Center Manifold Theory for Functional Differential Equations of Mixed Type

Hermen Jan Hupkes
Mathematisch Instituut
Universiteit Leiden
P.O. Box 9512
2300 RA Leiden, Netherlands
hhupkes@math.leidenuniv.nl

Sjoerd Verduyn Lunel
Mathematisch Instituut
Universiteit Leiden
P.O. Box 9512
2300 RA Leiden, Netherlands
verduyn@math.leidenuniv.nl

FEBRUARY 2006

Abstract. We study the behaviour of solutions to nonlinear autonomous functional differential equations of mixed type in the neighbourhood of an equilibrium. We show that all solutions that remain sufficiently close to an equilibrium can be captured on a finite dimensional invariant center manifold, that inherits the smoothness of the nonlinearity. In addition, we provide a Hopf bifurcation theorem for such equations. We illustrate the application range of our results by discussing an economic life-cycle model that gives rise to functional differential equations of mixed type.

Keywords. Mixed type functional differential equations, lattice differential equations, life-cycle model, center manifold, Hopf bifurcation, finite dimensional reduction, advanced and retarded arguments.

1 Introduction

The purpose of this paper is to provide a tool to analyze the behaviour of solutions to a nonlinear functional differential equation of mixed type

$$\dot{x}(\xi) = G(x_\xi), \tag{1.1}$$

in the neighbourhood of an equilibrium $\overline{x}$. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \leq 0$ and $r_{\max} \geq 0$, hence the nonlinearity G may depend on advanced and retarded arguments simultaneously.

We establish a center manifold theorem for solutions to (1.1) close to $\overline{x}$, that is, we show that all sufficiently small solutions to the equation

$$\dot{u}(\xi) = DG(\overline{x})u_\xi + (G(\overline{x} + u_\xi) - DG(\overline{x})u_\xi) \tag{1.2}$$

can be captured on a finite dimensional invariant manifold and we explicitly describe the dynamics on this manifold. This reduction allows us to establish a Hopf bifurcation theorem for (1.1), yielding a very powerful tool to perform a bifurcation analysis on parameter dependent versions of this equation. If the linearization $\dot{u}(\xi) = DG(\overline{x})u_\xi$ has no bounded solutions on the line, we say that the equilibrium $\overline{x}$ is hyperbolic and in this case the center manifold contains only the zero function. We will thus be particularly interested in situations where the linear operator $DG(\overline{x})$ has eigenvalues on the imaginary axis, implying that $\overline{x}$ is a nonhyperbolic equilibrium.

The study of center manifolds in infinite dimensions forms one of the cornerstones of the theory of dynamical systems. During the last two decades, many authors have contributed towards developing the general theory. We mention specially the comprehensive overview by Iooss and Vanderbauwhede [41] and the work of Mielke on elliptic partial differential equations [38, 39], in which linear unbounded operators that have infinite spectrum to the right and left of the imaginary axis were analyzed. This type of operator also arises when studying (1.1), but our aproach in this paper is more closely related to the ideas developed in [17], where the theory of semigroups was used to succesfully construct center manifolds for delay equations.

Recent developments in the area of economic research have led to an increased interest in mixed type functional differential equations of the form (1.1). As an interesting example, we discuss here in some detail the work of Albis et al. [16], who analyze the dynamic behaviour of the capital growth rate in a market economy by using a continuous overlapping-generations model. In particular, they consider a population that consists of individuals that all live for a fixed time $T = 1$. They introduce the quantity $c(s,t) \geq 0$ as the consumption at time t of an individual born at time s and similarly $a(s,t)$ for the assests that such an invidual owns. Everybody earns an age-independent income $w(t)$ and receives interest on their assets at the rate $r(t)$, which leads to the following budget contraint,

$$\frac{\partial a(s,t)}{\partial t} = r(t)a(s,t) + w(t) - c(s,t). \tag{1.3}$$

The goal of every individual born at time s is to maximize his total life-time welfare, which is related to his consumption and is given by $\int_s^{s+1} \ln(c(s,\tau))d\tau$. Every individual except those that already exist at the start of the economy at $t = 0$, is born with zero assets and may not die in debt, i.e., $a(s, s+1) \geq a(s,s) = 0$ for all $s \geq 0$. Solving the above optimization problem shows that for any $s \geq 0$ and $t \in [s, s+1]$ the optimal amount of assets $a^*(s,t)$ is a function of the interest rates r_{s+} and wages w_{s+} during the lifetime of an individual. Here $r_{s+} \in C([0,1], \mathbb{R})$ is given by $r_{s+}(\tau) = r(s+\tau)$ and w_{s+} is similarly defined. We can thus write for some $F : C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$,

$$a^*(s,t) = F(r_{s+}, w_{s+}, t-s). \tag{1.4}$$

The dynamics of the interest rates and wages is governed by the capital and labour market. We write $l(t)$ for the amount of labour available at any time t and similarly $k(t)$ for the amount of available capital. Since the population is of fixed size we have $l(t) = 1$. The total capital is given by the sum of the assets of all individuals alive at time t, i.e.,

$$k(t) = \int_{t-1}^{t} a^*(\sigma, t) d\sigma. \tag{1.5}$$

There is a unique material good of unit price, which can be used for both consumption and investments and is produced at the rate Q given by

$$Q(k(t), e(t), l(t)) = Ak(t)^\alpha (e(t)l(t))^\beta, \tag{1.6}$$

for some $A > 0$ and exponents $\alpha > 0$ and $\beta > 0$. Here $e(t)$ is a factor to correct for the increase in labour efficiency over time. Since the prices for capital and employment equal their respective marginal products, we can calculate the interest rate and the height of the wages at any time t by partial differentiation of Q, yielding

$$\begin{array}{rcl} r(t) & = & \alpha A k(t)^{\alpha-1}(e(t)l(t))^{\beta}, \\ w(t) & = & \beta A k(t)^{\alpha} e(t)^{\beta} l(t)^{\beta-1}. \end{array} \tag{1.7}$$

Inserting (1.7) into (1.4) and using the definition (1.5) for $k(t)$, one obtains the market equilibrium condition

$$k(t) = G(k_t, \alpha, \beta). \tag{1.8}$$

Here $k_t \in C([-1,1])$ is defined by $k_t(\tau) = k(t+\tau)$ and the nonlinearity $G : C([-1,1],\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ will be defined explicitly in the sequel. Differentiation of (1.8) yields a mixed-type functional differential equation of the form (1.1). In [16] the authors choose $\beta = 1-\alpha$ and $e(t) = k(t)$, which using $l(t) = 1$ yields a constant interest rate $r(t) = \alpha A$ and wages $w(t) = (1-\alpha)Ak(t)$ that depend linearly on the capital. This in turn leads to the fact that G introduced above is linear in k_t, which enables a global analysis of the long term market behaviour. However, in the literature many other choices for the production function Q are considered, which often lead to nonlinear equations that could be analyzed locally using the theory developed in this paper. We refer to Section 12 for further details.

The interesting feature of the model in [16] is that even in this relatively simple situation with only one market product, oscillatory transitional behaviour is exhibited as the market converges to its stable growth rate. In the absence of delayed and advanced terms, a more involved model is typically needed to obtain this type of dynamics, due to the Poincaré-Bendixson theorem. As an example, we mention the work of Benhabib and Nishimura [6], who obtain periodic solutions to a market model without delays, but with at least $n = 3$ distinct capital goods. In [40] Rustichini considers a similar model as an optimal control problem with time delays, in which case taking $n = 1$ suffices to obtain the desired oscillatory behaviour. The Euler Lagrange equations that arise from his analysis of the control problem are in fact functional differential equations of mixed type and again the Hopf bifurcation theorem plays an important role when establishing the presence of periodic solutions.

Another important application area in which equations of the form (1.1) arise naturally, is the study of travelling wave solutions to so-called lattice differential equations (LDEs). Such equations are infinite systems of ordinary differential equations indexed by points on a spatial lattice. As an example we mention the system

$$\dot{u}_{i,j} = \alpha\big((J * u)_{i,j} - u_{i,j}\big) - f(u_{i,j}, \rho) \tag{1.9}$$

on the lattice $\mathbb{Z}^2$, where

$$(J * u)_{i,j} = \sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} J(l,m) u_{i+l,j+m}, \tag{1.10}$$

with $\sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} J(l,m) = 1$. The function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ typically is a bistable nonlinearity of the form $f(u,\rho) = (u^2-1)(u-\rho)$ for some parameter $-1 < \rho < 1$. Typically the support of the discrete kernel J is limited to close neighbours of $0 \in \mathbb{Z}^2$, but we specifically mention here the work of Bates [3], who analyzed a model incorporating infinite range interactions. Similarly as in the study of partial differential equations (see e.g. the classic work by Fife and McLeod [21]), travelling wave solutions play an important role in the analysis of the discrete system (1.9). Indeed, in [13] results are given concerning the asymptotic stability of travelling wave solutions to (1.9), illustrating that it is worth the effort to study such solutions.

Substituting the travelling wave ansatz $u_{i,j}(t) = \phi(ik_1 + jk_2 - ct)$ into (1.9), where the unit vector $k = (k_1, k_2)$ denotes the direction of propagation and c is the wavespeed, we obtain

$$-c\phi'(\xi) = \alpha\big((J_k\phi)(\xi) - \phi(\xi)\big) - f(\phi(\xi), \rho). \tag{1.11}$$

Here we have introduced the notation $(J_k\phi)(\xi) = \sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} J(l,m)\phi(\xi + k_1 l + k_2 m)$, from which we see that the travelling wave equation (1.11) is indeed of the form (1.1).

Setting $\alpha = 4h^{-2}$ and $J(0,\pm1) = J(\pm1,0) = \frac{1}{4}$ in (1.9), we arrive at the discrete Nagumo equation, which arises when one discretizes the two dimensional reaction diffusion equation,

$$u_t = \Delta u - f(u) \tag{1.12}$$

on a rectangular lattice with spacing h. In the literature, the discrete Nagumo equation has served as a prototype system for investigating the properties of lattice differential equations. We mention here the work of Mallet-Paret [35, 36], who analyzed travelling wave solutions to this equation connecting the two equilibria $\overline{u} = \pm 1$ and found that in general there exist nontrivial intervals of the detuning parameter ρ for which the wavespeed satisfies $c = 0$ and hence the waves fail to propagate. This feature, which distinguishes the discrete version of (1.12) from its continuous counterpart, is called propagation failure and it has been extensively studied both theoretically [11, 26, 35, 36] and numerically [1, 18, 26]. A second distinguishing feature is immediately visible from the discrete travelling wave equation (1.11): the waves depend upon the direction of propagation through the lattice. The consequences of this lattice anisotropy have been numerically illustrated in [18, 26].

The ability to incorporate nonlocal interactions into a model, together with the relatively rich structure of lattice differential equations, presents a strong motivation for the study of such systems. At present, models involving LDEs can be found in many scientific disciplines, including chemical reaction theory [20, 32], image processing and pattern recognition [14], material science [4, 10] and biology [5, 30]. Early papers on the subject by Chi, Bell and Hassard [12] and by Keener [31] were followed by many others which developed the basic theory; see, for example, [11, 13, 23, 27, 28, 33, 35, 37, 44, 45, 46]. We also note the natural occurence of these equations when one studies numerical methods to solve continuous PDEs and analyzes the effects of the employed spatial discretization [19]. In this context we specially mention the work of Benzoni-Gavage et al. [7, 8, 9], where the numerical computation of shock waves is considered in the setting of LDEs and nonhyperbolic functional differential equations of mixed type are encountered.

In [4] Bates shows how an Ising spin model from material science leads to lattice equations (1.9) in which the coefficients of J may be both positive and negative. As long as the convolution operator J is symmetric around zero, i.e. $J(i,j) = J(-i,-j)$, all equilibria of the cubic f lead to hyperbolic systems that admit no nontrivial small solutions around the equilibria. However, as soon as one considers asymmetric kernels J, which arise for example when studying inhomogeneous lattices, or studies a different class of nonlinearities f, the equilibria will in general no longer be hyperbolic and thus small solutions around these fixed points may arise.

When studying the nonlinear mixed-type functional differential equation (1.1), it is essential to have results for linear systems

$$\dot{x}(\xi) = L(\xi)x_\xi + f(\xi). \tag{1.13}$$

Mallet-Paret provided the basic theory in [34], showing that a Fredholm alternative theorem holds for hyperbolic systems (1.13) and providing exponential estimates for solutions to such equations. Later, the existence of exponential dichotomies for (1.13) was established independently by Mallet-Paret and Verduyn Lunel [37] and Härterich et al. [24]. In the present work, we extend the framework developed in [34] to nonhyperbolic but autonomous versions of (1.13), which allows us to generalize the center manifold theory developed for delay equations in [17] to equations of mixed type.

Our main results are stated in Section 2 and proved in Sections 3 through 10, where the necessary theory is developed. In particular, in Section 3 we discuss and apply the results of Mallet Paret to linear systems (1.13) that violate the hyperbolicity condition needed in [34]. In Section 4 we introduce an operator associated with (1.13) on the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$, that in the case of delay equations reduces to the generator of the semigroup associated with the homogeneous version of (1.13). Laplace transform techniques are used in Section 5 to combine the results from the previous two sections in order to define a pseudo-inverse $\mathcal{K}$ for (1.13), in the sense that inhomogeneities f are mapped to their corresponding solutions $x = \mathcal{K}f$ modulo a finite dimensional set of solutions to (1.13) with $f = 0$. This set is isomorphic to a finite dimensional subspace $X_0 \subset X$ and the operator $\mathcal{K}$ is used in Section 6 in a fixed point argument to construct small solutions $u^*\phi$ to the nonlinear equation (1.2) for any small $\phi \in X_0$. This map u^* is shown to be of class C^k in Section 7, while Section 8 shows that these small solutions can in fact be described as solutions to a finite dimensional ordinary differential equation. This reduction is used in Section 10 to establish a Hopf bifurcation theorem for parameter dependent versions of (1.2). In Section 11 we discuss some examples and explicitly describe the dynamics on the center manifold for a functional differential equation of mixed type that admits a double eigenvalue at zero after linearization. In particular, we exhibit a Takens-Burganov bifurcation and show that for delay equations the results from [17] can be recovered from our framework. We conclude in Section 12 by returning to the model of the capital market discussed here and using this example to illustrate the application range of our results.

2 Main Results

Consider for some $N \geq 0$ the functional differential equation of mixed type

$$\dot{x}(\xi) = \sum_{j=0}^{N} A_j x(\xi + r_j) + R(x(\xi + r_0), \ldots, x(\xi + r_N)), \tag{2.1}$$

in which x is a mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$ and each A_j is a $n \times n$ matrix with complex entries. The shifts $r_j \in \mathbb{R}$ may be both positive and negative and for convenience we assume that they are ordered and distinct, i.e., $r_0 < r_1 < \ldots < r_N$. Defining $r_{\min} = r_0$ and $r_{\max} = r_N$, we require $r_{\min} \leq 0 \leq r_{\max}$.

The space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ of continuous $\mathbb{C}^n$-valued functions defined on the interval $[r_{\min}, r_{\max}]$ will serve as a state space when analyzing (2.1). In particular, for any $x \in C(\mathbb{R}, \mathbb{C}^n)$ and any $\xi \in \mathbb{R}$, we define the state $x_\xi \in X$ as the function $x_\xi(\theta) = x(\xi + \theta)$ for any $r_{\min} \leq \theta \leq r_{\max}$. Introducing the bounded linear operator $L : X \to \mathbb{C}^n$ given by

$$L\phi = \sum_{j=0}^{N} A_j \phi(r_j), \tag{2.2}$$

one can rewrite (2.1) as

$$\dot{x}(\xi) = Lx_\xi + R(x_\xi). \tag{2.3}$$

In our analysis of (2.3) we will be particularly interested in the scale of Banach spaces

$$BC_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in C(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |x(\xi)| < \infty \right\}, \tag{2.4}$$

parametrized by $\eta \in \mathbb{R}$. The corresponding norm is given by $\|x\|_\eta = \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |x(\xi)|$. We also need the Banach spaces

$$BC^1_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n) \mid \dot{x} \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \right\}, \tag{2.5}$$

with corresponding norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \|\dot{x}\|_\eta$. Notice that we have continuous inclusions $BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ and $BC^1_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ for any pair $\eta_2 \geq \eta_1$. We will write $\mathcal{J}_{\eta_2\eta_1}$ and $\mathcal{J}^1_{\eta_2\eta_1}$ respectively for the corresponding embedding operators.

In the analysis of (2.3), it is essential to study the behaviour of the homogeneous linear equation

$$\dot{x}(\xi) = Lx_\xi. \tag{2.6}$$

Associated with this system (2.6) one has the characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$, given by

$$\Delta(z) = zI - \sum_{j=0}^{N} A_j e^{zr_j}. \tag{2.7}$$

A value of z such that $\det \Delta(z) = 0$ is called an eigenvalue for the system (2.6). In order to formulate our main results, we need the following proposition.

Proposition 2.1. *For any homogeneous linear equation of the form (2.6), there exists a finite dimensional linear subspace $X_0 \subset X$ with the following properties.*

(i) Suppose $x \in \bigcap_{\eta>0} BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ *is a solution of (2.6). Then for any* $\xi \in \mathbb{R}$ *we have* $x_\xi \in X_0$.

(ii) For any $\phi \in X_0$, *we have* $\dot{\phi} \in X_0$.

(iii) For any $\phi \in X_0$, *there is a unique solution* $x = T\phi \in \bigcup_{\eta>0} BC_\eta(\mathbb{R},\mathbb{C}^n)$ *of (2.6) such that* $x_0 = \phi$. *Moreover, we have that* $x \in BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ *for any* $\eta > 0$.

We will write Q_0 for the projection operator from X onto X_0, which will be defined precisely in the sequel. The following two assumptions on the nonlinearity $R : X \to \mathbb{C}^n$ will be needed in our results.

(HR1) The nonlinearity R is C^k-smooth for some $k \geq 1$.

(HR2) We have $R(0) = 0$ and $DR(0) = 0$.

We remark here that the smoothness requirement in condition (HR1) in fact refers to the Fréchet differentiability of R, since this operator is defined on the infinite dimensional space X. This technicality should be implicitly understood throughout the remainder of this paper. However, one should note that this issue becomes irrelevant when considering nonlinearities R as in (2.1), which have a finite dimensional domain.

Theorem 2.2. *Consider the nonlinear equation (2.3) and assume that (HR1) and (HR2) are satisfied. Then there exists* $\gamma > 0$ *such that the characteristic equation* $\det \Delta(z) = 0$ *has no roots with* $0 < |\mathrm{Re}\, z| < \gamma$. *Fix an interval* $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ *such that* $\eta_{\max} > k\eta_{\min}$, *with* k *as introduced in (HR1). Then there exists a mapping* $u^* : X_0 \to \bigcap_{\eta>0} BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ *and constants* $\epsilon > 0$, $\epsilon^* > 0$ *such that the following statements hold.*

(i) For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, *the function* u^* *viewed as a map from* X_0 *into* $BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ *is* C^k*-smooth.*

(ii) Suppose for some $\zeta > 0$ *that* $x \in BC^1_\zeta(\mathbb{R},\mathbb{C}^n)$ *is a solution of (2.3) with* $\sup_{\xi\in\mathbb{R}} |x(\xi)| < \epsilon^*$. *Then we have* $x = u^*(Q_0 x_0)$. *In addition, the function* $\Phi : \mathbb{R} \to X_0$ *defined by* $\Phi(\xi) = Q_0 x_\xi \in X_0$ *is of class* C^{k+1} *and satisfies the ordinary differential equation*

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi)), \tag{2.8}$$

in which $A : X_0 \to X_0$ *is the linear operator* $\phi \to \dot{\phi}$ *for* $\phi \in X_0$*. The function* $f : X_0 \to X_0$ *is* C^k*-smooth with* $f(0) = 0$ *and* $Df(0) = 0$ *and is explicitly given by*

$$f(\psi) = Q_0\big(L(u^*\psi - T\psi)_\theta + R((u^*\psi)_\theta)\big), \tag{2.9}$$

in which the projection Q_0 *is taken with respect to the variable* θ*. Finally, we have* $x_\xi = (u^*\Phi(\xi))_0$ *for all* $\xi \in \mathbb{R}$*.*

(iii) For any $\phi \in X_0$ *such that* $\sup_{\xi\in\mathbb{R}} |(u^*\phi)(\xi)| < \epsilon^*$*, the function* $u^*\phi$ *satisfies (2.3).*

(iv) For any continuous function $\Phi : \mathbb{R} \to X_0$ *that satisfies (2.8) and has* $\|\Phi(\xi)\| < \epsilon$ *for all* $\xi \in \mathbb{R}$*, we have that* $x = u^*\Phi(0)$ *is a solution of (2.3). In addition, we have* $x_\xi = (u^*\Phi(\xi))_0$ *for any* $\xi \in \mathbb{R}$*.*

(v) Consider the interval $I = (\xi_-, \xi_+)$*, where* $\xi_- = -\infty$ *and* $\xi_+ = \infty$ *are allowed. Let* $\Phi : I \to X_0$ *be a continuous function that satisfies (2.8) for every* $\xi \in I$ *and in addition has* $\|\Phi(\xi)\| < \epsilon$ *for all such* ξ*. Then for any* $\zeta \in (\xi_-, \xi_+)$ *we have that* $x(\xi) = (u^*\Phi(\zeta))(\xi - \zeta)$ *satisfies (2.3) for all* $\xi \in I$*. In addition, we have* $x_\xi = (u^*\Phi(\xi))_0$ *for all* $\xi \in I$*.*

The results above should be compared to similar results for delay differential equations, see e.g. [17, Chp. VIII and IX]. When considering delay equations, it is possible to capture all sufficiently small solutions defined only on the half lines $\mathbb{R}_\pm$ on invariant manifolds. This feature is absent when considering mixed type equations, due to the fact that (2.6) is ill-posed as an initial value problem. We believe that the same ill-posedness can be used to explain the fact that the nonlinearity (2.9) on the center manifold cannot immediately be simplified to its delay equation counterpart [17, (IX.8.3)].

An interesting application of statement (v) above arises when one considers functional differential equations of mixed type on finite intervals. This situation arises for example when studying numerical methods to solve such equations on the line [26]. These methods typically truncate the problem to a finite interval, possibly introducing extra solutions. The center manifold reduction will allow us to at least partially analyze the presence of such solutions. Other preliminary research in this area can be found in [37].

In order to state the Hopf bifurcation theorem, it is necessary to include parameter dependance into our framework. In particular, we introduce an open parameter set $\Omega \subset \mathbb{R}^d$ for some integer

$d \geq 1$ and consider for $\mu \in \Omega$ the equation

$$\dot{x}(\xi) = L(\mu)x_\xi + R(x_\xi, \mu), \tag{2.10}$$

in which R is a nonlinear mapping from $X \times \Omega$ into $\mathbb{C}^n$ and

$$L(\mu)\phi = \sum_{j=0}^{N} A_j(\mu)\phi(r_j). \tag{2.11}$$

We will need the following assumptions on the system (2.10).

(HLμ) The mapping $(\mu, \phi) \to L(\mu)\phi$ from $\Omega \times X$ into $\mathbb{C}^n$ is C^k-smooth for some $k \geq 1$.

(HRμ1) The nonlinearity $R : X \times \Omega \to \mathbb{C}^n$ is C^k-smooth for some $k \geq 1$.

(HRμ2) We have $R(0, \mu) = 0$ and $D_1 R(0, \mu) = 0$ for all $\mu \in \Omega$.

These assumptions are sufficient in order to rewrite the parameter dependent equation (2.10) as an equation of the form (2.3) that satisfies the assumptions of Theorem 2.2. This implies that for fixed $\mu_0 \in \Omega$ and corresponding subspace $X_0 = X_0(\mu_0) \subset X$, it is possible to define a mapping $u^* : X_0 \times \Omega \to \bigcap_{\zeta > 0} BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ that is C^k-smooth when considered as a map into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ for suitable values of η. To establish the Hopf bifurcation theorem, we also need the following.

(Hζ1) The parameter space is one-dimensional, i.e., $d = 1$. In addition, the matrices $A_j(\mu)$ have real valued coefficients and the nonlinearity R maps into $\mathbb{R}^n$. Finally, in (HLμ) and (HRμ1) we have $k \geq 2$.

(Hζ2) For some $\mu_0 \in \Omega$ and $\omega_0 > 0$, the characteristic equation $\det \Delta(z, \mu_0) = 0$ has simple roots at $z = \pm i\omega_0$ and no other root belongs to $i\omega_0\mathbb{Z}$.

(Hζ3) Letting $p, q \in \mathbb{C}^n$ be non-zero vectors such that $\Delta(i\omega_0, \mu_0)p = 0$ and $\Delta(i\omega_0, \mu_0)^T q = 0$, normalized such that $q^T D_1\Delta(i\omega_0, \mu_0)p = 1$, we have that $\operatorname{Re} q^T D_2\Delta(i\omega_0, \mu_0)p \neq 0$.

With p as in (Hζ3), we can define the functions $\phi = pe^{i\omega_0 \cdot}$ and $\overline{\phi} = \overline{p}e^{-i\omega_0 \cdot}$ and it is easy to see that both functions are solutions to the homogeneous equation $\dot{x} = L(\mu_0)x_\xi$.

