Invariant Manifolds and Applications for Functional Differential Equations of Mixed Type

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Contents

Preface

Throughout a significant portion of recorded history, mankind has expressed a fascination for the concepts of motion and change. In 1600 BC the Babylonians had already constructed star charts based on detailed observations of the rising of celestial bodies. They used these charts to determine the best harvesting and planting times. Ever since the Jewish people emerged from their wanderings through the Sinai desert, they needed to keep track of lunar cycles to calculate the exact dates for their numerous feasts. They knew that God had promised Noah (in Genesis 8:22, King James Bible):

> While the earth remaineth, seedtime and harvest, and cold and heat,
> and summer and winter, and day and night shall not cease.

The ancient Greeks started studying the subject from a more philosophical point of view. The famous quote $\pi\alpha\nu\tau\alpha\ \rho\epsilon\iota$ is due to Heraclitus (540 BC), who argued that the world around us is always in motion. This assertion was pulled apart masterfully by Zeno of Elea (450 BC), who laid down the groundwork for modern calculus through his devious paradoxes. The most well-known of these is probably the story of Achilles who should never be able to overtake the tortoise, a mind-teaser that still sharpens the minds of children, students and scholars alike.

Mathematical Models

In *The Almagest,* Ptolemaeus (150 AD) proposed the first comprehensive mathematical system to describe the planetary motions. His predictions were actually quite accurate, in spite of the fact that they were based upon a stationary earth, fixed at the center of the cosmos. His model found widespread favour for more than a thousand years, until increasingly accurate observations and work by Oresme, Copernicus, Galileo and Kepler led to the acceptance of the heliocentric point of view in the seventeenth century. The crowning achievement of this golden age of astronomy was undoubtedly the formulation by Newton of the differential equations that describe the laws of gravity and the subsequent development of calculus by Newton and Leibniz. The realization that very complex and puzzling behaviour over long periods of time could be described by simple rules governing rates of change on extremely small timescales, led to the birth of dynamical systems theory as we know it today.

The eighteenth and nineteenth century witnessed the development of a relatively complete theory for linear ordinary differential equations. In addition, perturbation methods were developed and applied to systems with weak nonlinear interactions. The study of general nonlinear systems far from equilibria however long remained a barren area. At the end of the nineteenth century Poincaré and Lyapunov both added new impetus to the subject by abandoning the search for explicit solutions to differential equations in favour of a more qualitative approach. In particular, Poincaré introduced topological methods to the theory, treating the full trajectory traversed by the components of a dynamical system as a single geometrical object. He was the first to use Poincaré sections to analyze the behaviour of systems near periodic orbits and fixed points, locally reducing the continuous-time dynamics to a discrete iteration map. His subsequent research on the behaviour of intersections of stable and unstable manifolds allowed him to prove that the solar system is highly unstable and marked the birth of the modern theory of chaos. Lyapunov on the other hand laid the basis for the current theory of stability, by providing definitions that are still important today and pioneering the use of energy methods.

The marriage between geometry and analysis thus initiated proved to be particularly fruitful. Major contributions to the current relatively complete theory for planar systems were made by Birkhoff [22] and Andronov et al. [2, 3, 4, 5]. Import advances in chaotic systems were sparked by the oscillators studied by Duffing [48] and van der Pol [157], the meteorological problem considered by Lorentz [107] and important results for integrable systems obtained by Arnold [6, 7, 8]. Readers that are interested in detailed accounts of the development of the finite dimensional theory should consult the books by Guckenheimer and Holmes [71] and Katok and Hasselblatt [91].

Infinite dimensional systems

In the later part of the twentieth century there has been an increasing tendency to fit partial differential equations into the framework of dynamical systems. For elliptic PDEs this was initiated by Kirchgassner [94], who studied nonlinear boundary value problems in infinite elliptic cylinders, treating the unbounded spatial direction as a temporal coordinate. These developments have lead to the formation of an active research community in the area of infinite dimensional systems. We refer to [138, Chapter 1] for a nice light-weight overview of the history of this subject, which touches upon most of the major tools and techniques that have been developed. By contrast, our short presentation here will merely highlight some aspects that have a direct connection to the subject of this thesis. Hopefully, this will assist the reader in viewing the developments described throughout this work in the broad context of infinite dimensional evolution equations.

