Counting and ordering periodic stationary solutions of lattice Nagumo equations

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Abstract

We study the rich structure of periodic stationary solutions of Nagumo reaction diffusion equation on lattices. By exploring the relationship with Nagumo equations on cyclic graphs we are able to divide these periodic solutions into equivalence classes that can be partially ordered and counted. In order to accomplish this, we use combinatorial concepts such as necklaces, bracelets and Lyndon words.

Keywords: reaction diffusion equation, lattice differential equation, graph differential equations, periodic solutions, travelling waves
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1. Introduction

In this paper we explore the structure of periodic stationary solutions of the lattice Nagumo equation

$$\dot{u}_i(t) = d(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + g(u_i(t); a), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}. \tag{LDE}$$

We assume $d > 0$ and consider the cubic bistable nonlinearity $g(u; a) := u(1-u)(u-a)$, $a \in (0, 1)$. This equation has been extensively studied as the simplest model describing the competition between two stable states $u = 0$ and $u = 1$ in a spatially discrete environment. One of its key features is the existence of nondecreasing travelling waves $u_j(t) = \Phi(j - ct)$, and the fact that these waves do not move ($c = 0$) for small values of d. This phenomenon (called pinning) is caused by the existence of heterogeneous stationary solutions, which prevent the dominance of the two stable homogeneous states $u = 0$ and $u = 1$. Our goal is to show that the periodic stationary solutions of (LDE), which exist mainly inside the pinning region, form equivalence classes that can be partially ordered and counted.

Equation (LDE) is a discrete-space version of the famous Nagumo reaction-diffusion PDE $u_t = du_{xx} + g(u; a)$, with $x \in \mathbb{R}$. The lattice counterpart (LDE) has a richer set of equilibria [11] which in turn implies more complex behaviour of travelling and standing front solutions [12, 17]. Mallet-Paret [12] established that for each $a \in [0, 1]$ and $d > 0$ there exists a unique $c = c(a, d)$ for which the wave Φ exists. However, if we fix $a \in (0, 1) \setminus \{\frac{1}{2}\}$, Zinner [17] showed that $c(a, d) \neq 0$ for $d \gg 1$ and Keener [9] proved that $c(a, d) = 0$ for $0 < d \ll 1$. Moreover, for fixed $d > 0$ the results in [5] suggest the existence of $\delta(d) > 0$ so that $c(a, d) = 0$ whenever $\left|a - \frac{1}{2}\right| \leq \delta(d)$. This above mentioned pinning is typical for lattice equations [3, 4, 8]. Since the pinning region is dominated by heterogeneous (periodic and aperiodic) stationary solutions, our paper contributes to the understanding of this important feature (see Fig. 1 for a simple illustration).

A second important motivation for understanding the periodic stationary solutions of (LDE) is that this knowledge aids us in the search for so-called multichromatic waves. These non-monotone traveling waves connect two or more n-periodic stationary solutions of (LDE) (in contrast to standard monochromatic waves which are monotone). In our companion papers [6, 7] we have shown that these waves exist mainly inside the pinning region, appearing and disappearing as d increases. The waves that exist outside of the pinning region can be combined to form complex collision waves that involve direction changes.

In this paper we name, partially order and count the equivalence classes of periodic stationary solutions of (LDE) based on the connection between (LDE) and the Nagumo equation posed on cyclic graphs. For an arbitrary undirected graph $\mathcal{G} = (V, E)$ with the set of vertices $V = \{1, 2, 3, \ldots, n\}$ and a set of edges E, the Nagumo equation