Theorem 2.3. *Consider the nonlinear equation (2.10) and assume that (HLμ), (HRμ1), (HRμ2) and (Hζ1) -(Hζ3) all hold. There exist C^{k-1}-smooth functions $\tau \to \mu^*(\tau)$, $\tau \to \rho^*(\tau)$ and $\tau \to \omega^*(\tau)$*

taking values in $\mathbb{R}$ *and a mapping* $\tau \to \psi^*(\tau)$ *taking values in* X_0*, all defined for* τ *sufficiently small, such that* $x^*(\tau) = u^*(\rho^*(\tau)(\phi + \overline{\phi} + \psi^*(\tau)), \mu^*(\tau))$ *is a periodic solution of (2.10) at* $\mu = \mu^*(\tau)$ *with period* $\frac{2\pi}{\omega^*(\tau)}$*. Moreover,* $\mu^*(\tau)$ *and* $\omega^*(\tau)$ *are even in* τ*,* $\mu^*(0) = \mu_0$ *and if* x *is any sufficiently small periodic solution of (2.10) with* μ *close to* μ_0 *and period close to* $\frac{2\pi}{\omega_0}$*, then* $\mu = \mu^*(\tau)$ *for some* τ *and there exists* $\xi_0 \in [0, \frac{2\pi}{\omega^*(\tau)})$ *such that* $x(\cdot + \xi_0) = x^*(\tau)(\cdot)$*. Finally, we have* $\rho^*(\tau) = \tau + o(\tau)$ *and* $\psi^*(\tau) = o(1)$ *as* $\tau \to 0$*.*

We wish to emphasize here that the corresponding result for delay equations [17, Chp X] can be recovered almost verbatim from the conditions and statement above by making the appropriate restrictions. Our last main theorem establishes a result on the direction of the Hopf bifurcation.

Theorem 2.4. *Consider the nonlinear equation (2.10) and assume that (HLμ), (HRμ1), (HRμ2) and (Hζ1) -(Hζ3) all hold, but with* $k \geq 3$ *in (Hζ1). Let* $\mu^*(\tau)$ *be as defined in Theorem 2.3. Then we have* $\mu^*(\tau) = \mu_0 + \mu_2\tau^2 + o(\tau^2)$*, with*

$$\mu_2 = \frac{\operatorname{Re} c}{\operatorname{Re} q^T D_2 \Delta(i\omega_0, \mu_0) p}, \tag{2.12}$$

in which

$$\begin{array}{rcl} c & = & \frac{1}{2} q^T D_1^3 R(0, \mu_0)(\phi, \phi, \overline{\phi}) \\ & & + q^T D_1^2 R(0, \mu_0)(\phi, \mathbf{1}\Delta(0, \mu_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi})) \\ & & + \frac{1}{2} q^T D_1^2 R(0, \mu_0)(\overline{\phi}, e^{2i\omega_0 \cdot}\Delta(2i\omega_0, \mu_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \phi)). \end{array} \tag{2.13}$$

We conclude this section by remarking that the restriction to point delays in (2.1) is merely a notational convenience to improve the readability of our arguments. In fact, all results carry over almost verbatim to the more general system (2.3) with arbitrary linear $L : X \to \mathbb{C}^n$ and nonlinear $R : X \to \mathbb{C}^n$.

3 Linear Inhomogeneous Equations

In this section we develop some results for linear inhomogeneous functional differential equations of mixed type,

$$\dot{x}(\xi) = L x_\xi + f(\xi). \tag{3.1}$$

The techniques used here should be compared to similar ones employed in the context of delay equations, see e.g. [2, 29].

For the moment we take $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, with the bounded linear operator L as defined in (2.2). Associated to the system (3.1) we define a linear operator $\Lambda : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda x)(\xi) = \dot{x}(\xi) - L x_\xi. \tag{3.2}$$

We recall the characteristic matrix $\Delta(z)$ associated to (3.1) as defined in (2.7),

$$\Delta(z) = zI - \sum_{j=0}^{N} A_j e^{z r_j}. \tag{3.3}$$

The following result establishes some elementary properties concerning the behaviour of $\Delta(z)$ on vertical strips in the complex plane.

Lemma 3.1. *Consider any closed vertical strip $S = \{z \in \mathbb{C} \mid \gamma_- \le \operatorname{Re} z \le \gamma_+\}$ and for any $\rho > 0$ define $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. Then there exist $K, \rho > 0$ such that $\det \Delta(z) \neq 0$ for all $z \in S_\rho$ and in addition $\left|\Delta(z)^{-1}\right| < \frac{K}{|\operatorname{Im} z|}$ for each such z. In particular, there are only finitely many zeroes of $\det \Delta(z)$ in S. Furthermore, if $\det \Delta(z) \neq 0$ for all $z \in S$, then for any $\alpha \notin S$ the function $R_\alpha(z) = \Delta(z)^{-1} - (z - \alpha)^{-1} I$ is holomorphic in an open neighbourhood of S and in addition there exists $K > 0$ such that $|R_\alpha(z)| \le \frac{K}{1 + |\operatorname{Im} z|^2}$ for all $z \in S$.*

Proof. For any $z \in S$, define $A(z) = \sum_{j=0}^{N} A_j e^{z r_j}$ and $\overline{A} = \sup_{z \in S} |A(z)| < \infty$. For any $z \in S$ with $|z| > 2\overline{A}$, we have that $\Delta(z) = z(I - \frac{A(z)}{z})$ is invertible. The inverse is given by

$$\Delta(z)^{-1} = \frac{1}{z} \sum_{j=0}^{\infty} \frac{A(z)^j}{z^j}, \tag{3.4}$$

and satisfies the bound

$$\left|\Delta(z)^{-1}\right| \le \frac{1}{|z| \left(1 - \frac{1}{|z|} |A(z)|\right)} \le \frac{2}{|z|}. \tag{3.5}$$

Now consider the case that $\det \Delta(z) \neq 0$ for all $z \in S$. Since all zeroes of $\det \Delta(z)$ are isolated, there exists an open neighbourhood of S on which $\Delta(z)^{-1}$ and hence $R_\alpha(z)$ is holomorphic. Note that for $|z| > 2\overline{A}$ we have

$$|R_\alpha(z)| = \left| \frac{\alpha}{z(\alpha - z)} I + \frac{1}{z} \sum_{j=1}^{\infty} \frac{A(z)^j}{z^j} \right| \le \frac{|\alpha|}{|z(z-\alpha)|} + \frac{|A(z)|}{|z|^2} \frac{1}{1 - \frac{|A(z)|}{|z|}} \le \frac{|\alpha|}{|z(z-\alpha)|} + \frac{2\overline{A}}{|z|^2}, \tag{3.6}$$

which yields the final estimate using the fact that $R_\alpha(z)$ is bounded on the set $\left\{z \in S \mid |z| < 2\overline{A}\right\}$. □

The inhomogeneous system (3.1) has been analyzed with respect to the space

$$W^{1,\infty}(\mathbb{R},\mathbb{C}^n) = \{x \in L^\infty(\mathbb{R},\mathbb{C}^n) \mid x \text{ is absolutely continuous and } \dot{x} \in L^\infty(\mathbb{R},\mathbb{C}^n)\} \tag{3.7}$$

by Mallet-Paret in [34], where he obtained the following result.

Theorem 3.2 (Mallet-Paret). *Consider the operator L in (2.2) and suppose that the characteristic equation $\det \Delta(z) = 0$ has no roots on the imaginary axis. Then the operator Λ defined in (3.2) is a bounded linear isomorphism from $W^{1,\infty}(\mathbb{R},\mathbb{C}^n)$ onto $L^\infty(\mathbb{R},\mathbb{C}^n)$. In particular, there exists a Green's function $G : \mathbb{R} \to \mathbb{C}^{n\times n}$ such that the equation $\Lambda x = f$ has the unique solution*

$$x(\xi) = \int_{-\infty}^{\infty} G(\xi - s) f(s) ds. \tag{3.8}$$

In addition, we have $G \in L^p(\mathbb{R},\mathbb{C}^{n\times n})$ for any $1 \le p \le \infty$ and the following identity holds for the Fourier transform (B.1) of G,

$$\widehat{G}(\eta) = \Delta(i\eta)^{-1}. \tag{3.9}$$

Corollary 3.3. *Fix an $a_- < 0$ and $a_+ > 0$ such that $\det \Delta(z) \neq 0$ for all $a_- \le \operatorname{Re} z \le a_+$ and choose an $\alpha < a_-$. Then we have*

$$|G(\xi)| \le \begin{cases} (1 + K(a_-))e^{a_-\xi} & \text{for all } \xi \ge 0, \\ K(a_+)e^{a_+\xi} & \text{for all } \xi < 0, \end{cases} \tag{3.10}$$

in which

$$K(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |R_\alpha(a + i\omega)|\, d\omega. \tag{3.11}$$

In particular, we have the estimate

$$\left\|\Lambda^{-1}\right\| \le 1 + (\frac{1 + K(a_-)}{-a_-} + \frac{K(a_+)}{a_+})(1 + \sum_{j=0}^{N} |A_j|). \tag{3.12}$$

Finally, suppose f satisfies a growth condition $f(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$ for some $0 < \lambda < -a_-$. Then also $x = \Lambda^{-1}f$ satisfies $x(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$. The analogous statement also holds for $\xi \to -\infty$.

Proof. Write $\Delta(z)^{-1} = (z - \alpha)^{-1}I + R_\alpha(z)$. Writing $E(\xi)$ for the inverse transform of $(z - \alpha)^{-1}$, we have that $E(\xi) = e^{\alpha\xi}$ for $\xi > 0$ while $E(\xi) = 0$ for $\xi < 0$. We thus obtain for $\xi > 0$

$$G(\xi) = e^{\alpha\xi}I + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi\omega}R_\alpha(i\omega)d\omega = e^{\alpha\xi}I + \frac{e^{a_-\xi}}{2\pi}\int_{-\infty}^{\infty} e^{i\xi\omega}R_\alpha(a_- + i\omega)d\omega, \tag{3.13}$$

where the integration contour was shifted to the line $\operatorname{Re} z = a_-$ in the last step. A similar estimate can be obtained for $\xi < 0$ by shifting the integration contour to $\operatorname{Re} z = a_+$. Lemma 3.1 ensures that both $R_\alpha(a_- + i\omega)$ and $R_\alpha(a_+ + i\omega)$ are integrable and this concludes the proof of the exponential decay of G.

Consider the equation $\Lambda x = f$ and notice that $\|x\|_{L^\infty} \le \|G\|_{L^1} \|f\|_{L^\infty}$. Using the estimates above we compute $\|G\|_{L^1} \le \frac{1+K(a_-)}{-a_-} + \frac{K(a_+)}{a_+}$. The differential equation (3.1) now implies

$$\|x\|_{W^{1,\infty}} = \|x\|_{L^\infty} + \|\dot{x}\|_{L^\infty} \le \|x\|_{L^\infty} + \|f\|_{L^\infty} + \sum_{j=0}^{N} |A_j| \|x\|_{L^\infty}, \tag{3.14}$$

from which the bound for $\|\Lambda^{-1}\|$ follows.

Finally, if $f(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$, there exists $M > 0$ such that $|f(\xi)| \le Me^{-\lambda\xi}$ for all $\xi > 0$. Hence for all such ξ we compute

$$\begin{array}{rcl} x(\xi) & = & \int_{-\infty}^{\infty} G(\xi - s) f(s) ds \le \frac{1+K(a_-)}{-a_-} e^{a_-\xi} \|f\|_\infty + \int_0^\infty G(\xi - s) f(s) ds \\ & \le & \frac{1+K(a_-)}{-a_-} e^{a_-\xi} \|f\|_\infty + (1 + K(a_-)) e^{a_-\xi} \frac{M}{-a_- - \lambda} \big(e^{(-a_- - \lambda)\xi} - 1 \big) + \frac{M}{\lambda + a_+} K(a_+) e^{-\lambda\xi}, \end{array} \tag{3.15}$$

which concludes the proof. □

In order to proceed, we need to generalize the results above to the situation where the characteristic equation does have roots on the imaginary axis. The key observation which we shall use is that one can shift the roots of the characteristic equation by multiplying the functions in (3.1) by a suitable exponential. In order to make this precise, we introduce the notation $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and any $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$.

Taking any $y \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$, one can compute

$$\big(e_{-\eta} \Lambda e_\eta y\big)(\xi) = \dot{y}(\xi) + \eta y(\xi) - \sum_{j=0}^{N} A_j e^{\eta r_j} y(\xi + r_j). \tag{3.16}$$

Upon defining the linear operator $\Lambda_\eta : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda_\eta x)(\xi) = \dot{x}(\xi) - \eta x(\xi) - \sum_{j=0}^{N} A_j e^{-\eta r_j} x(\xi + r_j) \tag{3.17}$$

and writing $\Delta_\eta(z)$ for the corresponding characteristic matrix, we see that for any $x \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ we have

$$\Lambda_\eta e_\eta x = e_\eta \Lambda x \quad \text{and} \quad \Delta_\eta(z) = (z - \eta)I - \textstyle\sum_{j=0}^{N} A_j e^{(z-\eta) r_j} = \Delta(z - \eta). \tag{3.18}$$

In view of these observations we introduce for any $\eta \in \mathbb{R}$ the Banach spaces

$$\begin{array}{rcl} L^\infty_\eta(\mathbb{R},\mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n) \mid e_{-\eta}x \in L^\infty(\mathbb{R},\mathbb{C}^n)\right\}, \\ W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n) \mid e_{-\eta}x \in W^{1,\infty}(\mathbb{R},\mathbb{C}^n)\right\}, \end{array} \tag{3.19}$$

with norms given by $\|x\|_{L^\infty_\eta} = \|e_{-\eta}x\|_{L^\infty}$ and similarly $\|x\|_{W^{1,\infty}_\eta} = \|e_{-\eta}x\|_{W^{1,\infty}}$. The next proposition provides the appropriate generalization of Theorem 3.2.

Proposition 3.4. *Fix $\eta \in \mathbb{R}$. Consider the operator L in (2.2) and suppose that the characteristic function $\Delta(z)$ has no eigenvalues with $\mathrm{Re}\, z = \eta$. The operator Λ is a bounded linear isomorphism from $W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ onto $L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$, with inverse given by $\Lambda^{-1}f = e_\eta \Lambda^{-1}_{-\eta} e_{-\eta} f$. In particular, we have $\left\|\Lambda^{-1}\right\| = \left\|\Lambda^{-1}_{-\eta}\right\|$. In addition, there exists $\epsilon_0 > 0$ such that $\Delta(z)$ has no eigenvalues in the strip $\eta - \epsilon_0 < \mathrm{Re}\, z < \eta + \epsilon_0$. Finally, for any $0 < \epsilon < \epsilon_0$ and $f \in L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$, we have the following explicit expression for $x = \Lambda^{-1}f$,*

$$x(\xi) = \frac{1}{2\pi i}\int_{\eta+\epsilon-i\infty}^{\eta+\epsilon+i\infty} e^{\xi z}\Delta(z)^{-1}\widetilde{f}_+(z)dz + \frac{1}{2\pi i}\int_{\eta-\epsilon-i\infty}^{\eta-\epsilon+i\infty} e^{\xi z}\Delta(z)^{-1}\widetilde{f}_-(z)dz, \tag{3.20}$$

where the Laplace transforms $\widetilde{f}_+$ and $\widetilde{f}_-$ are as defined in Appendix B.

Proof. Note that $\Delta_{-\eta}$ has no eigenvalues on the imaginary axis and hence $\Lambda_{-\eta}$ is an isomorphism from $W^{1,\infty}(\mathbb{R},\mathbb{C}^n)$ onto $L^\infty(\mathbb{R},\mathbb{C}^n)$. Since e_η is an isometric isomorphism between $W^{1,\infty}(\mathbb{R},\mathbb{C}^n)$ and $W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$ and also between $L^\infty(\mathbb{R},\mathbb{C}^n)$ and $L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$, this proves that Λ is an isomorphism and yields the supplied bound for the norm of the inverse.

Now let $f \in L^\infty_\eta(\mathbb{R},\mathbb{C}^n)$ and consider $x = \Lambda^{-1}f \in W^{1,\infty}_\eta(\mathbb{R},\mathbb{C}^n)$. Write $f = f_+ + f_-$ with $f_+(\xi) = 0$ for $\xi < 0$ and $f_-(\xi) = 0$ for $\xi \geq 0$. Let $x_\pm = \Lambda^{-1}f_\pm = e_\eta\Lambda^{-1}_{-\eta}e_{-\eta}f_\pm$. Using the exponential decay (3.10) of G for $a = \frac{\epsilon_0+\epsilon}{2}$, we easily see that $x_+(\xi) = O(e^{(\eta+a)\xi})$ as $\xi \to -\infty$, and similarly $x_-(\xi) = O(e^{(\eta-a)\xi})$ as $\xi \to \infty$. Using the differential equation (3.1) one sees that similar asymptotic estimates apply for $\dot{x}_\pm$. This implies that both $\overline{x}_\pm = e_{-(\eta\pm\epsilon)}x_\pm$ and their first derivatives have exponential decay at both $\pm\infty$ and in particular satisfy $\overline{x}_\pm \in W^{1,\infty}(\mathbb{R},\mathbb{C}^n) \cap W^{1,2}(\mathbb{R},\mathbb{C}^n) \cap W^{1,1}(\mathbb{R},\mathbb{C}^n)$. Similarly, upon defining $\overline{f}_\pm = e_{-(\eta\pm\epsilon)}f_\pm$, we easily see $\overline{f}_\pm \in L^\infty(\mathbb{R},\mathbb{C}^n) \cap L^1(\mathbb{R},\mathbb{C}^n) \cap L^2(\mathbb{R},\mathbb{C}^n)$. Using the identity (3.18) and the fact that both $\Lambda_{-(\eta\pm\epsilon)}$ are isomorphisms from $W^{1,\infty}(\mathbb{R},\mathbb{C}^n)$ onto $L^\infty(\mathbb{R},\mathbb{C}^n)$, we have $\overline{x}_\pm = \Lambda^{-1}_{-(\eta\pm\epsilon)}\overline{f}_\pm$. Since $\overline{x}_\pm, \overline{f}_\pm \in L^2(\mathbb{R},\mathbb{C}^n) \cap L^1(\mathbb{R},\mathbb{C}^n)$ we may take the Fourier transform and obtain

$$\widehat{\overline{x}}_\pm(k) = \Delta^{-1}_{-(\eta\pm\epsilon)}(ik)\widehat{\overline{f}}_\pm(k). \tag{3.21}$$

Inversion yields

$$\overline{x}_{\pm}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}\Delta^{-1}_{-(\eta\pm\epsilon)}(ik)\widehat{\overline{f}}_{\pm}(k)dk. \tag{3.22}$$

Writing $z = \eta \pm \epsilon + ik$ and noting $\Delta_{-(\eta\pm\epsilon)}(ik) = \Delta(z)$ together with

$$\begin{array}{rclcl} \widehat{\overline{f}}_{+}(k) & = & \int_0^{\infty} e^{-ik\xi}e^{-(\eta+\epsilon)\xi}f_{+}(\xi)d\xi & = & \widetilde{f}_{+}(z), \\ \widehat{\overline{f}}_{-}(k) & = & \int_{-\infty}^{0} e^{-ik\xi}e^{-(\eta-\epsilon)\xi}f_{-}(\xi)d\xi & = & \widetilde{f}_{-}(z), \end{array} \tag{3.23}$$

we obtain

$$x_{\pm}(\xi) = \frac{1}{2\pi i}\int_{\eta\pm\epsilon-i\infty}^{\eta\pm\epsilon+i\infty} e^{z\xi}\Delta(z)^{-1}\widetilde{f}_{\pm}(z)dz. \tag{3.24}$$

□

4 The State Space

In this section we focus our attention on the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$. We define a closed and densely defined operator $A : \mathcal{D}(A) \subset X \to X$, via

$$\begin{array}{rcl} \mathcal{D}(A) & = & \left\{\phi \in X \cap C^1([r_{\min}, r_{\max}], \mathbb{C}^n) \mid \dot{\phi}(0) = L\phi = \sum_{j=0}^{N} A_j\phi(r_j)\right\}, \\ A\phi & = & \dot{\phi}. \end{array} \tag{4.1}$$

Note that the closedness of A can be easily established using the fact that differentiation is a closed operation, together with the continuity of L. The density of the domain $\mathcal{D}(A)$ follows from the density of C^1-smooth functions in X, together with the fact that for any $\epsilon > 0$ and any neighbourhood of zero, one can modify an arbitrary C^1 function ϕ in such a way that $\dot{\phi}(0)$ can be set at will, while $\phi(0)$ remains unchanged and $\|\phi\|_X$ changes by at most ϵ. The first lemma of this section shows that X is indeed a state space for the homogeneous equation $\Lambda x = 0$ in some sense, even though one cannot view this equation as an initial value problem.

Lemma 4.1. *Suppose that for some $x \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ we have the identity $\Lambda x = 0$ with $x_{\xi_0} = 0$ for some $\xi_0 \in \mathbb{R}$. If x satisfies the growth condition $x(\xi) = O(e^{b\xi})$ as $\xi \to \infty$ for any $b \in \mathbb{R}$, then $x(\xi) = 0$ for all $\xi \geq \xi_0 + r_{\min}$. Similarly, if $x(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then $x(\xi) = 0$ for all $\xi \leq \xi_0 + r_{\max}$.*

Proof. Without loss of generality take $\xi_0 = 0$ and assume that the growth condition at $+\infty$ holds. Introducing the function y with $y(\xi) = 0$ for all $\xi \leq 0$ and $y(\xi) = x(\xi)$ for all $\xi > 0$, we see that

$\Lambda y = 0$. Consider any $\eta > b$ such that $\det \Delta(z) \neq 0$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z = \eta$ and notice that $y \in W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. It now follows from Proposition 3.4 that $y = 0$. □

The next lemma establishes the relationship between the characteristic equation $\det \Delta(z) = 0$ and the spectrum of A.

Lemma 4.2. *The operator A has only point spectrum, with $\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}$. In addition, for $z \in \rho(A)$, the resolvent of A is given by*

$$(zI - A)^{-1}\psi = e^{\cdot z} K(\cdot, z, \psi), \tag{4.2}$$

in which $K : [r_{\min}, r_{\max}] \times \mathbb{C} \times X \to \mathbb{C}^n$ is given by

$$K(\theta, z, \psi) = \int_{\theta}^{0} e^{-z\sigma}\psi(\sigma)d\sigma + \Delta(z)^{-1}\big(\psi(0) + \sum_{j=0}^{N} A_j e^{zr_j} \int_{r_j}^{0} e^{-z\sigma}\psi(\sigma)d\sigma\big). \tag{4.3}$$

Proof. Fix $\psi \in X$ and consider the equation $(zI - A)\phi = \psi$ for $\phi \in \mathcal{D}(A)$, which is equivalent to the system

$$\begin{array}{rcl} \dot{\phi} & = & z\phi - \psi, \\ \dot{\phi}(0) & = & \sum_{j=0}^{N} A_j \phi(r_j). \end{array} \tag{4.4}$$

Suppose $\det \Delta(z) \neq 0$. Solving the first equation yields

$$\phi(\theta) = e^{\theta z}\phi(0) + e^{\theta z}\int_{\theta}^{0} e^{-z\sigma}\psi(\sigma)d\sigma \tag{4.5}$$

and hence using the second equation

$$\dot{\phi}(0) = z\phi(0) - \psi(0) = \sum_{j=0}^{N} A_j e^{zr_j}(\phi(0) + \int_{r_j}^{0} e^{-z\sigma}\psi(\sigma)d\sigma). \tag{4.6}$$

Thus if we set

$$\phi(0) = \Delta(z)^{-1}\big(\psi(0) + \sum_{j=0}^{N} A_j e^{zr_j} \int_{r_j}^{0} e^{-z\sigma}\psi(\sigma)d\sigma\big), \tag{4.7}$$

we see that (4.5) yields a solution to (4.4), showing that indeed $z \in \rho(A)$. On the other hand, consider any $z \in \mathbb{C}$ such that $\det \Delta(z) = 0$. Choosing a non-zero $v \in \mathbb{R}^n$ such that $\Delta(z)v = 0$, one sees that the function $\phi(\theta) = e^{z\theta}v$ satisfies $\phi \in \mathcal{D}(A)$ and $A\phi = z\phi$. This shows that $z \in \sigma_p(A)$, completing the proof. □

The next lemma enables us to compute spectral projections corresponding to sets of eigenvalues in vertical strips in the complex plane. We will particularly be interested in the projection operator corresponding to all eigenvalues on the imaginary axis.

Lemma 4.3. *For any pair $\mu, \nu \in \mathbb{R}$, set $\Sigma = \Sigma_{\mu,\nu} = \{z \in \sigma(A) \mid \mu < \operatorname{Re} z < \nu\}$. Then Σ is a finite set, consisting of poles of $(zI - A)^{-1}$ that all have finite order. Furthermore, we have the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$, where $\mathcal{M}_\Sigma$ is the generalized eigenspace corresponding to the eigenvalues in Σ. For any $\mu < \gamma_- < \gamma_+ < \nu$ such that $\Sigma_{\gamma_-,\gamma_+} = \Sigma$, the spectral projection Q_Σ onto $\mathcal{M}_\Sigma$ along $\mathcal{R}_\Sigma$ is given by*

$$(Q_\Sigma \phi)(\theta) = \frac{1}{2\pi i} \int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\theta z} K(\theta, z, \phi) dz + \frac{1}{2\pi i} \int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\theta z} K(\theta, z, \phi) dz. \tag{4.8}$$

If there are no $z \in \sigma(A)$ with $\operatorname{Re} z = \mu$, then $\gamma_- = \mu$ is allowed. Similary, one may choose $\gamma_+ = \nu$ if there are no $z \in \sigma(A)$ with $\operatorname{Re} z = \nu$.