The most important obstacle that has to be overcome in an infinite dimensional setting, is the fact that Banach spaces lack many of the desirable properties that are taken for granted in finite dimensional spaces. This causes many geometric arguments that work beautifully in $\mathbb{R}^n$ to break down. Furthermore, one needs to worry about domains of operators, regularity of solutions and ill-posedness of initial value problems, which all tend to make many arguments very technical. A logical first step in the development of the theory would of course be to identify the parts of the powerful finite dimensional toolbox that can be salvaged for use in Banach space settings. Indeed, this is still one of the main themes of research in this area.

Early on in the twentieth century the foundations for linear semigroup theory were already being laid, in an effort to generalize the matrix exponentials that appear ubiquitously when studying ODEs. The theory reached maturity in 1948 with the formulation of the Hille Yosida generation theorem [79, 169], which provides criteria to determine if a given linear operator can be exponentiated in a sensible fashion. Since then the use of semigroups has branched out considerably and they now play an important role in many applications, including stochastic processes, partial differential equations, quantum mechanics, infinite-dimensional control theory and integro-differential equations [55].

Even though the use of semigroups has proved to be extremely successful, there is still a wide class of systems in which the machinery cannot be so readily applied. As an important example, we mention situations where the linear operator describing the infinitesimal change of a system has unbounded spectrum both to the left and right of the imaginary axis. One cannot define a strongly continuous semigroup that behaves as the exponential of such an operator. This difficulty can often be circumvented by splitting the state space of the system into two separate parts, that both do allow the construction of a semigroup. One of these will however only be defined in backward time. Such a splitting is referred to as an exponential dichotomy. Work on this subject in finite dimensions can be traced back to Lyapunov [109] and Perron [124], but Coppel established the important fact that such splittings are robust under perturbations [37]. Results on exponential splittings in infinite dimensional systems were obtained by Sacker and Sell [133], Henry [77], Pliss and Sell [125] and Sandstede and Scheel [135].

As in the finite dimensional situation, invariant manifolds play a fundamental role in the study of nonlinear systems. A very important structure in this respect is the so-called center manifold, which according to Vanderbauwhede and Iooss forms one of the cornerstones of the theory of infinite dimensional dynamical systems [158]. The reason for this is that small amplitude variations near non-hyperbolic equilibria can be captured by a flow on a smooth invariant center manifold, which typically is finite dimensional. In addition, this flow can often be explicitly computed up to arbitrary order. In view of the considerations above it should be clear that such a reduction from an infinite to a finite dimensional setting can be extremely powerful. As a consequence many different authors have worked on the subject from many different perspectives. We mention here the constructions for elliptic PDEs due to Mielke [118, 119], the results on semilinear PDEs by Bates and Jones [14] and the work by Diekmann and van Gils [44] on Volterra integral equations. To be fair, we should also note that infinite dimensional center manifolds can also be encountered, see e.g. a paper by Scarpellini [136].

When considering a dissipative evolution equation, it becomes feasible to study the global attractor associated to the system. This object attracts all bounded sets and hence captures the long-term behaviour of any orbit. It has been established in quite some generality that this global object has finite Hausdorff dimension [110, 116], although little is known about its geometry, which often has a fractal nature. Important topics in this area include smooth approximations of these attractors [64], classifications based on connection equivalence using Morse index theory [60, 61] and estimates of attractor dimensions from system parameters [154].

Retarded Functional Differential Equations

In the study of evolution equations, the underlying principal of causality states that the future state of the system is independent of the past states and is determined solely by the present. Many physical systems however feature feedback mechanisms with a non-negligible time lag. Of course, this can still be fitted into the evolution equation framework by extending the state space to include the relevant portion of the system's past. The price one has to pay is that this extended state space will be infinite dimensional, even if the original state space is finite dimensional. In addition, a naive application of this approach raises major technical complications if one wishes to add small perturbations to the original equations.

