\eta_3 \in [\eta_{\min},\eta_{\max}]$ *that have* $\eta_1+\eta_2 \leq \eta_3$*, any two functions* $\theta_1,\theta_2 \in BC_{\eta_1}(\mathbb{R},\mathbb{R})$*, any two* $\overline{\omega}_1,\overline{\omega}_2 \in \Omega$ *and any* $f \in BY_{\eta_2}(\mathbb{R}, H^3\times H^2)$*, we have the estimate*

$$\left\|\mathcal{K}^{\mathrm{gb}}_{\eta_2}(\theta_1,\overline{\omega}_1)f - \mathcal{K}^{\mathrm{gb}}_{\eta_2}(\theta_2,\overline{\omega}_2)f\right\|_{BC_{\eta_3}\cap BY_{\eta_3}} \leq C\big[\,\|\theta_1-\theta_2\|_{\eta_1} + |\overline{\omega}_1-\overline{\omega}_2|\,\big]\,\|f\|_{BY_{\eta_2}}, \tag{6.5}$$

in which we have introduced the shorthands $BY_\eta = BY_\eta(\mathbb{R}, H^3\times H^2)$ *and* $BC_\eta = BC_\eta(\mathbb{R}, H^2 \times H^1)$.

(v) Consider a pair $\eta_0,\eta_1 \in [\eta_{\min},\eta_{\max}]$ *together with a function*

$$f \in BY_{\eta_0}(\mathbb{R}, H^3 \times H^2) \cap BY_{\eta_1}(\mathbb{R}, H^3 \times H^2). \tag{6.6}$$

Then for any $\theta \in C(\mathbb{R},\mathbb{R})$ *and* $\overline{\omega} \in \Omega$ *we have*

$$\mathcal{K}^{\mathrm{gb}}_{\eta_0}(\theta,\overline{\omega})f = \mathcal{K}^{\mathrm{gb}}_{\eta_1}(\theta,\overline{\omega})f. \tag{6.7}$$

(vi) Recall the integer r *that appears in (Hg). Consider any* $\ell \leq r-3$ *and pick* $\eta_1,\eta_2,\eta_3 \in [\eta_{\min},\eta_{\max}]$ *in such a way that* $\eta_3 > \ell\eta_1 + \eta_2$*. Then the map* $(\theta,\overline{\omega}) \mapsto \mathcal{K}^{\mathrm{gb}}(\theta,\overline{\omega})$ *is* C^ℓ*-smooth when considered as a map from* $BC_{\eta_1}(\mathbb{R},\mathbb{R}) \times \Omega$ *into* $\mathcal{L}(BY_{\eta_2}, BY_{\eta_3} \cap BC_{\eta_3})$*. In addition, for any pair of integers* $p_1,p_2 \geq 0$ *with* $p_1+p_2 \leq \ell$*, the derivative* $D_1^{p_1}D_2^{p_2}\mathcal{K}^{\mathrm{gb}}$ *can be interpreted as a map*

$$D_1^{p}D_2^{p_2}\mathcal{K}^{\mathrm{gb}} : BC_{\eta_1}(\mathbb{R},\mathbb{R}) \times \Omega \to \mathcal{L}^{(p_1+p_2)}\big(BC_{\eta_1}(\mathbb{R},\mathbb{R})^{p_1} \times \mathbb{R}^{p_2}, \mathcal{L}(BY_{\eta_2}, BY_\eta \cap BC_\eta)\big) \tag{6.8}$$

for all $\eta \geq p_1\eta_1 + \eta_2$*. This map is continuous in the first variable if* $\eta > p_1\eta_1+\eta_2$.

Proposition 6.2. *Consider the linear system (6.1) and suppose that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Then there exist a pair* $0 < \eta_{\min} < \eta_{\max}$*, a constant* $\epsilon > 0$ *and an open set* $\Omega \subset \mathbb{R}$ *with* $0 \in \Omega$*, together with a map*

$$E^{\mathrm{gb}} : C(\mathbb{R},\mathbb{R}) \times \Omega \to \mathcal{L}\big(\mathbb{R}^2, BC_{\eta_{\min}}(\mathbb{R}, H^2 \times H^1) \cap BY_{\eta_{\min}}(\mathbb{R}, H^3 \times H^2)\big), \tag{6.9}$$

such that the following properties are satisfied.

(i) If $\theta \in C^1(\mathbb{R},\mathbb{R})$ *has* $|\theta'(\xi)| < \epsilon$ *for all* $\xi \in \mathbb{R}$*, then* $v = E^{\mathrm{gb}}(\theta,\overline{\omega})a$ *satisfies* $\Lambda(\theta,\overline{\omega})v = 0$ *for any* $\overline{\omega} \in \Omega$ *and* $a \in \mathbb{R}^2$.

(ii) We have $\Pi T_{-\theta(0)}\mathrm{pev}_0 E^{\mathrm{gb}}(\theta,\overline{\omega})a = a$ *for all* $\theta \in C(\mathbb{R},\mathbb{R})$, $\overline{\omega} \in \Omega$ *and* $a \in \mathbb{R}^2$.

(iii) If $\theta \in C^1(\mathbb{R},\mathbb{R})$ *has* $|\theta'(\xi)| < \epsilon$ *for all* $\xi \in \mathbb{R}$ *and* $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ *satisfies* $\Lambda(\theta,\overline{\omega})v = 0$ *for some* $\overline{\omega} \in \Omega$ *and* $\eta \in [\eta_{\min},\eta_{\max}]$*, then*

$$v = E^{\mathrm{gb}}(\theta,\overline{\omega})\Pi T_{-\theta(0)}\mathrm{pev}_0 v. \tag{6.10}$$

We recall the four constants $\eta^*_{\min} < \eta_{\min} < \eta_{\max} < \eta^*_{\max}$ that were introduced in Section 5 and fix these for use throughout the current section. Without loss of generality, we will assume that the results in Lemma 4.3 hold for this choice of $[\eta_{\min},\eta_{\max}]$. To prevent confusion, for any $\vartheta \in \mathbb{R}$ we will use the notation $[\vartheta]$ to represent the constant function $\theta = \vartheta\mathbf{1}$. We start by solving (6.1) for such constant functions $\theta = [\vartheta]$. The following result shows how this is closely related to solving the local equation (5.1).

Lemma 6.3. *Consider the linear system (6.1) and suppose that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Fix any sufficiently small open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$. Then for every $\vartheta \in \mathbb{R}$, $\eta \in [\eta_{\min}, \eta_{\max}]$ and $\overline{\omega} \in \Omega$ there is a linear operator*

$$\mathcal{K}_{\eta}^{\perp}(\vartheta, \overline{\omega}) : BY_{\eta}(\mathbb{R}, H^3 \times H^2) \to BY_{\eta}(\mathbb{R}, H^3 \times H^2) \cap BC_{\eta}(\mathbb{R}, H^2 \times H^1), \tag{6.11}$$

that satisfies the following properties.

(i) For any $f \in BY_{\eta}(\mathbb{R}, H^3 \times H^2)$, the function $x = \mathcal{K}_{\eta}^{\perp}(\vartheta, \overline{\omega}) f$ solves $\Lambda([\vartheta], \overline{\omega}) = f$.

(ii) The operators $\mathcal{K}_{\eta}^{\perp}(\vartheta, \overline{\omega})$ can be bounded independently of $\overline{\omega} \in \Omega$, $\vartheta \in \mathbb{R}$ and $\eta \in [\eta_{\min}, \eta_{\max}]$.

(iii) Recall the integer r defined in (Hg). For every $\eta \in [\eta_{\min}, \eta_{\max}]$, the map $(\vartheta, \overline{\omega}) \mapsto \mathcal{K}_{\eta}^{\perp}(\vartheta, \overline{\omega})$ is C^{r-3}-smooth.

(iv) For each $\overline{\omega} \in \Omega$ and $\eta \in [\eta_{\min}, \eta_{\max}]$, the set $\mathcal{N}_{\eta}^{\perp}(\vartheta, \overline{\omega})$ that contains all $v \in BC_{\eta}(\mathbb{R}, H^2 \times H^1)$ that have $\Lambda([\vartheta], \overline{\omega}) v = 0$ is two dimensional. For each $a \in \mathbb{R}^2$, there is a unique $v \in \mathcal{N}_{\eta}^{\perp}(\vartheta, \overline{\omega})$ that has $\Pi T_{-\vartheta} \mathrm{pev}_0 v = a$.

Proof. Fix any $\eta \in [\eta_{\min}, \eta_{\max}]$. Without loss of generality, assume that $\vartheta = 0$. As a preparation, let us note that for any $\alpha \in BC_{\eta}(\mathbb{R}, \mathbb{R})$ we have

$$L(\overline{\omega}) \mathrm{vev}_{\xi} \alpha \mathbf{u}_0' = -\overline{\omega}\big(0, u''(k_0)\big) \alpha(\xi) + QM \mathrm{cev}_{\xi} \alpha. \tag{6.12}$$

Consider any $f \in BY_{\eta}(\mathbb{R}, H^3 \times H^2)$ and introduce the function $\beta \in BC_{\eta}(\mathbb{R}, \mathbb{R})$ that is given by

$$\beta(\xi) = -\frac{1}{\gamma} \int_0^{\xi} Qf(\zeta) d\zeta. \tag{6.13}$$

Since $\eta_{\min} > 0$, the map $f \mapsto \beta$ defined this way can be bounded uniformly for $\eta \in [\eta_{\min}, \eta_{\max}]$. Let us also introduce the function $g \in BY_{\eta}(\mathbb{R}, H^3 \times H^2)$ that is given by

$$g(\xi) \quad = \quad [I - P] f(\xi) + [I - P] M \mathrm{cev}_{\xi} \mathcal{K}_{\eta} Q L(\overline{\omega}) \beta \mathbf{u}_0' + [I - P] L(\overline{\omega}) \mathrm{vev}_{\xi} \beta \mathbf{u}_0'. \tag{6.14}$$

Note that after decreasing the size of Ω, it is possible to find $w \in BC_{\eta}(\mathbb{R}, H^2 \times H^1) \cap BY_{\eta}(\mathbb{R}, H^3 \times H^2)$ that has

$$w = \mathcal{K}_{\eta}^{\mathrm{lc}}(0, \overline{\omega})[g(\xi') + \overline{\omega}[I - P] M \mathrm{cev}_{\xi'} \mathcal{K}_{\eta}[Q(0, u''(k_0))] Q w + \overline{\omega}[I - P](0, u''(k_0)) Q w(\xi')]. \tag{6.15}$$

Related to w, we also consider the functions $v \in BC_{\eta}(\mathbb{R}, H^2 \times H^1) \cap BY_{\eta}(\mathbb{R}, H^3 \times H^2)$ and $\alpha \in BC_{\eta}(\mathbb{R}, \mathbb{R})$ that are given by

$$\begin{array}{rcl} \alpha(\xi) & = & Q w(\xi), \\ v(\xi) & = & [I - P] w(\xi) + \beta(\xi) \mathbf{u}_0'. \end{array} \tag{6.16}$$

Notice that $w = v + (\alpha - \beta) \mathbf{u}_0'$. A short computation now yields

$$\begin{array}{rcl} -\gamma \alpha'(\xi) & = & -\gamma Q \partial_{\xi} w(\xi) = Q L(\overline{\omega}) \mathrm{vev}_{\xi}[v + (\alpha - \beta) \mathbf{u}_0'] + Q g \\ & = & Q L(\overline{\omega}) \mathrm{vev}_{\xi} v + Q M \mathrm{cev}_{\xi} \alpha - Q L(\overline{\omega}) \mathrm{vev}_{\xi} \beta \mathbf{u}_0' - \overline{\omega} Q(0, u''(k_0)) \alpha(\xi), \end{array} \tag{6.17}$$

from which we find, using $\alpha(0) = 0$,

$$\alpha(\xi) = \Theta_{\mathrm{L}}([0], \overline{\omega}) v - \mathcal{K} Q L(\overline{\omega}) \beta \mathbf{u}_0' - \overline{\omega} \mathcal{K} Q(0, u''(k_0)) \alpha. \tag{6.18}$$

We can now calculate

$$\begin{array}{rcl} -\gamma\partial_\xi v(\xi) & = & -\gamma[I-P]\partial_\xi w(\xi) + Pf(\xi) \\ & = & [I-P]L(\overline{\omega})\mathrm{vev}_\xi[v + (\alpha-\beta)\mathbf{u}_0'] + Pf(\xi) \\ & & +[I-P]g(\xi) + \overline{\omega}[I-P]M\mathrm{cev}_\xi\mathcal{K}Q(0,u''(k_0))\alpha + \overline{\omega}[I-P](0,u''(k_0))\alpha(\xi) \\ & = & [I-P]L(\overline{\omega})\mathrm{vev}_\xi v + [I-P]M\mathrm{cev}_\xi\alpha - [I-P]L(\overline{\omega})\mathrm{vev}_\xi\beta\mathbf{u}_0' + Pf(\xi) \\ & & +[I-P]g(\xi) + \overline{\omega}[I-P]M\mathrm{cev}_\xi\mathcal{K}Q(0,u''(k_0))\alpha \\ & = & [I-P]L(\overline{\omega})\mathrm{vev}_\xi v + [I-P]M\mathrm{cev}_\xi\Theta_{\mathrm{L}}([0],\overline{\omega})v + f(\xi), \end{array} \tag{6.19}$$

which implies that we may write $\mathcal{K}_\eta^\perp(0,\overline{\omega})f = v$. Since the auxiliary functions g and β can both be bounded uniformly with respect to the norm of f, item (ii) follows directly from Proposition 5.1. In addition, the auxiliary function g depends smoothly on the pair $(\vartheta, \overline{\omega})$, hence (iii) also follows from Proposition 5.1.

To see (iv), suppose that $\Lambda([0],\overline{\omega})v = 0$ for some $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ and write

$$\begin{array}{rcl} \alpha & = & \Theta_{\mathrm{L}}([0],\overline{\omega})v = \mathcal{K}QL(\overline{\omega})v, \\ w & = & v + \alpha\mathbf{u}_0'. \end{array} \tag{6.20}$$

Since $\alpha(0) = 0$ and $Pv(\xi) = Pv(0)$ for all $\xi \in \mathbb{R}$, we find $w(0) = v(0)$ and

$$v(\xi) = Pv(0) + [I-P]v(\xi) = Pv(0) + [I-P]w(\xi). \tag{6.21}$$

Due to our choice of α, we may use (6.12) to compute

$$\begin{array}{rcl} -\gamma\partial_\xi w(\xi) & = & [I-P]L(\overline{\omega})\mathrm{vev}_\xi[w - \alpha\mathbf{u}_0'] + [I-P]M\mathrm{cev}_\xi\Theta_{\mathrm{L}}([0],\overline{\omega})v - \gamma\alpha'(\xi)\mathbf{u}_0' \\ & = & [I-P]L(\overline{\omega})\mathrm{vev}_\xi w + \overline{\omega}[I-P](0,u''(k_0))\alpha(\xi) - \gamma\alpha'(\xi)\mathbf{u}_0'. \end{array} \tag{6.22}$$

In addition, notice that

$$\begin{array}{rcl} QL(\overline{\omega})\mathrm{vev}_\xi w & = & QL(\overline{\omega})\mathrm{vev}_\xi v + QL(\overline{\omega})\mathrm{vev}_\xi\alpha\mathbf{u}_0' \\ & = & QL(\overline{\omega})\mathrm{vev}_\xi v + QM\mathrm{cev}_\xi\alpha - \overline{\omega}Q(0,u''(k_0))\alpha(\xi) \\ & = & -\gamma\alpha'(\xi) - \overline{\omega}Q(0,u''(k_0))\alpha(\xi), \end{array} \tag{6.23}$$

which shows that

$$\begin{array}{rcl} -\gamma w'(\xi) & = & L(\overline{\omega})\mathrm{vev}_\xi w + \overline{\omega}(0,u''(k_0))\alpha(\xi) \\ & = & L(\overline{\omega})\mathrm{vev}_\xi w + \overline{\omega}(0,u''(k_0))Qw(\xi) - \overline{\omega}(0,u''(k_0))Qv(0). \end{array} \tag{6.24}$$