Proof. Lemma 3.1 shows that Σ is finite. Since $\det \Delta(z)$ is a non-zero entire function all zeroes are of finite order, hence the representation (4.2) implies that $(zI - A)^{-1}$ has a pole of order $k \leq k_0$ at λ_0 if λ_0 is a zero of $\det \Delta(z)$ of order k_0. It now follows from standard spectral theory (see e.g. [17, Theorem IV.2.5]) that we have the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$, for some closed linear subspace $\mathcal{M}_\Sigma$. Using Dunford calculus, it follows that for any Jordan path $\Gamma \subset \rho(A)$ with $\operatorname{int}(\Gamma) \cap \sigma(A) = \Sigma$, we have

$$Q_\Sigma = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} dz. \tag{4.9}$$

For any $\rho > 0$ such that $|\operatorname{Im} \lambda| < \rho$ for any $\lambda \in \Sigma$, we introduce the path $\Gamma_\rho = \Gamma_\rho^\uparrow \cup \Gamma_\rho^\leftarrow \cup \Gamma_\rho^\downarrow \cup \Gamma_\rho^\rightarrow$, in which we have introduced the line segments

$$\begin{aligned} \Gamma_\rho^\uparrow &= \operatorname{seg}[\gamma_+ - i\rho, \gamma_+ + i\rho], & \Gamma_\rho^\downarrow &= \operatorname{seg}[\gamma_- + i\rho, \gamma_- - i\rho], \\ \Gamma_\rho^\leftarrow &= \operatorname{seg}[\gamma_+ + i\rho, \gamma_- + i\rho], & \Gamma_\rho^\rightarrow &= \operatorname{seg}[\gamma_- - i\rho, \gamma_+ - i\rho]. \end{aligned} \tag{4.10}$$

Note that the proof is completed if we show that for every $\theta \in [r_{\min}, r_{\max}]$, we have

$$\lim_{\rho \to \infty} \int_{\Gamma_\rho^\rightleftarrows} e^{\theta z} \Big(\int_\theta^0 e^{-z\sigma} \phi(\sigma) d\sigma + \Delta(z)^{-1} \big(\phi(0) + \sum_{j=0}^N A_j e^{z r_j} \int_{r_j}^0 e^{-z\sigma} \phi(\sigma) d\sigma \big) \Big) dz = 0. \tag{4.11}$$

We treat the case for $\Gamma^\leftarrow$, as the other case is analogous. First note that for some $K > 0$ we have the uniform bound

$$\left| e^{\theta z} \big(\phi(0) + \sum_{j=0}^N A_j e^{z r_j} \int_{r_j}^0 e^{-z\sigma} \phi(\sigma) d\sigma \big) \right| \leq K \tag{4.12}$$

in the strip $\gamma_- \leq \operatorname{Re} z \leq \gamma_+$, while by Lemma 3.1 $\Delta(z)^{-1} = O(|\operatorname{Im} z|^{-1})$ uniformly in this strip. In addition, using Fubini to change the order of integration and applying Lemma B.1, we compute

$$\lim_{\rho \to \infty} \int_{\gamma_+}^{\gamma_-} \int_\theta^0 e^{(i\rho + w)(\theta - \sigma)} \phi(\sigma) d\sigma dw = \lim_{\rho \to \infty} \int_\theta^0 e^{i\rho v} \frac{e^{\gamma_- v} - e^{\gamma_+ v}}{v} \phi(\theta - v) dv = 0, \tag{4.13}$$

which concludes the proof. ∎

In order to show that $\mathcal{M}_\Sigma$ is finite dimensional, we introduce a new operator $\widehat{A}$ on the larger space $\widehat{X} = \mathbb{C}^n \times X$,

$$\begin{array}{rcl} \mathcal{D}(\widehat{A}) & = & \left\{(c,\phi) \in \widehat{X} \mid \dot{\phi} \in X, c = \phi(0)\right\}, \\ \widehat{A}(c,\phi) & = & (L\phi, \dot{\phi}). \end{array} \tag{4.14}$$

Writing $j : X \to \widehat{X}$ for the continuous embedding $\phi \to (\phi(0), \phi)$, we see that the part of $\widehat{A}$ in jX is equivalent to A and that the closure of $\mathcal{D}(\widehat{A})$ is given by jX. Hence the spectral analysis of A and $\widehat{A}$ is one and the same. The next lemma shows that $\Delta(z)$ is a characteristic matrix for $\widehat{A}$, in the sense of [17, Def. IV.4.17].

Lemma 4.4. *Consider the holomorphic functions $E : \mathbb{C} \to \mathcal{L}(\widehat{X}, \mathcal{D}(\widehat{A}))$ and $F : \mathbb{C} \to \mathcal{L}(\widehat{X}, \widehat{X})$ given by*

$$\begin{array}{rcl} E(z)(c,\psi)(\theta) & = & (c, e^{\theta z}c + e^{\theta z}\int_\theta^0 e^{-z\sigma}\psi(\sigma)d\sigma), \\ F(z)(c,\psi)(\theta) & = & (c + \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\psi(\sigma)d\sigma, \psi(\theta)), \end{array} \tag{4.15}$$

in which $\mathcal{D}(\widehat{A})$ is considered as a Banach space with the graph norm. Then $E(z)$ and $F(z)$ are bijective for every $z \in \mathbb{C}$ and we have the identity

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I \end{pmatrix} = F(z)(zI - \widehat{A})E(z). \tag{4.16}$$

Proof. Writing $E_2(z)$ for the projection of $E(z)$ onto the X component of $\widehat{X}$, we compute

$$\psi(\theta) = zE_2(z)(c,\psi)(\theta) - D(E_2(z)(c,\psi))(\theta). \tag{4.17}$$

On the other hand, using partial integration we find

$$E_2(z)(\psi(0),(zI - D)\psi) = e^{\theta z}\psi(0) + e^{\theta z}\int_\theta^0 e^{-z\sigma}(z\psi(\sigma) - \dot{\psi}(\sigma))d\sigma = \psi(\theta), \tag{4.18}$$

from which it easily follows that $E(z)$ is bijective for all $z \in \mathbb{C}$. The bijectivity of $F(z)$ is almost immediate. The last identity in the statement of the lemma follows easily by using the definition of $\Delta(z)$ and computing

$$(zI - \widehat{A})E(z)(c,\psi) = ((z - \sum_{j=0}^N A_j e^{zr_j})c - \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\psi(\sigma)d\sigma, \psi). \tag{4.19}$$

∎

Using the theory of characteristic matrices (see e.g. [17, Theorem IV.4.18], one now obtains the following result.

Corollary 4.5. *For any Σ as in the statement of Lemma 4.3, the generalized eigenspace $\mathcal{M}_\Sigma$ is finite dimensional.*

We conclude this section by referring the reader to [2, 22], where similar results are obtained in the framework of delay equations.

5 Pseudo-Inverse for Linear Inhomogeneous Equations

The goal of this section is to define a pseudo-inverse $\mathcal{K} : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ for the linear inhomogeneous equation (3.1) in the spirit of Theorem 3.2, that however can still be defined when the system has eigenvalues on the imaginary axis.

We first need to introduce two families of Banach spaces, parametrized by $\mu, \nu \in \mathbb{R}$, that describe distributions that have controlled exponential growth at $\pm\infty$.

$$\begin{aligned} BX_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) &= \Big\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX_{\mu,\nu}} := \sup_{\xi<0} e^{-\mu\xi} |x(\xi)| + \sup_{\xi \geq 0} e^{-\nu\xi} |x(\xi)| < \infty\Big\}, \\ BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) &= \Big\{x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX^1_{\mu,\nu}} := \|x\|_{BX_{\mu,\nu}} + \|\dot{x}\|_{BX_{\mu,\nu}} < \infty\Big\}. \end{aligned} \tag{5.1}$$

For any $\eta > 0$, we have continuous inclusions

$$i_{\pm\eta} : W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BX^1_{-\eta,\eta}(\mathbb{R}, \mathbb{C}^n), \tag{5.2}$$

with $\|i_{\pm\eta}\| \leq 2 + |\eta|$. Indeed, this can be seen by considering $x \in W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$, defining $y = e_{\mp\eta} x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and noting that

$$\left|e^{-\eta|\xi|}\dot{x}(\xi)\right| = \left|e^{-\eta|\xi|}D(e^{\pm\eta\xi}y(\xi))\right| = \left|e^{\pm\eta\xi-\eta|\xi|}(\dot{y}(\xi) \pm \eta y(\xi))\right| \leq (1+\eta)\,\|y\|_{W^{1,\infty}}. \tag{5.3}$$

The following important result allows us to relate the projection operators Q_Σ as defined in (4.8) to the solution operator (3.20).

Proposition 5.1. *Consider any $x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and write $\Lambda x = f \in BX_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$. Then for any $\gamma_+ > \nu$ and $\gamma_- < \mu$ such that the characteristic equation $\det \Delta(z) = 0$ has no roots with*

$\operatorname{Re} z = \gamma_\pm$, *and for any* $\xi \in \mathbb{R}$, *we have*

$$\begin{array}{lll} x(\xi) & = & \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z}\big(K(\xi, z, x) + \Delta(z)^{-1}\widetilde{f}_+(z)\big)dz \\ & & + \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z}\big(K(\xi, z, x) - \Delta(z)^{-1}\widetilde{f}_-(z)\big)dz, \end{array} \tag{5.4}$$

in which the operator K *defined in (4.3) has been canonically extended to* $\mathbb{R} \times \mathbb{C} \times BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$. *The Laplace transforms* $\widetilde{f}_+$ *and* $\widetilde{f}_-$ *are again as defined in Appendix B.*

Proof. An application of Lemma B.2 shows that

$$\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z}\Big(\int_{\xi}^{0} e^{-z\sigma}x(\sigma)d\sigma + \widetilde{x}_+(z)\Big)dz. \tag{5.5}$$

Taking the Laplace transform of (3.1) yields

$$\begin{array}{lll} z\widetilde{x}_+(z) - x(0) & = & \sum_{j=0}^{N} A_j \int_0^\infty e^{-zu}x(u + r_j)du + \widetilde{f}_+(z) \\ & = & \sum_{j=0}^{N} A_j e^{zr_j}\big(\widetilde{x}_+(z) + \int_{r_j}^{0} e^{-z\sigma}x(\sigma)d\sigma\big) + \widetilde{f}_+(z) \end{array} \tag{5.6}$$

and thus after rearrangement we have

$$\widetilde{x}_+(z) = \Delta(z)^{-1}\Big(x(0) + \sum_{j=0}^{N} A_j e^{zr_j}\int_{r_j}^{0} e^{-z\sigma}x(\sigma)d\sigma + \widetilde{f}_+(z)\Big). \tag{5.7}$$

Now define $y(\xi) = x(-\xi)$ and notice that y satisfies the following equation on $[0, \infty)$,

$$\dot{y}(\xi) = -f(-\xi) - \sum_{j=0}^{N} A_j y(\xi - r_j). \tag{5.8}$$

Taking the Laplace transform of this identity yields

$$z\widetilde{y}_+(z) - y(0) = -\sum_{j=0}^{N} A_j e^{-zr_j}\big(\widetilde{y}_+(z) + \int_{-r_j}^{0} e^{-z\sigma}y(\sigma)d\sigma\big) - \widetilde{f}_-(-z) \tag{5.9}$$

and thus after rearrangement

$$\widetilde{y}_+(z) = \Delta(-z)^{-1}\Big(-y(0) + \sum_{j=0}^{N} A_j e^{-zr_j}\int_{-r_j}^{0} e^{-z\sigma}y(\sigma)d\sigma + \widetilde{f}_-(-z)\Big). \tag{5.10}$$

Reasoning as in the derivation of (5.5) we obtain the identity

$$\frac{1}{2}y(\xi) = \frac{1}{2\pi i}\int_{-\gamma_- - i\infty}^{-\gamma_- + i\infty} e^{\xi z}\Big(\int_{\xi}^{0} e^{-z\sigma}y(\sigma)d\sigma + \widetilde{y}_+(z)\Big)dz \tag{5.11}$$

and thus $\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{-\gamma_- - i\infty}^{-\gamma_- + i\infty} e^{-\xi z}(\Psi(\xi, z) + \Delta(-z)^{-1}\widetilde{f}_-(-z))dz$, with

$$\Psi(\xi, z) = \int_{-\xi}^{0} e^{-z\sigma}x(-\sigma)d\sigma - \Delta(-z)^{-1}\big(x(0) - \sum_{j=0}^{N} A_j e^{-zr_j}\int_{-r_j}^{0} e^{-z\sigma}x(-\sigma)d\sigma\big). \tag{5.12}$$

Substituting $z \to -z$, we obtain $\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z}(\Psi(\xi, -z) - \Delta(z)^{-1}\widetilde{f}_-(z))dz$ with

$$\Psi(\xi, -z) = \int_{\xi}^{0} e^{-z\sigma}x(\sigma)d\sigma + \Delta(z)^{-1}\big(x(0) + \sum_{j=0}^{N} A_j e^{zr_j}\int_{r_j}^{0} e^{-z\sigma}x(\sigma)d\sigma\big), \tag{5.13}$$

which follows from (5.12) after the substitution $\sigma \to -\sigma$ and concludes the proof. □

Using Lemma 3.1 one sees that there exists $\gamma > 0$ such that (3.1) has no eigenvalues z with $0 < |\mathrm{Re}\, z| < \gamma$. Throughout the rest of this section we fix an arbitrary $\eta \in (0, \gamma)$. We introduce $L^\infty(\mathbb{R}, \mathbb{C}^{n\times n})$ functions $\chi_\pm$ such that $\chi_+(\xi) = I$ for $\xi \geq 0$, $\chi_-(\xi) = I$ for $\xi < 0$ and $\chi_+ + \chi_- = I$. Associated with these functions we define bounded linear cutoff operators $\Phi_\pm : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ by $\Phi_\pm x(\xi) = \chi_\pm(\xi)x(\xi)$ and notice that $\Phi_+ + \Phi_- = I_{BC_\eta(\mathbb{R},\mathbb{C}^n)}$.

Using Proposition 3.4 we can define the isomorphisms $\Lambda_\pm = \Lambda_\pm^{(\eta)} : W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \to L^\infty_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ and a linear operator $P_\eta : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ by

$$P_\eta x = \Lambda_+^{-1}\Phi_+\Lambda x + \Lambda_-^{-1}\Phi_-\Lambda x. \tag{5.14}$$

Notice that P_η is well defined, since $\Lambda P_\eta x = \Phi_+\Lambda x + \Phi_-\Lambda x = \Lambda x \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$, which together with the differential equation (3.1) implies that the derivative of $P_\eta x$ is continous, yielding $P_\eta x \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ instead of merely $P_\eta x \in BX^1_{-\eta,\eta}(\mathbb{R}, \mathbb{C}^n)$. Define the space $\mathcal{R}_\eta \subset BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ as the range of P_η and the space $\mathcal{N}_0 \subset BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ as the kernel of P_η. Notice that the set of eigenvalues $\Sigma = \Sigma_{-\zeta,\zeta}$ is independent of ζ for $0 < \zeta < \gamma$. We introduce the projection $Q_0 : X \to X$ with $Q_0 = Q_\Sigma$ and define the finite dimensional linear subspace $X_0 = \mathcal{M}_\Sigma \subset X$.

Proposition 5.2. *The operator P_η defined above is bounded and in addition is a projection, i.e., it satisfies $P_\eta^2 = P_\eta$. The range $\mathcal{R}_\eta$ is a closed linear subspace of $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ and for any $x \in \mathcal{R}_\eta$ we have $Q_0 x_0 = 0$. The kernel $\mathcal{N}_0$ is finite dimensional and does not depend on η, with $\dim \mathcal{N}_0 = \dim X_0$. In particular, for any $x \in \mathcal{N}_0$ we have $x_\xi \in X_0$ for all $\xi \in \mathbb{R}$ and conversely, for any $\phi \in X_0$ there exists a unique $x = T\phi$ in $\mathcal{N}_0$ with $x_0 = \phi$. For any $\zeta_0 > 0$, we have that T viewed as a linear operator from X_0 into $BC^1_{\zeta_0}(\mathbb{R}, \mathbb{C}^n)$ is bounded with norm $\|T\|_{\zeta_0}$ that satisfies $\|T\|_{\zeta_1} \leq \|T\|_{\zeta_0}$ for $\zeta_1 \geq \zeta_0$.*

Proof. The boundedness of P_η follows from the boundedness of Λ, $\Phi_\pm$ and $\Lambda_\pm^{-1}$, together with the continuous embeddings $i_{\pm\eta} : W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BX^1_{-\eta,\eta}(\mathbb{R}, \mathbb{C}^n)$. For all $x \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, we notice

$$\Lambda P_\eta x = \Phi_+\Lambda x + \Phi_-\Lambda x = \Lambda x, \tag{5.15}$$

which yields

$$P_\eta^2 x = \Lambda_+^{-1}\Phi_+\Lambda P_\eta x + \Lambda_-^{-1}\Phi_-\Lambda P_\eta x = \Lambda_+^{-1}\Phi_+\Lambda x + \Lambda_-^{-1}\Phi_-\Lambda x = P_\eta x. \tag{5.16}$$

The range $\mathcal{R}_\eta$ can now immediately be seen to be closed, since if $P_\eta x_n \to z$, then $P_\eta^2 x_n = P_\eta x_n \to z$, but also $P_\eta x_n \to P_\eta z$, yielding $P_\eta z = z$ and thus $z \in \mathcal{R}_\eta$. Consider any $x \in \mathcal{R}_\eta$ and write $f = \Phi_+\Lambda x$ and $g = \Phi_-\Lambda x$. It is clear that $\widetilde{f}_-(z) = 0$ and similarly $\widetilde{g}_+(z) = 0$. Combining Propositions 3.4 and 5.1, we conclude that $Q_0 x_0 = 0$.

Now consider any $x \in \mathcal{N}_0$. It follows from Proposition 5.1 that $x_0 = Q_0 x_0$ and since $\mathcal{N}_0$ is invariant under translation, we see $x_\xi = Q_0 x_\xi$ for any $\xi \in \mathbb{R}$. Let $y_0 \in \mathcal{N}_0$ be such that $y_0 = x_0$, then $x - y \in \mathcal{N}_0$ with $(x-y)_0 = 0$, but then Lemma 4.1 implies that $x = y$. We thus have $\dim \mathcal{N}_0 \leq \dim X_0$. On the other hand, any $\phi \in X_0$ has the form $\phi(\theta) = \sum_{j=0}^{M} p_j(\theta) e^{\lambda_j \theta}$ with $\operatorname{Re}\lambda_j = 0$ and polynomials p_j and can thus be extended to a function $x = T\phi$ on the line, with $x \in \mathcal{N}_0$ and $x_0 = \phi$. Thus $\dim \mathcal{N}_0 = \dim X_0$ and the properties of T easily follow from the specific form of $\phi(\xi)$. This completes the proof. □

We remark here that all the statements in Proposition 2.1 have now been proved. Furthermore, we currently have all the ingredients we need to define a bounded pseudo-inverse for Λ. We thus introduce the operator $\mathcal{K}_\eta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to \mathcal{R}_\eta$, given by

$$\mathcal{K}_\eta x = \Lambda_+^{-1}\Phi_+ x + \Lambda_-^{-1}\Phi_- x. \tag{5.17}$$

Notice that the range of $\mathcal{K}_\eta$ is indeed contained in $\mathcal{R}_\eta$, since $x = \Lambda\mathcal{K}_\eta x$ and hence $\mathcal{K}_\eta x = P_\eta \mathcal{K}_\eta x$. This also immediately shows the injectivity of $\mathcal{K}_\eta$, since if $\mathcal{K}_\eta x = 0$, we have $x = \Lambda(0) = 0$. The surjectivity of $\mathcal{K}_\eta$ follows from the identity $y = P_\eta y = \mathcal{K}_\eta \Lambda y$ for any $y \in \mathcal{R}_\eta$. The following result shows that $\mathcal{K}_\eta$ behaves nicely on the scale of Banach spaces $BC_\zeta(\mathbb{R}, \mathbb{C}^n)$.

Lemma 5.3. *Consider any pair $0 < \eta_1 < \eta_2 < \gamma$. Then for any $f \in BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n)$ we have*

$$\mathcal{K}_{\eta_1} f = \mathcal{K}_{\eta_2} f. \tag{5.18}$$

Proof. Note that $\overline{f}_+ = e_{-\eta_2}\Phi_+ f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ satisfies a growth condition $\overline{f}_+(\xi) = O(e^{-(\eta_2-\eta_1)\xi})$ as $\xi \to \infty$ and hence $\overline{x}_+ = \Lambda_{-\eta_2}^{-1}\overline{f}_+$ shares this growth condition by Corollary 3.3. This implies that the function $x_+ = e_{\eta_2}\overline{x}_+$ satisfies $x_+ = O(e^{\eta_1\xi})$ as $\xi \to \infty$ and since $\overline{x}_+$ is bounded on

$\mathbb{R}$, we have $x_+ = O(e^{-\eta_2|\xi|})$ as $\xi \to -\infty$. Using the differential equation (3.1) it follows that $x_+ \in W^{1,\infty}_{\eta_1}(\mathbb{R},\mathbb{C}^n) \cap W^{1,\infty}_{\eta_2}(\mathbb{R},\mathbb{C}^n)$. Since $\Lambda x_+ = \Phi_+ f \in L^1_{\eta_1}(\mathbb{R},\mathbb{C}^n) \cap L^1_{\eta_2}(\mathbb{R},\mathbb{C}^n)$, we see that $x_+ = (\Lambda_+^{(\eta_1)})^{-1}\Phi_+ f = (\Lambda_+^{(\eta_2)})^{-1}\Phi_+ f$. A similar argument for $\Phi_- f$ completes the proof. □

The next lemma shows that $\mathcal{K}_\eta$ and the translation operator do not commute.

Lemma 5.4. *For any $f \in BC_\eta(\mathbb{R},\mathbb{C}^n)$ and $\xi_0 \in \mathbb{R}$, define the function $y \in BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ by*

$$y(\xi) = (\mathcal{K}_\eta f)(\xi_0 + \xi) - (\mathcal{K}_\eta f(\xi_0 + \cdot))(\xi). \tag{5.19}$$

Then we have $y \in \mathcal{N}_0$. In particular, we have the identity

$$(I - Q_0)(\mathcal{K}_\eta f)_{\xi_0} = (\mathcal{K}_\eta f(\xi_0 + \cdot))_0. \tag{5.20}$$

Proof. Define functions $x_0(\xi) = (\mathcal{K}_\eta f)(\xi_0 + \xi)$ and $x_1(\xi) = (\mathcal{K}_\eta f(\xi_0 + \cdot))(\xi)$. Notice that for all $\xi \in \mathbb{R}$ we have $(\Lambda x_0)(\xi) = f(\xi_0 + \xi)$ but also $(\Lambda x_1)(\xi) = f(\xi_0 + \xi)$. This implies $\Lambda(x_0 - x_1) = 0$ and hence $y = x_0 - x_1 \in \mathcal{N}_0$. The final statement follows from the fact that $Q_0 y_\xi = y_\xi$ for any $y \in \mathcal{N}_0$ together with the identity $Q_0(\mathcal{K}_\eta f)_0 = 0$ for any $f \in BC_\eta(\mathbb{R},\mathbb{C}^n)$. □

For notational convenience, we introduce the quantity

$$w = \max(e^{-r_{\min}}, e^{r_{\max}}) \geq 1 \tag{5.21}$$

and note that for any $\eta > 0$, $\xi \in \mathbb{R}$ and $r_{\min} \leq \theta \leq r_{\max}$, we have $e^{-\eta|\xi|}e^{\eta|\xi+\theta|} \leq w^\eta$. This in turn implies that for any $x \in BC_\eta(\mathbb{R},\mathbb{C}^n)$ and any $\xi \in \mathbb{R}$, we have

$$\|x_\xi\| = \sup_{r_{\min} \leq \theta \leq r_{\max}} e^{\eta|\xi+\theta|}e^{-\eta|\xi+\theta|}\,|x(\xi+\theta)| \leq e^{\eta|\xi|}w^\eta \|x\|_\eta . \tag{5.22}$$

The following corollary to Lemma 5.4 shows that the hyperbolic component of $\mathcal{K}f$ remains bounded whenever f is bounded, which in the sequel will allow us to restrict our attention to the growth rate on the center component.

Corollary 5.5. *Suppose that $f \in BC_0(\mathbb{R},\mathbb{C}^n)$. Then for any $\xi \in \mathbb{R}$ we have*

$$\|(I - Q_0)(\mathcal{K}_\eta f)_\xi\| \leq w^\eta \|K_\eta\| \|f\|_0 . \tag{5.23}$$

Proof. Using Lemma 5.4 we compute

$$\|(I - Q_0)(\mathcal{K}_\eta f)_\xi\| = \|(\mathcal{K}_\eta f(\xi + \cdot))_0\| \leq w^\eta \|\mathcal{K}_\eta\| \|f(\xi + \cdot)\|_\eta. \tag{5.24}$$

The statement now follows from the observation $\|f(\xi + \cdot)\|_\eta \leq \|f(\xi + \cdot)\|_0 = \|f\|_0$. □

Finally, we show that we can bound the norm of $\mathcal{K}_\eta$ uniformly on closed intervals.