These issues are addressed by the theory of retarded functional differential equations, which was pioneered by Volterra [161]. Many authors have since contributed to the theory and a comprehensive overview can now be found in the monographs by Hale and Verduyn Lunel [72] and Diekmann et al. [45]. The main technical tool exhibited in the latter work, is the development of a sun-star semigroup calculus that allows the (finite dimensional) original state space to be separated in a sense from the part of the extended state space that keeps track of the "past" of the system. This technique paved the way for the construction of invariant manifolds and consequently opened up the development of the nonlinear theory.

Functional Differential Equations of Mixed Type

Functional differential equations of mixed type (MFDEs) generalize the retarded equations mentioned above, in the sense that the rate of change of a system is allowed to depend on future states as well as past states. MFDEs have attracted considerable attention over the past two decades. This interest has been sparked chiefly due to the importance of MFDEs in the study of travelling wave solutions to differential equations posed on lattices (LDEs). These lattice based systems arise naturally when modelling systems that possess a discrete spatial structure. In addition, MFDEs play a major role in a number of applications from economic theory. We refer to Chapter 1 for an extensive discussion on these modelling aspects.

The Fredholm theory for linear MFDEs was developed by Mallet-Paret [112], while important results concerning exponential dichotomies were obtained by Rustichini [130] for autonomous systems. The latter work was later extended to nonautonomous systems simultaneously by Mallet-Paret and Verduyn Lunel [115] on the one hand and Härterich and Sandstede [75] on the other.

This thesis should be seen as a continuation of these efforts to prepare the rich concepts and techniques currently available in infinite dimensional systems theory for use in the context of MFDEs and LDEs. In particular, we focus heavily on the construction of invariant manifolds for MFDEs. Major difficulties that need to be overcome in this respect are the absence of a semiflow and the ill-posedness of the natural initial value problem. This precludes the direct application of the ideas developed for retarded functional differential equations, which at first sight would appear to be closely related to MFDEs. A lengthier discussion concerning these dissimilarities can be found in Chapter 1. As a consequence, the methods employed here differ somewhat from those in [45]. They may best be described as a mixture of the classical Lyapunov-Perron techniques with those that were used by Mielke for elliptic PDEs [118]. In particular, in Chapter 2 we provide a center manifold framework for autonomous MFDEs, while the same is done for autonomous differential-algebraic functional equations in Chapter 3. In a similar spirit, Chapter 4 is concerned with the development of Floquet theory for periodic MFDEs. In Chapter 6 we move on to study homoclinic bifurcations. We also pay a considerable amount of attention to the application range of these results, discussing and numerically analyzing models from economic theory, solid state physics and biology in Chapters 1 and 5.

Chapter 1

Introduction

This thesis is focussed entirely on the study of functional differential equations of mixed type. Such equations can be written in the form

$$x'(\xi) = G(x_\xi), \tag{1.1}$$

in which x is a continuous function, G is a nonlinear mapping from $C([-1, 1], \mathbb{C}^n)$ into $\mathbb{C}^n$ and the state $x_\xi \in C([-1, 1], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$ for all $\xi \in \mathbb{R}$. The nonlinearity G thus typically depends on both advanced and retarded arguments of x, which distinguishes our setting from the by now extensively studied area of delay differential equations.

We will be specially interested in versions of (1.1) that depend on one or more parameters. In particular, we wish to study changes in the behaviour of (1.1) that arise as these parameters are varied. Such changes are commonly referred to as bifurcations and throughout the present work they will be explored from both a theoretical and a numerical point of view. A significant portion of the research described here was motivated directly by problems encountered in the modelling community. To illustrate this, we will demonstrate the application range of our results by discussing several such examples.

Although equations of the form (1.1) have appeared haphazardly in the literature for at least forty years, active interest in these functional differential equations of mixed type (MFDEs) has been limited to the last two decades. Surprisingly enough, this increase in activity was sparked more or less simultaneously by developments in two at first sight completely unrelated subject areas, namely physical and biological modelling on the one side and economic theory on the other. We will explain both developments here in some detail.