This implies that we must have

$$w = [I - \overline{\omega}\mathcal{K}_\eta^{\mathrm{lc}}(0,\overline{\omega})(0,u''(k_0))Q]^{-1}\big[E^{\mathrm{lc}}(\overline{\omega})\Pi v(0) - \overline{\omega}\mathcal{K}_\eta^{\mathrm{lc}}(0,\overline{\omega})(0,u''(k_0))Qv(0)\big]. \tag{6.25}$$

Every $v \in \mathcal{N}_\eta^\perp(0,\overline{\omega})$ is thus uniquely determined by $\Pi v(0) \in \mathbb{R}^2$. Conversely, let us fix $a \in \mathbb{R}^2$ and define w according to (6.25) with $\Pi v(0)$ replaced by a and $Qv(0)$ replaced by a_2. Upon writing $v = Pw(0) + [I-P]w$, the computations above can be repeated to show that $v \in \mathcal{N}_\eta^\perp(0,\overline{\omega})$ with $\Pi v(0) = a$. □

Let us now consider a constant $\eta \in [\eta_{\min}, \eta_{\max}]$ together with a function $\theta \in C(\mathbb{R},\mathbb{R})$. As a second step towards solving $\Lambda(\theta,\overline{\omega}) = f$, we define an approximate inverse

$$\mathcal{K}_\eta^{\mathrm{apx}}(\theta,\overline{\omega}) : BY_\eta(\mathbb{R}, H^3 \times H^2) \to BY_\eta(\mathbb{R}, H^3 \times H^2) \cap BC_\eta(\mathbb{R}, H^2 \times H^1) \tag{6.26}$$

by means of

$$[\mathcal{K}_\eta^{\mathrm{apx}}(\theta,\overline{\omega})f](\xi) = \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}}\int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} \mathrm{pev}_\xi\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f\,d\zeta'\,d\zeta. \tag{6.27}$$

The integrals are necessary to ensure that $\mathcal{K}^{\mathrm{apx}}(\theta, \overline{\omega})$ is well-defined as a map into $BY_\eta(\mathbb{R}, H^3 \times H^2)$. Indeed, writing $v = \mathcal{K}^{\mathrm{apx}}_\eta(\theta, \overline{\omega}) f$, we can use Cauchy-Schwartz to compute

$$\begin{array}{lcl} \int_{\xi-1}^{\xi+1} |v(\zeta)|^2_{H^3 \times H^2} \, d\zeta & = & \int_{\xi-1}^{\xi+1} \Big| \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} \int_{\zeta'-\frac{1}{2}}^{\zeta'+\frac{1}{2}} \mathrm{pev}_\zeta \mathcal{K}^\perp_\eta(\theta(\zeta''), \overline{\omega}) f d\zeta'' d\zeta' \Big|^2_{H^3 \times H^2} d\zeta \\ & \leq & \int_{\xi-1}^{\xi+1} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} \int_{\zeta'-\frac{1}{2}}^{\zeta'+\frac{1}{2}} \big| \mathrm{pev}_\zeta \mathcal{K}^\perp_\eta(\theta(\zeta''), \overline{\omega}) f \big|^2_{H^3 \times H^2} d\zeta'' d\zeta' d\zeta. \end{array} \tag{6.28}$$

Upon slightly extending the integration region and using Fubini to change the order of integration, we obtain

$$\begin{array}{lcl} \int_{\xi-1}^{\xi+1} |v(\zeta)|^2_{H^3 \times H^2} \, d\zeta & \leq & \int_{\xi-\frac{3}{2}}^{\xi+\frac{3}{2}} \int_{\zeta'-\frac{1}{2}}^{\zeta'+\frac{1}{2}} \int_{\zeta'-\frac{1}{2}}^{\zeta'+\frac{1}{2}} \big| \mathrm{pev}_\zeta \mathcal{K}^\perp_\eta(\theta(\zeta''), \overline{\omega}) f \big|^2_{H^3 \times H^2} d\zeta d\zeta'' d\zeta' \\ & \leq & C e^{2\eta|\xi|} \, \|f\|^2_{BY_\eta(\mathbb{R}, H^3 \times H^2)} . \end{array} \tag{6.29}$$

Note that a similar computation would have been possible in the presence of only a single integral in (6.27). The double integral will however be needed in the sequel.

To turn this approximate inverse into a full inverse, let us analyze the remainder

$$S_{\mathrm{rm}}(\theta, \overline{\omega}) f = \Lambda(\theta, \overline{\omega}) \mathcal{K}^{\mathrm{apx}}_\eta(\theta, \overline{\omega}) f - f \tag{6.30}$$

that is given by

$$\begin{array}{lcl} [S_{\mathrm{rm}}(\theta, \overline{\omega}) f](\xi) & = & -\gamma \int_{\xi-1}^{\xi} \mathrm{pev}_\xi [\mathcal{K}^\perp_\eta(\theta(\zeta'+1), \overline{\omega}) - \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega})] f d\zeta' \\ & & + \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} s_1(\theta, \overline{\omega}, \xi, \zeta') d\zeta' d\zeta \\ & & + \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} [I - P_{\theta(\xi)}] T_{\theta(\xi)} M \mathrm{cev}_\xi \mathcal{K}_\eta s_2(\theta, \overline{\omega}, \xi, \zeta', \cdot) d\zeta' d\zeta. \end{array} \tag{6.31}$$

Here we have introduced

$$\begin{array}{lcl} s_1(\theta, \overline{\omega}, \xi, \zeta') & = & [P_{\theta(\xi)} - P_{\theta(\zeta')}] L(\theta(\zeta'), \overline{\omega}) \mathrm{pev}_\xi \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f \\ & & + [I - P_{\theta(\xi)}] \big(0, (B(\theta(\zeta')) - B(\theta(\xi))) \pi_1 \mathrm{pev}_\xi \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f \big) \\ & & + [I - P_{\theta(\xi)}] \big(0, \sum_{j=0}^N A_j T_{-n_j k_0} \pi_1 \mathrm{pev}_{\xi+n_j} \\ & & \quad [\mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f - \mathcal{K}^\perp_\eta(\theta(\zeta'+n_j), \overline{\omega}) f] \big) \\ & & + [P_{\theta(\xi)} T_{\theta(\xi)} - P_{\theta(\zeta')} T_{\theta(\zeta')}] M \mathrm{cev}_\xi \mathcal{K}_\eta Q_{\theta(\zeta')} L(\theta(\zeta'), \overline{\omega}) \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f \end{array} \tag{6.32}$$

and

$$\begin{array}{lcl} s_2(\theta, \overline{\omega}, \xi, \zeta', \xi') & = & [Q_{\theta(\zeta')} - Q_{\theta(\xi')}] L(\theta(\zeta'), \overline{\omega}) \mathrm{vev}_{\xi'} \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f \\ & & + Q_{\theta(\xi')} \big(0, (B(\theta(\zeta')) - B(\theta(\xi'))) \pi_1 \mathrm{pev}_\xi \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) f \big) \\ & & + Q_{\theta(\xi')} L(\theta(\xi'), \overline{\omega}) \mathrm{vev}_{\xi'} [\mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega}) - \mathcal{K}^\perp_\eta(\theta(\zeta' + \xi' - \xi), \overline{\omega})] f \\ & & + Q_{\theta(\xi')} \big(0, \sum_{j=0}^N A_j T_{-n_j k_0} \pi_1 \mathrm{pev}_{\xi'+n_j} \\ & & \quad [\mathcal{K}^\perp_\eta(\theta(\zeta' + \xi' - \xi), \overline{\omega}) - \mathcal{K}^\perp_\eta(\theta(\zeta' + \xi' - \xi + n_j), \overline{\omega})] f \big). \end{array} \tag{6.33}$$

Using a computation similar to (6.29) one may verify that $S_{\mathrm{rm}} \in \mathcal{L}(BY_\eta(\mathbb{R}, H^3 \times H^2))$, but the size of this remainder will in general be too large for our purposes. To control the size of S_{rm}, we add cut-offs to this operator and write

$$\begin{array}{lcl} [S^{\mathrm{c}}_{\mathrm{rm}}(\theta, \overline{\omega}) f](\xi) & = & -\gamma \int_{\xi-1}^{\xi} \chi_\delta(|\theta(\zeta'+1) - \theta(\zeta')|) \mathrm{pev}_\xi [\mathcal{K}^\perp_\eta(\theta(\zeta'+1), \overline{\omega}) - \mathcal{K}^\perp_\eta(\theta(\zeta'), \overline{\omega})] f d\zeta' \\ & & + \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} s^{\mathrm{c}}_1(\theta, \overline{\omega}, \xi, \zeta') d\zeta' d\zeta \\ & & + \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} [I - P_{\theta(\xi)}] T_{\theta(\xi)} M \mathrm{cev}_\xi \mathcal{K}_\eta [s^{\mathrm{lc}}_2(\theta, \overline{\omega}, \xi, \zeta', \cdot) + s^{\mathrm{nlc}}_2(\theta, \overline{\omega}, \xi, \zeta', \cdot)] d\zeta' d\zeta, \end{array} \tag{6.34}$$

in which

$$
\begin{array}{rcl}
s_1^c(\theta,\overline{\omega},\xi,\zeta') & = & \chi_\delta(|\theta(\xi)-\theta(\zeta')|)[P_{\theta(\xi)}-P_{\theta(\zeta')}]L(\theta(\zeta'),\overline{\omega})\mathrm{pev}_\xi\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f \\
& & +\chi_\delta(|\theta(\xi)-\theta(\zeta')|)[I-P_{\theta(\xi)}]\big(0,[B(\theta(\zeta'))-B(\theta(\xi))]\pi_1\mathrm{pev}_\xi\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f\big) \\
& & +\chi_\delta(|\theta(\zeta'+n_j)-\theta(\zeta')|)[I-P_{\theta(\xi)}]\big(0,\sum_{j=0}^N A_jT_{-n_jk_0}\pi_1\mathrm{pev}_{\xi+n_j} \\
& & \quad[\mathcal{K}_\eta^\perp(\theta(\zeta'))f-\mathcal{K}_\eta^\perp(\theta(\zeta'+n_j),\overline{\omega})f]\big) \\
& & +\chi_\delta(|\theta(\xi)-\theta(\zeta')|)[P_{\theta(\xi)}T_{\theta(\xi)}-P_{\theta(\zeta')}T_{\theta(\zeta')}] \\
& & \quad M\mathrm{cev}_\xi\mathcal{K}_\eta Q_{\theta(\zeta')}L(\theta(\zeta'),\overline{\omega})\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f.
\end{array}
\tag{6.35}
$$

The non-local cut-off s_2^{nlc} is defined by

$$
s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\xi')=s_2(\theta,\overline{\omega},\xi,\zeta',\xi') \tag{6.36}
$$

whenever $|\xi'-\xi|\geq\xi_{\mathrm{co}}$ and $s_2^{\mathrm{nlc}}=0$ otherwise. By contrast, the local cut-off s_2^{lc} is defined by

$$
\begin{array}{rcl}
s_2^{\mathrm{lc}}(\theta,\overline{\omega},\xi,\zeta',\xi') & = & \chi_\delta(|\theta(\xi')-\theta(\zeta')|)[Q_{\theta(\zeta')}-Q_{\theta(\xi')}]L(\theta(\zeta'),\overline{\omega})\mathrm{vev}_{\xi'}\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f \\
& & +\chi_\delta(|\theta(\xi')-\theta(\zeta')|)Q_{\theta(\xi')}\big(0,[B(\theta(\zeta'))-B(\theta(\xi'))]\pi_1\mathrm{pev}_\xi\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})f\big) \\
& & +\chi_\delta(|\theta(\zeta')-\theta(\zeta'+\xi'-\xi)|)Q_{\theta(\xi')}L(\theta(\xi'),\overline{\omega})\mathrm{vev}_{\xi'} \\
& & \quad[\mathcal{K}_\eta^\perp(\theta(\zeta'),\overline{\omega})-\mathcal{K}_\eta^\perp(\theta(\zeta'+\xi'-\xi),\overline{\omega})]f \\
& & +\chi_\delta(|\theta(\zeta'+\xi'-\xi+n_j)-\theta(\zeta'+\xi'-\xi)|)Q_{\theta(\xi')}\big(0,\sum_{j=0}^N A_jT_{-n_jk_0}\pi_1\mathrm{pev}_{\xi'+n_j} \\
& & \quad[\mathcal{K}_\eta^\perp(\theta(\zeta'+\xi'-\xi),\overline{\omega})-\mathcal{K}_\eta^\perp(\theta(\zeta'+\xi'-\xi+n_j),\overline{\omega})]f\big),
\end{array}
\tag{6.37}
$$

whenever $|\xi'-\xi|<\xi_{\mathrm{co}}$ and $s_2^{\mathrm{lc}}=0$ otherwise.

Let us recall the following identity, which holds for every $\eta\in[\eta_{\min},\eta_{\max}]$,

$$
\mathrm{cev}_\xi\mathcal{K}_\eta s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\cdot)=\mathrm{cev}_0\mathcal{K}_\eta T_\xi^{(1)}s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\cdot). \tag{6.38}
$$

Using the fact that $\mathcal{K}_{\eta^*_{\max}}$ and $\mathcal{K}_\eta$ agree on $BC_\eta(\mathbb{R},\mathbb{R})$, we may compute

$$
\begin{array}{rcl}
\left|\mathrm{cev}_\xi\mathcal{K}_\eta s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\cdot)\right| & \leq & C\left\|T_\xi^{(1)}s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\cdot)\right\|_{\eta^*_{\max}} \\
& \leq & Ce^{-(\eta^*_{\max}-\eta)\xi_{\mathrm{co}}}\left\|T_\xi^{(1)}s_2^{\mathrm{nlc}}(\theta,\overline{\omega},\xi,\zeta',\cdot)\right\|_\eta \\
& \leq & Ce^{-(\eta^*_{\max}-\eta)\xi_{\mathrm{co}}}e^{\eta|\xi|}\left\|f\right\|_{BY_\eta}.
\end{array}
\tag{6.39}
$$

We hence find

$$
\|S^c_{\mathrm{rm}}\|_{\mathcal{L}(BY_\eta(\mathbb{R},H^3\times H^2))}=O(\delta+e^{-(\eta^*_{\max}-\eta)\xi_{\mathrm{co}}}) \tag{6.40}
$$

as $\delta\to0$ and $\xi_{\mathrm{co}}\to\infty$, which allows us to define the full inverse

$$
\widetilde{\mathcal{K}}_\eta(\theta,\overline{\omega})=\mathcal{K}_\eta^{\mathrm{apx}}(\theta,\overline{\omega})[I+S^c_{\mathrm{rm}}(\theta,\overline{\omega})]^{-1}, \tag{6.41}
$$

after fixing a sufficiently small $\delta>0$ and sufficiently large ξ_{co}.