Lemma 5.6. *Consider any interval* $I = [\eta_-, \eta_+] \subset (0, \gamma)$. *Then* $\|\mathcal{K}_\eta\|$ *is uniformly bounded for* $\eta \in I$.

Proof. In view of the bounds $\|i_{\pm\eta}\| \leq 2 + |\eta|$ for the embedding operators introduced in (5.2), it is enough to show that we can uniformly bound $B_{\pm\eta} = \left\|\Lambda_\pm^{-1}\right\|_{\mathcal{L}(L^\infty_{\pm\eta}(\mathbb{R},\mathbb{C}^n), W^{1,\infty}_{\pm\eta})(\mathbb{R},\mathbb{C}^n)}$. We here concentate on the $+$ case, as the remaining case follows analogously. From Proposition 3.4 we know $B_\eta = \|\Lambda_{-\eta}\|$. Fix $a = \min(\frac{1}{2}\eta_-, \frac{1}{2}(\ \gamma - \eta_+))$ and choose any $\alpha < -a$. Using Corollary 3.3 and the fact that the norms $|A_j e^{\eta r_j}|$ are uniformly bounded, we see it is enough to show that the quantities $K^\pm_\eta$ are uniformly bounded for $\eta \in I$, where

$$K^\pm_\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\Delta_{-\eta}(\pm a + iz)^{-1} - \frac{1}{z - \alpha} I\right| dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\Delta(\pm a + \eta + iz)^{-1} - \frac{1}{z - \alpha} I\right| dz. \tag{5.25}$$

This however follows immediately from Lemma 3.1. □

6 A Lipschitz Smooth Center Manifold

Using the pseudo-inverse $\mathcal{K}$ defined in the previous section for the inhomogeneous linear equation (3.1), we are now in a position to construct a Lipschitz smooth center manifold for the nonlinear equation (2.3). Throughout this section we consider a fixed nonlinearity $R : X \to \mathbb{C}^n$ that satisfies the assumptions (HR1) and (HR2). In order to employ the Banach contraction theorem, we need to modify the nonlinearity R so that it becomes globally Lipschitz continuous with a sufficiently small Lipschitz constant. To this end, we let $\chi : [0, \infty) \to \mathbb{R}$ be any C^∞-smooth function that satisfies $\chi(\xi) = 0$ for $\xi \geq 2$, $\chi(\xi) = 1$ for $\xi \leq 1$ and $0 \leq \chi(\xi) \leq 1$ for all $1 \leq \xi \leq 2$. For any $\delta > 0$, we define $\chi_\delta : [0, \infty) \to \mathbb{R}$ by $\chi_\delta(\xi) = \chi(\frac{\xi}{\delta})$. Following the approach in [17], we modify the nonlinearity separately in the hyperbolic and nonhyperbolic directions and define $R_\delta : X \to \mathbb{C}^n$ by

$$R_\delta(\phi) = \chi_\delta(\|Q_0\phi\|)\chi_\delta(\|(I - Q_0)\phi\|)R(\phi). \tag{6.1}$$

Lemma 6.1. *Let E and F be Banach spaces and let $f : E \to F$ with $f(0) = 0$ be a Lipschitz continuous mapping with Lipschitz constant $L(\delta)$ on on the ball of radius δ. Let $V, W \in \mathcal{L}(E, E)$ with $V + W = I$. Then there exists $C > 0$ such that for all $\delta > 0$ the mapping $x \to \chi_\delta(\|Vx\|)\chi_\delta(\|Wx\|)f(x)$ is globally Lipschitz continuous with Lipschitz constant $(4C\|V\| + 4C\|W\| + 1)L(4\delta)$.*

Proof. There exists $C > 0$ such that χ_δ is globally Lipschitz continuous with Lipschitz constant C/δ. Introduce the shorthands $f_x = f(x)$, $\chi_x^V = \chi_\delta(\|Vx\|)$ and $\chi_x^W = \chi_\delta(\|Wx\|)$ and the corresponding notations for y. We obtain the following estimate,

$$\begin{aligned} \Delta &= \|f(x)\chi_\delta(\|Vx\|)\chi_\delta(\|Wx\|) - f(y)\chi_\delta(\|Vy\|)\chi_\delta(\|Wy\|)\| = \left\|f_x\chi_x^V\chi_x^W - f_y\chi_y^V\chi_y^W\right\| \\ &\leq \|f_x - f_y\|\chi_y^V\chi_y^W + \|f_x\|\left|\chi_x^V - \chi_y^V\right|\chi_y^W + \|f_x\|\chi_x^V\left|\chi_x^W - \chi_y^W\right|. \end{aligned} \tag{6.2}$$

We now treat the three different cases. Suppose that both $\chi_x^V\chi_x^W = 0$ and $\chi_y^V\chi_y^W = 0$, then it immediately follows that $\Delta = 0$. Now suppose that both $\chi_x^V\chi_x^W \neq 0$ and $\chi_y^V\chi_y^W \neq 0$, which implies $\|x\|, \|y\| \leq 4\delta$. This means $\|f_x\|, \|f_y\| \leq 4\delta L(4\delta)$ and hence

$$\begin{aligned} \Delta &\leq L(4\delta)\|x - y\| + 4\delta L(4\delta)\tfrac{C}{\delta}\|V\|\|x - y\| + 4\delta L(4\delta)\tfrac{C}{\delta}\|W\|\|x - y\| \\ &= (4C\|V\| + 4C\|W\| + 1)L(4\delta)\|x - y\|. \end{aligned} \tag{6.3}$$

Notice the only case left to consider is the situation where $\chi_x^V\chi_x^W \neq 0$ but $\chi_y^V\chi_y^W = 0$, since x and y are interchangeable. We obtain

$$\Delta \leq 4\delta L(4\delta)\frac{C}{\delta}\|V\|\|x - y\| + 4\delta L(4\delta)\frac{C}{\delta}\|W\|\|x - y\| = (4C\|V\| + 4C\|W\|)L(4\delta)\|x - y\|. \tag{6.4}$$

□

Corollary 6.2. *The mappings $R_\delta : X \to \mathbb{C}^n$ are globally Lipschitz continuous with Lipschitz constants L_{R_δ} that go to zero as δ goes to zero. In addition, $\|R_\delta(\phi)\| \leq 4\delta L_{R_\delta}$ for all $\phi \in X$.*

Proof. The first statement follows from assumption (HR2). The second statement follows by noting that if $R_\delta(\phi) \neq 0$, then $\|\phi\| \leq \|Q_0\phi\| + \|(I - Q_0)\phi\| \leq 2\delta + 2\delta = 4\delta$. □

We observe here that the nonlinearity R_δ induces a map $\widetilde{R}_\delta : C(\mathbb{R}, \mathbb{C}^n) \to C(\mathbb{R}, \mathbb{C}^n)$ via substitution, i.e.,

$$\widetilde{R}_\delta x(\xi) = R_\delta x_\xi. \tag{6.5}$$

Notice that $\widetilde{R}_\delta$ is well-defined, since $i_x : \mathbb{R} \to X$ which sends $\xi \to x_\xi$ is a continuous mapping for any continuous x and hence the same holds for $\widetilde{R}_\delta x = R_\delta \circ i_x$. The next lemma shows that $\widetilde{R}_\delta$ inherits the global Lipschitz continuity of R_δ.

Lemma 6.3. *For any $\eta \in \mathbb{R}$, the substitution operator $\widetilde{R}_\delta$ viewed as an operator from $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ into $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ is globally Lipschitz continuous with Lipschitz constant $w^\eta L_{R_\delta}$.*

Proof. Write $x = \widetilde{R}_\delta u$, $y = \widetilde{R}_\delta v$ and compute

$$e^{-\eta|\xi|} |y(\xi) - x(\xi)| \quad = \quad e^{-\eta|\xi|} |R_\delta u_\xi - R_\delta v_\xi| \leq e^{-\eta|\xi|} L_{R_\delta} \|u_\xi - v_\xi\| \leq w^\eta L_{R_\delta} \|u - v\|_\eta . \tag{6.6}$$

□

We are now ready to construct solutions to the system (2.3) with the modified nonlinearity R_δ substituted for R. This will be done by employing a fixed point argument. To this end, we recall the extension operator $T : X_0 \to \bigcap_{\zeta > 0} BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ introduced in Proposition 5.2 and define an operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ via

$$\mathcal{G}(u, \phi) = T\phi + \mathcal{K}_\eta \widetilde{R}_\delta(u). \tag{6.7}$$

Choose $\delta > 0$ small enough to guarantee

$$w^\eta L_{R_\delta} \|\mathcal{K}_\eta\| < \frac{1}{2}. \tag{6.8}$$

Note that if $\|T\|_\eta \|\phi\| < \frac{\rho}{2}$, then $\mathcal{G}(\cdot, \phi)$ leaves the ball with radius ρ in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ invariant. Notice in addition that $\mathcal{G}(\cdot, \phi)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. Since ρ can be chosen arbitrarily, the following theorem can be established using standard arguments.

Theorem 6.4. *Consider the system (2.3) and suppose that the conditions (HR1) and (HR2) are satisfied. Fix $\gamma > 0$ such that the characteristic equation $\det \Delta(z) = 0$ has no roots with $0 < |\operatorname{Re} z| < \gamma$. Fix any $\eta \in (0, \gamma)$ and choose $\delta > 0$ such that (6.8) is satisfied. Then there exists a globally Lipschitz continuous mapping u^*_η from X_0 into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ such that $u = u^*_\eta \phi$ is the unique solution in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ of the equation*

$$u = \mathcal{G}(u, \phi). \tag{6.9}$$

The following results show that the family of mappings u^*_ζ defined above behaves appropriately under translations and under shifts of the parameter ζ.

Lemma 6.5. *Consider the setting of Theorem 6.4 and let $\phi \in X_0$. Then for any $\xi_0 \in \mathbb{R}$ we have the identity*

$$(u_\eta^* \phi)(\xi_0 + \cdot) = (u_\eta^* Q_0 (u_\eta^* \phi)_{\xi_0})(\cdot). \tag{6.10}$$

Proof. Using Lemma 5.4 we compute

$$\psi := Q_0 (u_\eta^* \phi)_{\xi_0} = (T\phi)_{\xi_0} + (\mathcal{K}_\eta \widetilde{R}_\delta (u_\eta^* \phi))_{\xi_0} - (\mathcal{K}_\eta \widetilde{R}_\delta ((u_\eta^* \phi)(\xi_0 + \cdot)))_0, \tag{6.11}$$

hence upon defining

$$y(\xi) = T\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{R}_\delta (u_\eta^* \phi)(\xi_0 + \xi) - \mathcal{K}_\eta \widetilde{R}_\delta ((u_\eta^* \phi)(\xi_0 + \cdot))(\xi), \tag{6.12}$$

we conclude that $y \in \mathcal{N}_0$ by Lemma 5.4 and in addition that $y = T\psi$. Upon calculating

$$\begin{array}{rcl} \mathcal{G}((u_\eta^* \phi)(\xi_0 + \cdot), \psi)(\xi) & = & y(\xi) + \mathcal{K}_\eta \widetilde{R}_\delta ((u_\eta^* \phi)(\xi_0 + \cdot))(\xi) \\ & = & T\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{R}_\delta (u_\eta^* \phi)(\xi_0 + \xi) = (u_\eta^* \phi)(\xi_0 + \xi), \end{array} \tag{6.13}$$

we see that due to uniquess of solutions we must have

$$(u_\eta^* \psi)(\xi) = (u_\eta^* \phi)(\xi_0 + \xi), \tag{6.14}$$

from which the claim follows. □

Combining Lemma 5.3 and Corollary 6.2 immediately yields the final result of this section.

Lemma 6.6. *Consider any pair $0 < \eta_1 < \eta_2 < \gamma$ and suppose that (6.8) holds for both η_1 and η_2. Then we have $u_{\eta_2}^* = \mathcal{J}_{\eta_2 \eta_1}^1 u_{\eta_1}^*$.*

7 Smoothness of the center manifold

In the previous section we saw that the mapping $u_\eta^* : X_0 \to BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$ is Lipschitz continuous. In this section we will extend this result and show that u_η^* inherits the C^k-smoothness of the nonlinearity R. More precisely, we shall establish the following theorem.

Theorem 7.1. *Consider the system (2.3) and suppose that the conditions (HR1) and (HR2) are satisfied. Fix $\gamma > 0$ such that the characteristic equation $\det \Delta(z) = 0$ has no roots with $0 < |\mathrm{Re}\, z| < \gamma$ and consider any interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ with $k\eta_{\min} < \eta_{\max}$, where k is as defined in (HR1). Then there exists $\delta > 0$ such that the following statements hold.*

(i) For any $\eta \in [\eta_{\min}, \eta_{\max}]$, we have the inequality

$$w^{\eta} L_{R_\delta} \|\mathcal{K}_\eta\| < \frac{1}{4}. \tag{7.1}$$

(ii) For each integer $1 \le p \le k$ and for each $\eta \in (p\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is of class C^p, where u^*_ζ for $\zeta \in [\eta_{\min}, \eta_{\max}]$ is as defined in Theorem 6.4 with the above value for δ.

We remark here that the arguments in this section follow closely the lines of [17, Section IX.7]. Throughout this entire section we consider a fixed system (2.3) that satisfies the conditions (HR1) and (HR2), i.e., we shall use the corresponding integer k and C^k-smooth nonlinearity R without further comment in our results.

As a first step towards proving the above theorem, we need to find a suitable domain of definition for $\widetilde{R}_\delta$ to ensure that this operator becomes sufficiently smooth. Due to the presence of the cutoff function on the infinite dimensional complement of X_0, the nonlinearity R_δ loses the C^k-smoothness on X and becomes merely Lipschitz continuous. In view of these observations, we introduce for any $\eta > 0$ the space

$$V^1_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \infty \right\}, \tag{7.2}$$

in which $Q_h = (I - Q_0)$ is the projection onto the hyperbolic part of X. We provide the above space with the norm

$$\|u\|_{V^1_\eta} = \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} \|Q_0 u_\xi\| + \sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| + \|\dot{u}\|_\eta , \tag{7.3}$$

with which $V^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is a Banach space that has continuous inclusions $V^1_\eta(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$. In addition, for any $\delta > 0$ we define the open set

$$V^{1,\delta}_\eta = \left\{ u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \delta \right\} \subset V^1_\eta(\mathbb{R}, \mathbb{C}^n). \tag{7.4}$$

Since X_0 is finite dimensional, we have that R_δ is of class C^k on the set $B^h_\delta = \{\phi \in X \mid \|Q_h \phi\| < \delta\}$. In addition, the norms $\|D^p R_\delta \phi\|$ are uniformly bounded on B^h_δ for all $0 \le p \le k$. Thus, for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ for which $\sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \delta$ and any $0 \le p \le k$, we can define a map $\widetilde{R}^{(p)}_\delta(u) \in \mathcal{L}^{(p)}(C(\mathbb{R}, \mathbb{C}^n), C(\mathbb{R}, \mathbb{C}^n))$ by

$$\widetilde{R}^{(p)}_\delta(u)(v_1, \ldots, v_p)(\xi) = D^p R_\delta(u_\xi)\big((v_1)_\xi, \ldots, (v_p)_\xi\big). \tag{7.5}$$

Here the symbol $\mathcal{L}^{(p)}(Y_1 \times \ldots \times Y_p, Z)$ denotes the space of p-linear mappings from $Y_1 \times \ldots \times Y_p$ into Z. If $Y_1 = \ldots = Y_p = Y$, we use the shorthand $\mathcal{L}^{(p)}(Y, Z)$. Note that the map $\widetilde{R}_\delta^{(p)}(u)$ defined above is well defined, since $D^p R_\delta$ is a continuous map from $B_\delta^h \times X^p$ into $\mathbb{C}^n$, as is the map $i_x : \mathbb{R} \to X$ which sends $\xi \to x_\xi$, for any $x \in C(\mathbb{R}, \mathbb{C}^n)$.

The next lemma shows that for sufficiently small δ, the operator u_η^* maps precisely into the region on which the modification of R in the hyperbolic direction is trivial, which means that R_δ is C^k-smooth on this region.

Lemma 7.2. *Let $\delta > 0$ be so small that for some $0 < \eta_0 < \gamma$,*

$$w^{\eta_0} L_{R_\delta} < (4 \|\mathcal{K}_{\eta_0}\|)^{-1}. \tag{7.6}$$

Then for any $\phi \in X_0$ and $0 < \eta < \gamma$, we have that for all $\xi \in \mathbb{R}$,

$$\left\|Q_h(u_\eta^* \phi)_\xi\right\| < \delta. \tag{7.7}$$

Proof. Note first that the cutoff function ensures that

$$\left\|\widetilde{R}_\delta(u_\eta^* \phi)\right\|_0 \leq 4\delta L_{R_\delta}. \tag{7.8}$$

Since Lemma 5.3 guarantees that $\mathcal{K}_\eta$ and $\mathcal{K}_{\eta_0}$ agree on $BC_0(\mathbb{R}, \mathbb{C}^n)$, we can use Corollary 5.5 to compute

$$\left\|Q_h(u_\eta^* \phi)_\xi\right\| = \left\|Q_h \mathcal{K}_{\eta_0}(\widetilde{R}_\delta(u_\eta^* \phi))_\xi\right\| \leq w^{\eta_0} \|\mathcal{K}_{\eta_0}\| 4\delta L_{R_\delta}. \tag{7.9}$$

□

The next series of results establishes conditions under which the maps $\widetilde{R}_\delta : V_\sigma^{1,\delta}(\mathbb{R}, \mathbb{C}^n) \to BC_\zeta^1(\mathbb{R}, \mathbb{C}^n)$ are smooth. In the remainder of this section we will for convenience adopt the shorthand $BC_\zeta^1 = BC_\zeta^1(\mathbb{R}, \mathbb{C}^n)$, together with similar ones for the other function spaces.

Lemma 7.3. *Let $1 \leq p \leq k$, $\zeta_i > 0$ for $1 \leq i \leq p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta \geq \zeta$. Then for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ such that $\sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \delta$, we have*

$$\widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(BC_{\zeta_1}^1 \times \ldots \times BC_{\zeta_p}^1, BC_\eta), \tag{7.10}$$

where the norm is bounded by

$$\left\|\widetilde{R}_\delta^{(p)}\right\|_{\mathcal{L}^{(p)}} \leq w^\zeta \sup_{\xi \in \mathbb{R}} e^{-(\eta - \zeta)|\xi|} \left\|D^p R_\delta(u_\xi)\right\| < \infty. \tag{7.11}$$

If $\eta > \zeta$ and $\sigma > 0$, then in addition $u \to \widetilde{R}_\delta^{(p)}(u)$ is continuous as a map from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$.

Finally, in the statements above, any subset of the $BC^1_{\zeta_i}$ spaces may be replaced by $V^1_{\zeta_i}$.

Proof. We define $r = \sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \delta$. The bound for $\left\|\widetilde{R}_\delta^{(p)}\right\|_{\mathcal{L}^{(p)}}$ follows from the estimates $\|(v_i)_\xi\| \le w^{\zeta_i} e^{\zeta_i |\xi|} \|v_i\|_{\zeta_i}$ and $\|v_i\|_{\zeta_i} \le \|v_i\|_{BC^1_{\zeta_i}}$. Since $\|D^p R_\delta\|$ is uniformly bounded on B^h_δ, the norm above can be seen to be finite and hence $\widetilde{R}_\delta^{(p)}(u)$ is well defined.

We now consider the case that $\eta > \zeta$ and prove the continuity of $u \to \widetilde{R}_\delta^{(p)}(u)$. Let $\widetilde{B} \subset V^1_\sigma$ be the open ball of radius $\delta - r$ and note that for any $0 < \epsilon < 1$, we have

$$\begin{array}{l} \sup_{g \in \widetilde{B}} \left\| \widetilde{R}_\delta^{(p)}(u + \epsilon g) - \widetilde{R}_\delta^{(p)}(u) \right\|_{\mathcal{L}^{(p)}} \\ \le \sup_{g \in \widetilde{B}} \sup_{\xi \in \mathbb{R}} e^{-(\eta - \zeta)|\xi|} \left\| D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi) \right\|. \end{array} \tag{7.12}$$

Fix an arbitrary $\kappa > 0$. Exploiting the fact that $D^p R_\delta$ is uniformly bounded on B^h_δ, we choose an $A > 0$ such that

$$2e^{-(\eta-\zeta)A} \sup_{\phi \in B^h_\delta} \|D^p R_\delta(\phi)\| \le \kappa, \tag{7.13}$$

which implies

$$\sup_{g \in \widetilde{B}} \sup_{|\xi| \ge A} e^{-(\eta-\zeta)|\xi|} \|D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi)\| \le \kappa. \tag{7.14}$$

Due to the compactness of the interval $[-A, A]$, we can choose a finite open covering $\mathrm{Cov} = \bigcup_{j=1}^M B_{\rho_j}(u_{\xi_j}) \subset B^h_\delta \subset X$, with standard open balls $B_\rho(\psi) \subset X$, such that $u_\xi \in \mathrm{Cov}$ for all $\xi \in [-A, A]$ and in addition $\left\|D^p R_\delta(\phi) - D^p R_\delta(u_{\xi_j})\right\| \le \frac{\kappa}{2}$ for all $\phi \in B_{2\rho_j}(u_{\xi_j})$. Choose any $\epsilon > 0$ such that $\epsilon w^\sigma e^{\sigma A}(\delta - r) < \min\{\rho_j \mid 1 \le j \le M\}$. This implies that for every $g \in \widetilde{B}$ and any $1 \le j \le M$ we have $\|\epsilon g_\xi\| \le \epsilon w^\sigma e^{\sigma|\xi|} \|g\|_{V^1_\sigma} < \rho_j$ and hence

$$\begin{array}{rcl} \|D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi)\| & \le & \left\|D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_{\xi_{j_0}})\right\| \\ & + & \left\|D^p R_\delta(u_{\xi_{j_0}}) - D^p R_\delta(u_\xi)\right\| \\ & \le & \frac{\kappa}{2} + \frac{\kappa}{2} = \kappa, \end{array} \tag{7.15}$$

where we have chosen j_0 such that $u_\xi \in B_{\rho_{j_0}}(u_{\xi_{j_0}})$. Since $\xi > 0$ was arbitrary, we have that $u \to \widetilde{R}_\delta^{(p)}(u)$ is indeed continuous as a map from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$. Finally, note that the arguments above carry over upon replacing any subset of the $BC^1_{\zeta_i}$ spaces by their corresponding $V^1_{\zeta_i}$ spaces. □

Lemma 7.4. *Let $0 \le p < k$, $\zeta_i > 0$ for $1 \le i \le p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + \sigma$. Then the map $\widetilde{R}_\delta^{(p)} : V_\sigma^{1,\delta} \to \mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$ is of class C^1 with derivative*

$$D\widetilde{R}_\delta^{(p)}(u) = \widetilde{R}_\delta^{(p+1)}(u) \in \mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p} \times V^1_\sigma, BC_\eta). \tag{7.16}$$

In addition, the same statement holds upon replacing any subset of the $BC^1_{\zeta_i}$ spaces with the corresponding $V^1_{\zeta_i}$ spaces.

Proof. Pick an arbitrary $u \in V_\sigma^{1,\delta}$ and write $r = \sup_{\xi \in \mathbb{R}} \|Q_h u_\xi\| < \delta$. Write $\widetilde{B} \subset V^1_\sigma$ for the open ball with radius $\delta - r$ and note that for any $0 < \epsilon < 1$, we have

$$\begin{aligned}
&\sup_{g \in \widetilde{B}} \tfrac{1}{\epsilon} \left\| \widetilde{R}_\delta^{(p)}(u + \epsilon g) - \widetilde{R}_\delta^{(p)}(u) - \epsilon \widetilde{R}_\delta^{(p+1)}(u) g \right\| \\
&= \sup_{g \in \widetilde{B}} \sup_{\xi \in \mathbb{R}} \sup_{\|v_1\|_{BC^1_{\zeta_1}} = 1} \cdots \sup_{\|v_p\|_{BC^1_{\zeta_p}} = 1} \tfrac{1}{\epsilon} || D^p R_\delta(u_\xi + \epsilon g_\xi)((v_1)_\xi, \ldots (v_p)_\xi) \\
&-D^p R_\delta(u_\xi)((v_1)_\xi, \ldots, (v_m)_\xi) - \epsilon D^{p+1} R_\delta(u_\xi)((v_1)_\xi, \ldots, (v_p)_\xi, g_\xi) ||_\eta \\
&\le \sup_{\xi \in \mathbb{R}} \sup_{\phi \in B^h_{(\xi)}} w^{\zeta + \sigma} e^{(-\eta + \zeta + \sigma)|\xi|} \left\| D^{p+1} R_\delta(u_\xi + \epsilon\phi) - D^{p+1} R_\delta(u_\xi) \right\|,
\end{aligned} \tag{7.17}$$

where we have introduced $B^h_{(\xi)} = \{\phi \in X \mid \|\phi\| < (\delta - r) w^\sigma e^{\sigma|\xi|} \text{ and } \|Q_h \phi\| < \delta - r\}$. Since the exponent $-\eta + \zeta + \sigma$ is negative, one can reason as in the proof of Lemma 7.3 to conclude that the last expression tends to zero as $\epsilon \to 0$. This implies $D\widetilde{R}_\delta^{(p)}(u) = \widetilde{R}_\delta^{(p+1)}(u)$ as an operator in $\mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times BC^1_{\zeta_p} \times V^1_\sigma, BC_\eta)$. Lemma 7.3 ensures that this derivative $u \to \widetilde{R}_\delta^{(p+1)}(u)$ is continuous. Again, the arguments above carry over upon replacing any subset of the $BC^1_{\zeta_i}$ spaces by their corresponding $V^1_{\zeta_i}$ spaces. □

Corollary 7.5. *Let $\eta_2 > k\eta_1 > 0$ and $1 \le p \le k$. Then the mapping $\widetilde{R}_\delta : V^{1,\delta}_{\eta_1} \to BC_{\eta_2}$ is of class C^k with*

$$D^p \widetilde{R}_\delta(u) = \widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(V^1_{\eta_1}, BC_{\eta_2}) \cap \mathcal{L}^{(p)}(BC^1_{\eta_1}, BC_{\eta_2}). \tag{7.18}$$

Proof. The fact that $\widetilde{R}_\delta$ is of class C^k follows by repeated application of Lemma 7.4. In addition, Lemma 7.3 implies that the derivatives $\widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(V^1_{\eta_1}, BC_{\eta_2})$ can be naturally extended to elements in $\mathcal{L}^{(p)}(BC^1_{\eta_1}, BC_{\eta_2})$.