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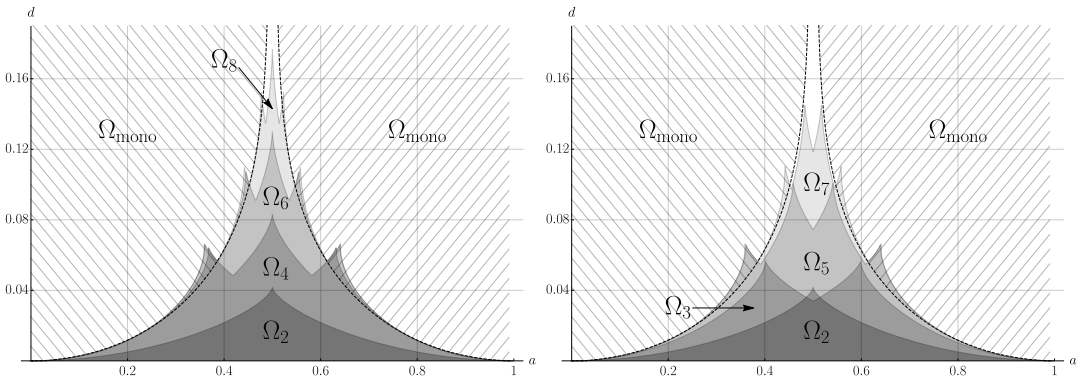


Figure 1: Existence regions Ω_n of stable n-perioidic stationary solutions of (LDE). The left panel depicts even periods, the right one odd periods. The hatched regions Ω_{mono} correspond to parameter values for which the monochromatic waves of (LDE) travel.

on a graph $\mathcal{G}$ is[1]

$$\dot{\mathrm{u}}_i(t) = d \sum_{j \in \mathcal{N}(i)} \big(\mathrm{u}_j(t) - \mathrm{u}_i(t)\big) + g(\mathrm{u}_i(t); a), \quad i \in V, t \in \mathbb{R}, \tag{GDE}$$

where $\mathcal{N}(i)$ denotes the 1-neighbourhood of vertex $i \in V$.

In §2 we establish the connection between the stationary solutions of (GDE) and the periodic stationary solutions of (LDE). In §3 we use this connection and the implicit function theorem to build a naming scheme for periodic stationary solutions of (LDE). In §4 we discuss their symmetries, which allows us to define and count their equivalence classes in §5. This is achieved by establishing a link with combinatorial concepts such as necklaces, bracelets and Lyndon words. Our main result is formulated in Theorem 4 and illustrated by simple examples.

2. Periodic solutions and solutions of graph Nagumo equation

We here consider $\mathcal{G} = \mathcal{C}_n$, where $\mathcal{C}_n$ is a cycle graph on n vertices. The equation (GDE) can now be written as $\dot{\mathrm{u}}(t) = G(\mathrm{u}(t); a, d)$, where $G : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$G(\mathrm{u}; a, d) := \begin{pmatrix} d(\mathrm{u}_n - 2\mathrm{u}_1 + \mathrm{u}_2) + g\big(\mathrm{u}_1; a\big) \\ d(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3) + g\big(\mathrm{u}_2; a\big) \\ \vdots \\ d(\mathrm{u}_{n-1} - 2\mathrm{u}_n + \mathrm{u}_1) + g\big(\mathrm{u}_n; a\big) \end{pmatrix}. \tag{1}$$

Our key results are based on the correspondence of stationary solutions of (GDE) on $\mathcal{G} = \mathcal{C}_n$ and periodic stationary solutions of (LDE)[2]. We say that a double sequence $u = (u_i)_{i \in \mathbb{Z}}$ is a periodic extension of a vector $\mathrm{u} \in \mathbb{R}^n$ if $u_i = \mathrm{u}_{\mathrm{mod}(i,n)}$ (we assume that the modulo operator takes values $\mathrm{mod}(i, n) \in \{1, \dots, n\}$). We remark that (LDE) is well-posed as an evolution equation on the space $\ell^\infty(\mathbb{Z}; \mathbb{R})$. However, we caution the reader that lattice equations do not necessarily have unique solutions if one drops this boundedness condition, even in the linear case [2, 14].

Lemma 1. *Let $\mathcal{G} = \mathcal{C}_n$, $n \geq 3$, be a cycle graph on n vertices. The vector $\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2, \dots, \mathrm{u}_n)$ is a stationary solution of* (GDE) *on $\mathcal{G} = \mathcal{C}_n$ if and only if its periodic extension u is an n-periodic stationary solution of* (LDE)*. Moreover,* u *is an asymptotically stable solution of* (GDE) *if and only if u is an asymptotically stable solution of* (LDE) *with respect to the ℓ^∞-norm.*

Proof. A short inspection readily yields the desired equivalence between solutions of (LDE) and (GDE)[3]. Turning

[1] We use italic letters for double sequences (e.g., u for solutions of (LDE)) and roman ones for vectors (e.g., u for solutions of (GDE)).