We now proceed to find $v\in BC_\eta(\mathbb{R},H^2\times H^1)$ that have $\Lambda(\theta,\overline{\omega})v=0$ for some $\theta\in C(\mathbb{R},\mathbb{R})$ and $\overline{\omega}\in\Omega$. As a preparation, let us write

$$
E^\perp(\vartheta,\overline{\omega}):\mathbb{R}^2\to BC_{\eta_{\min}}(\mathbb{R},H^2\times H^1)\cap BY_{\eta_{\min}}(\mathbb{R},H^3\times H^2) \tag{6.42}
$$

for the linear operator induced by item (iv) of Lemma 6.3. This means that for any $a\in\mathbb{R}^2$ the function $v=E^\perp(\vartheta,\overline{\omega})a$ solves $\Lambda([\vartheta],\overline{\omega})v=0$ and has $\Pi T_{-\vartheta}\mathrm{pev}_0v=a$. We also introduce for $\theta_1,\theta_2\in C(\mathbb{R},\mathbb{R})$ and $\overline{\omega}\in\Omega$, the operator

$$
\mathcal{E}(\theta_1,\theta_2,\overline{\omega}):BC_\eta(\mathbb{R},H^2\times H^1)\to BY_\eta(\mathbb{R},H^3\times H^2) \tag{6.43}
$$

that acts as

$$\mathcal{E}(\theta_1, \theta_2, \overline{\omega})v = [\Lambda(\theta_1, \overline{\omega}) - \Lambda(\theta_2, \overline{\omega})]v. \tag{6.44}$$

To see that this operator is well-defined, one may use the representation

$$[\mathcal{E}(\theta_1, \theta_2)v](\xi) \quad = \quad s_3(\theta_1, \theta_2, \overline{\omega}, \xi)v + [I - P_{\theta_1(\xi)}]T_{\theta_1(\xi)}M\mathrm{cev}_\xi\mathcal{K}s_4(\theta_1, \theta_2, \overline{\omega}, \cdot)v, \tag{6.45}$$

in which

$$\begin{array}{rcl} s_3(\theta_1, \theta_2, \overline{\omega}, \xi)v & = & [P_{\theta_1(\xi)} - P_{\theta_2(\xi)}]L(\theta_2(\xi), \overline{\omega})\mathrm{vev}_\xi v \\ & & +[I - P_{\theta_1(\xi)}]\big(0, [B(\theta_2(\xi)) - B(\theta_1(\xi))]\pi_1\mathrm{pev}_\xi v\big) \\ & & +[P_{\theta_1(\xi)}T_{\theta_1(\xi)} - P_{\theta_2(\xi)}T_{\theta_2(\xi)}]M\mathrm{cev}_\xi\Theta_{\mathrm{L}}(\theta_2, \overline{\omega})v \end{array} \tag{6.46}$$

and

$$\begin{array}{rcl} s_4(\theta_1, \theta_2, \overline{\omega}, \xi)v & = & [Q_{\theta_2(\xi)} - Q_{\theta_1(\xi)}]L(\theta_2(\xi), \overline{\omega})\mathrm{vev}_\xi v \\ & & +Q_{\theta_1(\xi)}\big(0, [B(\theta_2(\xi)) - B(\theta_1(\xi))]\pi_1\mathrm{pev}_\xi v\big). \end{array} \tag{6.47}$$

As before, we will need to put cut-offs on these functions. Let us write

$$\begin{array}{rcl} s_3^{\mathrm{lc}}(\theta_1, \theta_2, \overline{\omega}, \xi)v & = & \chi_{\delta_\theta}(|\theta_1(\xi) - \theta_2(\xi)|)s_3(\theta_1, \theta_2, \overline{\omega}, \xi)v, \\ s_4^{\mathrm{lc}}(\theta_1, \theta_2, \overline{\omega}, \xi)v & = & \chi_{\delta_\theta}(|\theta_1(\xi) - \theta_2(\xi)|)s_4(\theta_1, \theta_2, \overline{\omega}, \xi)v, \end{array} \tag{6.48}$$

whenever $|\xi| < \xi_{\mathrm{co}}$ and $s_3^{\mathrm{lc}} = s_4^{\mathrm{lc}} = 0$ otherwise. Conversely, we write $s_3^{\mathrm{nlc}}(\theta_1, \theta_2, \overline{\omega}, \xi)v = s_3(\theta_1, \theta_2, \overline{\omega}, \xi)v$ and $s_4^{\mathrm{nlc}}(\theta_1, \theta_2, \overline{\omega}, \xi)v = s_4(\theta_1, \theta_2, \overline{\omega}, \xi)v$ whenever $|\xi| \geq \xi_{\mathrm{co}}$ and $s_3^{\mathrm{nlc}} = s_4^{\mathrm{nlc}} = 0$ otherwise.

After applying these cut-offs to $\mathcal{E}$, we obtain the operator

$$\mathcal{E}^c(\theta_1, \theta_2, \overline{\omega}) : BC_\eta(\mathbb{R}, H^2 \times H^1) \to BY_\eta(\mathbb{R}, H^3 \times H^2) \tag{6.49}$$

that is given by

$$\begin{array}{rcl} [\mathcal{E}^c(\theta_1, \theta_2, \overline{\omega})v](\xi) & = & [s_3^{\mathrm{lc}}(\theta_1, \theta_2, \overline{\omega}, \xi) + s_3^{\mathrm{nlc}}(\theta_1, \theta_2, \overline{\omega}, \xi)]v \\ & & +[I - P_{\theta_1(\xi)}]T_{\theta_1(\xi)}M\mathrm{cev}_\xi\mathcal{K}_\eta[s_4^{\mathrm{lc}}(\theta_1, \theta_2, \overline{\omega}, \cdot) + s_4^{\mathrm{nlc}}(\theta_1, \theta_2, \overline{\omega}, \cdot)]v. \end{array} \tag{6.50}$$

A computation similar to (6.39) now yields

$$\|\mathcal{E}^c(\theta, [\theta(0)], \overline{\omega})\|_{\mathcal{L}(BC_\eta(\mathbb{R}, H^2 \times H^1), BY_{\eta^*_{\max}}(\mathbb{R}, H^3 \times H^2))} = O(\delta_\theta + e^{-(\eta^*_{\max} - \eta)\xi_{\mathrm{co}}}). \tag{6.51}$$

After choosing $\delta_\theta > 0$ to be sufficiently small and ξ_{co} to be sufficiently large, we may define

$$\widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega}) : \mathbb{R}^2 \to BC_{\eta_{\min}}(\mathbb{R}, H^2 \times H^1) \cap BY_{\eta_{\min}}(\mathbb{R}, H^3 \times H^2) \tag{6.52}$$

by means of

$$\begin{array}{rcl} \widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega})a & = & [E^\perp(\theta(0), \overline{\omega}) - \widetilde{\mathcal{K}}(\theta, \overline{\omega})\mathcal{E}^c(\theta, [\theta(0)], \overline{\omega})E^\perp(\theta(0), \overline{\omega})] \\ & & [I - \Pi T_{-\theta(0)}\mathrm{pev}_0\widetilde{\mathcal{K}}(\theta, \overline{\omega})\mathcal{E}^c(\theta, [\theta(0)], \overline{\omega})E^\perp(\theta(0), \overline{\omega})]^{-1}a. \end{array} \tag{6.53}$$

By construction we have $\Pi T_{-\theta(0)}\widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega})a = a$ and $\Lambda(\theta, \overline{\omega})\widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega})a = 0$, whenever $\|\theta'\|_\infty$ is sufficiently small to ensure that the cut-offs have no effect. We can now use these operators $\widetilde{E}^{\mathrm{gb}}$ to define the final inverses

$$\mathcal{K}_\eta^{\mathrm{gb}}(\theta, \overline{\omega})f = \widetilde{\mathcal{K}}(\theta, \overline{\omega})f - \widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega})\Pi T_{-\theta(0)}\mathrm{pev}_0\widetilde{\mathcal{K}}_\eta(\theta, \overline{\omega})f. \tag{6.54}$$

Note that this choice ensures that the normalization condition (ii) in Proposition 6.1 will hold.

The definition (6.53) is somewhat awkward to use in computations. However, we can use (6.54) to construct a more convenient alternative for the operators $\widetilde{E}^{\mathrm{gb}}(\theta, \overline{\omega})$. Indeed, let us consider any $\theta \in C(\mathbb{R}, \mathbb{R})$ and $\overline{\omega} \in \Omega$. For any $a \in \mathbb{R}^2$, we introduce the function

$$\begin{array}{rcl}[\mathcal{E}^c(\theta, \overline{\omega})a](\xi) & = & [s_3^{\mathrm{lc}}(\theta, [\theta(0)], \overline{\omega}, \xi) + s_3^{\mathrm{nlc}}(\theta, [\theta(0)], \overline{\omega}, \xi)]E^{\perp}(\theta(0), 0)a \\ & & +[I - P_{\theta(\xi)}]T_{\theta(\xi)}M\mathrm{cev}_{\xi}\mathcal{K}[s_4^{\mathrm{lc}}(\theta, [\theta(0)], \overline{\omega}, \cdot) + s_4^{\mathrm{nlc}}(\theta, [\theta(0)], \overline{\omega}, \cdot)]E^{\perp}(\theta(0), 0)a \\ & & +\overline{\omega}[I - P_{\theta(\xi)}](0, \partial_{\sigma}\pi_1 \mathrm{pev}_{\xi})E^{\perp}(\theta(0), 0)a \\ & & +\overline{\omega}[I - P_{\theta(\xi)}]T_{\theta(\xi)}M\mathrm{cev}_{\xi}\mathcal{K}Q_{\theta(\xi')}\overline{\omega}(0, \partial_{\sigma}\pi_1 \mathrm{pev}_{\xi'})E^{\perp}(\theta(0), 0)a.\end{array} \tag{6.55}$$

Using the fact that

$$E^{\perp}(\vartheta, 0)(a_1, a_2) = a_1 T_{\vartheta}\mathbf{u}_0' + a_2 T_{\vartheta}\mathbf{u}_1, \tag{6.56}$$

it is not hard to see that for any $a \in \mathbb{R}^2$ and any $\eta \in [\eta_{\min}, \eta_{\max}]$ we have

$$\mathcal{E}^c(\theta, \overline{\omega})a \in BC_{\eta}(\mathbb{R}, H^3 \times H^2). \tag{6.57}$$

In addition, if $|\theta'(\xi)|$ is sufficiently small for all $\xi \in \mathbb{R}$, then by construction

$$\Lambda(\theta, \overline{\omega})E^{\perp}(\theta(0), 0)a = \mathcal{E}^c(\theta, \overline{\omega})a. \tag{6.58}$$

We may now write

$$E^{\mathrm{gb}}(\theta, \overline{\omega})a \quad = \quad E^{\perp}(\theta(0), 0)a - \mathcal{K}^{\mathrm{gb}}(\theta, \overline{\omega})\mathcal{E}^c(\theta, \overline{\omega})a. \tag{6.59}$$

This construction ensures that the normalization condition (ii) in Proposition 6.2 holds. The next result shows that the operators $E^{\mathrm{gb}}(\theta, \overline{\omega})$ capture all solutions of (6.1) with $f = 0$, provided that $\|\theta'\|_{\infty}$ is sufficiently small.

Lemma 6.4. *Consider the linear system (6.1) and suppose that (Hg), (H$\mathcal{F}$), (HD) and (HL) are satisfied. Fix a sufficiently small $\epsilon > 0$ and choose a sufficiently small open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$. Then for any $\theta \in C^1(\mathbb{R}, \mathbb{R})$ that has $|\theta'(\xi)| < \epsilon$ for all $\xi \in \mathbb{R}$, the set*

$$\mathcal{N}_{\eta}^{\mathrm{gb}}(\theta, \overline{\omega}) = \{v \in BC_{\eta}(\mathbb{R}, H^2 \times H^1) \mid \Lambda(\theta, \overline{\omega})v = 0\} \tag{6.60}$$

is two dimensional for all $\eta \in [\eta_{\min}, \eta_{\max}]$.

Proof. Fix any $\eta \in [\eta_{\min}, \eta_{\max}]$ and $\overline{\omega} \in \Omega$. Consider any $\theta \in C^1(\mathbb{R}, \mathbb{R})$ that has $\theta(0) = 0$ and $|\theta'(\xi)| < \epsilon$ for all $\xi \in \mathbb{R}$. For any integer $j \in \mathbb{Z}_{\geq 0}$, we write

$$\theta^{(j)}(\xi) = \begin{cases} \theta(\xi) & \text{for } \xi \in [-j, j], \\ \theta(j) & \text{for } \xi \geq j, \\ \theta(-j) & \text{for } \xi \leq -j. \end{cases} \tag{6.61}$$

For any integer $\ell \in \mathbb{Z}_{\geq 0}$ and operator $E \in \mathcal{L}(\mathbb{R}^2, BC_{\eta}(\mathbb{R}, H^2 \times H^1)$, we introduce the following properties.

(I1) For every $a \in \mathbb{R}^2$, the identity $\Pi \mathrm{pev}_0 Ea = a$ holds.

(I2) For all $a \in \mathbb{R}^2$ we have $\Lambda(\theta^{(\ell)}, \overline{\omega})Ea = 0$.

(I3) Any $v \in BC_{\eta}(\mathbb{R}, H^2 \times H^1)$ that has $\Lambda(\theta^{(\ell)}, \overline{\omega})v = 0$ must satisfy $v = E\Pi \mathrm{pev}_0 v$.

We will inductively construct operators $E_\ell : \mathbb{R}^2 \to BC_\eta(\mathbb{R}, H^2 \times H^1)$ for all integers $\ell \geq 0$ that satisfy (I1) through (I3). Lemma 6.3 guarantees that the choice $E_0 = E^\perp(0, \overline{\omega})$ satisfies these properties. Let us now consider for $j \geq 1$ the linear problem

$$E = E_{j-1} - \mathcal{K}(\theta^{(j-1)}, \overline{\omega})\mathcal{E}(\theta^{(j)}, \theta^{(j-1)}, \overline{\omega})E, \tag{6.62}$$

in which we take $E \in \mathcal{L}(\mathbb{R}^2, BC_\eta(\mathbb{R}, H^2 \times H^1))$. One may easily verify that if E_{j-1} satisfies (I1) through (I3) with $\ell = j-1$, then any solution E of (6.62) will satisfy these properties for $\ell = j$. Notice that

$$\mathcal{E}(\theta^{(j)}, \theta^{(j-1)}, \overline{\omega})_{\mathcal{L}(BC_\eta(\mathbb{R},H^2\times H^1), BY_\eta(\mathbb{R},H^3\times H^2))} = O(\epsilon). \tag{6.63}$$

After choosing $\epsilon > 0$ to be sufficiently small, we can hence write

$$E_\ell = [I + \mathcal{K}(\theta^{(\ell-1)}, \overline{\omega})\mathcal{E}(\theta^{(\ell)}, \theta^{(\ell-1)}, \overline{\omega})]^{-1} E_{\ell-1}, \tag{6.64}$$

which completes the construction of the operators $\{E_\ell\}_{\ell \geq 0}$ that satisfy (I1) through (I3).

Now, suppose that for some nonzero $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ we have $\Lambda(\theta, \overline{\omega})v = 0$ but $\Pi \mathrm{pev}_0 v = 0$. On account of (I1) and (I3), we must have

$$v = \mathcal{K}_\eta(\theta^{(\ell)}, \overline{\omega})\mathcal{E}(\theta^{(\ell)}, \theta, \overline{\omega})v \tag{6.65}$$

for any integer $\ell \geq 0$. Notice that $s_3(\theta^{(\ell)}, \theta, \xi) = 0$ for all $|\xi| \leq \ell$, while an analogous identity holds for s_4. We hence find the bound

$$\mathcal{E}(\theta^{(\ell)}, \theta, \overline{\omega})_{\mathcal{L}(BC_\eta(\mathbb{R},H^2\times H^1), BY_{\eta^*_{\max}}(\mathbb{R},H^3\times H^2))} = O(e^{-(\eta^*_{\max} - \eta)\ell}), \tag{6.66}$$

as $\ell \to \infty$. This implies that for some constant $C > 0$ and all large ℓ we have

$$\|v\|_{BC_{\eta^*_{\max}}(\mathbb{R},H^2\times H^1)} \leq Ce^{-(\eta^*_{\max}-\eta)\ell} \|v\|_{BC_\eta(\mathbb{R},H^2\times H^1)}, \tag{6.67}$$

which implies $v = 0$ and completes the proof. □

For future use, we define the shorthands

$$\begin{aligned} \mathcal{E}^c_1(\theta, \overline{\omega}) &= \mathcal{E}^c(\theta, \overline{\omega})(1, 0), \\ \mathcal{E}^c_2(\theta, \overline{\omega}) &= \mathcal{E}^c(\theta, \overline{\omega})(0, 1). \end{aligned} \tag{6.68}$$

Inspection of (6.55) and (6.56) shows that for $i \in \{1, 2\}$ and any $\eta \in [\eta_{\min}, \eta_{\max}]$ we have

$$\|\mathcal{E}^c_i(\theta, \overline{\omega})\|_{BC_\eta(\mathbb{R},H^3\times H^2)} = O\big(\delta_\theta + e^{-\eta\xi_{\mathrm{co}}} + |\overline{\omega}|\,\big) \tag{6.69}$$

as $\delta_\theta, \overline{\omega} \to 0$ and $\xi_{\mathrm{co}} \to \infty$. We are now ready to provide the proof of the main results of this section.