□

Corollary 7.6. *Let $1 \le p \le k$, $\zeta_i > 0$ for $1 \le i \le p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + (k-p)\sigma$ for some $\sigma > 0$. Then the mapping $\widetilde{R}_\delta^{(p)} : V_\sigma^{1,\delta} \to \mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$ is of class C^{k-p}.*

Lemma 7.7. *Let $1 \leq p < k$, $\zeta_i > 0$ for $1 \leq i \leq p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + \sigma$ for some $\sigma > 0$. Let Φ be a mapping of class C^1 from X_0 into $V_\sigma^{1,\delta}$. Then the mapping $\widetilde{R}_\delta^{(p)} \circ \Phi$ from X_0 into $\mathcal{L}^{(p)}(BC_{\zeta_1}^1, \ldots, BC_{\zeta_p}^1, BC_\eta)$ is of class C^1 with*

$$D(\widetilde{R}_\delta^{(p)} \circ \Phi)(\phi)(v_1, \ldots v_p, \psi) = \widetilde{R}^{(p+1)}(\Phi(\phi))(v_1, \ldots, v_p, \Phi'(\phi)\psi). \tag{7.19}$$

Proof. Let $M = \sup_{\phi \in B_\delta^h} \left\| D^{(p+1)} R_\delta(\phi) \right\|$. Fix $\mathbf{v} = (v_1, \ldots, v_p)$, with $\|v_i\|_{\eta_i} = 1$. Observe that if

$$S(\xi) = \widetilde{R}_\delta^{(p)}(\Phi(\phi))(\mathbf{v})(\xi) - \widetilde{R}_\delta^{(p)}(\Phi(\psi))(\mathbf{v})(\xi) - \widetilde{R}^{(p+1)}(\Phi(\psi))(\mathbf{v}, \Phi'(\psi)(\phi - \psi))(\xi), \tag{7.20}$$

then S can be written as $S(\xi) = S_1(\xi) + S_2(\xi)$, with

$$\begin{array}{rcl} S_1(\xi) & = & \int_0^1 \left(D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) - D^{p+1} R_\delta(\Phi(\psi)_\xi) \right) \left(\mathbf{v}_\xi, (\Phi'(\psi)(\phi - \psi))_\xi \right) d\theta, \\ S_2(\xi) & = & \int_0^1 D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) \left(\mathbf{v}_\xi, \Phi(\phi)_\xi - \Phi(\psi)_\xi - (\Phi'(\psi)(\phi - \psi))_\xi \right) d\theta. \end{array} \tag{7.21}$$

Define $I(\xi) = \int_0^1 \left\| D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) - D^{p+1} R_\delta(\Phi(\psi)_\xi) \right\| d\theta$ and calculate

$$\begin{array}{rcl} e^{-\eta|\xi|} \left| S_1(\xi) \right| & \leq & w^{\zeta+\sigma} e^{(-\eta+\zeta+\sigma)|\xi|} \left\| \phi - \psi \right\| \left\| \Phi'(\psi) \right\|_{V_\sigma^1} I(\xi) \\ & \leq & w^{\zeta+\sigma} \left\| \phi - \psi \right\| \left\| \Phi'(\psi) \right\|_{V_\sigma^1} \max \left\{ 2M e^{(-\eta+\zeta+\sigma)A}, \sup_{\xi \in [-A,A]} I(\xi) \right\}, \\ e^{-\eta|\xi|} \left| S_2(\xi) \right| & \leq & M w^{\zeta+\sigma} e^{(-\eta+\zeta+\sigma)|\xi|} \left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1} \\ & \leq & M w^{\zeta+\sigma} \left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1}. \end{array} \tag{7.22}$$

Fixing some $\epsilon > 0$ and letting $A > 0$ be such that $2Me^{(-\eta+\zeta+\sigma)A} < \epsilon$, we define

$$\Omega = \{ \Phi(\psi)_\xi \mid \xi \in [-A, A] \} \subset X. \tag{7.23}$$

We can argue as in the proof of Lemma 7.3 to show that there exists $\delta_1 > 0$ such that

$$\left\| D^{p+1} R_\delta(\overline{\phi} + \overline{\psi}) - D^{p+1} R_\delta(\overline{\phi}) \right\| < \epsilon \tag{7.24}$$

for any $\overline{\phi} \in \Omega$ and $\left\| \overline{\psi} \right\| < \delta_1$. Since $\sup_{\xi \in [-A,A]} \left\| \Phi(\phi)_\xi - \Phi(\psi)_\xi \right\| \to 0$ as $\phi \to \psi$, there exists $\delta_2 > 0$ such that $\|\phi - \psi\| < \delta_2$ implies $\left\| \Phi(\phi)_\xi - \Phi(\psi)_\xi \right\| < \delta_1$ for all $\xi \in [-A, A]$. In addition, as Φ is differentiable at ψ, there exists $\delta_3 > 0$ such that $\left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1} \leq \|\phi - \psi\| \epsilon$ whenever $\|\phi - \psi\| < \delta_3$. Together this implies that if $\|\phi - \psi\| < \min(\delta_2, \delta_3)$, we have

$$\left\| S(\cdot) \right\|_\eta \leq \|\phi - \psi\| w^{\sigma+\zeta} (M + \left\| \Phi'(\psi) \right\|_{V_\sigma^1}) \epsilon, \tag{7.25}$$

which proves that $\widetilde{R}^{(p)} \circ \Phi$ is differentiable. The continuity of this derivative follows from the fact that Φ is of class C^1 together with the continuity of the mapping $u \to \widetilde{R}^{(p+1)}(u)$ from $V^{1,\delta}_{\sigma}$ into $\mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p} \times V^1_{\sigma}, BC_{\eta})$. $\square$

Corollary 7.8. *Consider any pair $0 < \eta_1 < \eta_2 < \gamma$. Then the map from $V^{1,\delta}_{\eta_1}$ into $BC^1_{\eta_2}$ defined by*

$$u \to \mathcal{J}^1_{\eta_2\eta_1}\mathcal{K}_{\eta_1}\widetilde{R}_{\delta}(u) \tag{7.26}$$

is of class C^1 with derivative $u \to \mathcal{J}^1_{\eta_2\eta_1} \circ \mathcal{K}_{\eta_1} \circ \widetilde{R}^{(1)}_{\delta}(u) \in \mathcal{L}(V^1_{\eta_1}, BC^1_{\eta_2}) \cap \mathcal{L}(BC^1_{\eta_1}, BC^1_{\eta_2})$.

Proof. Using Lemma 5.3 and Corollary 6.2 we observe that $\mathcal{J}^1_{\eta_2\eta_1}\mathcal{K}_{\eta_1}\widetilde{R}_{\delta}(u) = \mathcal{K}_{\eta_2}\widetilde{R}_{\delta}(u)$. This last map is C^1-smooth by Lemma 7.4 and the fact that $\mathcal{K}_{\eta_2}$ is a bounded linear operator from BC_{η_2} into $BC^1_{\eta_2}$, with derivative $\mathcal{K}_{\eta_2} \circ \widetilde{R}^{(1)}_{\delta}(u)$. The proof is completed upon noting that $\widetilde{R}^{(1)}_{\delta}(u)$ in fact maps $BC^1_{\eta_1}$ into BC_{η_1} by Lemma 7.3. $\square$

Proof of Theorem 7.1. In view of Lemma 5.6 we can choose the constant $\delta > 0$ in such a way that both (7.1) and (7.6) are satisfied. We start with the case $k = 1$. Let $\eta \in (\eta_{\min}, \eta_{\max}]$. We will apply Lemma A.2 with the Banach spaces $Y_0 = V^1_{\eta_{\min}}$, $Y = BC^1_{\eta_{\min}}$, $Y_1 = BC^1_{\eta}$ with the corresponding natural inclusions and $\Lambda = X_0$. We fix $\Omega_0 = V^{1,\delta}_{\eta_{\min}} \subset V^1_{\eta_{\min}}$, recall the extension operator $T : X_0 \to \bigcap_{\zeta>0} BC^1_{\zeta}(\mathbb{R}, \mathbb{C}^n)$ introduced in Proposition 5.2 and choose

$$\begin{aligned}
F(u,\phi) &= T\phi + \mathcal{K}_{\eta_{\min}}\widetilde{R}_{\delta}(u), && \phi \in X_0, \quad u \in BC^1_{\eta_{\min}}, \\
F^{(1)}(u,\phi) &= \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(1)}_{\delta}(u) \in \mathcal{L}(BC^1_{\eta_{\min}}), && \phi \in X_0, \quad u \in V^{1,\delta}_{\eta_{\min}}, \\
F^{(1)}_1(u,\phi) &= \mathcal{K}_{\eta} \circ \widetilde{R}^{(1)}_{\delta}(u) \in \mathcal{L}(BC^1_{\eta}), && \phi \in X_0, \quad u \in V^{1,\delta}_{\eta_{\min}}.
\end{aligned} \tag{7.27}$$

In the context of Lemma A.2, we have that $G : V^1_{\eta_{\min}} \times X_0 \to BC^1_{\eta}$ is defined by

$$G(u,\phi) = T\phi + \mathcal{J}^1_{\eta\eta_{\min}}\mathcal{K}_{\eta_{\min}}\widetilde{R}_{\delta}(u), \tag{7.28}$$

and hence using Corollary 7.8 and Lemma 7.3 we see that condition (HC1) is satisfied. Since $\sup_{\phi \in B^h_{\delta}} \|DR_{\delta}(\phi)\| \le L_{R_{\delta}}$, we see that (7.1) in combination with Lemma 7.3 implies condition (HC2). Condition (HC3) follows from Corollary 7.8, (HC4) is evident since $D_2G(u,\phi)\psi = T\psi \in BC^1_{\eta_{\min}}$, (HC5) follows from (7.1) and finally (HC6) follows from Lemma 7.2. We conclude that $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^1 and that $D(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}})(\phi) = \mathcal{J}^1_{\eta\eta_{\min}} \circ u^{*(1)}_{\eta_{\min}}(\phi) \in \mathcal{L}(X_0, BC^1_{\eta})$, where

$u^{*(1)}_{\eta_{\min}}(\phi)$ is the unique solution of the equation

$$u^{(1)} = \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(1)}(u^*_{\eta_{\min}}(\phi))u^{(1)} + T \tag{7.29}$$

in the space $\mathcal{L}(X_0, BC^1_{\eta_{\min}})$.

We now assume that $k \geq 2$ and use induction on p. Let $1 \leq p < k$ and suppose that for all $1 \leq q \leq p$ and all $\eta \in (q\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^q with $D^q(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}) = \mathcal{J}^1_{\eta\, q\eta_{\min}} u^{*(q)}_{\eta_{\min}}$, for some map $u^{*(q)}_{\eta_{\min}} : X_0 \to \mathcal{L}^{(q)}(X_0, BC^1_{q\eta_{\min}})$. In addition, assume that $u^{*(p)}_{\eta_{\min}}(\phi)$ is the unique solution at $\overline{\eta} = \eta_{\min}$ of an equation of the form

$$u^{(p)} = \mathcal{K}_{p\overline{\eta}} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))u^{(p)} + H^{(p)}_{\overline{\eta}}(\phi) = F^{(p)}_{\overline{\eta}}(u^{(p)}, \phi), \tag{7.30}$$

in $\mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}})$. Here we have $H^{(1)}(\phi) = T$ and for $p \geq 2$ we can write $H^{(p)}_{\overline{\eta}}(\phi)$ as a finite sum of terms of the form

$$\mathcal{K}_{p\overline{\eta}} \circ \widetilde{R}^{(q)}_{\delta}(u^*_{\eta_{\min}}(\phi))(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)) \tag{7.31}$$

with $2 \leq q \leq p$ and integers $e_i \geq 1$ such that $e_1 + \ldots + e_q = p$. Notice that these conditions ensure that $F^{(p)}_{\overline{\eta}} : \mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}}) \times X_0 \to \mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}})$ is well-defined for all $\overline{\eta} \in [\eta_{\min}, \frac{1}{p}\eta_{\max}]$ and, in addition, is a uniform contraction for these values of $\overline{\eta}$. We now fix $\eta \in ((p+1)\eta_{\min}, \eta_{\max}]$ and choose σ and ζ such that $\eta_{\min} < \sigma < (p+1)\sigma < \zeta < \eta$. We wish to apply Lemma A.2 in the setting $\Omega_0 = Y_0 = \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma})$, $Y = \mathcal{L}^{(p)}(X_0, BC^1_{\zeta})$, $Y_1 = \mathcal{L}^{(p)}(X_0, BC^1_{\eta})$ with the corresponding natural inclusions and $\Lambda = X_0$. We use the functions

$$\begin{array}{rcll} F(u^{(p)}, \phi) & = & \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))u^{(p)} + H^{(p)}_{\zeta/p}(\phi), & \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{\zeta}), \\ F^{(1)}(u^{(p)}, \phi) & = & \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi)) \in \mathcal{L}(\mathcal{L}^{(p)}(X_0, BC^1_{\zeta})), & \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}), \\ F^{(1)}_1(u^{(p)}, \phi) & = & \mathcal{K}_{\eta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi)) \in \mathcal{L}(\mathcal{L}^{(p)}(X_0, BC^1_{\eta})), & \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}). \end{array} \tag{7.32}$$

To check (HC1), we need to show that the map $G : \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}) \times X_0 \to \mathcal{L}^{(p)}(X_0, BC^1_{\eta})$ given by

$$G(u^{(p)}, \phi) = \mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))u^{(p)} + \mathcal{J}^1_{\eta\zeta} H^{(p)}_{\zeta/p}(\phi) \tag{7.33}$$

is of class C^1. In view of the linearity of this map with respect to $u^{(p)}$, it is sufficient to show that $\phi \to \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))$ is of class C^1 as a map from X_0 into $\mathcal{L}(BC^1_{p\sigma}, BC^1_{\zeta})$ and, in addition, that $\phi \to H^{(p)}_{\zeta/p}(\phi)$ is of class C^1 as a map from X_0 into $\mathcal{L}^{(p)}(X_0, BC^1_{\zeta})$. The first fact follows from Lemma

7.7 using $\zeta > (p+1)\sigma$ and the C^1-smoothness of the map $\phi \to \mathcal{J}^1_{\sigma\eta_{\min}} u^*_{\eta_{\min}}\phi$. To verify the second fact, we use Lemma 7.7 and the chain rule to compute

$$\begin{aligned} &D_\phi \mathcal{K}_\zeta \circ \widetilde{R}^{(q)}_\delta(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)) \\ &= \mathcal{K}_\zeta \circ \widetilde{R}^{(q+1)}_\delta(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi), u^{*(1)}_{\eta_{\min}}(\phi)) \\ &+ \textstyle\sum_{j=1}^q \mathcal{K}_\zeta \circ \widetilde{R}^{(q)}(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_j+1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)), \end{aligned} \tag{7.34}$$

in which each occurrence of $u^{*(j)}_{\eta_{\min}}$ is understood to map into $BC^1_{j\sigma}$. An application of Lemma 7.3 with $\zeta > (p+1)\sigma$, shows that the above map is indeed continuous from X_0 into $\mathcal{L}^{(p+1)}(X_0, BC^1_\zeta)$. These arguments immediately show that also (HC4) is satisfied. Condition (HC3) can be verified by writing $\mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_\zeta \circ \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}\phi) = \mathcal{K}_\eta \circ \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}\phi)$ and applying Lemma 7.3 to conclude that $\phi \to \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}\phi) \in \mathcal{L}(BC^1_\zeta, BC_\eta)$ is continuous. Conditions (HC2) and (HC5) again follow from (7.1) and Lemma 7.3 and (HC6) follows from the fact that the fixed point of (7.30) lies in $\mathcal{L}^{(p)}(X_0, BC^1_{p\sigma})$ since $p\sigma > p\eta_{\min}$. We thus conclude from Lemma A.2 that $\mathcal{J}^1_{\eta\, p\eta_{\min}} \circ u^{*(p)}_{\eta_{\min}}$ is of class C^1 with $D(\mathcal{J}^1_{\eta\, p\eta_{\min}} \circ u^{*(p)}_{\eta_{\min}})(\phi) = \mathcal{J}^1_{\eta\zeta} \circ u^{*(p+1)}(\phi)$, in which $u^{*(p+1)}(\phi)$ is the unique solution of the equation

$$u^{(p+1)} = \mathcal{K}_\zeta \circ \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}(\phi))u^{(p+1)} + H^{(p+1)}_{\zeta/(p+1)}(\phi) \tag{7.35}$$

in $\mathcal{L}^{(p+1)}(X_0, BC^1_\zeta)$, with

$$H^{(p+1)}_{\zeta/(p+1)}(\phi) = \mathcal{K}_\zeta \circ \widetilde{R}^{(2)}_\delta(u^*_{\eta_{\min}}\phi)(u^{*(p)}(\phi), u^{*(1)}(\phi)) + DH^{(p)}_{\zeta/p}(\phi). \tag{7.36}$$

The arguments in the first part of this proof show that the fixed point $u^{*(p+1)}(\phi)$ is also contained in $\mathcal{L}^{(p+1)}(X_0, BC^1_{(p+1)\eta_{\min}})$. We can hence write $u^{*(p+1)}_{\eta_{\min}} = u^{*(p+1)}(\phi) \in \mathcal{L}^{(p+1)}(X_0, BC^1_{(p+1)\eta_{\min}})$, upon which the proof is completed. □

Corollary 7.9. *Consider the setting of Theorem 7.1. Then for any $\zeta \in [\eta_{\min}, \eta_{\max}]$ and any $\xi \in \mathbb{R}$, the mapping $\phi \to (u^*_\zeta\phi)_\xi$ from X_0 into X is C^k-smooth.*

Proof. For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, we have $(u^*_\zeta\phi)_\xi = (u^*_\eta\phi)_\xi$. The latter mapping is C^k-smooth as a consequence of Theorem 7.1 and the fact that the evaluation at ξ is a bounded linear mapping. □

As a conclusion of this section, we use the explicit expression (7.35) for the derivatives of u^*, together with the fact that $u^*_\eta(0) = 0$, to compute the Taylor expansion of $u^*_\eta\phi$ around $\phi = 0$ up to second order. This can be done if $k \geq 2$ and yields

$$u_\eta^* \phi = T\phi + \frac{1}{2}\mathcal{K}_\eta D^2 R_\delta(0)((T\phi)_\xi, (T\phi)_\xi) + o(\|\phi\|^2), \tag{7.37}$$

in which the operator $\mathcal{K}_\eta$ acts with respect to the variable ξ.

8 Dynamics on the Center Manifold

In this section we show that the dynamics on the center manifold can be described by an ordinary differential equation. In addition, this reduction will be used to supply the proof of Theorem 2.2.

Theorem 8.1. *Consider the setting of Theorem 7.1 and choose $\eta \in (k\eta_{\min}, \eta_{\max}]$. Then for any $\phi \in X_0$, the function $\Phi : \mathbb{R} \to X_0$ given by $\Phi(\xi) = Q_0(u_\eta^* \phi)_\xi$ is C^{k+1}-smooth and satisfies an ordinary differential equation*

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi)). \tag{8.1}$$

Here the function $f : X_0 \to X_0$ is C^k-smooth and is explicitly given by

$$f(\psi) = Q_0(L(u_\eta^* \psi - T\psi)_\theta + R_\delta((u_\eta^* \psi)_\theta)), \tag{8.2}$$

where the projection Q_0 is taken with respect to the variable θ. Finally, we have $f(0) = 0$ and $Df(0) = 0$.

Proof. Notice first that Φ is a continuous function, since $\xi \to (u_\eta^* \phi)_\xi$ is continuous. We calculate

$$\begin{array}{rcl} \dot{\Phi}(\xi)(\theta) & = & \lim_{h \to 0} \frac{1}{h}(\Phi(\xi + h)(\theta) - \Phi(\xi)(\theta)) \\ & = & \lim_{h \to 0} \frac{1}{h}(Q_0(u_\eta^* \phi)_{\xi+h}(\theta) - Q_0(u_\eta^* \phi)_\xi(\theta)) \\ & = & Q_0(D(u_\eta^* \phi)(\xi + \cdot))(\theta), \end{array} \tag{8.3}$$

where the continuity of the projection Q_0 together with the fact that $\mathcal{K}_\eta$ maps into $C^1(\mathbb{R}, \mathbb{C}^n)$ was used in the last step. Using the definition of $\mathcal{K}_\eta$ we compute

$$D(u_\eta^* \phi)(\xi + \theta) = L(u_\eta^* \phi)_{\xi+\theta} + R_\delta((u_\eta^* \phi)_{\xi+\theta}). \tag{8.4}$$

For convenience, define $\psi = \Phi(\xi)$. Lemma 6.5 implies that $(u_\eta^* \phi)_{\xi+\theta} = (u_\eta^* \psi)_\theta$. The ODE (8.1) now follows upon noting that

$$Q_0(L(T\psi)_\theta) = Q_0(\dot{\psi}(\theta)) = Q_0((A\psi)(\theta)) = A\psi. \tag{8.5}$$

The fact that f is C^k-smooth follows from the fact that the C^k-smooth function $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ maps into a region on which $\widetilde{R}_\delta$ is itself C^k-smooth by Corollary 7.5. It is easy to see that $f(0) = 0$ and from (HR2) and the Taylor expansion (7.37), it follows that $Df(0) = 0$. The fact that Φ is C^{k+1}-smooth follows from repeated differentiation of (8.1). □

In order to lift solutions of (8.1) back to the original equation (2.3), we need to establish that the nonlinearity in (2.9) agrees with the version in (8.2) in a small neighbourhood of zero. The next lemma shows that this can indeed be realized.

Lemma 8.2. *Let $\delta > 0$ and $\epsilon > 0$ be so small that for some $0 < \eta_0 < \gamma$,*

$$\begin{array}{lcl} L_{R_\delta}(w^{2\eta_0} + w^{\eta_0}) & < & (8\,\|\mathcal{K}_{\eta_0}\|)^{-1}, \\ \epsilon w^{2\eta_0}\,\|T\|_{\eta_0} & < & \tfrac{1}{2}\delta, \end{array} \tag{8.6}$$

with the Lipschitz constant L_{R_δ} as introduced in Corollary 6.2 and the extension operator T as defined in Proposition 5.2. Then for any $0 < \eta < \gamma$ and any $\phi \in X_0$ with $\|\phi\| < \epsilon$, we have for all $r_{\min} \leq \theta \leq r_{\max}$ that

$$\left\|Q_0(u^*_\eta\phi)_\theta\right\| < \delta. \tag{8.7}$$

Proof. Similarly as in the proof of Lemma 7.2, we compute

$$Q_0(u^*_\eta\phi)_\theta = (T\phi)_\theta + (\mathcal{K}_{\eta_0}\widetilde{R}_\delta(u^*_\eta\phi))_\theta - (\mathcal{K}_{\eta_0}\widetilde{R}_\delta((u^*_\eta\phi)(\theta + \cdot)))_0 \tag{8.8}$$

and hence using (5.22) we obtain

$$\left\|Q_0(u^*_\eta\phi)_\theta\right\| \leq w^{\eta_0}w^{\eta_0}\,\|T\|_{\eta_0}\,\|\phi\| + 4\delta L_{R_\delta}(w^{\eta_0}w^{\eta_0}\,\|\mathcal{K}_{\eta_0}\| + w^{\eta_0}\,\|\mathcal{K}_{\eta_0}\|) < \delta, \tag{8.9}$$

which completes the proof. □

Proof of Theorem 2.2. Choose $\delta > 0$ such that (7.1), (7.6) and (8.6) are all satisfied and fix the constant $\epsilon^* > 0$ such that $\epsilon^* \max(\|Q_0\|, \|I - Q_0\|) < \delta$. Fix $0 < \epsilon < \delta$ such that (8.6) is satisfied, pick any $\eta \in (k\eta_{\min}, \eta_{\max}]$ and write $u^* = u^*_\eta$.

(i) This follows from Theorem 7.1 together with $u^* = u^*_\zeta = \mathcal{J}^1_{\zeta\eta_{\min}} u^*_{\eta_{\min}}$ for any $\zeta \in (k\eta_{\min}, \eta_{\max}]$.