Lattice-based Modelling

Motivated by the study of physical structures such as crystals, grids of neurons and population patches, an increasing demand has arisen over the last few decades for mathematical modelling techniques that reflect the spatial discreteness that such systems possess. In the past, the additional complexity of the resulting equations often posed as a deterrent to deviate from the classical models, which were most often based on ordinary and partial differential equations. The increase of computer power during the last few decades however has served to remove this obstacle. As a consequence, a wave of numerical investigations has been initiated, focussing on the evolution of patterns that live on discrete lattices. The spectacular results that have been obtained have in fact opened up some thriving new areas in the field of dynamical systems theory.

As an informative illustration for these developments, we take the opportunity here to briefly discuss an early paper by Chi et al. [32]. In this paper the authors analyze a model for the propagation of signals through myelinated nerve fibres. The key feature of this model is that the nerve fibre is almost entirely surrounded by a myeline coating, that effectively insulates the nerve completely. The coating however admits small gaps at regular intervals and these gaps are known as nodes of Ranvier. The insulation induced by the myeline causes excitations of the nerve at these nodes to effectively jump from one node to the next, through a process called saltatory conduction [120]. The fibre is assumed to have infinite length and the nodes of Ranvier are indexed by $j \in \mathbb{Z}$. The dynamical behaviour can then be described by the following differential equation, posed on the integer lattice $\mathbb{Z}$,

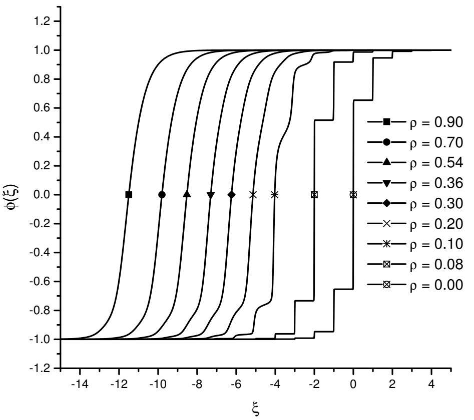


Figure 1.1: Wave profiles for (1.2) at different values of ρ, with $\alpha = 0.1$.

$$\dot{v}_j(t) = \alpha[v_{j+1}(t) + v_{j-1}(t) - 2v_j(t)] - \tfrac{1}{4}(v_j(t)+1)(v_j(t)-1)(v_j(t)-\rho), \quad j \in \mathbb{Z}. \tag{1.2}$$

This equation is a one-dimensional example of a so-called lattice differential equation (LDE), which in general is an infinite system of ordinary differential equations, indexed by points on a discrete spatial lattice. The quantity v_j in (1.2) represents the potential at the node j, while $\alpha \sim h^{-2}$ is related to the distance h between the nodes. The parameter ρ satisfies $-1 < \rho < 1$ and models the various impedances and activation energies connected with the signal propagation through the nerve.

From a biological point of view, it is interesting to study how signals propagate from one end of the nerve to the other. Figure 1 depicts a special class of solutions to (1.2), that propagate through the nerve at a speed c while retaining a fixed shape ϕ. Such solutions are called travelling waves. Notice that as the parameter ρ is decreased, the waveprofiles lose their smoothness and turn into step functions. These latter profiles have the special property that they fail to propagate through the nerve. Put differently, the identity $c = 0$ holds for the associated wavespeed. This feature is called propagation failure and poses many computational [1] and theoretical [113] challenges when studying (1.2). From the modelling perspective, this phenomenon can be understood in terms of an energy barrier caused by the gaps, which must be overcome in order to allow propagation. Indeed, the effect disappears when passing to the PDE version of (1.2), where one takes the limit $h \to 0$ for the internode distance h. These issues will be explored in depth in Chapter 5, where techniques are contributed that aid the numerical computations in the regime where $c \sim 0$.