Proof of Proposition 6.1. The operators $\mathcal{K}^{\mathrm{gb}}_\eta$ have been defined in (6.54). Statements (i), (ii) and (v) follow directly from this construction, while (iii) is a consequence of Lemma 6.3. To see (iv), let us write $w = \widetilde{\mathcal{K}}(\theta_1, \overline{\omega}_1)f - \widetilde{\mathcal{K}}(\theta_2, \overline{\omega}_2)f$ and estimate

$$\begin{array}{lcl} \|w\|_{BY_{\eta_3}} & \leq & \big\|[\mathcal{K}^{\mathrm{apx}}(\theta_1, \overline{\omega}_1) - \mathcal{K}^{\mathrm{apx}}(\theta_2, \overline{\omega}_2)][I + S^c_{\mathrm{rm}}(\theta_1, \overline{\omega}_1)]^{-1} f\big\|_{BY_{\eta_3}} \\ & & + \big\|\mathcal{K}^{\mathrm{apx}}(\theta_2, \overline{\omega})[[I + S^c_{\mathrm{rm}}(\theta_1, \overline{\omega}_1)]^{-1} - [I + S^c_{\mathrm{rm}}(\theta_2, \overline{\omega}_2)]^{-1}]f\big\|_{BY_{\eta_3}}. \end{array} \tag{6.70}$$

Let us consider $h \in BY_{\eta_2}(\mathbb{R}, H^3 \times H^2)$ and write $z = [\mathcal{K}^{\mathrm{apx}}(\theta_1, \overline{\omega}_1) - \mathcal{K}^{\mathrm{apx}}(\theta_2, \overline{\omega}_2)]h$. In view of item (iii) in Lemma 6.3, we obtain the estimate

$$\begin{array}{lcl} |z(\xi)|_{H^2\times H^1} & = & \Big|\int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} \mathrm{pev}_\xi[\mathcal{K}^\perp(\theta_1(\zeta'), \overline{\omega}_1)h - \mathcal{K}^\perp(\theta_2(\zeta'), \overline{\omega}_2)h]d\zeta' d\zeta\Big| \\ & \leq & \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} \int_{\zeta-\frac{1}{2}}^{\zeta+\frac{1}{2}} C_1\big[\,|\theta_1(\zeta') - \theta_2(\zeta')| + |\overline{\omega}_1 - \overline{\omega}_2|\,\big] e^{\eta_2|\xi|} \|h\|_{BY_{\eta_2}} d\zeta' d\zeta \\ & \leq & C_2 e^{\eta_1|\xi|}\big[\,\|\theta_1 - \theta_2\|_{\eta_1} + |\overline{\omega}_1 - \overline{\omega}_2|\,\big] e^{\eta_2|\xi|} \|h\|_{BY_{\eta_2}}. \end{array} \tag{6.71}$$

This shows that

$$\|z\|_{BC_{\eta_3}(\mathbb{R},H^2\times H^1)} \leq C_2\big[\,\|\theta_1-\theta_2\|_{\eta_1} + |\overline{\omega}_1-\overline{\omega}_2|\,\big]\,\|h\|_{BY_{\eta_2}(\mathbb{R},H^3\times H^2)}\,. \tag{6.72}$$

An analogous computation involving an application of Fubini as in (6.29) shows that $\|z\|_{BY_{\eta_3}(\mathbb{R},H^3\times H^2)}$ shares this estimate. In order to bound the second line of (6.70), we write

$$\Delta = [I + S^c_{\mathrm{rm}}(\theta_1,\overline{\omega}_1)]^{-1} - [I + S^c_{\mathrm{rm}}(\theta_2,\overline{\omega}_2)]^{-1} \tag{6.73}$$

and observe

$$\Delta = [I + S^c_{\mathrm{rm}}(\theta_2,\overline{\omega}_2)]^{-1}[S^c_{\mathrm{rm}}(\theta_2,\overline{\omega}_2) - S^c_{\mathrm{rm}}(\theta_1,\overline{\omega}_1)][I + S^c_{\mathrm{rm}}(\theta_1,\overline{\omega}_1)]^{-1}. \tag{6.74}$$

Inspection of (6.34) shows that we may use a similar computation as above to compute

$$\|S^c_{\mathrm{rm}}(\theta_2,\overline{\omega}_2)w - S^c_{\mathrm{rm}}(\theta_1,\overline{\omega}_1)w\|_{BY_{\eta_3}} \leq C\big[\,\|\theta_1-\theta_2\|_{\eta_1} + |\overline{\omega}_1-\overline{\omega}_2|\,\big]\,\|w\|_{BY_{\eta_2}(\mathbb{R},H^3\times H^2)}\,. \tag{6.75}$$

We thus see that the estimate (iv) holds for the operators $\widetilde{\mathcal{K}}$. The constructions in (6.53) and (6.54) ensure that (iv) carries over to the final inverses $\mathcal{K}^{\mathrm{gb}}$.

Finally, the smoothness property (vi) can be established using the representation (6.70) together with the smoothness of $(\theta,\overline{\omega}) \mapsto \mathcal{K}^{\perp}(\theta,\overline{\omega})$. Very similar arguments can be found in [53, Lemma 2.5] and in [56]. □

Proof of Proposition 6.2. The results follow from Lemma 6.4 and the construction of E^{gb} in (6.59). □

7 The Center Manifold

In this section, we set out to construct a center manifold for the system (4.47) and prove Theorem 4.5. This construction will proceed in three steps, which we roughly outline here in simplified form. In the first step, we will fix a function $h : \mathbb{R}^3 \to H^2 \times H^1$ and assume that solutions to (4.47) can be written in the form

$$v(\xi) = \alpha(\xi)T_{\theta(\xi)}\mathbf{u}_0' + \beta(\xi)T_{\theta(\xi)}\mathbf{u}_1 + h(\alpha(\xi),\beta(\xi),\theta(\xi)), \tag{7.1}$$

for $\mathbb{R}$-valued functions α, β and θ. Plugging this Ansatz into (4.47) will allow us to derive a differential equation for the triple (α,β,θ). This equation will turn out to be a functional differential equation of mixed type. Using techniques developed in [32], we will see that for each suitable h, the functions α, β and θ are uniquely determined after fixing values for $\alpha(0)$, $\beta(0)$ and $\theta(0)$.

In the second step of the construction, we will consider the identity

$$\mathrm{pev}_0 v = \mathrm{pev}_0 E^{\mathrm{gb}}(\theta,\overline{\omega})\Pi T_{-\theta(0)}\mathrm{pev}_0 v + \mathrm{pev}_0\mathcal{K}^{\mathrm{gb}}(\theta,\overline{\omega})\mathcal{R}^c(\theta,v,\overline{\omega}). \tag{7.2}$$

In view of the previous step, the right hand side of this equation will depend solely on $\alpha(0)$, $\beta(0)$, $\theta(0)$ and h. Plugging the Ansatz (7.1) into the left hand side of this identity, we uncover a fixed point equation for the function h that will admit a unique fixed point h^*. Establishing that the Ansatz (7.1) with $h = h^*$ indeed captures all sufficiently small solutions to (4.47) will be the final step in the construction of the center manifold.

Throughout this entire section, it will be a standing assumption that the conditions (Hg), (H$\mathcal{F}$), (HD), (HL), (H$\mathcal{T}$1) and (H$\mathcal{T}$2) are all satisfied. In addition, we fix two constants $0 < \eta_{\min} < \eta_{\max}$ and an open set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ in such a way that the results in Lemma 4.3 and Propositions 5.1, 6.1 and 6.2 all hold. Unless explicitly stated otherwise, the inner products that appear in this section are those defined on $H^1 \times H^0$.

We start our analysis by introducing the domain $\mathcal{D}_{\delta_v} \subset \mathbb{R}^3$ given by

$$\mathcal{D}_{\delta_v} = [-2\delta_v, 2\delta_v] \times [-2\delta_v, 2\delta_v] \times [0, 2\pi], \tag{7.3}$$

together with the function space

$$\mathcal{H}_{\delta_v} = \{h : \mathcal{D}_{\delta_v} \to (H^2)^{N+2} \times H^1 \mid \left|h(\psi^1) - h(\psi^2)\right| \le \left|\psi^1 - \psi^2\right| \text{ and } |h(\psi)| \le \delta_v\}, \tag{7.4}$$

equipped with the norm

$$\|h\|_\infty = \sup_{\psi \in \mathcal{D}_{\delta_v}} |h(\psi)|_{(H^2)^{N+2} \times H^1} . \tag{7.5}$$

Throughout the sequel, we will use the notation $\psi = (\psi_\alpha, \psi_\beta, \psi_\theta)$ for vectors $\psi \in \mathbb{R}^3$ and $\Psi = (\Psi_\alpha, \Psi_\beta, \Psi_\theta)$ for functions $\Psi \in C(\mathbb{R}, \mathbb{R}^3)$. We modify the Ansatz (7.1) slightly and look for functions $v \in C(\mathbb{R}, H^2 \times H^1)$ that satisfy

$$\mathrm{vev}_\xi v = \Psi_\alpha(\xi) T_{\Psi_\theta(\xi)} \mathbf{u}_0' \mathbf{1} + \Psi_\beta(\xi) T_{\Psi_\theta(\xi)} \mathbf{u}_1 \mathbf{1} + h(\Psi(\xi)), \tag{7.6}$$

for some $\Psi \in C(\mathbb{R}, \mathbb{R}^3)$ and $h \in \mathcal{H}_{\delta_v}$. To accommodate this way of writing v, we introduce the new cut-off nonlinearities

$$\begin{array}{lcl} G^c(\psi, h) & = & \chi_{\delta_v}\big(|(\psi_\alpha, \psi_\beta)|\big) G\big(\psi_\theta, T_{\psi_\theta}[\psi_\alpha \mathbf{u}_0' + \psi_\beta \mathbf{u}_1] + \mathrm{pev}_0 h(\psi)\big), \\ H^c(\mathrm{ev}_\xi \Psi_\theta) & = & \chi_{\delta_\theta}(|\mathrm{cev}_\xi \Psi_\theta|) H(\mathrm{ev}_\xi \Psi_\theta), \\ V^c(\mathrm{ev}_\xi \Psi_\theta, \psi, h) & = & \chi_{\delta_\theta}(|\mathrm{cev}_\xi \Psi_\theta|) \chi_{\delta_v}(|(\psi_\alpha, \psi_\beta)|) \\ & & \quad V\big(\psi_\theta, T_{\psi_\theta}[\psi_\alpha \mathbf{u}_0' + \psi_\beta \mathbf{u}_1] + \mathrm{pev}_0 h(\psi)\big). \end{array} \tag{7.7}$$

In view of (7.6) it should be clear how the expression $\mathrm{pev}_0 h$ in the definitions above should be interpreted. Note also that the cut-offs ensure that these expressions are well-defined even for $\psi, \Psi(\xi) \notin \mathcal{D}_{\delta_v}$. Throughout this section it will be a standing assumption that $\delta_v \le \delta_\theta$. For quantities a and b that depend on these cut-offs δ_θ and δ_v, we will use the notation

$$a \le_* b \tag{7.8}$$

to express the fact that there exists a $C > 0$ that does not depend on δ_θ and δ_v, such that $a \le Cb$ holds for all $\delta_\theta \le 1$ and $\delta_v \le 1$. Note that for $\psi^1, \psi^2 \in \mathcal{D}_{\delta_v}$ and $h^1, h^2 \in \mathcal{H}_{\delta_v}$ we have the bound

$$|\Delta|_{(H^2)^{N+2} \times H^1} \le_* \left|\psi^1 - \psi^2\right| + \left\|h^1 - h^2\right\| \tag{7.9}$$

for the quantity

$$\Delta = T_{\psi_\theta^1}[\psi_\alpha^1 \mathbf{u}_0' + \psi_\beta^1 \mathbf{u}_1]\mathbf{1} + h^1(\psi^1) - T_{\psi_\theta^2}[\psi_\alpha^2 \mathbf{u}_0' + \psi_\beta^2 \mathbf{u}_1]\mathbf{1} - h^2(\psi^2). \tag{7.10}$$

Using the estimates in Lemmas 4.1 and 4.2 and computations similar to those in the proof of Lemma 4.4, we obtain the estimates

$$\begin{array}{lcl} |G^c(\psi, h)|_{H^3 \times H^2} & \le_* & \delta_v^2, \\ \left|G^c(\psi^1, h^1) - G^c(\psi^2, h^2)\right|_{H^3 \times H^2} & \le_* & \delta_v \left|\psi^1 - \psi^2\right| + \delta_v \left\|h^1 - h^2\right\|, \\ |H^c(\mathrm{ev}_\xi \Psi_\theta)|_{H^3 \times H^2} & \le_* & \delta_\theta^2, \\ \left|H^c(\mathrm{ev}_\xi \Psi_\theta^1) - H^c(\mathrm{ev}_\xi \Psi_\theta^2)\right|_{H^3 \times H^2} & \le_* & \delta_\theta \left|\mathrm{ev}_\xi \Psi_\theta^1 - \mathrm{ev}_\xi \Psi_\theta^2\right|_{\mathbb{R}^{N+2}}, \end{array} \tag{7.11}$$

which hold for all $h, h^1, h^2 \in \mathcal{H}_{\delta_v}$, $\psi, \psi^1, \psi^2 \in \mathbb{R}^3$ and $\Psi, \Psi^1, \Psi^2 \in C(\mathbb{R}, \mathbb{R}^3)$. It is also not hard to see that

$$\begin{array}{lcl} |V^c(\mathrm{ev}_\xi \Psi, \psi, h)| & \le_* & \delta_v, \\ \left|V^c(\mathrm{ev}_\xi \Psi_\theta^1, \psi^1, h^1) - V^c(\mathrm{ev}_\xi \Psi_\theta^2, \psi^2, h^2)\right| & \le_* & \left|\mathrm{ev}_\xi \Psi_\theta^1 - \mathrm{ev}_\xi \Psi_\theta^2\right|_{\mathbb{R}^{N+2}} + \left|\psi^1 - \psi^2\right| + \left\|h^1 - h^2\right\|. \end{array} \tag{7.12}$$

Let us also define, for $\Psi \in C(\mathbb{R}, \mathbb{R}^3)$ and $h \in \mathcal{H}_{\delta_v}$, the function

$$\begin{array}{rcl}\mathcal{S}^c(\Psi, h, \overline{\omega})(\xi) & = & Q_{\Psi_\theta(\xi)} G^c(\Psi(\xi), h) + Q_{\Psi_\theta(\xi)} H^c(\mathrm{ev}_\xi \Psi_\theta) \\ & & + V^c(\mathrm{ev}_\xi \Psi_\theta, \Psi(\xi), h) \Big[QM \mathrm{cev}_\xi \Psi_\theta + QL(\overline{\omega})[\Psi_\alpha(\xi')\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi')\mathbf{u}_1\mathbf{1}] \\ & & + Q_{\Psi_\theta(\xi)} L(\Psi_\theta(\xi), \overline{\omega}) h(\Psi(\xi)) + Q_{\Psi_\theta(\xi)} G^c(\Psi(\xi), h) + Q_{\Psi_\theta(\xi)} H^c(\mathrm{ev}_\xi \Psi_\theta) \Big],\end{array} \tag{7.13}$$

together with

$$\Theta^c_{\mathrm{NL}}(\Psi, h, \overline{\omega}) = \mathcal{K}\mathcal{S}^c(\Psi, h, \overline{\omega}). \tag{7.14}$$

We obtain the estimates

$$\begin{array}{lcl}\left|\mathcal{S}^c(\Psi, h, \overline{\omega})(\xi)\right| & \leq_* & (\delta_v + \delta_\theta)^2, \\ \left|\mathcal{S}^c(\Psi^1, h^1, \overline{\omega})(\xi) - \mathcal{S}^c(\Psi^2, h^2, \overline{\omega})(\xi)\right| & \leq_* & (\delta_v + \delta_\theta)\big[\left|\mathrm{ev}_\xi \Psi^1_\theta - \mathrm{ev}_\xi \Psi^2_\theta\right| \\ & & \quad + \left|\Psi^1(\xi) - \Psi^2(\xi)\right| + \left\|h^1 - h^2\right\| \big].\end{array} \tag{7.15}$$