(ii) First note that (i) and the conditions (HR1)-(HR2) imply that f is C^k-smooth with $f(0) = 0$ and $Df(0) = 0$. Since $\xi \mapsto x_\xi$ maps into the subset of X on which R and R_δ agree, we have $\Lambda x = \widetilde{R}_\delta(x)$ and hence $Px = \mathcal{K}_\eta \Lambda x = \mathcal{K}_\eta \widetilde{R}_\delta(x)$. Since $Px = x - TQ_0 x_0$ we see that $\mathcal{G}(x, Q_0 x_0) = x$ and hence due to uniqueness of solutions we indeed have $x = u^* Q_0 x_0$. Note that for all $\xi \in \mathbb{R}$ we have $\|\Phi(\xi)\| < \delta$, which by Lemma 6.5 implies that $\|Q_0(u^*\Phi(\xi))_\theta\| < \delta$ for any $\xi \in \mathbb{R}$ and $\theta \in [r_{\min}, r_{\max}]$. Thus the function f defined in (8.2) agrees with (2.9) and hence an application of Theorem 8.1 shows that Φ satisfies the ODE (2.8). An application of Lemma 6.5 completes the proof.

(iii) This is clear from the fact that $\xi \to (u^*\phi)_\xi$ maps into the subset of X on which R and R_δ agree.

(iv) See (v) with $\xi_- = -\infty$ and $\xi_+ = +\infty$.

(v) Define the function $\Psi(\xi) = Q_0(u^*\Phi(\zeta))_{\xi-\zeta}$ and note that it satisfies (8.1) on $\mathbb{R}$, with $\Psi(\zeta) = \Phi(\zeta)$. Note further that Lemmas 7.2 and 8.2 imply that the nonlinearities (2.9) and (8.2) agree on the set $\{\phi \in X_0 \mid \|\phi\| < \epsilon\}$. Since both nonlinearities are locally Lipschitz continuous, this implies that in fact $\Psi(\xi) = \Phi(\xi)$ for all $\xi \in (\xi_{\min}, \xi_{\max})$. Thus defining $x(\xi) = (u^*\Phi(\zeta))(\xi - \zeta)$, we see that $Q_0 x_\xi = \Psi(\xi)$ and hence $\|Q_0 x_\xi\| < \epsilon < \delta$ for all $\xi \in (\xi_{\min}, \xi_{\max})$. Since $(\Lambda x)(\xi) = R_\delta(x_\xi) = R(x_\xi)$ for all such ξ, we see that x indeed satisfies (2.3) on the interval $(\xi_{\min}, \xi_{\max})$. Finally, Lemma 6.5 shows that for any ξ in this interval we have $x_\xi = (u^*\Psi(\xi))_0 = (u^*\Phi(\xi))_0$.

□

9 Parameter Dependence

We now wish to incorporate parameter dependent equations into our framework. In particular, we will study equations of the form

$$\dot{x}(\xi) = L(\mu)x_\xi + R(x_\xi, \mu) \tag{9.1}$$

for parameters $\mu \in \Omega \subset \mathbb{C}^d$ in some open subset Ω and linearities

$$L(\mu)\phi = \sum_{j=0}^{N} A_j(\mu)\phi(r_j). \tag{9.2}$$

We assume here that the conditions (HLμ), (HRμ1) and (HRμ2) all hold. Suppose that for some $\mu_0 \in \Omega$ we have that $\det \Delta_{L(\mu_0)}(z) = 0$ has roots on the imaginary axis. Introducing new coordinates

$\nu = \mu - \mu_0$ and defining $\mathbf{y} = (x, \nu)$, we obtain the system

$$\dot{\mathbf{y}}(\xi) = \mathbf{L}\mathbf{y}_\xi + \mathbf{R}(\mathbf{y}_\xi), \tag{9.3}$$

in which $\mathbf{L} = (L(\mu_0), 0)$ and $\mathbf{R}((\phi, \nu)) = ((L(\mu_0 + \nu) - L(\mu_0))\phi + R(\phi, \mu_0 + \nu), 0)$. Notice that $\mathbf{R}$ satisfies the assumptions (HR1) and (HR2), which enables the application of the theory developed above. Notice that for any $x \in \mathcal{N}_0$, we have that $\mathbf{y} = (x, \nu)$ satisfies $\dot{\mathbf{y}}(\xi) = \mathbf{L}\mathbf{y}_\xi$ and hence we have the identity $\mathbf{X}_0 = X_0 \times \mathbb{C}^d$ for the respective center spaces.

From now on we will simply write $\mathbf{u}^*$ for the function $\mathbf{u}^*_\eta$ defined in Theorem 6.4. We split off the part of this operator which acts on the state space for the parameter ν and write $\mathbf{u}^* = (u_1^*, u_2^*)$, with $u_2^*(\phi, \nu) = \nu$. The first component of the differential equation (2.8) on the center manifold in our setting becomes

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi), \nu), \tag{9.4}$$

for $\Phi : \mathbb{R} \to X_0$, where $f : X_0 \times \mathbb{C}^d \to X_0$ is given by

$$f(\psi, \nu) = Q_0(L(u_1^*(\psi, \nu) - T\psi)_\theta + (L(\mu_0 + \nu) - L(\mu_0))(u_1^*(\psi, \nu))_\theta + R((u_1^*(\psi, \nu))_\theta, \mu_0 + \nu)), \tag{9.5}$$

in which the projection Q_0 is taken with respect to the variable θ. We finish by computing the Taylor expansion of u_1^* to second order, which is possible if $k \geq 2$. We have

$$u_1^*(\phi, \nu) = T\phi + \mathcal{K}\big(L'(\mu_0)\nu(T\phi)_\xi + \frac{1}{2}D_1^2 R(0, \mu_0)((T\phi)_\xi, (T\phi)_\xi)\big) + o((|\nu| + |\phi|)^2), \tag{9.6}$$

in which $\mathcal{K}$ acts with respect to the variable ξ.

10 Hopf Bifurcation

In this section we use the projection on the center manifold to apply the finite dimensional Hopf bifurcation theorem to our infinite dimensional setting. In particular, we will consider a system of the form (2.10) that depends on a parameter $\mu \in \mathbb{R}$. We will assume that for some $\mu_0 \in \mathbb{R}$ the linear operator $L = L(\mu_0)$ has simple eigenvalues at $\pm i\omega_0$ for some $\omega_0 > 0$ and we write X_0 for the center subspace at this parameter value μ_0. We will look for small continuous periodic solutions $\Phi : \mathbb{R} \to X_0$ to the equation

$$\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi), \nu), \tag{10.1}$$

for small values of ν, with f as in (9.5). Using Theorem 2.2 these solutions can be lifted to periodic solutions of the original equation (2.10).

Before we can apply Theorem C.1, we need to study the generalized eigenspace of A for simple eigenvalues.

Lemma 10.1. *Consider the system (2.3) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has a simple root at* $z = i\omega_0$*. Then the matrix valued function*

$$H(z) = (z - i\omega_0)\Delta(z)^{-1} \tag{10.2}$$

is analytic in a neighbourhood of $z = i\omega_0$*. In addition, there exist* $p, q \in \mathbb{C}^n$ *such that* $\Delta(i\omega_0)p = \Delta(i\omega_0)^T q = 0$*, while* $q^T \Delta'(i\omega_0)p \neq 0$*. For any such pair the function* $\phi = e^{i\omega_0 \cdot} p$ *is an eigenvector of the operator* A *defined in (4.1) corresponding to the algebraically simple eigenvalue* $i\omega_0$ *and in addition we have the identities*

$$\begin{aligned} H(i\omega_0) &= pq^T(q^T\Delta'(i\omega_0)p)^{-1}, \\ Q_\phi \psi &= e^{i\omega_0 \cdot} H(i\omega_0)(\psi(0) + \textstyle\sum_{j=0}^N A_j e^{i\omega_0 r_j} \int_{r_j}^0 e^{-i\omega_0 \sigma}\psi(\sigma)d\sigma). \end{aligned} \tag{10.3}$$

Here $Q_\phi : X_0 \to X_0$ *denotes the spectral projection onto the generalized eigenspace of* A *for the eigenvalue* $i\omega_0$*.*

Proof. Since $\Delta(z)$ is a characteristic matrix for A and $\det \Delta(z) = 0$ has a simple root at $z = i\omega_0$, it follows from the theory of characteristic matrices (see e.g. [17, Theorem IV.4.18]) that $\Delta(z)$ has a pole of order one at $z = i\omega_0$ and A has a simple eigenvalue at $z = i\omega_0$. This proves that $H(z)$ is analytic in a neigbourhood of $z = i\omega_0$. It also follows that the nullspace $\mathcal{N}(\Delta(i\omega_0))$ is the one dimensional span of some $p \in \mathbb{C}^n$. Similarly, we have $\mathcal{N}(\Delta(i\omega_0)^T) = \text{span}\{q\}$ for some $q \in \mathbb{C}^n$. It is easy to check that $\phi = e^{i\omega_0 \cdot} p$ is indeed a corresponding eigenvector for A. Using residue calculus and the formula (4.2) for the resolvent of A to simplify the Dunford integral (4.9), the expression (10.3) for the spectral projection follows easily.

It remains to derive the explicit expression (10.3) for $H(i\omega_0)$. To this end, observe that

$$\Delta(z)H(z) = H(z)\Delta(z) = (z - i\omega_0)I, \tag{10.4}$$

which implies $\mathcal{R}H(i\omega_0) \subset \mathcal{N}\Delta(i\omega_0)$ and $\mathcal{R}H(i\omega_0)^T \subset \mathcal{N}\Delta(i\omega_0)^T$. From this it follows that $H(i\omega_0) =$

Cpq^T for some constant C. Expanding (10.4) in a Taylor series we obtain

$$\begin{array}{rcl} I & = & H'(i\omega_0)\Delta(i\omega_0) + H(i\omega_0)\Delta'(i\omega_0) = \Delta'(i\omega_0)H(i\omega_0) + \Delta(i\omega_0)H'(i\omega_0), \\ 0 & = & H''(i\omega_0)\Delta(i\omega_0) + 2H'(i\omega_0)\Delta'(i\omega_0) + H(i\omega_0)\Delta''(i\omega_0). \end{array} \tag{10.5}$$

Noting that $p = H(i\omega_0)\Delta'(i\omega_0)p = Cpq^T\Delta'(i\omega_0)p$ completes the proof. □

Since we are interested in real valued functions, we need to treat the two complex eigenvalues at $\pm i\omega_0$ together. To this end, we introduce the real valued functions $\psi_\pm \in X_0$ via

$$\begin{array}{rcl} \psi_+(\theta) & = & \frac{1}{2}(\phi(\theta) + \overline{\phi}(\theta)), \\ \psi_-(\theta) & = & -\frac{i}{2}(\phi(\theta) - \overline{\phi}(\theta)) \end{array} \tag{10.6}$$

and we note that the part of A on this basis takes the form $\left(\begin{smallmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{smallmatrix}\right)$. On the other hand, we consider the two dimensional real ODE

$$\begin{pmatrix} \dot{y}_+ \\ \dot{y}_- \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \tag{10.7}$$

and observe that under the complexification $z = y_+ + iy_-$, this system is transformed into

$$\dot{z} = \frac{1}{2}(a_{11} + a_{22} + i(a_{21} - a_{12}))z + \frac{1}{2}(a_{11} + a_{22} + i(a_{12} - a_{21}))\overline{z}. \tag{10.8}$$

The only nontrivial hypothesis we need to check before we can apply Theorem C.1 is the condition (HH3), i.e. $\mathrm{Re}\, D\sigma(\mu_0) \neq 0$ for the branch $\sigma(\mu)$ of eigenvalues of $D_1 g(0, \mu)$ through $i\omega_0$ at $\mu = \mu_0$. The following lemma indicates how this quantity can be explicitly calculated.

Lemma 10.2. *Consider real $m \times m$ matrices M_0 and $M_1(\nu)$ for some integer $m \geq 2$, where each entry $M_1^{(ij)}(\nu)$ of $M_1(\nu)$ is a C^1-smooth function of the real parameter ν with $M_1^{(ij)}(0) = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq m$. Suppose that for some $\omega_0 \in \mathbb{R}$ and $(m-2) \times (m-2)$ matrix B we have $M_0 = \mathrm{diag}(A(\omega_0), B)$ with $A(\omega_0) = \left(\begin{smallmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{smallmatrix}\right)$. Suppose further that the matrices $B \pm i\omega_0 I$ are both invertible, i.e., M_0 has simple eigenvalues $\pm i\omega_0$. Write $\sigma(\nu)$ for the branch of eigenvalues of $M = M_0 + M_1(\nu)$ through $i\omega_0$ at $\nu = 0$. Then we have $\mathrm{Re}\, D\sigma(0) = \frac{1}{2}(\dot{M}_1^{(11)}(0) + \dot{M}_1^{(22)}(0))$, in which the dot denotes differentiation with respect to ν.*

Proof. We define the function $\Delta(\nu, \lambda) = \det(M_0 + M_1(\nu) - (i\omega_0 + \lambda)I)$ and note that we have the identity $\Delta(\nu, \sigma(\nu) - i\omega_0) = 0$ for small ν. Using implicit differentiation it follows that

$$D\sigma(0) = -D_1\Delta(0,0)/D_2\Delta(0,0) \tag{10.9}$$

and hence it suffices to compute

$$\begin{aligned}
D_1\Delta(0,0) &= (-i\omega_0\dot{M}_1^{(11)}(0) - i\omega_0\dot{M}_1^{(22)}(0) - \omega_0\dot{M}_1^{(12)}(0) + \omega_0\dot{M}_1^{(21)}(0))\det(B - i\omega_0 I),\\
D_2\Delta(0,0) &= 2i\omega_0\det(B - i\omega_0 I),
\end{aligned} \tag{10.10}$$

from which the claim immediately follows. □

Thus in order to calculate $D\sigma(\mu_0)$, it suffices to expand (10.1) up to terms involving $O(\nu\phi)$, i.e.,

$$\dot{\Phi} = A\Phi + Q_0 h(\Phi,\nu) + O(|\Phi|^2 + |\nu|^2 + (|\Phi| + |\nu|)^3), \tag{10.11}$$

where $h : X_0 \times \mathbb{R} \to X$ is the bilinear operator

$$h(\psi,\nu)(\theta) = (L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)\big(\nu L'(\mu_0)\mathrm{ev}_{(\cdot)}T\psi\big), \tag{10.12}$$

in which we have introduced the evaluation function $\mathrm{ev}_{\theta'}f(\cdot) = f_{\theta'}$ and the point evaluation $\mathrm{pev}_{\theta'}f(\cdot) = f(\theta')$. In view of Lemma 10.2, the specific form of the transformation of the real ODE (10.7) into (10.8) and the fact that $\phi = \psi_+ + i\psi_-$, it is clear that

$$\mathrm{Re}\,D\sigma(0) = \mathrm{Re}\,\widetilde{Q}_\phi Q_0 h(\phi/\nu,\nu), \text{ with } Q_\phi = \phi\widetilde{Q}_\phi. \tag{10.13}$$

In order to evaluate (10.13), we need to calculate $\mathcal{K}e^{i\omega_0\cdot}v$ for arbitrary $v \in \mathbb{C}^n$. As a preparation, we compute

$$\begin{aligned}
Q_\phi e^{i\omega_0\theta}v &= e^{i\omega_0\cdot}H(i\omega_0)\Delta'(i\omega_0)v,\\
Q_\phi \theta e^{i\omega_0\theta}v &= \tfrac{1}{2}e^{i\omega_0\cdot}H(i\omega_0)\Delta''(i\omega_0)v,
\end{aligned} \tag{10.14}$$

in which the projections Q_ϕ were taken with respect to the variable θ.

Lemma 10.3. *Consider (2.3) and suppose that the characteristic equation* $\det\Delta(z) = 0$ *has a simple root at* $z = i\omega_0$. *Let* $H(z)$, p *and* q *be as in Lemma 10.1. Then for arbitrary* $v \in \mathbb{C}^n$ *we have*

$$(\mathcal{K}e^{i\omega_0\cdot}v)(\xi) = e^{i\omega_0\xi}(H(i\omega_0)\xi + H'(i\omega_0))v + (T\psi)(\xi), \tag{10.15}$$

for some $\psi \in X_0$ *with* $Q_\phi\psi = 0$. *In addition, we have*

$$Q_\phi((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\omega_0\cdot}v) = \phi q^T v(q^T\Delta'(i\omega_0)p)^{-1}. \tag{10.16}$$

Proof. For convenience, define $\Psi(\xi) = e^{i\omega_0\xi}(H(i\omega_0)\xi + H'(i\omega_0))v$. We first check that the function above indeed satisfies the differential equation. We compute

$$\dot{\Psi}(\xi) = e^{i\omega_0\xi}((i\omega_0\xi + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0))v. \tag{10.17}$$

Similarly, we compute

$$\begin{aligned} L\Psi_\xi &= e^{i\omega_0\xi}\big((i\omega_0 - \Delta(i\omega_0))H(i\omega_0)\xi + (I - \Delta'(i\omega_0))H(i\omega_0) + (i\omega_0 - \Delta(i\omega_0))H'(i\omega_0)\big)v \\ &= e^{i\omega_0\xi}\big((i\omega_0\xi + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0) - I\big)v, \end{aligned} \tag{10.18}$$

from which we see that indeed $(\Lambda\Psi)(\xi) = \dot{\Psi}(\xi) - L\Psi_\xi = e^{i\omega_0\xi}v$. In addition, using (10.5) we can calculate

$$\begin{aligned} e^{-i\omega_0\cdot}Q_\phi\Psi_0 &= (\tfrac{1}{2}H(i\omega_0)\Delta''(i\omega_0)H(i\omega_0) + H(i\omega_0)\Delta'(i\omega_0)H'(i\omega_0))v \\ &= -\tfrac{1}{2}H(i\omega_0)\Delta(i\omega_0)H''(i\omega_0)v = 0, \end{aligned} \tag{10.19}$$

as required. Finally, we compute

$$\begin{aligned} Q_\phi(L\Psi_\theta + e^{i\omega_0\theta}v) &= Q_\phi(e^{i\omega_0\theta}((i\omega_0\theta + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0))v) \\ &= e^{i\omega_0\cdot}(\tfrac{1}{2}i\omega_0 H(i\omega_0)\Delta''(i\omega_0)H(i\omega_0) + H(i\omega_0)\Delta'(i\omega_0)(H(i\omega_0) + i\omega_0 H'(i\omega_0)))v \\ &= e^{i\omega_0\cdot}H(i\omega_0)\Delta'(i\omega_0)H(i\omega_0)v \\ &= \phi q^T v(q^T\Delta'(i\omega_0)p)^{-1}. \end{aligned} \tag{10.20}$$

□

Using the above lemma we can now calculate

$$\begin{aligned} \operatorname{Re} D\sigma(0) &= \operatorname{Re}\widetilde{Q}_\phi Q_0((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)L'(\mu_0)\phi) \\ &= -\operatorname{Re}\widetilde{Q}_\phi((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)D_2\Delta(i\omega_0,\mu_0)e^{i\omega_0\cdot}p) \\ &= -\operatorname{Re} q^T D_2\Delta(i\omega_0,\mu_0)p(q^T\Delta'(i\omega_0)p)^{-1}. \end{aligned} \tag{10.21}$$

Proof of Theorem 2.3. We apply Theorem C.1 to the ODE (10.1). Conditions (HH1)-(HH2) are immediate from the assumptions on (2.10) and (HH3) follows from (Hζ3) and (10.21). Restricting the allowed values of τ in Theorem C.1 to a small interval I around zero such that $|\mu^*(\tau) - \mu_0| < \frac{\epsilon}{2}$ and $|x^*(\tau)(\xi)| < \frac{\epsilon}{2}$ for all $\xi \in \mathbb{R}$ and $\tau \in I$, with ϵ as in the statement of Theorem 2.2, it follows from part (iv) of this theorem that each $x^*(\tau)$ can be lifted to a periodic solution of (2.10). Similarly, every small periodic solution of (2.10) corresponds to a small periodic solution of (10.1), which is captured by Theorem C.1. □

We now set out to compute the direction of bifurcation using Theorem C.2. Notice first that $(T\phi)(\xi) = pe^{i\omega_0\xi}$ and similarly $(T\overline{\phi})(\xi) = \overline{p}e^{-i\omega_0\xi}$. In particular, this implies that $(T\phi)_\xi = e^{i\omega_0\xi}\phi \in X_0$ and similarly $(T\overline{\phi})_\xi = e^{-i\omega_0\xi}\overline{\phi} \in X_0$. In order to evaluate the constant c appearing in Theorem C.2, we need to calculate $\mathcal{K}e^{i\zeta\omega_0\theta}v$ for arbitrary $v \in \mathbb{C}^n$ and $\zeta \in \mathbb{R}$ such that $\det \Delta(i\zeta\omega_0) \neq 0$. We obtain the following result.

Lemma 10.4. *Consider (2.3) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has a simple root at* $z = i\omega_0$*. Let* $H(z)$*,* p *and* q *be as in Lemma 10.1. Then for arbitrary* $v \in \mathbb{C}^n$ *and* $\zeta \in \mathbb{R}$ *such that* $\det \Delta(i\zeta\omega_0) \neq 0$*, we have*

$$(\mathcal{K}e^{i\zeta\omega_0\cdot}v)(\xi) = e^{i\zeta\omega_0\xi}\Delta(i\zeta\omega_0)^{-1}v - Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v). \tag{10.22}$$

In addition, we have the identity

$$Q_0((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\zeta\omega_0\cdot}v) = (i\zeta\omega_0 - A)Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v). \tag{10.23}$$

Proof. For convenience, define $\Psi(\xi) = e^{i\zeta\omega_0\xi}\Delta(i\zeta\omega_0)^{-1}v$. First note that

$$L\Psi_\xi = e^{i\zeta\omega_0\xi}(i\zeta\omega_0 - \Delta(i\zeta\omega_0))\Delta(i\zeta\omega_0)^{-1}v = i\zeta\omega_0\Psi(\xi) - e^{i\zeta\omega_0\xi}v, \tag{10.24}$$

from which it follows that

$$(\Lambda\Psi)(\xi) = i\zeta\omega_0\Psi(\xi) - L\Psi_\xi = e^{i\zeta\omega_0\xi}v, \tag{10.25}$$

which implies the first claim. To substantiate the second claim, note that

$$\begin{aligned}
Q_0((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\zeta\omega_0\cdot}v) &= Q_0(L\Psi_\theta + e^{i\zeta\omega_0\theta}v - L(TQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v))_\theta) \\
&= Q_0(i\zeta\omega_0e^{i\zeta\omega_0\theta}\Delta(i\zeta\omega_0)^{-1}v - AQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v)) \\
&= (i\zeta\omega_0)Q_0(e^{i\zeta\omega_0\theta}\Delta(i\zeta\omega_0)^{-1}v) \\
&\quad - AQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v) \\
&= (i\zeta\omega_0 - A)Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v).
\end{aligned} \tag{10.26}$$

□

To explicitly calculate c, we write the nonlinearity $f : X_0 \times \mathbb{R} \to X_0$ in (10.1) in the form

$$f(\psi, \nu) = Q_0((\mathrm{Lev}_\theta \mathcal{K} + \mathrm{pev}_\theta)\widetilde{\mathbf{R}}_1(u_1^*(\psi, \nu), \nu)), \tag{10.27}$$

in which $\widetilde{\mathbf{R}}_1$ is the substitution operator associated with the first component of the compound operator $\mathbf{R}$ defined in (9.3). We thus need to compute

$$\begin{aligned} &D_1^3(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,0)(\psi_1, \psi_2, \psi_3)(\xi) \\ &= D_1^3 R(0, \mu_0)((T\psi_1)_\xi, (T\psi_2)_\xi, (T\psi_3)_\xi) \\ &\quad + D_1^2 R(0, \mu_0)((T\psi_1)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0, \mu_0)((T\psi_2)_{(\cdot)}, (T\psi_3)_{(\cdot)})) \\ &\quad + D_1^2 R(0, \mu_0)((T\psi_2)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0, \mu_0)((T\psi_3)_{(\cdot)}, (T\psi_1)_{(\cdot)})) \\ &\quad + D_1^2 R(0, \mu_0)((T\psi_3)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0, \mu_0)((T\psi_1)_{(\cdot)}, (T\psi_2)_{(\cdot)})) \end{aligned} \tag{10.28}$$

and hence substituting $\psi_1 = \psi_2 = \phi$ and $\psi_3 = \overline{\phi}$, we obtain

$$\begin{aligned} &D_1^3(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,0)(\phi, \phi, \overline{\phi})(\xi) \\ &= e^{i\omega_0 \xi} D_1^3 R(0, \mu_0)(\phi, \phi, \overline{\phi}) \\ &\quad + 2e^{i\omega_0 \xi} D_1^2 R(0, \mu_0)\big(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi})\big) \\ &\quad - 2D_1^2 R(0, \mu_0)\big((T\phi)_\xi, \mathrm{ev}_\xi T Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi}))\big) \\ &\quad + e^{i\omega_0 \xi} D_1^2 R(0, \mu_0)\big(\overline{\phi}, \Delta(2i\omega_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \phi)\big) \\ &\quad - D_1^2 R(0, \mu_0)\big((T\overline{\phi})_\xi, \mathrm{ev}_\xi T Q_0(e^{2i\omega_0 \cdot} \Delta(2i\omega_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \phi))\big). \end{aligned} \tag{10.29}$$

In addition, using Lemma 10.4 we calculate,

$$\begin{aligned} D_1^2 f(0, \mu_0)(\phi, \phi) &= Q_0((\mathrm{Lev}_\theta \mathcal{K} + \mathrm{pev}_\theta) e^{2i\omega_0 \cdot} D_1^2 R(0, \mu)(\phi, \phi)) \\ &= (2i\omega_0 - A) Q_0(e^{2i\omega_0 \cdot} \Delta(2i\omega_0)^{-1} D_1^2 R(0, \mu)(\phi, \phi)). \end{aligned} \tag{10.30}$$

A similar computation shows that

$$D_1^2 f(0, \mu_0)(\phi, \overline{\phi}) = -A Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi})). \tag{10.31}$$

Using these identities we can write

$$\begin{aligned} &D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0, \mu_0)(\phi, -A^{-1} D_1^2 f(0, \mu_0)(\phi, \overline{\phi}))(\xi) \\ &= D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0, \mu_0)\big(\phi, Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi}))\big) \\ &= D_1^2 R(0, \mu_0)\big((T\phi)_\xi, \mathrm{ev}_\xi T Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi}))\big) \end{aligned} \tag{10.32}$$

and similarly

$$
\begin{aligned}
&D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,\mu_0)(\overline{\phi}, (2i\omega_0 - A)^{-1} D_1^2 f(0,\mu_0)(\phi,\phi))(\xi) \\
&= D_1^2 R(0,\mu_0)\big((T\overline{\phi})_\xi, \mathrm{ev}_\xi T Q_0(e^{2i\omega_0 \cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi}))\big).
\end{aligned} \tag{10.33}
$$

Putting all our calculations together, we arrive at

$$
c\phi = Q_\phi Q_0((L\mathrm{ev}_\theta \mathcal{K} + \mathrm{pev}_\theta)\Psi(\cdot)), \tag{10.34}
$$

in which

$$
\begin{aligned}
\Psi(\xi) \quad = \quad & \tfrac{1}{2} e^{i\omega_0\xi} D_1^3 R(0,\mu_0)(\phi,\phi,\overline{\phi}) \\
& + e^{i\omega_0\xi} D_1^2 R(0,\mu_0)(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})) \\
& + \tfrac{1}{2} e^{i\omega_0\xi} D_1^2 R(0,\mu_0)(\overline{\phi}, e^{2i\omega_0 \cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi)).
\end{aligned} \tag{10.35}
$$

Finally, an application of Lemma 10.3 yields

$$
\begin{aligned}
(q^T \Delta'(i\omega_0) p) c \quad = \quad & \tfrac{1}{2} q^T D_1^3 R(0,\mu_0)(\phi,\phi,\overline{\phi}) \\
& + q^T D_1^2 R(0,\mu_0)(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})) \\
& + \tfrac{1}{2} q^T D_1^2 N(0,\mu_0)(\overline{\phi}, e^{2i\omega_0 \cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi)).
\end{aligned} \tag{10.36}
$$

Proof of Theorem 2.4. Using Theorem C.2, the statement follows immediately from the formulas (10.21) and (10.36). □

11 Example: Double Eigenvalue At Zero

We here give a concrete example of the power of the finite dimensional reduction by considering a functional differential equation of mixed type that depends on four parameters. For certain values of the parameters the equation reduces to a delay equation, which has already been studied in [17]. This example hence allows us to check that our framework yields reproducable results when restricting to delay equations. The equation we consider has the origin as an equilibrium and in addition has a double eigenvalue at zero with geometric multiplicity one, for certain critical parameter values. This means that the origin is a Takens-Bogdanov point and it is known that for such equilibria only the second order terms are needed to determine the local phase portrait.