[2] Additionally, it is well-known that the problem of finding stationary solutions of graph differential equations on cycles $\mathcal{G} = \mathcal{C}_n$ is actually equivalent to periodic discrete boundary value problems [15, 16].

[3] We omit the case of $n = 2$. In this case, a slightly modified version of Lem. 1 holds. The reduced version of (1) for $n = 2$

$$G(\mathrm{u}; a, d) := \begin{pmatrix} d(\mathrm{u}_2 - 2\mathrm{u}_1 + \mathrm{u}_2) + g\big(\mathrm{u}_1; a\big) \\ d(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_1) + g\big(\mathrm{u}_2; a\big) \end{pmatrix} = \begin{pmatrix} 2d(\mathrm{u}_2 - \mathrm{u}_1) + g\big(\mathrm{u}_1; a\big) \\ 2d(\mathrm{u}_1 - \mathrm{u}_2) + g\big(\mathrm{u}_2; a\big) \end{pmatrix}.$$

implies that solutions of (LDE) and (GDE) are equivalent if one considers the double value of d in (LDE).

to their stability, let us assume that $\mathrm{u}^* = (\mathrm{u}_1^*, \mathrm{u}_2^*, \ldots, \mathrm{u}_n^*)$ is an asymptotically stable solution of (GDE). There hence exists $\gamma > 0$ such that for each $\mathrm{u}_0 \in \mathbb{R}^n$ with $\|\mathrm{u}_0 - \mathrm{u}^*\| < \gamma$ we have

$$\lim_{t\to\infty} \mathrm{u}(t, \mathrm{u}_0) = \mathrm{u}^*,$$

in which $\mathrm{u}(t, \mathrm{u}_0)$ denotes the solution of (GDE) with the initial condition u_0. Consequently, there exists $\delta > 0$ so that the vectors $\mathrm{w}_0, \mathrm{z}_0 \in \mathbb{R}^n$ defined by

$$(\mathrm{w}_0)_i = \mathrm{u}_i^* + \delta, \quad (\mathrm{z}_0)_i = \mathrm{u}_i^* - \delta \quad \text{for all } i = 1, 2, \ldots, n,$$

satisfy $\lim_{t\to\infty} \mathrm{w}(t, \mathrm{w}_0) = \mathrm{u}^*$, $\lim_{t\to\infty} \mathrm{z}(t, \mathrm{z}_0) = \mathrm{u}^*$.

Let us now consider the periodic extensions u^*, w_0 and z_0 of the vectors u^*, w_0 and z_0. Then the corresponding solutions $w(t, w_0)$, $z(t, w_0)$ of (LDE) satisfy

$$w_i(t, w_0) = \mathrm{w}_{\mathrm{mod}(i,n)}(t, \mathrm{w}_0), \quad z_i(t, w_0) = \mathrm{z}_{\mathrm{mod}(i,n)}(t, \mathrm{z}_0),$$

for each $t \geq 0$, which implies

$$\lim_{t\to\infty} w(t, w_0) = u^*, \quad \lim_{t\to\infty} z(t, z_0) = u^*.$$

Using the comparison principle (e.g., Chen et al. [2, Lemma 1]) we can hence conclude that all solutions u of (LDE) with an initial condition u_0 that satisfies $\|u_0 - u^*\|_\infty < \delta$ indeed have $\lim_{t\to\infty} u(t, u_0) = u^*$, since

$$u^* \leftarrow z(t, z_0) \leq u(t, u_0) \leq w(t, w_0) \to u^*.$$

The opposite implication can be proved similarly. □

3. Naming scheme for stationary periodic solutions

The equivalence between n-periodic solutions of (LDE) and solutions of (GDE) on $\mathcal{G} = \mathcal{C}_n$ (see Lem. 1) allows us to focus on the latter in order to establish our naming scheme for the former solutions. First, let us observe that $G(\mathrm{u}; a, 0) = 0$ for any $a \in (0, 1)$ and $\mathrm{u} \in \{0, a, 1\}^n$. Moreover, the fact that