We are now ready to introduce our final nonlinearity

$$\begin{array}{rcl}\mathcal{R}^c(\Psi, h, \overline{\omega})(\xi) & = & [I - P_{\Psi_\theta(\xi)}] G^c(\Psi(\xi), h) + [I - P_{\Psi_\theta(\xi)}] H^c(\mathrm{ev}_\xi \Psi_\theta) - \overline{\omega} T_{\Psi_\theta(\xi)}(0, u'(k_0)) \\ & & + [I - P_{\Psi_\theta(\xi)}] T_{\Psi_\theta(\xi)} M \mathrm{cev}_\xi \Theta^c_{\mathrm{NL}}(\Psi, h, \overline{\omega}) \\ & & - P_{\Psi_\theta(\xi)} V^c(\mathrm{ev}_\xi \Psi_\theta, \Psi(\xi), h) \Big[T_{\Psi_\theta(\xi)} M \mathrm{cev}_\xi \Psi_\theta + G^c(\Psi(\xi), h) + H^c(\mathrm{ev}_\xi \Psi_\theta) \\ & & + L(\Psi_\theta(\xi), \overline{\omega}) T_{\Psi_\theta(\xi)}[\Psi_\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi)\mathbf{u}_1\mathbf{1}] + L(\Psi_\theta(\xi), \overline{\omega}) h(\Psi(\xi)) \Big].\end{array} \tag{7.16}$$

Using (7.11) and (7.12) we obtain

$$\begin{array}{lcl}\left|\mathcal{R}^c(\Psi, h, \overline{\omega})(\xi)\right|_{H^3 \times H^2} & \leq_* & |\overline{\omega}| + (\delta_v + \delta_\theta)^2, \\ \left\|\mathcal{R}^c(\Psi^1, h^1, \overline{\omega}) - \mathcal{R}^c(\Psi^2, h^2, \overline{\omega})\right\|_{BC_\eta(\mathbb{R}, H^3 \times H^2)} & \leq_* & (|\overline{\omega}| + \delta_v + \delta_\theta) \left\|\Psi^1 - \Psi^2\right\|_{BC_\eta(\mathbb{R}, \mathbb{R}^3)} \\ & & + (\delta_v + \delta_\theta) \left\|h^1 - h^2\right\|.\end{array} \tag{7.17}$$

Consider an $\eta \in [\eta_{\min}, \eta_{\max}]$, fix a function $\Psi \in BC_\eta(\mathbb{R}, \mathbb{R}^3)$ together with a map $h \in \mathcal{H}_{\delta_v}$ and write $\psi = \Psi(0)$. Let us introduce the function $w \in BC_\eta(\mathbb{R}, H^2 \times H^1) \cap BY_\eta(\mathbb{R}, H^3 \times H^2)$ that is given by

$$w = E^{\mathrm{gb}}(\Psi_\theta, \overline{\omega})(\psi_\alpha, \psi_\beta) + \mathcal{K}^{\mathrm{gb}}(\Psi_\theta, \overline{\omega}) \mathcal{R}^c(\Psi, h, \overline{\omega}). \tag{7.18}$$

In addition, we introduce two functions $\alpha, \beta \in BC_\eta(\mathbb{R}, \mathbb{R})$ by way of

$$(\alpha(\xi), \beta(\xi)) = \Pi T_{-\Psi_\theta(\xi)} \mathrm{pev}_\xi w. \tag{7.19}$$

Now, let us suppose that $\|\Psi_\theta'\|_\infty < \epsilon$, with ϵ as introduced in item (i) of Proposition 6.1. We then have $\Lambda(\Psi_\theta, \overline{\omega}) w = \mathcal{R}^c(\Psi, h, \overline{\omega})$, which allows us to compute

$$\begin{array}{rcl}-\gamma\langle \mathbf{u}_0', \mathbf{u}_0' \rangle \alpha'(\xi) & = & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0'', w(\xi) \rangle + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0', -\gamma \partial_\xi w(\xi) \rangle \\ & = & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0'', w(\xi) \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0', \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle, \\ -\gamma\langle \mathbf{u}_1, \mathbf{u}_1 \rangle \beta'(\xi) & = & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1', w(\xi) \rangle + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, -\gamma \partial_\xi w(\xi) \rangle \\ & = & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1', w(\xi) \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, [I - P_{\Psi_\theta(\xi)}] L(\Psi_\theta(\xi), \overline{\omega}) \mathrm{vev}_\xi w \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, [I - P_{\Psi_\theta(\xi)}] T_{\Psi_\theta(\xi)} M \mathrm{cev}_\xi \mathcal{K} Q_{\Psi_\theta(\xi')} L(\Psi_\theta(\xi'), \overline{\omega}) \mathrm{vev}_{\xi'} w \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle.\end{array} \tag{7.20}$$

On the other hand, still under the assumption that $\|\Psi_\theta'\|_\infty < \epsilon$, item (iii) of Proposition 6.2 implies that for any pair $\xi_0, \xi \in \mathbb{R}$ we have the identity

$$\mathrm{pev}_{\xi_0+\xi} w = \mathrm{pev}_\xi E^{\mathrm{gb}}(T^{(1)}_{\xi_0}\Psi_\theta, \overline{\omega})(\alpha(\xi_0), \beta(\xi_0)) + \mathrm{pev}_\xi \mathcal{K}^{\mathrm{gb}}(T^{(1)}_{\xi_0}\Psi_\theta, \overline{\omega})\mathcal{R}^c(T^{(1)}_{\xi_0}\Psi, h, \overline{\omega}). \tag{7.21}$$

We recall from (6.68) that for any $\theta \in C(\mathbb{R}, \mathbb{R})$ and $a = (a_1, a_2) \in \mathbb{R}^2$ we have

$$E^{\mathrm{gb}}(\theta, \overline{\omega})a \quad = \quad T_{\theta(0)} a_1 \mathbf{u}_0' \mathbf{1} + T_{\theta(0)} a_2 \mathbf{u}_1 \mathbf{1} - a_1 \mathcal{K}^{\mathrm{gb}}(\theta, \overline{\omega})\mathcal{E}_1^c(\theta, \overline{\omega}) - a_2 \mathcal{K}^{\mathrm{gb}}(\theta, \overline{\omega})\mathcal{E}_2^c(\theta, \overline{\omega}), \tag{7.22}$$

together with the bounds

$$\|\mathcal{E}_i^c(\theta, \overline{\omega})\|_{BC_\eta(\mathbb{R}, H^3 \times H^2)} \leq_* \delta_\theta + e^{-\eta \xi_{\mathrm{co}}} + |\overline{\omega}|, \tag{7.23}$$

for $i = 1, 2$. Plugging (7.21) back into (7.20), we find the differential equations

$$\begin{array}{rcl} -\gamma \langle \mathbf{u}_0', \mathbf{u}_0' \rangle \alpha'(\xi) & = & -\gamma \Psi_\theta'(\xi) \langle \mathbf{u}_0'', \alpha(\xi)\mathbf{u}_0' + \beta(\xi)\mathbf{u}_1 \rangle \\ & & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0'', \mathcal{W}_1(\Psi, h, \overline{\omega})(\xi) + \mathcal{W}_2(\Psi, \overline{\omega})[\alpha, \beta](\xi) \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_0', \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle, \\ -\gamma \langle \mathbf{u}_1, \mathbf{u}_1 \rangle \beta'(\xi) & = & -\gamma \Psi_\theta'(\xi) \langle \mathbf{u}_1', \alpha(\xi)\mathbf{u}_0' + \beta(\xi)\mathbf{u}_1 \rangle \\ & & -\gamma \Psi_\theta'(\xi) \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1', \mathcal{W}_1(\Psi, h, \overline{\omega})(\xi) + \mathcal{W}_2(\Psi, \overline{\omega})[\alpha, \beta](\xi) \rangle \\ & & + \langle \mathbf{u}_1, \alpha(\xi) L(\overline{\omega}) \mathbf{u}_0' \mathbf{1} + \beta(\xi) L(\overline{\omega}) \mathbf{u}_1 \mathbf{1} \\ & & \qquad + M \mathrm{cev}_\xi \mathcal{K} Q[\alpha(\xi') L(\overline{\omega}) \mathbf{u}_0' \mathbf{1} + \beta(\xi') L(\overline{\omega}) \mathbf{u}_1 \mathbf{1}] \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, [\mathrm{pev}_\xi + T_{\Psi_\theta(\xi)} M \mathrm{cev}_\xi \mathcal{K} Q_{\Psi_\theta(\cdot)}] \\ & & \qquad [\mathcal{W}_3(\Psi, h, \overline{\omega}) + \mathcal{W}_4(\Psi, \overline{\omega})[\alpha, \beta]] \rangle \\ & & + \langle T_{\Psi_\theta(\xi)} \mathbf{u}_1, \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle. \end{array} \tag{7.24}$$

Here we have introduced the notation

$$\begin{array}{rcl} \mathcal{W}_1(\Psi, h, \overline{\omega})(\xi) & = & \mathrm{pev}_0 \mathcal{K}^{\mathrm{gb}}(T^{(1)}_\xi \Psi_\theta, \overline{\omega}) T^{(1)}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}), \\ \mathcal{W}_2(\Psi, \overline{\omega})[\alpha, \beta](\xi) & = & -\mathrm{pev}_0 \mathcal{K}^{\mathrm{gb}}(T^{(1)}_\xi \Psi_\theta, \overline{\omega})[\alpha(\xi)\mathcal{E}_1^c(T^{(1)}_\xi \Psi_\theta, \overline{\omega}) + \beta(\xi)\mathcal{E}_2^c(T^{(1)}_\xi \Psi_\theta, \overline{\omega})], \\ \mathcal{W}_3(\Psi, h, \overline{\omega})(\xi) & = & L(\Psi_\theta(\xi), \overline{\omega}) \mathrm{vev}_0 \mathcal{K}^{\mathrm{gb}}(T^{(1)}_\xi \Psi_\theta, \overline{\omega}) T^{(1)}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}), \\ \mathcal{W}_4(\Psi, \overline{\omega})[\alpha, \beta](\xi) & = & -L(\Psi_\theta(\xi), \overline{\omega}) \mathrm{vev}_0 \mathcal{K}^{\mathrm{gb}}(T^{(1)}_\xi \Psi_\theta, \overline{\omega})[\alpha(\xi)\mathcal{E}_1^c(T^{(1)}_\xi \Psi_\theta, \overline{\omega}) + \beta(\xi)\mathcal{E}_2^c(T^{(1)}_\xi \Psi_\theta, \overline{\omega})]. \end{array} \tag{7.25}$$

Let us emphasize here that there are no cut-offs on α and β in (7.24).

Lemma 7.1. *Fix $\eta \in [\eta_{\min}, \eta_{\max}]$. Choose δ_θ sufficiently small and ξ_{co} sufficiently large. Consider any $\Psi \in BC_\eta(\mathbb{R}, \mathbb{R}^3)$ that has $|\Psi'(\xi)| < \epsilon$ for some sufficiently small ϵ, that depends only on δ_θ and ξ_{co}. Then for each pair $(\alpha_0, \beta_0) \in \mathbb{R}^2$ and every sufficiently small $\overline{\omega} \in \mathbb{R}$, there exists a unique function $(\alpha, \beta) \in BC_\eta(\mathbb{R}, \mathbb{R}^2)$ that solves (7.24) and has $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$.*

Proof. Let us first write $\Phi = (\alpha, \beta)$. Observe that (7.24) has the form

$$-\gamma \Phi'(\xi) = [L_0 \Phi](\xi) + [L_1 \Phi](\xi) + f(\xi), \tag{7.26}$$

in which we have $f = (f_\alpha, f_\beta)$, together with

$$[L_0 \Phi](\xi) = \big(0, \langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{-1} \langle \mathbf{u}_1, \beta(\xi) L(0,0) \mathbf{u}_1 \mathbf{1} + M \mathrm{cev}_\xi \mathcal{K} Q \beta(\xi') L(0,0) \mathbf{u}_1 \mathbf{1} \rangle \big) \tag{7.27}$$

and

$$\|L_1\|_{\mathcal{L}(BC_\eta(\mathbb{R}, \mathbb{R}^2))} \leq_* \delta_\theta + e^{-\eta \xi_{\mathrm{co}}} + |\overline{\omega}| + \epsilon. \tag{7.28}$$

Let us remark here that $L(0,0)\mathbf{u}_0' = 0$ and

$$L(0,0)\mathbf{u}_1 \mathbf{1} = -\mathcal{T}(0)\mathbf{u}_1 = \mathcal{T}'(0)\mathbf{u}_0'. \tag{7.29}$$

This means that solving $-\gamma\Phi'(\xi) = [L_0\Phi](\xi) + f(\xi)$ is equivalent to solving

$$-\gamma\langle \mathbf{u}_1, \mathbf{u}_1\rangle \beta'(\xi) = \beta(\xi)\langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle + M\mathrm{cev}_\xi \mathcal{K}\beta(\xi')Q\mathcal{T}'(0)\mathbf{u}_0' + f_\beta(\xi). \tag{7.30}$$

Introducing an auxiliary variable θ, we find the equivalent system

$$\begin{array}{lcl} -\gamma\theta'(\xi) & = & QM\mathrm{cev}_\xi\theta + \beta(\xi)Q\mathcal{T}'(0)\mathbf{u}_0', \\ -\gamma\langle \mathbf{u}_1, \mathbf{u}_1\rangle\beta'(\xi) & = & \beta(\xi)\langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle + \langle \mathbf{u}_1, M\mathrm{cev}_\xi\theta\rangle + f_\beta(\xi). \end{array} \tag{7.31}$$

To find a characteristic equation, we substitute the Ansatz $(\theta, \beta)(\xi) = e^{z\xi}(\theta_0, \beta_0)$ and find, using the calculation in (4.34) and (4.35) together with $\langle \mathbf{u}_1, \mathbf{u}_0'\rangle = 0$,

$$\begin{pmatrix} Q\mathcal{T}(z)\mathbf{u}_0' & -Q\mathcal{T}'(0)\mathbf{u}_0' \\ \langle \mathbf{u}_1, \mathcal{T}(z)\mathbf{u}_0'\rangle & -\gamma z\langle \mathbf{u}_1, \mathbf{u}_1\rangle - \langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle \end{pmatrix} \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} = 0. \tag{7.32}$$

We thus conclude that the characteristic function $\Delta(z)$ is given by

$$\Delta(z) = -\gamma z\langle \mathbf{u}_1, \mathbf{u}_1\rangle Q\mathcal{T}(z)\mathbf{u}_0' - \langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle Q\mathcal{T}(z)\mathbf{u}_0' + \langle \mathbf{u}_1, \mathcal{T}(z)\mathbf{u}_0'\rangle Q\mathcal{T}'(0)\mathbf{u}_0'. \tag{7.33}$$

Using (H$\mathcal{T}$2) we now see that for $\eta \in [\eta_{\min}, \eta_{\max}]$, there exists $\mathcal{K}_\eta^{(1)} : BC_\eta(\mathbb{R}, \mathbb{R}) \to BC_\eta(\mathbb{R}, \mathbb{R}^2)$ such that for every $f_\beta \in BC_\eta(\mathbb{R}, \mathbb{R})$ and $(\theta_0, \beta_0) \in \mathbb{R}^2$, the function

$$(\theta, \beta)(\xi) = (\theta_0 + \beta_0\xi, \beta_0) + [\mathcal{K}_\eta^{(1)} f_2](\xi) \tag{7.34}$$

solves (7.31) and has $\theta(0) = \theta_0$ and $\beta(0) = \beta_0$. This operator can in turn be used to construct a map $\mathcal{K}_\eta^{(2)} : BC_\eta(\mathbb{R}, \mathbb{R}^2) \to BC_\eta(\mathbb{R}, \mathbb{R}^2)$ in such a way that the equation $-\gamma\Phi'(\xi) = [L_0\Phi](\xi) + f(\xi)$ supplemented with the initial condition $\Phi(0) = \Phi_0 \in \mathbb{R}^2$, has a unique solution in the class $BC_\eta(\mathbb{R}, \mathbb{R}^2)$ that is given by $\Phi = \Phi_0 + \mathcal{K}^{(2)} f$. Finally, for sufficiently small δ_θ and $\overline{\omega}$ and sufficiently large ξ_{co}, we may define the map $\mathcal{K}_\eta^{(3)} : BC_\eta(\mathbb{R}, \mathbb{R}^2) \to BC_\eta(\mathbb{R}, \mathbb{R}^2)$ by means of

$$\mathcal{K}_\eta^{(3)} = [I - \mathcal{K}_\eta^{(2)} L_1]^{-1}\mathcal{K}_\eta^{(2)}. \tag{7.35}$$