In particular, we consider the equation

$$
\dot{x}(\xi) = \alpha x(\xi) + \beta_- g(x(\xi-1),\mu) + \beta_+ g(x(\xi-1),\mu), \tag{11.1}
$$

for some $g \in C^3(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. We enforce the conditions $\beta_+ + \beta_- \neq 0$ and $\beta_+ - \beta_- \neq 0$. Suppose that $g(0, \mu) = 0$ for any $\mu \in \mathbb{R}$ and in addition $g'(0, \mu) = \mu$. Linearization around the zero equilibrium yields

$$\dot{x}(\xi) = \alpha x(\xi) + \beta_- \mu x(\xi - 1) + \beta_+ \mu x(\xi + 1) \tag{11.2}$$

and with a short calculation one can verify that this equation has a double eigenvalue zero at $(\alpha, \mu) = (\alpha_0, \mu_0) := (\frac{\beta_- + \beta_+}{\beta_- - \beta_+}, \frac{1}{\beta_+ - \beta_-})$, with corresponding eigenvectors $\phi_0 = \mathbf{1}$ and $\phi_1 = \{\theta \mapsto \theta\}$. The projection operator $Q_0 : X \to X_0$ onto the span of ϕ_0 and ϕ_1 can be calculated by using residue calculus on the resolvent equation (4.2). We find

$$\begin{aligned}(Q_0\phi)(\theta) \quad = \quad & \tfrac{2\theta(\beta_- - \beta_+)^2 + \frac{2}{3}(\beta_- - \beta_+)^2}{(\beta_+ + \beta_-)^2}\psi(0) \\ & + \tfrac{\beta_-}{(\beta_+ + \beta_-)^2}\int_{-1}^{0}\big(2(\sigma - \theta)(\beta_+ + \beta_-) + \tfrac{8}{3}\beta_+ + \tfrac{4}{3}\beta_-\big)\psi(\sigma)d\sigma \\ & + \tfrac{\beta_+}{(\beta_+ + \beta_-)^2}\int_{0}^{1}\big(2(\theta - \sigma)(\beta_+ + \beta_-) + \tfrac{8}{3}\beta_- + \tfrac{4}{3}\beta_+\big)\psi(\sigma)d\sigma.\end{aligned} \tag{11.3}$$

We introduce parameters $\lambda = \alpha - \alpha_0$ and $\nu = \mu - \mu_0$ and investigate (11.1) for small values of λ and ν, keeping β_+ and β_- fixed. Writing

$$\begin{aligned}R(\phi, \lambda, \nu) \quad = \quad & \beta_- g(\phi(-1), \mu_0 + \nu) - \beta_-(\mu_0 + \nu)\phi(-1) \\ & + \beta_+ g(\phi(+1), \mu_0 + \nu) - \beta_+(\mu_0 + \nu)\phi(+1) \\ = \quad & \tfrac{\beta_-}{2} g''(0, \mu_0 + \nu)(\phi(-1))^2 + \tfrac{\beta_+}{2} g''(0, \mu_0 + \nu)(\phi(+1))^2 + O(\|\phi\|^3),\end{aligned} \tag{11.4}$$

equation (11.1) transforms into the system

$$\begin{aligned}\dot{x}(\xi) \quad = \quad & \tfrac{\beta_+ + \beta_-}{\beta_- - \beta_+} x(\xi) + \tfrac{\beta_-}{\beta_+ - \beta_-} x(\xi - 1) + \tfrac{\beta_+}{\beta_+ - \beta_-} x(\xi + 1) \\ & + \lambda x(\xi) + \nu\beta_- x(\xi - 1) + \nu\beta_+ x(\xi + 1) + R(x_\xi, \lambda, \nu),\end{aligned} \tag{11.5}$$

which satisfies the conditions (HRμ1)-(HRμ2) and (HLμ). Using the explicit form of R and the linear part of (11.5), we see that the first component of the second order Taylor expansion (9.6) in our case becomes

$$\begin{aligned}u_1^*(\phi, \lambda, \nu) \quad = \quad & T\phi + \mathcal{K}\big(\lambda(T\phi)(\cdot) + \beta_-\nu(T\phi)(\cdot - 1) + \beta_+\nu(T\phi)(\cdot + 1) \\ + \quad & \tfrac{\beta_-}{2} g''(0, \mu_0)((T\phi)(\cdot - 1))^2\big) + \tfrac{\beta_+}{2} g''(0, \mu_0)((T\phi)(\cdot + 1))^2) \\ + \quad & O(|\phi|^3 + (|\nu| + |\lambda|)\,|\phi|\,(|\nu| + |\lambda| + |\phi|)).\end{aligned} \tag{11.6}$$

We now set out to calculate the differential equation that is satisfied on the center manifold up to and including second order terms. Using Theorem 8.1 we calculate

$$\dot{\Phi} = A\Phi + f(\Phi) + O(|\Phi|^3 + (|\lambda| + |\nu|)\,|\Phi|\,(|\lambda| + |\nu| + |\Phi|)), \tag{11.7}$$

in which $A\phi_1 = \phi_0$, $A\phi_0 = 0$ and $f : X_0 \to X_0$ is the function

$$\begin{array}{rcl} f(\psi) & = & Q_0\big(\frac{\beta_- - \beta_+}{\beta_- + \beta_+}\mathrm{pev}_\theta \mathcal{K} + \frac{\beta_-}{\beta_+ - \beta_-}\mathrm{pev}_{\theta-1}\mathcal{K} + \frac{\beta_+}{\beta_+ - \beta_-}\mathrm{pev}_{\theta+1}\mathcal{K} + \mathrm{pev}_\theta\big) \\ & & \big(\lambda(T\psi)(\cdot) + \beta_-\nu(T\psi)(\cdot - 1) + \beta_+\nu(T\psi)(\cdot + 1) \\ & & + \frac{\beta_-}{2}g''(0,\mu_0)((T\psi)(\cdot - 1))^2 + \frac{\beta_+}{2}g''(0,\mu_0)((T\psi)(\cdot + 1))^2\big), \end{array} \tag{11.8}$$

in which the projection Q_0 is taken with respect to the variable θ. We introduce coordinates $\Phi(\xi)(\theta) = u(\xi) + v(\xi)\theta$ on the center space X_0. Fixing a value of $\xi \in \mathbb{R}$ and writing $\psi = \Phi(\xi)$, $u = u(\xi)$ and $v = v(\xi)$, we compute

$$\begin{array}{l} \lambda(T\psi)(\xi') + \beta_-\nu(T\psi)(\xi' - 1) + \beta_+\nu(T\psi)(\xi' + 1) \\ \quad + \frac{\beta_-}{2}g''(0,\mu_0)((T\psi)(\xi' - 1))^2 \ + \frac{\beta_+}{2}g''(0,\mu_0)((T\psi)(\xi' + 1))^2 = C_0 + C_1\xi' + C_2(\xi')^2, \end{array} \tag{11.9}$$

in which

$$\begin{array}{rcl} C_0 & = & \lambda u + \beta_-\nu(u - v) + \beta_+\nu(u + v) + \frac{\beta_-}{2}g''(0,\mu_0)(u - v)^2 + \frac{\beta_+}{2}g''(0,\mu_0)(u + v)^2, \\ C_1 & = & (\lambda + (\beta_- + \beta_+)\nu)v + \beta_- g''(0,\mu_0)v(u - v) + \beta_+ g''(0,\mu_0)v(u + v), \\ C_2 & = & \frac{\beta_- + \beta_+}{2}g''(0,\mu_0)v^2. \end{array} \tag{11.10}$$

In order to proceed, we need to calculate the action of the pseudo-inverse $\mathcal{K}$ on the powers of ξ'. This can be done by using a polynomial ansatz and projecting out the X_0 component at zero. We obtain

$$\begin{array}{rcl} (\mathcal{K}\mathbf{1})(\xi) & = & \frac{\beta_- - \beta_+}{\beta_+ + \beta_-}\xi^2 + \frac{2(\beta_+ - \beta_-)^2}{3(\beta_+ + \beta_-)^2}\xi + \frac{(\beta_- - \beta_+)(-14\beta_-\beta_+ + \beta_-^2 + \beta_+^2)}{18(\beta_+ + \beta_-)^3}, \\ (\mathcal{K}\xi')(\xi) & = & \frac{\beta_- - \beta_+}{3(\beta_+ + \beta_-)}\xi^3 + \frac{(\beta_- - \beta_+)^2}{3(\beta_+ + \beta_-)^2}\xi^2 + \frac{(\beta_- - \beta_+)(-14\beta_-\beta_+ + \beta_-^2 + \beta_+^2)}{18(\beta_+ + \beta_-)^3}\xi \\ & & + \frac{(\beta_- - \beta_+)(81\beta_-\beta_+^2 - 81\beta_-^2\beta_+ - \beta_-^3 + \beta_+^3)}{270(\beta_+ + \beta_-)^4}, \\ (\mathcal{K}(\xi')^{\mathbf{2}})(\xi) & = & \frac{\beta_- - \beta_+}{6(\beta_+ + \beta_-)}\xi^4 + \frac{2(\beta_- - \beta_+)^2}{9(\beta_+ + \beta_-)^2}\xi^3 + \big(\frac{\beta_+ - \beta_-}{6(\beta_+ + \beta_-)} + \frac{2(\beta_- - \beta_+)^3}{9(\beta_+ + \beta_-)^3}\big)\xi^2 \\ & & + \frac{(\beta_- - \beta_+)(81\beta_-\beta_+^2 - 81\beta_-^2\beta_+ - \beta_-^3 + \beta_+^3)}{135(\beta_+ + \beta_-)^4}\xi \\ & & + \frac{(\beta_+ - \beta_-)(212\beta_-\beta_+^3 - 858\beta_-^2\beta_+^2 + 212\beta_+\beta_-^3 + \beta_-^4 + \beta_+^4)}{1620(\beta_+ + \beta_-)^5}. \end{array} \tag{11.11}$$

Inserting (11.9) and (11.11) into (11.7), calculating the relevant projections and performing some extensive formula manipulation now yields the system

$$\begin{array}{rcl} \dot{u} & = & v + \frac{2}{3}\frac{(\beta_- - \beta_+)^2}{(\beta_- + \beta_+)^2}h(u, v, \lambda, \nu) \\ \dot{v} & = & 2\frac{\beta_- - \beta_+}{\beta_- + \beta_+}h(u, v, \lambda, \nu), \end{array} \tag{11.12}$$

with

$$\begin{aligned} h(u,v,\lambda,\nu) &= \tfrac{\beta_-}{2} g''(0,\mu_0)(u-v)^2 + \tfrac{\beta_+}{2} g''(0,\mu_0)(u+v)^2 + \lambda u + \beta_- \nu(u-v) + \beta_+ \nu(u+v) \\ &+ O((|u|+|v|)^3 + (|\lambda|+|\nu|)(|u|+|v|)(|\lambda|+|\nu|+|u|+|v|)). \end{aligned} \tag{11.13}$$

If we choose $\beta_- = 1$ and $\beta_+ = 0$, equation (11.1) reduces to a delay equation that has been studied in [17]. The differential equation on the center manifold that was found there using specific delay equation techniques, matches the equation (11.12) derived here.

12 Capital Market Dynamics

We now return to the model for capital market dynamics that was discussed in the introduction. In this section we show that the model indeed leads to a functional differential equation of mixed type that is linear only for special choices of the model parameters. Furthermore, we illustrate how the results developed in this paper can be used in the analysis of the large time behaviour of the capital market.

We recall the definition of the Cobb-Douglas production function for the economy under consideration,

$$Q(k(t), e(t), l(t)) = Ak(t)^\alpha (e(t)l(t))^\beta, \tag{12.1}$$

for some $A > 0$ and exponents $\alpha > 0$ and $\beta > 0$. We fix $e(t) = k(t)$ and set $l(t) = 1$, since the work force has fixed size. These choices lead to the following interest rates $r(t)$ and wages $w(t)$,

$$\begin{aligned} r(t) &= \alpha A k(t)^{\alpha+\beta-1}, \\ w(t) &= \beta A k(t)^{\alpha+\beta}. \end{aligned} \tag{12.2}$$

In [16] the following expression was derived for the optimal amount of assets at time t for an individual born at time $s \geq t-1$,

$$a^*(s,t) = (s+1-t)\int_s^{s+1} w(\sigma) e^{-\int_t^\sigma r(\tau)d\tau} d\sigma - \int_t^{s+1} w(\sigma) e^{-\int_t^\sigma r(\tau)d\tau} d\sigma. \tag{12.3}$$

Substituting these equations into the capital equilibrium condition $k(t) = \int_{t-1}^t a^*(s,t)ds$, we obtain the following functional equation for $k(t)$,

$$\begin{aligned} k(t) &= \beta A \textstyle\int_{t-1}^t (s+1-t) \int_s^{s+1} k(\sigma)^{\alpha+\beta} e^{-\alpha A \int_t^\sigma k(\tau)^{\alpha+\beta-1} d\tau} d\sigma ds \\ &\quad - \beta A \textstyle\int_{t-1}^t \int_t^{s+1} k(\sigma)^{\alpha+\beta} e^{-\alpha A \int_t^\sigma k(\tau)^{\alpha+\beta-1} d\tau} d\sigma ds. \end{aligned} \tag{12.4}$$

This integral equation can be transformed into a mixed type functional differential equation by threefold differentiation. An involved computation leads to

$$\begin{aligned} k'''(t) \;=\; & [(\alpha+\beta)^2 A + 2\alpha A]k(t)^{\alpha+\beta-1}k''(t) + (\alpha+\beta-1)A((\alpha+\beta)^2+\alpha)k(t)^{\alpha+\beta-2}k'(t)^2 \\ & - \alpha A^2(3(\alpha+\beta)^2-\beta)k(t)^{2(\alpha+\beta-1)}k'(t) + (\alpha A)^2(\alpha+\beta)Ak(t)^{3(\alpha+\beta)-2} + 2\beta A k(t)^{\alpha+\beta} \\ & - \beta A k(t+1)^{\alpha+\beta}e^{-\alpha A\int_t^{t+1}k(\tau)^{\alpha+\beta-1}d\tau} - \beta A k(t-1)^{\alpha+\beta}e^{-\alpha A\int_t^{t-1}k(\tau)^{\alpha+\beta-1}d\tau}. \end{aligned} \tag{12.5}$$

Upon substitution of $\alpha+\beta=1$, (12.5) reduces to the linear functional differential equation,

$$\begin{aligned} k'''(t) \;=\; & A(1+2\alpha)k''(t) - \alpha A^2(2+\alpha)k'(t) - (1-\alpha)Ak(t-1)e^{\alpha A} \\ & + [2(1-\alpha)+(\alpha A)^2]Ak(t) - (1-\alpha)Ak(t+1)e^{-\alpha A}. \end{aligned} \tag{12.6}$$

This equation matches the expression derived in [16] by substituting $\alpha+\beta=1$ directly in (12.2). Since (12.6) is linear, the global behaviour of the capital market can be analyzed by studying the zeroes of the characteristic function

$$\Delta(z,\alpha,A) = (z-\alpha A)^3 - (1-\alpha)A[(z-\alpha A)^2 + 2 - e^{-(z-\alpha A)} - e^{z-\alpha A}]. \tag{12.7}$$

This was performed in [16], where the following result was found.

Lemma 12.1. *Fix $0<\alpha<1$ and consider the entire function*

$$\widetilde{\Delta}(z,A) = (z-\alpha A)^{-3}\Delta(z,\alpha,A). \tag{12.8}$$

For every $A>0$, the equation $\widetilde{\Delta}(z,A)=0$ has precisely one real root $\overline{g}(A)$, an infinite number of roots z with $\operatorname{Re} z < \overline{g}(A)$ and an infinite number of roots z with $\operatorname{Re} z > \overline{g}(A)$. In addition, there exists a constant $\overline{A}>0$, such that for all $A>\overline{A}$, the root $\overline{g}(A)$ satisfies $0<\overline{g}(A)<\alpha A$ and there are no other roots z with $\operatorname{Re} z = \overline{g}(A)$.

We remark here that insertion of $k(t)=Ce^{\alpha A t}$ into (12.4) yields $k(t)=0$, which is why this root needs to be excluded. Using the above result, one concludes the existence of a balanced growth path for the economy, namely $k(t)\sim e^{\overline{g}t}$. Demanding that $k(t)$ should remain strictly positive, one sees that the capital dynamics may exhibit oscillations at the start of the economy, but will finally converge to the balanced growth path [16].

We now shift our attention to the case that $\alpha+\beta\neq 1$ and look for non-zero equilibrium solutions to (12.4). For convenience, we introduce the new variable $y=e^{(\alpha+\beta-1)\ln k}$. Insertion into (12.4) yields the equilibrium condition $f(\overline{y}) = 0$, in which the function f is given by

$$f(y) = (\alpha A)^2(\alpha + \beta)y^2 + 2\beta(1 - \cosh \alpha A y). \tag{12.9}$$

Lemma 12.2. *For any $A > 0$ and parameters $\alpha > 0$ and $\beta > 0$, the equation $f(y) = 0$ has a unique strictly positive solution $\overline{y} = \overline{y}(A, \alpha, \beta) > 0$.*

Proof. Notice that $f(0) = f'(0) = 0$. In addition, we calculate

$$\begin{aligned} f''(y) &= 2(\alpha A)^2(\alpha + \beta(1 - \cosh \alpha A y)), \\ f'''(y) &= -2\beta(\alpha A)^3 \sinh \alpha A y, \end{aligned} \tag{12.10}$$

which implies that $f''(0) = 2\alpha(\alpha A)^2 > 0$ and $f'''(y) < 0$ for all $y > 0$. The claim now immediately follows upon observing that $\lim_{y \to \infty} f(y) = -\infty$. □

Please note that the equilibrium $\overline{y}$ found above does not translate into a valid equilibrium $\overline{k}$ for the capital when $\alpha + \beta = 1$, due to the nature of the corresponding variable transformation.

Linearizing (12.5) around an equilibrium $\overline{y}$ and using the condition $f(\overline{y}) = 0$ yields the following characteristic function,

$$\begin{aligned} \Delta(z) = \; & (z - \alpha A\overline{y})^3 + A\overline{y}(\alpha - (\alpha + \beta)^2)(z - \alpha A\overline{y})^2 \\ & - \alpha(\alpha + \beta)A^2\overline{y}^2(1 - (\alpha + \beta))(z - \alpha A\overline{y}) - (\alpha + \beta)A\overline{y}(\alpha^2(\alpha + \beta - 1)A^2\overline{y}^2 + 2\beta) \\ & + z^{-1}[2\beta A\overline{y}((\alpha + \beta)(z - \alpha A\overline{y}) + \alpha A\overline{y})\cosh(z - \alpha A\overline{y}) \\ & + \alpha A^2\overline{y}^2(\alpha + \beta - 1)(\alpha^2 A^2(\alpha + \beta)\overline{y}^2 + 2\beta)]. \end{aligned} \tag{12.11}$$

We remark here that the apparent singularity at $z = 0$ in the above expression can easily be seen to be removable by invoking the equilibrium condition $f(\overline{y}) = 0$. A short calculation shows that $\Delta(\alpha A\overline{y}) = 0$ and using standard arguments one can prove that the characteristic equation $\Delta(z) = 0$ has infinitely many roots to the right and left of $z = \alpha A\overline{y}$ in the complex plane. For any root z that has negative real part $\operatorname{Re} z = -\lambda < 0$, one can use the techniques in this paper to construct solutions $k(t)$ to (12.5) that exists for all $t > 0$ and satisfy $k(t) = \overline{k} + O(e^{(-\lambda+\epsilon)t})$ as $t \to \infty$, for all sufficiently small $\epsilon > 0$. In addition, if a complex conjugate pair of roots crosses the imaginary axis at some parameter pair $(\overline{\alpha}, \overline{\beta})$, the Hopf Bifurcation theorem could be used to establish the existence of capital paths that oscillate periodically around the equilibrium value for (α, β) near $(\overline{\alpha}, \overline{\beta})$.

We conclude by remarking that these observations show that the qualitative behaviour of the capital market critically depends on the choice of the exponents α and β. Thus even with this relatively simple economic model, very diverse dynamical behaviour can be obtained by varying the parameters. It is hence important to have accurate models for the production function Q.

A Embedded Contractions

In this appendix we develop a version of the embedded contraction theorem which we used to prove that the center manifold is C^k-smooth. The presentation given here contains slight adapations of results given in [42].

Let Y_0, Y, Y_1 and Λ be Banach spaces with norms denoted respectively by

$$\|\cdot\|_0, \quad \|\cdot\|, \quad \|\cdot\|_1 \quad \text{and} \quad |\cdot|, \tag{A.1}$$

and suppose that we have continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$. Let $\Omega_0 \subset Y_0$ be a convex open subset of Y_0. We consider the fixed-point equation

$$y = F(y, \lambda) \tag{A.2}$$

for some $F : Y \times \Lambda \to Y$. Associated to F we define a function $F_0 : Y_0 \times \Lambda \to Y$ via

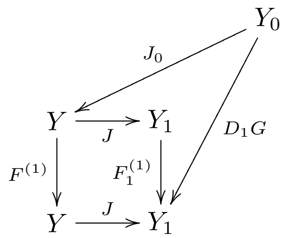
$$F_0(y_0, \lambda) = F(J_0 y_0, \lambda) \tag{A.3}$$

and also a function $G : Y_0 \times \Lambda \to Y_1$ by $G = J \circ F_0$. The situation is illustrated by the following commutative diagram.

$$\begin{array}{ccc} Y_0 \times \Lambda & \xrightarrow{G} & Y_1 \\ {\scriptstyle J_0 \times I} \downarrow & \searrow^{F_0} & \uparrow {\scriptstyle J} \\ Y \times \Lambda & \xrightarrow{F} & Y \end{array} \tag{A.4}$$

We shall need the following assumptions on F and G.