$$D_1 G(\mathrm{u}; a, 0) = \mathrm{diag}\Big(g'(\mathrm{u}_1; a), \ldots, g'(\mathrm{u}_n; a)\Big) \tag{2}$$

has non-zero entries allows us to employ the implicit function theorem to conclude that there are 3^n solution branches emanating out of the roots $\{0, a, 1\}^n$ for d small. These branches can be tracked up until the first collision with another branch. This justifies the use of the following naming scheme for n-periodic stationary solutions.

We introduce an alphabet $A = \{\mathfrak{0}, \mathfrak{a}, \mathfrak{1}\}$ and call $\mathrm{u}_\mathrm{w} \in [0, 1]^n$ a stationary solution of type $\mathrm{w} \in A^n = \{\mathfrak{0}, \mathfrak{a}, \mathfrak{1}\}^n$ if it satisfies $G(\mathrm{u}_\mathrm{w}; a, d) = 0$ and lies on the branch emanating from the root w_a at $d = 0$, where $\mathrm{w}_a : \{\mathfrak{0}, \mathfrak{a}, \mathfrak{1}\}^n \to \{0, a, 1\}^n$ is defined by

$$(\mathrm{w}_a)_i = \begin{cases} 0 & \text{if } \mathrm{w}_i = \mathfrak{0}, \\ a & \text{if } \mathrm{w}_i = \mathfrak{a}, \\ 1 & \text{if } \mathrm{w}_i = \mathfrak{1}. \end{cases}$$

Using this definition we introduce connected sets

$$\Omega_\mathrm{w} = \{(a, d) \in \mathcal{H} : \text{the system } G(\cdot\,; a, d) = 0 \text{ admits an equilibrium of type w}\}, \tag{3}$$

which are open in the half-strip $\mathcal{H} = [0, 1] \times [0, \infty)$. While a full analysis of the sets Ω_w can be very tricky (for a fixed $a \in (0, 1)$, solutions of type w can disappear and then reappear, see [7]), we are only interested in small values of d here in this paper.

Lemma 2. *Let $a \in (0, 1)$ and $d > 0$ be small enough. Then* (LDE) *has 3^n stationary n-periodic solutions. These solutions are asymptotically stable if and only if they belong to the 2^n solutions of type $\mathrm{w} \in \{\mathfrak{0}, \mathfrak{1}\}^n$.*

Proof. The existence of 3^n stationary n-periodic solutions follows from Lemma 1 and the 3^n solution branches for (GDE) supplied by the implicit function theorem. The stability properties follow from (2) and the fact that $g'(0; a) = -a < 0$, $g'(1; a) = a - 1 < 0$ and $g'(a; a) = a(1 - a) > 0$. □

Moreover, pairs of stable solutions can be ordered if the corresponding words are ordered.

Lemma 3. *Let $a \in (0, 1)$ and $d > 0$ be small enough and consider a distinct pair $\mathrm{w}_A, \mathrm{w}_B \in \{\mathfrak{0}, \mathfrak{a}, \mathfrak{1}\}^n$ with $(\mathrm{w}_A)_i \leq (\mathrm{w}_B)_i$ for all i. Suppose furthermore that at least one of these two words is contained in $\{\mathfrak{0}, \mathfrak{1}\}^n$. Then the solutions $u_{\mathrm{w}_A}, u_{\mathrm{w}_B}$ of* (LDE) *satisfy the strict component-wise inequality $(u_{\mathrm{w}_A})_i < (u_{\mathrm{w}_B})_i$, for all $i \in \mathbb{Z}$.*