It is not hard to verify that (7.26) is solved by

$$\Phi = \Phi_0 + \mathcal{K}_\eta^{(3)} L_1 \Phi_0 \mathbf{1} + \mathcal{K}_\eta^{(3)} f, \tag{7.36}$$

which completes the proof. □

We now augment the system (7.24) by appending the following equation for θ,

$$\begin{array}{lcl} -\gamma\theta'(\xi) & = & QM\mathrm{cev}_\xi\theta + QL(\overline{\omega})[\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \beta(\xi)\mathbf{u}_1\mathbf{1}] \\ & & +Q_{\Psi_\theta(\xi)}[\mathcal{W}_3(\Psi, h, \overline{\omega})(\xi) + \mathcal{W}_4(\Psi, \overline{\omega})[\alpha, \beta](\xi)] + \mathcal{S}^c(\Psi, h, \overline{\omega})(\xi). \end{array} \tag{7.37}$$

Our goal here is to find solutions to (7.24) and (7.37) that have $(\alpha, \beta, \theta) = \Psi$. Before we can do this, we need to add cut-offs to the $\mathcal{W}$ operators. We write $\mathcal{W}_1^c = \mathcal{W}_1$, $\mathcal{W}_3^c = \mathcal{W}_3$ and introduce

$$\begin{array}{lcl} \mathcal{W}_2^c(\Psi, \overline{\omega})(\xi) & = & -\chi_{\delta_v}(|(\Psi_\alpha(\xi), \Psi_\beta(\xi))|)\mathrm{pev}_0\mathcal{K}^{\mathrm{gb}}(T_\xi^{(1)}\Psi_\theta, \overline{\omega}) \\ & & [\Psi_\alpha(\xi)\mathcal{E}_1^c(T_\xi^{(1)}\Psi_\theta, \overline{\omega}) + \Psi_\beta(\xi)\mathcal{E}_2^c(T_\xi^{(1)}\Psi_\theta, \overline{\omega})], \\ \mathcal{W}_4^c(\Psi, \overline{\omega})(\xi) & = & -\chi_{\delta_v}(|(\Psi_\alpha(\xi), \Psi_\beta(\xi))|)L(\Psi_\theta(\xi), \overline{\omega})\mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(T_\xi^{(1)}\Psi_\theta, \overline{\omega}) \\ & & [\Psi_\alpha(\xi)\mathcal{E}_1^c(T_\xi^{(1)}\Psi_\theta, \overline{\omega}) + \Psi_\beta(\xi)\mathcal{E}_2^c(T_\xi^{(1)}\Psi_\theta, \overline{\omega})]. \end{array} \tag{7.38}$$

We proceed to study the system

$$\begin{array}{lcl} -\gamma\Psi_\alpha' & = & \mathcal{R}_{*,\alpha}^c(\Psi, h, \overline{\omega}), \\ -\gamma\Psi_\beta' & = & \mathcal{L}_\beta(\overline{\omega})\Psi + \mathcal{R}_{*,\beta}^c(\Psi, h, \overline{\omega}), \\ -\gamma\Psi_\theta' & = & \mathcal{L}_\theta(\overline{\omega})\Psi + \mathcal{R}_{*,\theta}^c(\Psi, h, \overline{\omega}). \end{array} \tag{7.39}$$

The linear parts of this equation are given by

$$
\begin{array}{rcl}
[\mathcal{L}_\beta(\overline{\omega})\Psi](\xi) & = & \langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{-1} \langle \mathbf{u}_1, L(\overline{\omega})[\Psi_\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi)\mathbf{u}_1\mathbf{1}] \rangle \\
 & & + \langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{-1} \langle \mathbf{u}_1, M\mathrm{cev}_\xi \mathcal{K} Q L(\overline{\omega})(\Psi_\alpha(\xi')\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi')\mathbf{u}_1\mathbf{1}) \rangle, \\
[\mathcal{L}_\theta(\overline{\omega})\Psi](\xi) & = & QM\mathrm{cev}_\xi \Psi_\theta + QL(\overline{\omega})(\Psi_\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi)\mathbf{u}_1\mathbf{1}),
\end{array}
\tag{7.40}
$$

while the nonlinear parts are defined by

$$
\begin{array}{rcl}
\langle \mathbf{u}_0', \mathbf{u}_0' \rangle \mathcal{R}^c_{*,\alpha}(\Psi, h, \overline{\omega})(\xi) & = & [\mathcal{L}^c_\theta(\overline{\omega})\Psi + \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})]\chi_{\delta_v}\big(\,|(\Psi_\alpha(\xi), \Psi_\beta(\xi))|\,\big)\langle \mathbf{u}_0'', \Psi_\beta(\xi)u_1 \rangle \\
 & & + [\mathcal{L}^c_\theta(\overline{\omega})\Psi + \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})]\langle T_{\Psi_\theta(\xi)}\mathbf{u}_0'', \mathcal{W}^c_1(\Psi, h, \overline{\omega})(\xi) + \mathcal{W}^c_2(\Psi, \overline{\omega})(\xi) \rangle \\
 & & + \langle T_{\Psi_\theta(\xi)}\mathbf{u}_0', \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle, \\
\langle \mathbf{u}_1, \mathbf{u}_1 \rangle \mathcal{R}^c_{*,\beta}(\Psi, h, \overline{\omega})(\xi) & = & [\mathcal{L}^c_\theta(\overline{\omega})\Psi + \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})]\chi_{\delta_v}\big(\,|(\Psi_\alpha(\xi), \Psi_\beta(\xi))|\,\big)\langle \mathbf{u}_1', \Psi_\alpha(\xi)\mathbf{u}_0' \rangle \\
 & & + [\mathcal{L}^c_\theta(\overline{\omega})\Psi + \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})]\langle T_{\Psi_\theta(\xi)}\mathbf{u}_1', \mathcal{W}^c_1(\Psi, h, \overline{\omega})(\xi) + \mathcal{W}^c_2(\Psi, \overline{\omega})(\xi) \rangle \\
 & & + \langle T_{\Psi_\theta(\xi)}\mathbf{u}_1, [\mathrm{pev}_\xi + T_{\Psi_\theta(\xi)} M\mathrm{cev}_\xi \mathcal{K} Q_{\Psi_\theta(\cdot)}][\mathcal{W}^c_3(\Psi, h, \overline{\omega}) + \mathcal{W}^c_4(\Psi, \overline{\omega})] \rangle \\
 & & + \langle T_{\Psi_\theta(\xi)}\mathbf{u}_1, \mathrm{pev}_\xi \mathcal{R}^c(\Psi, h, \overline{\omega}) \rangle, \\
\mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})(\xi) & = & Q_{\theta(\xi)}\mathcal{W}^c_3(\Psi, h, \overline{\omega})(\xi) + Q_{\theta(\xi)}\mathcal{W}^c_4(\Psi, \overline{\omega})(\xi) + \mathcal{S}^c(\Psi, h, \overline{\omega})(\xi),
\end{array}
\tag{7.41}
$$

in which

$$
\begin{array}{rcl}
[\mathcal{L}^c_\theta(\overline{\omega})\Psi](\xi) & = & \chi_{\delta_\theta}(|\mathrm{cev}_\xi \Psi_\theta|) QM\mathrm{cev}_\xi \Psi_\theta \\
 & & + \chi_{\delta_v}(|(\Psi_\alpha(\xi), \Psi_\beta(\xi))|) QL(\overline{\omega})(\Psi_\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi)\mathbf{u}_1\mathbf{1}).
\end{array}
\tag{7.42}
$$

Using Proposition 6.1 and (7.17) we find the estimates

$$
\begin{array}{rcl}
|\mathcal{W}^c_1(\Psi, h, \overline{\omega})(\xi)|_{H^1 \times H^0} & \leq_* & |\overline{\omega}| + (\delta_v + \delta_\theta)^2, \\
\left\| \mathcal{W}^c_1(\Psi^1, h^1, \overline{\omega}) - \mathcal{W}^c_1(\Psi^2, h^2, \overline{\omega}) \right\|_{BC_\eta(\mathbb{R}, H^1 \times H^0)} & \leq_* & (|\overline{\omega}| + \delta_\theta + \delta_v) \left\| \Psi^1 - \Psi^2 \right\|_\eta \\
 & & + (\delta_v + \delta_\theta) \left\| h^1 - h^2 \right\|.
\end{array}
\tag{7.43}
$$

The same estimates hold for $\mathcal{W}^c_3$. In addition, we obtain

$$
\begin{array}{rcl}
|\mathcal{W}^c_2(\Psi, \overline{\omega})(\xi)|_{H^1 \times H^0} & \leq_* & \delta_v(\delta_\theta + |\overline{\omega}| + e^{-\eta \xi_{\mathrm{co}}}), \\
\left\| \mathcal{W}^c_2(\Psi^1, \overline{\omega}) - \mathcal{W}^c_2(\Psi^2, \overline{\omega}) \right\|_{BC_\eta(\mathbb{R}, H^1 \times H^0)} & \leq_* & (\delta_v + \delta_\theta + |\overline{\omega}| + e^{-\eta \xi_{\mathrm{co}}}) \left\| \Psi^1 - \Psi^2 \right\|_\eta
\end{array}
\tag{7.44}
$$

and note that these estimates are shared by $\mathcal{W}^c_4$. Applying these estimates to (7.41), we find

$$
\begin{array}{rcl}
\left| \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega})(\xi) \right| & \leq_* & |\overline{\omega}| + (\delta_v + \delta_\theta)^2 + \delta_v e^{-\eta \xi_{\mathrm{co}}}, \\
\left\| \mathcal{R}^c_{*,\theta}(\Psi^1, h^1, \overline{\omega}) - \mathcal{R}^c_{*,\theta}(\Psi^2, h^2, \overline{\omega}) \right\|_\eta & \leq_* & (|\overline{\omega}| + \delta_\theta + \delta_v + e^{-\eta \xi_{\mathrm{co}}}) \left\| \Psi^1 - \Psi^2 \right\|_\eta \\
 & & + (\delta_v + \delta_\theta) \left\| h^1 - h^2 \right\|.
\end{array}
\tag{7.45}
$$

In addition, we have

$$
\begin{array}{rcl}
\left| \mathcal{R}^c_{*,\alpha}(\Psi, h, \overline{\omega})(\xi) \right| & \leq_* & |\overline{\omega}| + (\delta_v + \delta_\theta)^2, \\
\left\| \mathcal{R}^c_{*,\alpha}(\Psi^1, h^1, \overline{\omega}) - \mathcal{R}^c_{*,\alpha}(\Psi^2, h^2, \overline{\omega}) \right\|_\eta & \leq_* & (|\overline{\omega}| + \delta_\theta + \delta_v) \left\| \Psi^1 - \Psi^2 \right\|_\eta \\
 & & + (\delta_v + \delta_\theta) \left\| h^1 - h^2 \right\|.
\end{array}
\tag{7.46}
$$

Finally, using the identity $\mathrm{cev}_\xi \mathcal{K} = \mathrm{cev}_0 \mathcal{K} T^{(1)}_\xi$, we find

$$
\begin{array}{rcl}
\left| \mathcal{R}^c_{*,\beta}(\Psi, h)(\xi) \right| & \leq_* & |\overline{\omega}| + (\delta_v + \delta_\theta)^2 + \delta_v e^{-\eta \xi_{\mathrm{co}}}, \\
\left\| \mathcal{R}^c_{*,\beta}(\Psi^1, h^1) - \mathcal{R}^c_{*,\theta}(\Psi^2, h^2) \right\|_\eta & \leq_* & (|\overline{\omega}| + \delta_\theta + \delta_v + e^{-\eta \xi_{\mathrm{co}}}) \left\| \Psi^1 - \Psi^2 \right\|_\eta \\
 & & + (\delta_v + \delta_\theta) \left\| h^1 - h^2 \right\|.
\end{array}
\tag{7.47}
$$

Using (7.29), the linearization of (7.39) around $\Psi = 0$ can be written in the form

$$-\gamma\Psi' = L_0\Psi + L_1\Psi + f, \tag{7.48}$$

in which $f = (f_\alpha, f_\beta, f_\theta)$ and

$$\begin{array}{lcl} [L_0\Psi]_\alpha(\xi) & = & 0, \\ [L_0\Psi]_\beta(\xi) & = & \langle \mathbf{u}_1, \mathbf{u}_1\rangle^{-1}[\beta(\xi)\langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle + \langle \mathbf{u}_1, M\mathrm{cev}_\xi\mathcal{K}\beta(\xi')Q\mathcal{T}'(0)\mathbf{u}_0'\rangle], \\ [L_0\Psi]_\theta(\xi) & = & QM\mathrm{cev}_\xi\theta + \beta(\xi)Q\mathcal{T}'(0)\mathbf{u}_0'. \end{array} \tag{7.49}$$

The operator L_1 satisfies the bound

$$\|L_1\|_{\mathcal{L}(BC_\eta(\mathbb{R},\mathbb{R}^3))} \leq_* \delta_\theta + e^{-\eta\xi_{\mathrm{co}}} + |\overline{\omega}|\,. \tag{7.50}$$

Observe that solving $-\gamma\Psi'(\xi) = [L_0\Psi](\xi) + f(\xi)$ is equivalent to solving

$$\begin{array}{lcl} -\gamma\Psi_\alpha'(\xi) & = & f_\alpha(\xi), \\ -\gamma\Psi_\beta'(\xi) & = & \langle \mathbf{u}_1, \mathbf{u}_1\rangle^{-1}[\beta(\xi)\langle \mathbf{u}_1, \mathcal{T}'(0)\mathbf{u}_0'\rangle + \langle \mathbf{u}_1, M\mathrm{cev}_\xi\theta\rangle - \langle \mathbf{u}_1, M\mathrm{cev}_\xi\mathcal{K}f_\theta\rangle] + f_\beta(\xi), \\ -\gamma\Psi_\theta'(\xi) & = & QM\mathrm{cev}_\xi\theta + \beta(\xi)Q\mathcal{T}'(0)\mathbf{u}_0' + f_\theta(\xi). \end{array} \tag{7.51}$$

It is important to note that the map $f_\theta \mapsto M\mathrm{cev}_\xi\mathcal{K}f_\theta$ is a bounded linear map from $BC_\eta(\mathbb{R},\mathbb{R})$ into $BC_\eta(\mathbb{R},\mathbb{R})$ for $\eta \in [0, \eta_{\max}]$. Indeed, for any such η we have

$$\begin{array}{lcl} e^{-\eta|\xi|}\,|\mathrm{cev}_\xi\mathcal{K}f| & = & e^{-\eta|\xi|}\left|\mathrm{cev}_0\mathcal{K}_{\eta_{\max}}T_\xi^{(1)}f\right| \\ & \leq & e^{-\eta|\xi|}\,\|\mathcal{K}\|_{\eta_{\max}}\left\|T_\xi^{(1)}f\right\|_{\eta_{\max}} \\ & \leq & e^{-\eta|\xi|}\,\|\mathcal{K}\|_{\eta_{\max}}\left\|T_\xi^{(1)}f\right\|_{\eta} \\ & \leq & \|\mathcal{K}\|_{\eta_{\max}}\,\|f\|_\eta\,. \end{array} \tag{7.52}$$

Now, as in the proof of Lemma 7.1, (H$\mathcal{T}$2) implies that we can construct for $\eta \in [\eta_{\min}, \eta_{\max}]$ and $\overline{\omega} \in \Omega$, a linear operator

$$\mathcal{K}_\eta^{\mathrm{ct}}(\overline{\omega}) : BC_\eta(\mathbb{R},\mathbb{R}^3) \to BC_\eta(\mathbb{R},\mathbb{R}^3) \tag{7.53}$$

that has $\mathrm{pev}_0\mathcal{K}_\eta^{\mathrm{ct}}(\overline{\omega}) = 0$, such that for any $f \in BC_\eta(\mathbb{R},\mathbb{R}^3)$ and any $\psi \in \mathbb{R}^3$, the function

$$\Psi(\xi) = E^{\mathrm{ct}}(\overline{\omega})\psi + \mathcal{K}_\eta^{\mathrm{ct}}(\overline{\omega})f \tag{7.54}$$

is the unique solution in the class $BC_\eta(\mathbb{R},\mathbb{R}^3)$ of (7.48) with initial condition $\Psi(0) = \psi$. Here we have

$$[E^{\mathrm{ct}}(\overline{\omega})\psi](\xi) = (\psi_\alpha, \psi_\beta, \psi_\theta + \xi\psi_\beta) + \mathrm{pev}_\xi\mathcal{K}_\eta^{\mathrm{ct}}(\overline{\omega})L_1(\psi_\alpha, \psi_\beta, \psi_\theta + \xi'\psi_\beta). \tag{7.55}$$

Let us now consider for $\psi_0 \in \mathbb{R}^3$, $h \in \mathcal{H}_{\delta_v}$ and $\overline{\omega} \in \Omega$, the fixed point system

$$\Psi = E^{\mathrm{ct}}(\overline{\omega})\psi + \mathcal{K}_\eta^{\mathrm{ct}}(\overline{\omega})\mathcal{R}_*^c(\Psi, h, \overline{\omega}). \tag{7.56}$$

After adjusting the parameters δ_θ, δ_v and ξ_{co}, the right hand side of (7.56) becomes a contraction on Ψ. We hence obtain the following result.