(HC1) The function G is of class C^1. Fix any $\omega_0 \in \Omega_0$ and $\lambda \in \Lambda$ and consider the partial derivative $D_1 G(\omega_0, \lambda) \in \mathcal{L}(Y_0, Y_1)$. Then there exist $F^{(1)}(\omega_0, \lambda) \in \mathcal{L}(Y)$ and $F_1^{(1)}(\omega_0, \lambda) \in \mathcal{L}(Y_1)$ such that the following diagram is commutative,



(A.5)

i.e., for any $v_0 \in Y_0$ we have

$$\begin{aligned} D_1G(\omega_0,\lambda)v_0 &= JF^{(1)}(\omega_0,\lambda)J_0v_0, \\ JF^{(1)}(\omega_0,\lambda)y &= F_1^{(1)}(\omega_0,\lambda)Jy. \end{aligned} \tag{A.6}$$

(HC2) There exists some $\kappa_1 \in [0,1)$ such that for all $\omega_0 \in \Omega_0$ and $\lambda \in \Lambda$ we have

$$\left\|F^{(1)}(\omega_0,\lambda)\right\|_{\mathcal{L}(Y)} \leq \kappa_1 \quad \text{and} \quad \left\|F_1^{(1)}(\omega_0,\lambda)\right\|_{\mathcal{L}(Y_1)} \leq \kappa_1. \tag{A.7}$$

(HC3) The mapping $(\omega_0,\lambda) \to J \circ F^{(1)}(\omega_0,\lambda)$ is continuous as a map from $\Omega_0 \times \Lambda$ into $\mathcal{L}(Y,Y_1)$.

(HC4) The function F_0 has a continuous partial derivative

$$D_2F_0 : Y_0 \times \Lambda \to \mathcal{L}(\Lambda, Y). \tag{A.8}$$

(HC5) There exists some $\kappa_2 \in [0,1)$ such that for all $y, \overline{y} \in Y$ and all $\lambda \in \Lambda$ we have

$$\|F(y,\lambda) - F(\overline{y},\lambda)\| \leq \kappa_2 \|y - \overline{y}\|. \tag{A.9}$$

It follows from (HC4) that (A.2) has for each $\lambda \in \Lambda$ a unique solution $\Psi = \Psi(\lambda)$. We assume that

(HC6) For some continuous $\Phi : \Lambda \to \Omega_0$ we have $\Psi = J_0 \circ \Phi$.

We define $\kappa = \max(\kappa_1, \kappa_2)$.

Lemma A.1. *Assume that assumptions (HC1) through (HC6) hold, except possibly (HC3). Then Ψ is locally Lipschitz continuous.*

Proof. We calculate

$$\begin{aligned} \|\Psi(\lambda) - \Psi(\mu)\| &= \|F(\Psi(\lambda),\lambda) - F(\Psi(\mu),\mu)\| \\ &\leq \|F(\Psi(\lambda),\lambda) - F(\Psi(\mu),\lambda)\| + \|F_0(\Phi(\mu),\lambda) - F_0(\Phi(\mu),\mu)\| \\ &\leq \kappa\|\Psi(\lambda) - \Psi(\mu)\| + |\lambda - \mu| \sup_{s\in[0,1]} \|D_2F_0(\Phi(\mu), s\lambda + (1-s)\mu)\|. \end{aligned} \tag{A.10}$$

Now fix some $\lambda \in \Lambda$ and let $C(\lambda) > \|D_2 F_0(\Phi(\lambda), \lambda)\|$. Since both $D_2 F_0$ and Φ are continuous, there exists some $\delta > 0$ such that for all μ with $|\mu - \lambda| < \delta$ we have

$$\sup_{s \in [0,1]} \|D_2 F_0(\Phi(\mu), s\lambda + (1-s)\mu)\| \leq C(\lambda). \tag{A.11}$$

Using (A.10) we immediately conclude that for such μ we have

$$\|\Psi(\lambda) - \Psi(\mu)\| \leq C(\lambda)(1-\kappa)^{-1} |\lambda - \mu|, \tag{A.12}$$

which concludes the proof. □

Assuming that (HC1) through (HC6) hold, we can consider the following equation for $A \in \mathcal{L}(\Lambda, Y)$,

$$A = F^{(1)}(\Phi(\lambda), \lambda)A + D_2 F_0(\Phi(\lambda), \lambda). \tag{A.13}$$

Since $\left\|F^{(1)}\right\|_{\mathcal{L}(Y)} \leq \kappa < 1$ by (HC2), we see that $I - F^{(1)}(\Phi(\lambda), \lambda)$ is invertible in $\mathcal{L}(Y)$ and hence for each $\lambda \in \Lambda$ (A.13) has a unique solution $A = \mathcal{A}(\lambda) \in \mathcal{L}(\Lambda, Y)$.

Lemma A.2. *Assume that (HC1) through (HC6) hold. Then the mapping $J \circ \Psi$ is of class C^1 and $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$ for all $\lambda \in \Lambda$.*

Proof. Fix $\lambda \in \Lambda$. For any $\mu \in \Lambda$ write $S(\mu) = J\Psi(\mu) - J\Psi(\lambda) - J\mathcal{A}(\lambda)(\mu - \lambda)$ and calculate

$$\begin{aligned} S(\mu) &= JF(\Psi(\mu), \mu) - JF(\Psi(\lambda), \lambda) - JF^{(1)}(\Phi(\lambda), \lambda)\mathcal{A}(\lambda)(\mu - \lambda) - JD_2 F_0(\Phi(\lambda), \lambda)(\mu - \lambda) \\ &= G(\Phi(\mu), \mu) - G(\Phi(\lambda), \lambda) - JF^{(1)}(\Phi(\lambda), \lambda)\mathcal{A}(\lambda)(\mu - \lambda) - D_2 G(\Phi(\lambda), \lambda)(\mu - \lambda) \\ &= G(\Phi(\mu), \mu) - G(\Phi(\lambda), \mu) - JF^{(1)}(\Phi(\lambda), \lambda)\mathcal{A}(\lambda)(\mu - \lambda) \\ &\quad + G(\Phi(\lambda), \mu) - G(\Phi(\lambda), \lambda) - D_2 G(\Phi(\lambda), \lambda)(\mu - \lambda) \\ &= JF^{(1)}(\Phi(\lambda), \lambda)[\Psi(\mu) - \Psi(\lambda) - \mathcal{A}(\lambda)(\mu - \lambda)] + R(\lambda, \mu) \\ &= F_1^{(1)}(\Phi(\lambda), \lambda)[J\Psi(\mu) - J\Psi(\lambda) - J\mathcal{A}(\lambda)(\mu - \lambda)] + R(\lambda, \mu), \end{aligned} \tag{A.14}$$

where

$$\begin{aligned} R(\lambda, \mu) &= \textstyle\int_0^1 [JF^{(1)}\big(s\Phi(\mu) + (1-s)\Phi(\lambda), \mu\big) - JF^{(1)}(\Phi(\lambda), \lambda)][\Psi(\mu) - \Psi(\lambda)]ds \\ &\quad + \textstyle\int_0^1 [D_2 G(\Phi(\lambda), s\mu + (1-s)\lambda) - D_2 G(\Phi(\lambda), \lambda)][\mu - \lambda]ds. \end{aligned} \tag{A.15}$$

Using (HC3) and the continuity of $D_2 G$ and Φ, for each $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\begin{aligned} \sup_{s \in [0,1]} \left\|JF^{(1)}(s\Phi(\mu) + (1-s)\Phi(\lambda), \mu) - JF^{(1)}(\Phi(\lambda), \lambda)\right\| &< \epsilon, \\ \sup_{s \in [0,1]} \|D_2 G(\Phi(\lambda), s\mu + (1-s)\lambda) - D_2 G(\Phi(\lambda), \lambda)\| &< \epsilon, \end{aligned} \tag{A.16}$$

whenever $|\mu - \lambda| < \delta$. Letting $C(\lambda)$ be a Lipschitz constant for Ψ in a neighbourhood of λ, we obtain

$$\|R(\lambda,\mu)\| \leq \epsilon(C(\lambda)+1)\,|\mu-\lambda| \tag{A.17}$$

for $|\mu - \lambda| < \delta$. From (A.14) and (HC2) it now follows that for such values of μ we have

$$\|S(\mu)\|_1 \leq \epsilon\frac{C(\lambda)+1}{1-\kappa}\,|\mu-\lambda|\,, \tag{A.18}$$

which shows that $J \circ \Psi$ is differentiable at λ with $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$. It remains to show that $\lambda \to J \circ \mathcal{A}(\lambda)$ is continuous. Since

$$\begin{aligned} J\mathcal{A}(\lambda) - J\mathcal{A}(\mu) &= JF^{(1)}(\Phi(\lambda),\lambda)\mathcal{A}(\lambda) + D_2G(\Phi(\lambda),\lambda) \\ &\quad - JF^{(1)}(\Phi(\mu),\mu)\mathcal{A}(\mu) - D_2G(\Phi(\mu),\mu) \\ &= F_1^{(1)}(J_0\Phi(\lambda),\lambda)(J\mathcal{A}(\lambda) - J\mathcal{A}(\mu)) \\ &\quad + \big(JF^{(1)}(\Phi(\lambda),\lambda) - JF^{(1)}(\Phi(\lambda),\mu)\big)\mathcal{A}(\mu) \\ &\quad + \big(JF^{(1)}(\Phi(\lambda),\mu) - JF^{(1)}(\Phi(\mu),\mu)\big)\mathcal{A}(\mu) \\ &\quad + D_2G(\Phi(\lambda),\lambda) - D_2G(\Phi(\mu),\mu), \end{aligned} \tag{A.19}$$

it follows that

$$\begin{aligned} (1-\kappa)\,\|J\mathcal{A}(\lambda) - J\mathcal{A}(\mu)\| &\leq \big\|JF^{(1)}(\Phi(\lambda),\lambda) - JF^{(1)}(\Phi(\lambda),\mu)\big\|\,\|\mathcal{A}(\mu)\| \\ &\quad + \big\|JF^{(1)}(\Phi(\lambda),\mu) - JF^{(1)}(\Phi(\mu),\mu)\big\|\,\|\mathcal{A}(\mu)\| \\ &\quad + \|D_2G(\Phi(\lambda),\lambda) - D_2G(\Phi(\mu),\mu)\|\,. \end{aligned} \tag{A.20}$$

Using the continuity of Φ, D_2G and $JF^{(1)}$, the continuity of $\lambda \to J \circ \mathcal{A}(\lambda)$ now easily follows. □

B Fourier and Laplace Transform

We recall here the definitions of the Fourier transform $\widehat{f}(k)$ of an $L^2(\mathbb{R},\mathbb{C}^n)$ function f and the inverse Fourier transform $\check{g}(\xi)$ for any $g \in L^2(\mathbb{R},\mathbb{C}^n)$, given by

$$\widehat{f}(k) = \textstyle\int_{-\infty}^{\infty} e^{-ik\xi} f(\xi)d\xi, \qquad \check{g}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi} g(k)dk. \tag{B.1}$$

We remark here that the integrals above are well defined only if $f, g \in L^1(\mathbb{R},\mathbb{C}^n)$. If this is not the case, the integrals have to be understood as integrals in the Fourier sense, i.e., the functions

$$h_n(k) = \int_{-n}^{n} e^{-ik\xi} f(\xi)d\xi \tag{B.2}$$

satisfy $h_n(k) \to \widehat{f}$ in $L^2(\mathbb{R}, \mathbb{C}^n)$ and in addition there is a subsequence $\{n'\}$ such that $h_{n'}(k) \to \widehat{f}(k)$ almost everywhere. We recall that the Fourier transform takes convolutions into products, i.e., $(\widehat{f * g})(k) = \widehat{f}(k)\widehat{g}(k)$ for almost every k. As another useful tool, we state the Riemann Lebesgue lemma [25, Thm. 21.39].

Lemma B.1. *For any $f \in L^1(\mathbb{R}_+, \mathbb{C}^n)$, we have*

$$\lim_{\omega \to \pm\infty} \left| \int_0^\infty e^{i\omega\xi} f(\xi) d\xi \right| = 0. \tag{B.3}$$

Suppose $f : \mathbb{R} \to \mathbb{C}^n$ satisfies $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$. Then for any z with $\operatorname{Re} z > -a$, define the Laplace transform

$$\widetilde{f}_+(z) = \int_0^\infty e^{-z\xi} f(\xi) d\xi. \tag{B.4}$$

Similarly, if $f(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then for any z with $\operatorname{Re} z > -b$, define

$$\widetilde{f}_-(z) = \int_0^\infty e^{z\xi} f(-\xi) d\xi. \tag{B.5}$$

The inverse transformation is described in the next result, which can be found in the standard Laplace transform literature [43, 7.3-5].

Lemma B.2. *Let $f : \mathbb{R} \to \mathbb{C}^n$ satisfy a growth condition $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ and suppose that f is of bounded variation on bounded intervals. Then for any $\gamma > -a$ and $\xi > 0$ we have the inversion formula*

$$\frac{f(\xi+) + f(\xi-)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{-a-i\omega}^{-a+i\omega} e^{z\xi} \widetilde{f}_+(z) dz, \tag{B.6}$$

whereas for $\xi = 0$ we have

$$\frac{f(0+)}{2} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{-a-i\omega}^{-a+i\omega} e^{z\xi} \widetilde{f}_+(z) dz. \tag{B.7}$$

C Hopf Bifurcation Theorem

In this appendix we state the Hopf bifurcation theorem for the finite dimensional system of ODE's

$$\dot{x} = g(x, \mu), \tag{C.1}$$

for $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^n$, where g satisfies the following assumptions.

(HH1) For some integer $k \geq 2$ we have $g \in C^k(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, with $g(0,\mu) = 0$ for all $\mu \in \mathbb{R}$.

(HH2) For some $\mu_0 \in \mathbb{R}$ the matrix $A = D_1 g(0,\mu_0)$ has simple (i.e. of algebraic multiplicity one) eigenvalues at $\pm i\omega_0$, where $\omega_0 > 0$. In addition, no other eigenvalue of A belongs to $i\omega_0\mathbb{Z}$.

(HH3) Writing $\sigma(\mu)$ for the branch of eigenvalues of $D_1 g(0,\mu)$ through $i\omega_0$ at $\mu = \mu_0$, we have $\operatorname{Re} D\sigma(\mu_0) \neq 0$.

Finally, we define the non-zero vector $v \in \mathbb{R}^n$ to be an arbitrary eigenvector of the matrix A at the eigenvalue $i\omega_0$ and we let $w \in \mathbb{R}^n$ be an arbitrary eigenvector of A^T at $i\omega_0$ normalized such that $w^T v = 1$, i.e., the spectral projection $P_{i\omega_0}$ corresponding to the eigenvalue $i\omega_0$ is given by $P_{i\omega_0} x = v w^T x$. The following results are stated as in [17] and we refer to a paper by Crandall and Rabinowitz [15] for proofs and additional information.

Theorem C.1. *Consider (C.1) and suppose that (HH1)-(HH3) are satisfied. Then there exist C^{k-1}-smooth functions $\tau \to \mu^*(\tau) \in \mathbb{R}$, $\tau \to \omega^*(\tau) \in \mathbb{R}$ and $\tau \to x^*(\tau) \in C(\mathbb{R}, \mathbb{R}^n)$, all defined for τ sufficiently small, such that at $\mu = \mu^*(\tau)$, $x^*(\tau)$ is a $\frac{2\pi}{\omega^*(\tau)}$ periodic solution of (C.1). Moreover, μ^* and ω^* are even, $\mu(0) = \mu_0$, $\omega(0) = \omega_0$, $x^*(-\tau)(\xi) = x^*(\tau)(\xi + \frac{\pi}{\omega^*(\tau)})$ and $x^*(\tau)(\xi) = \tau \operatorname{Re}(e^{i\omega_0 \xi} v) + o(\tau)$, as $\tau \to 0$, uniformly on compact subsets of $\mathbb{R}$. In addition, if x is a small periodic solution of this equation with μ close to μ_0 and minimal period close to $\frac{2\pi}{\omega_0}$, then $x(\xi) = x^*(\tau)(\xi + \xi_0)$ and $\mu = \mu^*(\tau)$ for some τ and $\xi_0 \in [0, 2\pi/\omega^*(\tau))$, with τ unique modulo its sign.*

We conclude this appendix with a result on the direction of bifurcation.

Theorem C.2. *Consider (C.1) and suppose that (HH1)-(HH3) are satisfied, but with $k \geq 3$. Let μ^* be as defined in Theorem C.1. Then we have $\mu^*(\tau) = \mu_0 + \mu_2 \tau^2 + o(\tau^2)$, with*

$$\mu_2 = -\frac{\operatorname{Re} c}{\operatorname{Re} D\sigma(\mu_0)}. \tag{C.2}$$

The constant c is uniquely determined by the following identity

$$\begin{aligned} cv \;=\; & \tfrac{1}{2} P_{i\omega_0} D_1^3 g(0,\mu_0)(v, v, \overline{v}) \\ & + P_{i\omega_0} D_1^2 g(0,\mu_0)(v, -D_1 g(0,\mu_0)^{-1} D_1^2 g(0,\mu_0)(v, \overline{v})) \\ & + \tfrac{1}{2} P_{i\omega_0} D_1^2 g(0,\mu_0)(\overline{v}, (2i\omega_0 - D_1 g(0,\mu_0))^{-1} D_1^2 g(0,\mu_0)(v,v)). \end{aligned} \tag{C.3}$$

References

[1] Abell, K. A., Elmer, C. E., Humphries, A. R., and Vleck, E. S. V. (2005), Computation of Mixed Type Functional Differential Boundary Value Problems. *SIAM J. Appl. Dyn. Sys.* **4**, 755–781.

[2] Bart, H., Gohberg, I., and Kaashoek, M. A. (1986), Wiener-Hopf Factorization, Inverse Fourier Transforms and Exponentially Dichotomous Operators. *J. Funct. Anal.* **68**, 1–42.

[3] Bates, P. W., Chen, X., and Chmaj, A. (2003), Traveling Waves of Bistable Dynamics on a Lattice. *SIAM J. Math. Anal.* **35**, 520–546.

[4] Bates, P. W. and Chmaj, A. (1999), A discrete Convolution Model for Phase Transitions. *Arch. Rational Mech. Anal.* **150**, 281–305.

[5] Bell, J. (1981), Some Threshold Results for Models of Myelinated Nerves. *Math. Biosciences* **54**, 181–190.

[6] Benhabib, J. and Nishimura, K. (1979), The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth. *J. Econ. Theory* **21**, 421–444.

[7] Benzoni-Gavage, S. (1998), Semi-Discrete Shock Profiles for Hyperbolic Systems of Conservation Laws. *Physica D* **115**, 109–124.

[8] Benzoni-Gavage, S. and Huot, P. (2002), Existence of Semi-Discrete Shocks. *Discrete Contin. Dyn. Syst.* **8**, 163–190.

[9] Benzoni-Gavage, S., Huot, P., and Rousset, F. (2003), Nonlinear Stability of Semidiscrete Shock Waves. *SIAM J. Math. Anal.* **35**, 639–707.

[10] Cahn, J. W. (1960), Theory of Crystal Growth and Interface Motion in Crystalline Materials. *Acta Met.* **8**, 554–562.

[11] Cahn, J. W., Mallet-Paret, J., and Van Vleck, E. S. (1999), Traveling Wave Solutions for Systems of ODE's on a Two-Dimensional Spatial Lattice. *SIAM J. Appl. Math* **59**, 455–493.

[12] Chi, H., Bell, J., and Hassard, B. (1986), Numerical Solution of a Nonlinear Advance-Delay-Differential Equation from Nerve Conduction Theory. *J. Math. Bio.* **24**, 583–601.

[13] Chow, S. N., Mallet-Paret, J., and Shen, W. (1998), Traveling Waves in Lattice Dynamical Systems. *J. Diff. Eq.* **149**, 248–291.

[14] Chua, L. O. and Roska, T. (1993), The CNN paradigm. *IEEE Trans. Circuits Syst.* **40**, 147–156.

[15] Crandall, M. C. and Rabinowitz, P. H. (1978), The Hopf Bifurcation Theorem in Infinite Dimensions. *Arch. Rat. Mech. Anal.* **67**, 53–72.

[16] d'Albis, H. and Augeraud-Veron, E. (2004), Competitive Growth in a Life-Cycle Model: Existence and Dynamics. *preprint*.

[17] Diekmann, O., van Gils, S. A., Verduyn-Lunel, S. M., and Walther, H. O. (1995), *Delay Equations*. Springer-Verlag, New York.

[18] Elmer, C. E. and Van Vleck, E. S. (2002), A Variant of Newton's Method for the Computation of Traveling Waves of Bistable Differential-Difference Equations. *J. Dyn. Diff. Eqn.* **14**, 493–517.

[19] Elmer, C. E. and Van Vleck, E. S. (2005), Dynamics of Monotone Travelling Fronts for Discretizations of Nagumo PDEs. *Nonlinearity* **18**, 1605–1628.

[20] Erneux, T. and Nicolis, G. (1993), Propagating Waves in Discrete Bistable Reaction-Diffusion Systems. *Physica D* **67**, 237–244.

[21] Fife, P. and McLeod, J. (1977), The Approach of Solutions of Nonlinear Diffusion Equations to Traveling Front Solutions. *Arch. Rat. Mech. Anal.* **65**, 333–361.

[22] Frasson, M. V. S. and Verduyn-Lunel, S. M. (2003), Large Time Behaviour of Linear Functional Differential Equations. *Integral Eq. and Operator Theory* **47**, 91–121.

[23] Hankerson, D. and Zinner, B. (1993), Wavefronts for a Cooperative Tridiagonal System of Differential Equations. *J. Dyn. Diff. Eq.* **5**, 359–373.

[24] Härterich, J., Sandstede, B., and Scheel, A. (2002), Exponential Dichotomies for Linear Non-Autonomous Functional Differential Equations of Mixed Type. *Indiana Univ. Math. J.* **51**, 1081–1109.

[25] Hewitt, E. and Stromberg, K. (1965), *Real and Abstract Analysis.* Springer-Verlag, Berlin.

[26] Hupkes, H. J. and Verduyn-Lunel, S. M. (2005), Analysis of Newton's Method to Compute Travelling Waves in Discrete Media. *J. Dyn. Diff. Eqn.* **17**, 523–572.

[27] Iooss, G. (2000), Traveling Waves in the Fermi-Pasta-Ulam Lattice. *Nonlinearity* **13**, 849–866.

[28] Iooss, G. and Kirchgässner, K. (2000), Traveling Waves in a Chain of Coupled Nonlinear Oscillators. *Comm. Math. Phys.* **211**, 439–464.

[29] Kaashoek, M. and Verduyn-Lunel, S. M. (1994), An Integrability Condition on the Resolvent for Hyperbolicity of the Semigroup. *J. Diff. Eq.* **112**, 374–406.

[30] Keener, J. and Sneed, J. (1998), *Mathematical Physiology.* Springer–Verlag, New York.

[31] Keener, J. P. (1987), Propagation and its Failure in Coupled Systems of Discrete Excitable Cells. *SIAM J. Appl. Math.* **47**, 556–572.

[32] Laplante, J. P. and Erneux, T. (1992), Propagation Failure in Arrays of Coupled Bistable Chemical Reactors. *J. Phys.Chem.* **96**, 4931–4934.

[33] Mallet-Paret, J. (1996), Spatial Patterns, Spatial Chaos and Traveling Waves in Lattice Differential Equations. In: *Stochastic and Spatial Structures of Dynamical Systems*, Royal Netherlands Academy of Sciences. Proceedings, Physics Section. Series 1, Vol. 45. Amsterdam, pp. 105–129.

[34] Mallet-Paret, J. (1999), The Fredholm Alternative for Functional Differential Equations of Mixed Type. *J. Dyn. Diff. Eqn.* **11**, 1–48.

[35] Mallet-Paret, J. (1999), The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems. *J. Dyn. Diff. Eqn.* **11**, 49–128.

[36] Mallet-Paret, J. (2001), Crystallographic Pinning: Direction Dependent Pinning in Lattice Differential Equations. *preprint.*

[37] Mallet-Paret, J. and Verduyn-Lunel, S. M., Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differential Equations. *to appear in J. Diff. Eq.*

[38] Mielke, A. (1986), A Reduction Principle for Nonautonomous Systems in Infinite-Dimensional Spaces. *J. Diff. Eq.* **65**, 68–88.

[39] Mielke, A. (1994), Floquet Theory for, and Bifurcations from Spatially Periodic Patterns. *Tatra Mountains Math. Publ.* **4**, 153–158.

[40] Rustichini, A. (1989), Hopf Bifurcation for Functional-Differential Equations of Mixed Type. *J. Dyn. Diff. Equations* **1**, 145–177.

[41] Vanderbauwhede, A. and Iooss, G. (1992), Center Manifold Theory in Infinite Dimensions. *Dyn. Reported: Expositions in Dyn. Sys.* **1**, 125–163.

[42] Vanderbauwhede, A. and van Gils, S. A. (1987), Center Manifolds and Contractions on a Scale of Banach Spaces. *J. Functional Analysis* **72**, 209–224.

[43] Widder, D. V. (1946), *The Laplace Transform.* Princeton Univ. Press, Princeton, NJ.

[44] Wu, J. and Zou, X. (1997), Asymptotic and Periodic Boundary Value Problems of mixed FDE's and Wave Solutions of Lattice Differential Equations. *J. Diff. Eq.* **135**, 315–357.

[45] Zinner, B. (1992), Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation. *J. Diff. Eq.* **96**, 1–27.

[46] Zinner, B., Harris, G., and Hudson, W. (1993), Traveling Wavefronts for the Discrete Fisher's Equations. *J. Diff. Eq.* **105**, 46–62.