Proof. The proof follows from [7, Lemma 5.2] and Lemma 1. □

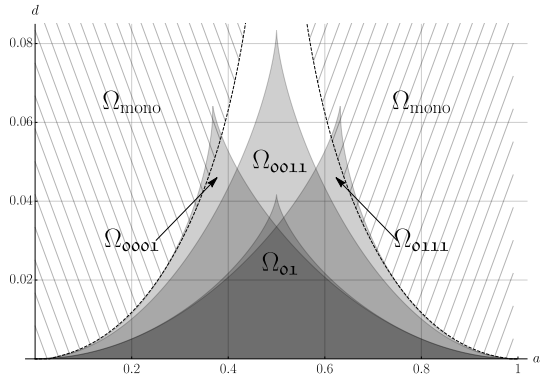


Figure 2: Regions Ω_{w} defined in (3) corresponding to asymptotically stable spatially heterogeneous 4-periodic solutions of (LDE): $u_{\mathfrak{o}\mathfrak{o}\mathfrak{o}\mathfrak{1}}, u_{\mathfrak{o}\mathfrak{o}\mathfrak{1}\mathfrak{1}}, u_{\mathfrak{o}\mathfrak{1}}, u_{\mathfrak{o}\mathfrak{1}\mathfrak{1}\mathfrak{1}}$. The hatched regions Ω_{mono} correspond to pairs (a, d) for which the monochromatic waves of (LDE) travel.

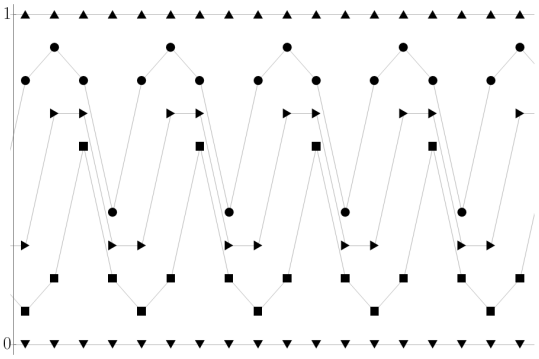


Figure 3: Lyndon representatives of 5 ordered classes of asymptotically stable stationary 4-periodic solutions of (LDE): ▼ - $u_{\mathfrak{o}}$, ■ - $u_{\mathfrak{o}\mathfrak{o}\mathfrak{o}\mathfrak{1}}$, ▶ - $u_{\mathfrak{o}\mathfrak{o}\mathfrak{1}\mathfrak{1}}$, ● - $u_{\mathfrak{o}\mathfrak{1}\mathfrak{1}\mathfrak{1}}$, ▲ - $u_{\mathfrak{1}}$ (the values are slightly modified for better visualisation).

4. Symmetries of stationary periodic solutions

The naming scheme introduced above allows us to study two key symmetries among the 3^n stationary solutions of (GDE) and the corresponding n-periodic stationary solutions of (LDE).

Translation (rotation). If (u_i) is an n-periodic stationary solution of (LDE) then this also holds for (u_{i+k}). We define the translation (rotation) operator on words (and more generally on any vectors of length n) by

$$(\mathcal{T}_\ell \mathrm{w})_i := \mathrm{w}_{\mathrm{mod}(i+\ell,n)}.$$

Reflection. If (u_i) is an n-periodic stationary solution of (LDE) then the same is true for (u_{1-i}). We define the reflection operator by

$$(\mathcal{R}\mathrm{w})_i := \mathrm{w}_{\mathrm{mod}(1-i,n)}.$$

Trivially, if u is a solution of type w for $G(\mathrm{u}; a, d) = 0$ then $\mathcal{T}_\ell \mathrm{u}$ is a solution of type $\mathcal{T}_\ell \mathrm{w}$ and $\mathcal{R}\mathrm{u}$ is a solution of type $\mathcal{R}\mathrm{w}$, which immediately implies that

$$\Omega_{\mathrm{w}} = \Omega_{\mathcal{T}_1 \mathrm{w}} = \ldots = \Omega_{\mathcal{T}_{n-1} \mathrm{w}} = \Omega_{\mathcal{R}\mathrm{w}}.$$

Naturally, other symmetries can be considered as well. For example, in the case of $a = 1/2$, it makes sense to consider symbol swapping $\mathfrak{o} \leftrightarrow \mathfrak{1}$. However, the existence of solution of type w for a given pair (a, d) does not imply the existence of solutions for the "swapped" word $\tilde{\mathrm{w}}$ for general $a \neq 1/2$, see Fig. 1. Therefore, we only focus on the simplest symmetries - translations $\mathcal{T}_\ell$ and reflections $\mathcal{R}$.