Lemma 7.2. *For all sufficiently large ξ_{co} and sufficiently small δ_θ, δ_v and $|\overline{\omega}|$, the fixed point equation (7.56) posed on $BC_\eta(\mathbb{R},\mathbb{R}^3)$ has a unique solution $\Psi = \Psi^*(\psi_0, h, \overline{\omega})$ for every $\eta \in [\eta_{\min}, \eta_{\max}]$, $\psi \in \mathbb{R}^3$ and $h \in \mathcal{H}_{\delta_v}$. These solutions satisfy the property*

$$T_\xi^{(1)}\Psi^*(\psi, h, \overline{\omega}) = \Psi^*\big(\Psi^*(\psi, h, \overline{\omega})(\xi), h, \overline{\omega}\big) \tag{7.57}$$

for any $\xi \in \mathbb{R}$ and $\psi \in \mathbb{R}^3$. In addition, we have Lipschitz estimate

$$\left\|\Psi^*(\psi^1, h^1, \overline{\omega}) - \Psi^*(\psi^2, h^2, \overline{\omega})\right\|_\eta \leq_* \left|\psi^1 - \psi^2\right| + (\delta_v + \delta_\theta)\left\|h^1 - h^2\right\|. \tag{7.58}$$

Finally, if $h \in \mathcal{H}_{\delta_v} \cap C^\ell(\mathcal{D}_{\delta_v}, H^2 \times H^1)$ for some integer $1 \leq \ell \leq r-3$, then for any $\eta \in (\ell\eta_{\min}, \eta_{\max}]$ the map

$$(\psi, \overline{\omega}) \mapsto \Psi^*(\psi, h, \overline{\omega}) \in BC_\eta(\mathbb{R}, \mathbb{R}^3) \tag{7.59}$$

is C^ℓ-smooth. Here we have recalled the integer r appearing in (Hg).

Proof. The existence of Ψ^* follows from the estimates obtained on $\mathcal{R}^c_*$. The identity (7.57) follows from the uniqueness of solutions to (7.56). The Lipschitz estimates follow directly using the fixed point problem (7.56) that Ψ^* satisfies. The smoothness of Ψ^* can be established using the fiber contraction mapping principle developed in [56]. For more details we refer to the center manifold theory that was developed in [32] for functional differential equations of mixed type. □

Note that we have now completed step one of the process outlined in the introduction of this section. Moving on to step two, we proceed to set up the fixed point system for the map h. For $\psi \in \mathbb{R}^3$, we write $E^\perp\psi$ to denote the function

$$E^\perp\psi = \psi_\alpha T_{\psi_\theta}\mathbf{u}_0'\mathbf{1} + \psi_\beta T_{\psi_\theta}\mathbf{u}_1. \tag{7.60}$$

Let us fix $\overline{\omega} \in \Omega$ and $\eta \in [\eta_{\min}, \eta_{\max}]$. For any $\psi \in \mathcal{D}_{\delta_v}$, we wish to have

$$\mathrm{vev}_0 E^\perp\psi + h(\psi) \;=\; \mathrm{vev}_0 E^{\mathrm{gb}}(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega})\psi + \mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega})\mathcal{R}^c(\Psi^*(\psi, h, \overline{\omega}), h, \overline{\omega}). \tag{7.61}$$

This can be reformulated as the fixed point problem

$$h = \mathcal{F}(h) \tag{7.62}$$

in which

$$\begin{array}{rcl}\mathcal{F}(h)(\psi) & = & -\psi_\alpha \mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega})\mathcal{E}^c_1(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega}) \\ & & -\psi_\beta \mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega})\mathcal{E}^c_2(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega}) \\ & & +\mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi, h, \overline{\omega}), \overline{\omega})\mathcal{R}^c(\Psi^*(\psi, h, \overline{\omega}), h, \overline{\omega}).\end{array} \tag{7.63}$$

Notice first that for any $\psi \in \mathcal{D}_{\delta_v}$ we have

$$|\mathcal{F}(h)(\psi)| \quad \leq_* \quad \delta_v(\delta_\theta + |\overline{\omega}| + e^{-\eta\xi_{\mathrm{co}}}) + |\overline{\omega}| + (\delta_v + \delta_\theta)^2. \tag{7.64}$$

In addition, for $\psi^1, \psi^2 \in \mathcal{D}_{\delta_v}$ we may estimate

$$\begin{array}{rcl}\left|\mathcal{F}(h^1)(\psi^1) - \mathcal{F}(h^2)(\psi^2)\right| & \leq_* & [\delta_\theta + \delta_v + |\overline{\omega}| + e^{-\eta\xi_{\mathrm{co}}}]\left|\psi^1 - \psi^2\right| \\ & & +(\delta_v + \delta_\theta)\left\|h^1 - h^2\right\|.\end{array} \tag{7.65}$$

Let us now introduce the requirements

$$\begin{array}{rcl}|\overline{\omega}| & \leq & \delta_v^{8/7}, \\ e^{-\eta\xi_{\mathrm{co}}} & = & \delta_\theta^{\frac{1}{2}}, \\ \delta_v & = & \delta_\theta^{7/4}.\end{array} \tag{7.66}$$

This leads to the simplification

$$\begin{array}{rcl}\|\mathcal{F}(h)(\psi)\| & \leq_* & \delta_v^{8/7}, \\ \left\|\mathcal{F}(h^1)(\psi^1) - \mathcal{F}(h^2)(\psi^2)\right\| & \leq_* & \sqrt{\delta_\theta}\left|\psi^1 - \psi^2\right| + \delta_\theta\left\|h^1 - h^2\right\|.\end{array} \tag{7.67}$$

By choosing δ_θ small enough, we can hence ensure that $\mathcal{F}$ maps into $\mathcal{H}_{\delta_v}$. In addition, we can ensure that we have the bound

$$\left\|\mathcal{F}(h^1) - \mathcal{F}(h^2)\right\| \le \frac{1}{2}\left\|h^1 - h^2\right\|, \tag{7.68}$$

which shows that the fixed point problem $h = \mathcal{F}(h)$ posed on the space $\mathcal{H}_{\delta_v}$ has a unique solution $h^* = h^*(\overline{\omega})$ for all sufficiently small $\overline{\omega}$.

The scalings (7.66) have a further important consequence. In particular, fixing $\psi \in \mathbb{R}^3$ and writing $\Psi = \Psi^*(\psi, h, \overline{\omega})$, note that

$$\Psi_\theta = \psi_\theta + \mathcal{K}_\eta Q L(\overline{\omega})[\Psi_\alpha(\xi)\mathbf{u}_0'\mathbf{1} + \Psi_\beta(\xi)\mathbf{u}_1\mathbf{1}] + \mathcal{K}_\eta \mathcal{R}^c_{*,\theta}(\Psi, h, \overline{\omega}). \tag{7.69}$$

Let us now suppose that

$$|(\Psi_\alpha(\xi), \Psi_\beta(\xi))| < \delta_v \text{ for all } \xi \in \mathbb{R}. \tag{7.70}$$

Then we find that

$$|\Psi_\theta(\xi) - \psi_\theta| \le_* e^{\eta|\xi|}\delta_v, \tag{7.71}$$

which implies that for all $\xi \in [-\xi_{\mathrm{co}}, \xi_{\mathrm{co}}]$, we have

$$|\Psi_\theta(\xi) - \psi_\theta| \le_* e^{\eta\xi_{\mathrm{co}}}\delta_v = \delta_\theta^{7/4}/\sqrt{\delta_\theta} = \delta_\theta^{5/4}. \tag{7.72}$$

This shows that by choosing δ_θ to be sufficiently small, we can ensure that all the cut-offs involving Ψ_θ are automatically satisfied whenever (7.70) holds.

We are now ready to move on to the final step of our program. The next result shows that small solutions of (7.39) with $h = h^*(\overline{\omega})$ can indeed be lifted to solutions of (4.47).

Lemma 7.3. *Fix an $\overline{\omega} \in \Omega$ and suppose that for some $\psi \in \mathbb{R}^3$, the function $\Psi = \Psi^*(\psi, h^*(\overline{\omega}), \overline{\omega})$ satisfies*

$$|(\Psi_\alpha(\xi), \Psi_\beta(\xi))| < \delta_v \tag{7.73}$$

for all $\xi \in \mathbb{R}$. Then the function

$$w(\xi) = \Psi_\alpha(\xi) T_{\Psi_\theta(\xi)}\mathbf{u}_0' + \Psi_\beta(\xi) T_{\Psi_\theta(\xi)}\mathbf{u}_1 + \mathrm{pev}_0 h^*(\Psi(\xi)) \tag{7.74}$$

satisfies (4.47).

Proof. Introduce the function $v \in BC_\eta(\mathbb{R}, H^2 \times H^1)$ via

$$v = E^{\mathrm{gb}}(\Psi_\theta, \overline{\omega})(\psi_\alpha, \psi_\beta) + \mathcal{K}^{\mathrm{gb}}(\Psi_\theta, \overline{\omega})\mathcal{R}^c(\Psi, h^*(\overline{\omega}), \overline{\omega}) \tag{7.75}$$

and define $\widetilde{\Psi} \in BC_\eta(\mathbb{R}, \mathbb{R}^3)$ by

$$\widetilde{\Psi}(\xi) = (\Pi T_{-\Psi_\theta(\xi)}\mathrm{pev}_\xi v, \Psi_\theta(\xi)). \tag{7.76}$$

In view of the remarks above, we note that $\|\Psi_\theta'\|_\infty$ is sufficiently small to ensure that the pair $(\widetilde{\Psi}_\alpha, \widetilde{\Psi}_\beta)$ satisfies the equation (7.20). However, by construction the pair $(\Psi_\alpha, \Psi_\beta)$ also satisfies this equation. Using Lemma 7.1 we conclude that $\widetilde{\Psi} = \Psi$. We may therefore compute, for any $\xi \in \mathbb{R}$,

$$\begin{array}{rcl} \mathrm{vev}_\xi v & = & \mathrm{vev}_0 E^{\mathrm{gb}}(T_\xi^{(1)}\Psi_\theta, \overline{\omega})\widetilde{\Psi}(\xi) + \mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(T_\xi^{(1)}\Psi_\theta, \overline{\omega})\mathcal{R}^c(T_\xi^{(1)}\Psi, h^*, \overline{\omega}) \\ & = & \mathrm{vev}_0 E^{\mathrm{gb}}(\Psi_\theta^*(\Psi(\xi), h^*, \overline{\omega}), \overline{\omega})\Psi(\xi) \\ & & +\mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi_\theta^*(\Psi(\xi), h^*, \overline{\omega}), \overline{\omega})\mathcal{R}^c(\Psi^*(\Psi(\xi), h^*, \overline{\omega}), h^*, \overline{\omega}) \\ & = & \mathrm{vev}_0 E^\perp\Psi(\xi) + h^*(\Psi(\xi)), \end{array} \tag{7.77}$$

in which we have used the shorthand $h^* = h^*(\overline{\omega})$. We hence conclude that $v = w$, which shows that w indeed satisfies (4.47). ∎

To complete our program, we need to show that any sufficiently small solution to (4.47) can be written in the form (7.6). This is established in the next result.

Lemma 7.4. *Suppose that for some* $\overline{\omega} \in \Omega$ *the system (4.47) admits a solution* (θ_*, v_*) *that has*

$$|v_*(\xi)|_{H^2 \times H^1} \leq \|\Pi\|^{-1} \delta_v \tag{7.78}$$

for all $\xi \in \mathbb{R}$*. Then upon writing*

$$\Psi^{(v)}(\xi) = (\Pi T_{-\theta_*(\xi)} \mathrm{pev}_\xi v_*, \theta_*(\xi)), \tag{7.79}$$

we have

$$v_*(\xi) = \mathrm{pev}_0 E^{\perp} \Psi^{(v)}(\xi) + \mathrm{pev}_0 h^*(\overline{\omega})(\Psi^{(v)}(\xi)) \tag{7.80}$$

for all $\xi \in \mathbb{R}$.

Proof. Let us first note that for any fixed $\theta \in C(\mathbb{R}, \mathbb{R})$ and $\overline{\omega} \in \Omega$, the fixed point system

$$w = E^{\mathrm{gb}}(\theta, \overline{\omega})\phi + \mathcal{K}^{\mathrm{gb}}(\theta, \overline{\omega})\mathcal{R}^c(\theta, w, \overline{\omega}) \tag{7.81}$$

has for every $\phi \in \mathbb{R}^2$ a unique solution $w = w^*(\theta, \phi) \in BC_\eta(\mathbb{R}, H^2 \times H^1)$, due to the estimates in Lemma 4.37. We will write these solutions as

$$w^*(\phi, \theta) = E^{\perp}(\phi, \theta(0)) + w_t^*(\phi, \theta) \tag{7.82}$$

and introduce the operator $\widetilde{h} : [-2\delta_v, 2\delta_v]^2 \times BC_\eta(\mathbb{R}, \mathbb{R}) \to (H^2)^{N+2} \times H^1$ via

$$\widetilde{h}(\phi, \theta) = \mathrm{vev}_0 w_t^*(\phi, \theta). \tag{7.83}$$

A computation involving item (iv) of Proposition 6.1 and Lemma 4.37 shows that $\widetilde{h}$ depends Lipschitz continuously on its two arguments θ and ϕ, with a Lipschitz constant $L_{\widetilde{h}}$ that behaves as

$$L_{\widetilde{h}} \leq_* |\overline{\omega}| + \delta_\theta + \delta_v + e^{-\eta \xi_{\mathrm{co}}}. \tag{7.84}$$

Let us now consider for $\Psi \in BC_\eta(\mathbb{R}, \mathbb{R}^3)$ the map $\widetilde{\mathcal{R}}^c(\Psi, \widetilde{h}, \overline{\omega})$ which is defined precisely as $\mathcal{R}^c$ is defined in (7.16) with (7.7), but with each occurrence of $h(\Psi(\xi))$ replaced by $\widetilde{h}(\Psi_\alpha(\xi), \Psi_\beta(\xi), T_\xi^{(1)} \Psi_\theta)$. Similarly, we introduce $\widetilde{\mathcal{R}}^c_*$ which is defined exactly as $\mathcal{R}^c_*$, but with each occurrence of $\mathcal{R}^c$ replaced by $\widetilde{\mathcal{R}}^c$. Let us now consider the equation

$$-\gamma \Psi'(\xi) = \mathcal{L}_\Psi \Psi(\xi) + \widetilde{\mathcal{R}}^c_*(\Psi, \overline{\omega}). \tag{7.85}$$

The estimates derived in this section for $\mathcal{R}^c_*$ all carry over to $\widetilde{\mathcal{R}}^c_*$ up to constants that do not depend on δ_v, δ_θ, ξ_{co} and $\overline{\omega}$. We may therefore conclude that (7.85) has for each $\psi \in \mathbb{R}^3$ a unique solution $\widetilde{\Psi}^*(\psi) \in BC_\eta(\mathbb{R}, \mathbb{R}^3)$ that has $\mathrm{pev}_0 \widetilde{\Psi}^*(\psi) = \psi$. Let us now introduce the map $h_1 \in \mathcal{H}_{\delta_v}$ via

$$h_1(\psi) = \widetilde{h}\big(\psi_\alpha, \psi_\beta, \widetilde{\Psi}^*_\theta(\psi)\big), \tag{7.86}$$

which is well-defined after sufficiently decreasing δ_v. Let us choose $\psi \in \mathcal{D}_{\delta_v}$ and write $\Psi = \widetilde{\Psi}^*(\psi)$. Due to the identity

$$T_\xi^{(1)} \widetilde{\Psi}^*(\psi) = \widetilde{\Psi}^*(\mathrm{pev}_\xi \widetilde{\Psi}^*(\psi)) \tag{7.87}$$

we see that

$$\widetilde{h}\big(\Psi_\alpha(\xi), \Psi_\beta(\xi), T_\xi^{(1)} \Psi_\theta\big) = \widetilde{h}\big(\Psi_\alpha(\xi), \Psi_\beta(\xi), \widetilde{\Psi}^*_\theta(\Psi(\xi))\big) = h_1\big(\Psi(\xi)\big). \tag{7.88}$$

This shows, by uniqueness of solutions, that $\Psi^*(\psi, h_1, \overline{\omega}) = \widetilde{\Psi}^*(\psi)$ for all $\psi \in \mathcal{D}_{\delta_v}$.