5. Counting equivalence classes of stationary periodic solutions

We can now define equivalence classes of n-periodic solutions to (LDE) by factoring out one or both of the symmetries discussed above. If we consider translations $\mathcal{T}_\ell$ and the word $\mathrm{w} = \mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}$ we have the following equivalence class of stationary 4-periodic solutions of (LDE):

$$[u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}]_{\mathcal{T}} = \{u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}, \mathcal{T}_1 u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}, \mathcal{T}_2 u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}, \mathcal{T}_3 u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}\} = \{u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}, u_{\mathfrak{o}\mathfrak{a}\mathfrak{1}\mathfrak{o}}, u_{\mathfrak{a}\mathfrak{1}\mathfrak{o}\mathfrak{o}}, u_{\mathfrak{1}\mathfrak{o}\mathfrak{o}\mathfrak{a}}\}.$$

If we consider both translations $\mathcal{T}_\ell$ and reflections $\mathcal{R}$, we have for example $\mathcal{R} u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}} = u_{\mathfrak{1}\mathfrak{a}\mathfrak{o}\mathfrak{o}}$ and thus

$$[u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}]_{\mathcal{T}\mathcal{R}} = \{u_{\mathfrak{o}\mathfrak{o}\mathfrak{a}\mathfrak{1}}, u_{\mathfrak{o}\mathfrak{a}\mathfrak{1}\mathfrak{o}}, u_{\mathfrak{a}\mathfrak{1}\mathfrak{o}\mathfrak{o}}, u_{\mathfrak{1}\mathfrak{o}\mathfrak{o}\mathfrak{a}}, u_{\mathfrak{1}\mathfrak{a}\mathfrak{o}\mathfrak{o}}, u_{\mathfrak{o}\mathfrak{1}\mathfrak{a}\mathfrak{o}}, u_{\mathfrak{o}\mathfrak{o}\mathfrak{1}\mathfrak{a}}, u_{\mathfrak{a}\mathfrak{o}\mathfrak{o}\mathfrak{1}}\}.$$

We always use the smallest word in the lexicographical sense (the so called Lyndon word) as a class representative. If the word is not primitive (i.e., it is periodic itself), we take the primitive (aperiodic) subword, e.g., $[\mathfrak{o}\mathfrak{1}\mathfrak{o}\mathfrak{1}]_{\mathcal{T}} = [\mathfrak{o}\mathfrak{1}]_{\mathcal{T}}$.

If one is simply interested in the periodic solutions themselves then it is reasonable to factor out both translation and reflection symmetries and consider the full alphabet $\{\mathfrak{o}, \mathfrak{a}, \mathfrak{1}\}$. However, in special circumstances (e.g., when connecting these solutions via multichromatic waves [7]) it only makes sense to factor out the translation symmetries and to consider the reduced alphabet $\{\mathfrak{o}, \mathfrak{1}\}$.

Period	All solutions	translation $\mathcal{T}$		translation $\mathcal{T}$+refection $\mathcal{R}$	
		All	Primitive	All	Primitive
n	k^n	$N_k(n)$	$L_k(n)$	$B_k(n)$	$BL_k(n)$
1	3(2)	3(2)	3(2)	3(2)	3(2)
2	9(4)	6(3)	3(1)	6(3)	3(1)
3	27(8)	11(4)	8(2)	10(4)	7(2)
4	81(16)	24(6)	18(3)	21(6)	15(3)
5	243(32)	51(8)	48(6)	39(8)	36(6)
6	729(64)	130(14)	116(9)	92(13)	79(8)
...					

Table 1: Number of equivalence classes of n-periodic stationary solutions of (LDE). In each pair, the former number corresponds to all equivalence classes and the latter number in the parentheses to the equivalence classes formed by asymptotically stable solutions.

We can now use the combinatorial theory of words [10, 13] to count the equivalence classes and to get the number of qualitatively different n-periodic stationary solutions. To this end, we define the quantities

$$N_k(n) = \frac{1}{n}\sum_{d:d|n} \varphi(d) k^{\frac{n}{d}}, \quad L_k(n) = \frac{1}{n}\sum_{d:d|n} \mu(d) k^{\frac{n}{d}}, \tag{4}$$

$$B_k(n) = \begin{cases} \frac{1}{2}\left(N_k(n) + \frac{k+1}{2}k^{n/2}\right) & \text{for } n \text{ even}, \\ \frac{1}{2}\left(N_k(n) + k^{(n+1)/2}\right) & \text{for } n \text{ odd}, \end{cases} \tag{5}$$

$$BL_k(n) = \sum_{d:d|n} \mu(d) B_k(n/d), \tag{6}$$

where $\varphi(d)$ is the Euler's totient function and $\mu(d)$ is the Möbius function [1, Chapter 2]. We can use these quantities to formulate our main result, which describes the number of equivalence classes of n-periodic stationary solutions.

Theorem 4. *Pick an integer $n \geq 2$ and a parameter $a \in (0,1)$. Then for $d > 0$ small enough* (LDE) *has exactly*

1. *3^n n-periodic stationary solutions which form*
   (a) *$N_3(n)$ equivalence classes with respect to translations $\mathcal{T}_\ell$. Moreover, $L_3(n)$ of these equivalence classes are formed by primitive periodic solutions.*
   (b) *$B_3(n)$ equivalence classes with respect to translations $\mathcal{T}_\ell$ and reflections $\mathcal{R}$. Moreover, $BL_3(n)$ of these equivalence classes are formed by primitive periodic solutions.*
2. *2^n asymptotically stable n-periodic stationary solutions which form*
   (a) *$N_2(n)$ equivalence classes with respect to translations $\mathcal{T}_\ell$. Moreover, $L_2(n)$ of these equivalence classes are formed by primitive periodic solutions.*
   (b) *$B_2(n)$ equivalence classes with respect to translations $\mathcal{T}_\ell$ and reflections $\mathcal{R}$. Moreover, $BL_2(n)$ of these equivalence classes are formed by primitive periodic solutions.*

Proof. A k-ary necklace of length n is an equivalence class of words of length n formed by k letters which are equivalent with respect to translations (rotations) $\mathcal{T}_\ell$. There are $N_k(n)$ different necklaces [13, Eq. (2.1)]. There are $L_k(n)$ distinct primitive (aperiodic) necklaces – Lyndon words [13, Eq. (2.2)].

A k-ary bracelet of length n is an equivalence class of words of length n formed by k letters which are equivalent with respect to translations (rotations) $\mathcal{T}_\ell$ and refection $\mathcal{R}$. There are $B_k(n)$ different bracelets [13, Eq. (2.4)]. A primitive (aperiodic) bracelet is called a Lyndon bracelet. There are $BL_k(n)$ distinct Lyndon bracelets, which can be proved by the direct application of the Möbius inversion formula (e.g., [1, Theorem 2.9]). The result for n-periodic stationary solutions of (LDE) then follows directly from Lemma 2. □

Table 1 provides a summary of these results for small periods. As an example, we give a detailed description of the equivalence classes for $n = 3$ and $n = 4$.

Example 5. There are 27 distinct 3-periodic stationary solutions of (LDE). Considering translations $\mathcal{T}_\ell$, they form 11 equivalence classes:

$[u_{\mathtt{o}}]_{\mathcal{T}} = \{u_{\mathtt{o}}\}$, $[u_{\mathtt{a}}]_{\mathcal{T}} = \{u_{\mathtt{a}}\}$, $[u_{\mathtt{1}}]_{\mathcal{T}} = \{u_{\mathtt{1}}\}$;