We now turn to the solution (θ_*, v_*) of (4.47). For every $\xi \in \mathbb{R}$ we must have

$$\mathrm{vev}_\xi v_* = \mathrm{vev}_0 E^\perp \Psi^{(v)}(\xi) + \widetilde{h}\big(\Psi^{(v)}_\alpha(\xi), \Psi^{(v)}_\beta(\xi), T^{(1)}_\xi \Psi^{(v)}_\theta\big). \tag{7.89}$$

By construction, this implies that $\Psi^{(v)}$ satisfies (7.85). We hence must have

$$T^{(1)}_\xi \Psi^{(v)} = \widetilde{\Psi}^*(\Psi^{(v)}(\xi)) = \Psi^*(\Psi^{(v)}(\xi), h_1, \overline{\omega}) \tag{7.90}$$

for all $\xi \in \mathbb{R}$. Now, let us consider any $\psi \in \mathcal{D}_{\delta_v}$ that has $\psi = \Psi^{(v)}(\xi_0)$ for some $\xi_0 \in \mathbb{R}$. We may compute

$$\begin{array}{lcl} \mathrm{vev}_0 E^\perp \psi + h_1(\psi) & = & \mathrm{vev}_0 E^\perp \psi + \widetilde{h}\big(\psi_\alpha, \psi_\beta, \widetilde{\Psi}^*_\theta(\psi)\big) \\ & = & \mathrm{vev}_0 E^\perp \psi + \mathrm{vev}_0 w^*_t\big(\psi_\alpha, \psi_\beta, \widetilde{\Psi}^*_\theta(\psi)\big) \\ & = & \mathrm{vev}_0 E^\perp \psi + \mathrm{vev}_0 w^*_t\big(\psi_\alpha, \psi_\beta, T^{(1)}_{\xi_0} \Psi^{(v)}_\theta\big) \\ & = & \mathrm{vev}_0 E^{\mathrm{gb}}(T^{(1)}_{\xi_0} \Psi^{(v)}_\theta, \overline{\omega})(\psi_\alpha, \psi_\beta) + \mathcal{K}^{\mathrm{gb}}(T^{(1)}_{\xi_0} \Psi^{(v)}_\theta, \overline{\omega}) \mathcal{R}^c(T^{(1)}_{\xi_0} \Psi^{(v)}_\theta, T^{(1)}_{\xi_0} v_*, \overline{\omega}) \\ & = & \mathrm{vev}_0 E^{\mathrm{gb}}(\Psi^*_\theta(\psi, h_1, \overline{\omega}), \overline{\omega})(\psi_\alpha, \psi_\beta) \\ & & + \mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi, h_1, \overline{\omega}), \overline{\omega}) \mathcal{R}^c(\Psi^*(\psi, h_1, \overline{\omega}), h_1, \overline{\omega}) \end{array} \tag{7.91}$$

and hence for any such ψ we have

$$h_1(\psi) = \mathcal{F}(h_1)(\psi). \tag{7.92}$$

This immediately implies that also $\Psi^*(\psi, h_1, \overline{\omega}) = \Psi^*(\psi, \mathcal{F}(h_1), \overline{\omega})$ for all such ψ. This allows us to conclude that for any integer $n \geq 0$ we have

$$\mathcal{F}^{(n)}(h_1)(\psi) = h_1(\psi), \tag{7.93}$$

where $\mathcal{F}^{(n)}$ denotes the n-th iterate of $\mathcal{F}$. Since $\mathcal{F}$ is a contraction mapping, these iterates converge to $h^*(\overline{\omega})$, which allows us to conclude that $h_1(\psi) = h^*(\overline{\omega})(\psi)$ for all $\psi \in \mathcal{D}_{\delta_v}$ that have $\psi = \Psi^{(v)}(\xi_0)$ for some $\xi_0 \in \mathbb{R}$. This completes the proof. □

Before we start to calculate the flow on the center manifold, we need to study in what fashion the smoothness of the original nonlinearity g_{nl} carries over to the map h^*.

Lemma 7.5. *Recall the integer r appearing in (Hg). The function $h^{**} : \mathcal{D}_{\delta_v} \times \Omega \to H^2 \times H^1$ given by*

$$h^{**}(\psi, \overline{\omega}) = h^*(\overline{\omega})(\psi), \tag{7.94}$$

depends C^{r-3}-smoothly on its arguments.

Proof. Let us first note that Lemma 7.2 in conjunction with Propositions 6.1 and 6.2 shows that for any $h \in \mathcal{H}_{\delta_v} \cap C^{r-3}(\mathcal{D}_{\delta_v}, (H^2)^{N+2} \times H^1)$, we also have

$$\mathcal{F}(h, \overline{\omega}) \in C^{r-3}(\mathcal{D}_{\delta_v}, (H^2)^{N+2} \times H^1). \tag{7.95}$$

Let us introduce for integers $\ell \geq 1$ the function spaces

$$\mathcal{H}^\ell_{\delta_v} = \{h \in C^\ell(\mathcal{D}_{\delta_v} \times \Omega, (H^2)^{N+2} \times H^1) \mid \|h\|_{C^\ell} < \delta_v\}. \tag{7.96}$$

We recall that the fixed point argument on the space $\mathcal{H}_{\delta_v}$ relied crucially on the fact that the linear part $E^\perp \psi$ on the left hand side of (7.98) was not included in the mapping h. The same trick can be used to obtain the desired higher order smoothness. For any $\ell \le r-3$, let us therefore write $E^{\perp}_{(\ell)}(\psi,\overline{\omega})$ for the formal Taylor expansion of

$$E^{\mathrm{gb}}(\Psi^*_\theta(\psi,h,\overline{\omega}),\overline{\omega})\psi + \mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi,h,\overline{\omega}),\overline{\omega})\mathcal{R}^c(\Psi^*(\psi,h,\overline{\omega}),h,\overline{\omega}) \tag{7.97}$$

in terms of ψ and $\overline{\omega}$, up to order ℓ. We can then study the equation

$$\begin{array}{rcl} \mathrm{vev}_0 E^{\perp}_{(\ell)}(\psi,\overline{\omega}) + h_\ell(\psi,\overline{\omega}) & = & \mathrm{vev}_0 E^{\mathrm{gb}}(\Psi^*_\theta(\psi,h,\overline{\omega}),\overline{\omega})\psi \\ & & +\mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}(\Psi^*_\theta(\psi,h,\overline{\omega}),\overline{\omega})\mathcal{R}^c(\Psi^*(\psi,h,\overline{\omega}),h,\overline{\omega}), \end{array} \tag{7.98}$$

in which we use $h = h_\ell + E^{\perp}_{(\ell)} - E^{\perp}$. We may subsequently set up a fixed point argument in the space $\mathcal{H}^{\ell}_{\delta_v}$ to find h_ℓ, but we omit the details here. By uniqueness of fixed points, we must have $h^*(\overline{\omega})(\psi) = E^{\perp}_{(\ell)}(\psi,\overline{\omega}) - E^{\perp}\psi + h_\ell(\psi)$, which completes the proof. □

Let us now take a closer look at the system

$$-\gamma\Psi'(\xi) = [\mathcal{L}_\Psi\Psi](\xi) + \mathcal{R}^c_*(\Psi,h^*(\overline{\omega}),\overline{\omega})(\xi). \tag{7.99}$$

The advanced and delayed terms in this equation can be eliminated by using the substitution

$$\Psi(\xi_0+\xi') = \Psi^*(\Psi(\xi_0),h^*(\overline{\omega}),\overline{\omega})(\xi'), \tag{7.100}$$

which on account of Lemma 7.2 does not affect the set of solutions in $BC_\eta(\mathbb{R},\mathbb{R}^3)$. This allows us to reduce (7.99) to the ODE

$$-\gamma\Psi'(\xi) = f(\Psi(\xi),\overline{\omega}), \tag{7.101}$$

for some $f = (f_\alpha, f_\beta, f_\theta) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$.

We set out to study f_α in greater detail. Let us write $h^* = h^*(\overline{\omega})$. Inspection of (7.63) shows that for any $\psi \in \mathcal{H}_{\delta_v}$ that satisfies $|(\psi_\alpha,\psi_\beta)| < \delta_v$, we have

$$\begin{array}{rcl} \mathrm{vev}_0 h^*(\psi) & = & \mathrm{vev}_0\mathcal{W}^c_1(\Psi^*(\psi,h^*,\overline{\omega}),h^*,\overline{\omega}) + \mathrm{vev}_0\mathcal{W}^c_2(\Psi^*(\psi,h^*,\overline{\omega}),\overline{\omega}), \\ L(\psi_\theta,\overline{\omega})\mathrm{vev}_0 h^*(\psi) & = & \mathrm{pev}_0\mathcal{W}^c_3(\Psi^*(\psi,h^*,\overline{\omega}),h^*,\overline{\omega}) + \mathrm{pev}_0\mathcal{W}^c_4(\Psi^*(\psi,h^*,\overline{\omega}),\overline{\omega}). \end{array} \tag{7.102}$$

Using these identities, a short computation shows that for any ψ that has $|(\psi_\alpha,\psi_\beta)| < \delta_v$, we have the identity

$$\mathcal{R}^c_{*,\alpha}\big(\Psi^*(\psi,h^*(\overline{\omega}),\overline{\omega}),h^*,\overline{\omega}\big) = 0, \tag{7.103}$$

which shows that also $f_\alpha(\psi) = 0$.

As long as we restrict our attention to solutions to (7.101) that have $|(\Psi_\alpha(\xi),\Psi_\beta(\xi))| < \delta_v$ for all $\xi \in \mathbb{R}$, we may thus safely drop the Ψ_α component. Let us write the resulting ODE in the form

$$\begin{array}{rcl} -\gamma\beta'(\xi) & = & c_\omega\overline{\omega} + c_\beta\beta(\xi)^2 + O(|\overline{\omega}|^2 + |\overline{\omega}\beta(\xi)| + |\beta(\xi)|^3), \\ -\gamma\theta'(\xi) & = & -\gamma\beta(\xi) + O(|\overline{\omega}| + |\beta(\xi)|^2). \end{array} \tag{7.104}$$

As a final preparation towards establishing Theorem 4.5, we explicitly compute the coefficient c_ω here. The only contribution comes from the loose $\overline{\omega}$ in $\mathcal{R}^c$. Inspection of (7.41) shows that c_ω can be defined implicitly by the linear equation

$$\begin{array}{rcl} c_\omega & = & -\langle\mathbf{u}_1,\mathbf{u}_1\rangle^{-1}\langle\mathbf{u}_1,(0,u'(k_0))\rangle \\ & & -\langle\mathbf{u}_1,\mathbf{u}_1\rangle^{-1}\langle\mathbf{u}_1,[I+M\mathrm{cev}_0\mathcal{K}Q\mathbf{1}]L(0,0)\mathrm{vev}_0\mathcal{K}^{\mathrm{gb}}([0],0)(0,u'(k_0))\rangle \\ & & -\frac{c_\omega}{\gamma}\langle\mathbf{u}_1,\mathbf{u}_1\rangle^{-1}\langle\mathbf{u}_1,M\mathrm{cev}_0\mathcal{K}Q\xi'L(0,0)\mathbf{u}_1\mathbf{1}\rangle. \end{array} \tag{7.105}$$

We set out to determine $v = \mathcal{K}^{\mathrm{gb}}([0], 0)(0, u'(k_0))$. Using the fact that $Q(0, u'(k_0)) = 0$, we find $v = [I - P]w$ for $w = \mathcal{K}^{\mathrm{lc}}(0, 0)(0, u'(k_0))$. Let us substitute an Ansatz of the form

$$w(\xi) = \mu\xi\mathbf{u}_1 + \frac{1}{2}\mu\xi^2\mathbf{u}_0' + \psi_1, \tag{7.106}$$

for some $\psi_1 \in H^3 \times H^2$. Using (3.24) we find that w solves (5.1) with the inhomogeneity

$$\begin{array}{rcl} f & = & [\mathcal{T}'(0)\mathbf{u}_1 + \frac{1}{2}\mathcal{T}''(0)\mathbf{u}_0']\mu + \mathcal{T}(0)\psi \\ & = & \mathcal{T}(0)\psi_2 - \frac{1}{2}\mu\lambda_{\mathrm{lin}}''(0)(0, u'(k_0)), \end{array} \tag{7.107}$$

for some $\psi_2 \in H^3 \times H^2$. After choosing

$$\mu = -2\lambda_{\mathrm{lin}}''(0)^{-1}, \tag{7.108}$$

we find, for some $\psi_3 \in H^3 \times H^2$, the identity

$$v(\xi) = -2\lambda_{\mathrm{lin}}''(0)^{-1}\xi\mathbf{u}_1 + \psi_3. \tag{7.109}$$

Let us observe that we have $\mathrm{vev}_{\xi'}v = \mathrm{vev}_0 v - 2\lambda_{\mathrm{lin}}''(0)^{-1}\xi'\mathrm{u}_1\mathbf{1}$, using which we may compute

$$\begin{array}{rcl} -\gamma\partial_\xi v(0) = 2\gamma\lambda_{\mathrm{lin}}''(0)^{-1}\mathbf{u}_1 & = & [I-P]L(0,0)\mathrm{vev}_0 v + [I-P]M\mathrm{cev}_0\mathcal{K}\mathbf{1}QL(0,0)\mathrm{vev}_0 v \\ & & -2\lambda_{\mathrm{lin}}''(0)^{-1}[I-P]M\mathrm{cev}_0\mathcal{K}\xi'QL(0,0)\mathbf{u}_1\mathbf{1} + (0, u'(k_0)). \end{array} \tag{7.110}$$

Plugging this back into (7.105), we find

$$[c_\omega + 2\gamma\lambda_{\mathrm{lin}}''(0)^{-1}][\gamma + \langle \mathbf{u}_1, \mathbf{u}_1\rangle^{-1}\langle \mathbf{u}_1, M\mathrm{cev}_0\mathcal{K}\xi'QL(0,0)\mathbf{u}_1\mathbf{1}\rangle] = 0. \tag{7.111}$$

Using (H$\mathcal{T}$2), the second factor above can be shown to be nonzero, which implies that

$$c_\omega = -2\gamma\lambda_{\mathrm{lin}}''(0)^{-1}. \tag{7.112}$$

Proof of Theorem 4.5. We define the function h via

$$h(\kappa, \theta, \overline{\omega}) = h^*(\overline{\omega})(0, \kappa, \theta). \tag{7.113}$$

Item (i) now follows from Lemma 7.5, while (ii) and (iii) follow from Lemmas 7.3 and 7.4, together with the identity (7.112). ☐

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