$[u_{\mathtt{oo a}}]_{\mathcal{T}} = \{u_{\mathtt{oo a}}, u_{\mathtt{oao}}, u_{\mathtt{aoo}}\}$, $[u_{\mathtt{oo 1}}]_{\mathcal{T}} = \{u_{\mathtt{oo 1}}, u_{\mathtt{o1o}}, u_{\mathtt{1oo}}\}$, $[u_{\mathtt{oaa}}]_{\mathcal{T}} = \{u_{\mathtt{oaa}}, u_{\mathtt{aao}}, u_{\mathtt{aoa}}\}$, $[u_{\mathtt{oa1}}]_{\mathcal{T}} = \{u_{\mathtt{oa1}}, u_{\mathtt{a1o}}, u_{\mathtt{1oa}}\}$,

$[u_{\mathtt{o1a}}]_{\mathcal{T}} = \{u_{\mathtt{o1a}}, u_{\mathtt{1ao}}, u_{\mathtt{ao1}}\}$, $[u_{\mathtt{o11}}]_{\mathcal{T}} = \{u_{\mathtt{o11}}, u_{\mathtt{11o}}, u_{\mathtt{1o1}}\}$, $[u_{\mathtt{aa1}}]_{\mathcal{T}} = \{u_{\mathtt{aa1}}, u_{\mathtt{a1a}}, u_{\mathtt{1aa}}\}$, $[u_{\mathtt{a11}}]_{\mathcal{T}} = \{u_{\mathtt{a11}}, u_{\mathtt{11a}}, u_{\mathtt{1a1}}\}$.

The former 3 classes correspond to constant solutions and are thus not primitive 3-periodic solutions, while the remaining 8 classes correspond to primitive stationary 3-periodic solutions. Upon taking reflections $\mathcal{R}$ into consideration, there are only 10 equivalence classes and 7 corresponding to primitive periodic solutions, since $[u_{\mathtt{0a1}}]_{\mathcal{T}}$ and $[u_{\mathtt{01a}}]_{\mathcal{T}}$ form one equivalence class

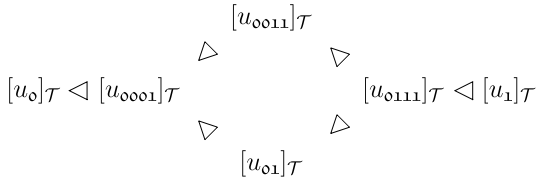
$$[u_{\mathtt{0a1}}]_{\mathcal{TR}} = \{u_{\mathtt{0a1}}, u_{\mathtt{a10}}, u_{\mathtt{10a}}, u_{\mathtt{1a0}}, u_{\mathtt{01a}}, u_{\mathtt{a01}}\}.$$

The 8 asymptotically stable solutions form 4 equivalence classes with respect to translations and 2 are primitive - $[u_{\mathtt{001}}]_{\mathcal{T}}$ and $[u_{\mathtt{011}}]_{\mathcal{T}}$. These classes are not affected by the reflection $\mathcal{R}$ and can be ordered as

$$[u_{\mathtt{0}}]_{\mathcal{T}} \lhd [u_{\mathtt{001}}]_{\mathcal{T}} \lhd [u_{\mathtt{011}}]_{\mathcal{T}} \lhd [u_{\mathtt{1}}]_{\mathcal{T}}.$$

where $[u_{\mathrm{w}_A}] \lhd [u_{\mathrm{w}_B}]$ means that the Lyndon representatives satisfy $u_{\mathrm{w}_A} < u_{\mathrm{w}_B}$.

The ordering of asymptotically stable n-periodic solutions is only partial for periods with $n > 3$. For example, Lemma 3 implies that 6 equivalence classes of 4-periodic solutions can be partially ordered in the following way.



The existence regions of the four spatially heterogeneous equivalence classes are depicted in Fig. 2 and the respective Lyndon representatives of five classes corresponding to the "upper path" in this diagram are sketched in Fig. 3.

Remark 6. Finally, let us note that analyzing the sums in (4)-(6) we arrive to the asymptotic estimates

$$N_k(n) \sim L_k(n) \sim \frac{k^n}{n}, \text{ and } B_k(n) \sim BL_k(n) \sim \frac{k^n}{2n} \text{ as } n \to \infty.$$

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