This biological example already hints towards the complex dynamical behaviour that LDEs may possess. The uncovering of this diverse behaviour has been a major driving issue in the early phases of the investigation into such equations. A pioneering example in this respect is formed by the work of Chua et al, who devised grid-based algorithms to identify edges and corners in pixelized digital images [36]. Using the original image as a starting point, they constructed electronic circuits that allowed each pixel to interact with its neighbours. By carefully selecting interactions that enhance only the required patterns, they were able to extract the outlines of shapes in noisy photographs quite successfully. The circuits used by Chua and his coworkers can be modelled by a lattice differential equation. Since a circuit-based approach is by nature massively parallel, they were able to obtain results which at the time would not have been possible using direct computer simulations of this underlying LDE.

The interesting features that these grid-like algorithms were thus shown to possess inspired many authors to work on LDEs, both from a numerical and theoretical point of view. As a result, numerous studies have by now firmly established that LDEs admit very rich dynamic and pattern-forming behaviour. Even the class of equilibrium solutions to an LDE may be full of interesting structure. Mallet-Paret for example proved that the balance between regular and chaotic spatial patterns in the set of equilibria for a simplified version of the circuit LDE described above may depend in a delicate fashion upon the parameters of the system [111]. In the sequel we will emphasize this point further, by discussing another property that distinguishes an LDE from its continuous counterpart, the partial differential equation.

The ability to include discrete effects into models, together with their interesting dynamical features, have been a tremendous stimulation for the development of lattice-based models. As a consequence, they can now be encountered in a wide variety of scientific disciplines, including chemical reaction theory [57, 104], image processing [36], material science [13, 25] and biology [15, 32, 92, 93]. We refer to Section 1.1 for a further discussion on LDEs and a detailed list of references.

Capital Market Dynamics

Optimal control problems are ubiquitous in economic theory, due to the simple fact that the behaviour of individuals and groups is almost always governed by a wish to maximize overall profit or welfare. As a very simple example to set the stage, let us consider an isolated country that comes into existence at time $t = 0$ and has an infinite life-span. Let us write $k(t)$ for the total production capacity at a certain point in time, which is a direct measure for the amount of economic output in the form of goods and services that can be produced. The crucial point is that at each moment in time, one must decide how to split the production capacity between investments $u(t)$ and consumption $c(t)$. On the one hand, consumption leads to the immediate satisfaction of the needs of the population. Investments on the other hand will increase the total production capacity, which will allow for increased consumption in the future. Mathematically, this can be formulated as an optimization problem,

$$\text{maximize} \int_0^\infty e^{-\rho t} W\big(c(t)\big) dt. \tag{1.3}$$

The function W measures the welfare that is attributed to a certain amount of consumption, while the discount factor ρ reflects how future welfare is rated relative to present welfare. The amount of consumption $c(t)$ that is possible is of course restricted in terms of $k(t)$ and $u(t)$.

This basic problem has appeared in all kinds of variants throughout the literature. As a specific example, Benhabib and Nishimura [17] considered models of this form with $n \geq 2$ distinct production goods. They were able to establish the existence of periodic cycles in the production capacity $k(t)$. The occurrence of oscillations is interesting from an economic point of view, since they are quite commonly observed in actual economic trends.

Already in the nineteenth century Böhm-Bawerk studied [162] the effects that time delays in a production process can have on the total economic production. In addition, in a seminal paper [101] Kydland and Prescott studied the oscillations in the production capacity $k(t)$ mentioned above, for the full post-war U.S. economy. They argued that in any such investigation it is crucial to take into account the presence of a time lag between the investment activity and the actual corresponding increase in the production capacity. Put in the terminology of the authors, ships and factories are not built in a day. Kydland and Prescott underlined this point by providing a detailed model that could be fitted quite reasonably to the actual post-war economy, in which these time lags play a major role. Further more recent results in this direction can be found in [9, 84].

Motivated by these considerations, Rustichini introduced [131] a time delay into the optimal control problems considered by Benhabib and Nishimura. Already in 1968 Hughes showed [81] that the Euler-Lagrange equations associated to an optimal control problem that involves delays are in fact MFDEs. Rustichini analyzed the characteristic equation associated to this variational MFDE for a model with only $n = 1$ production goods, thus considerably simplifying the earlier models in [17]. He gave conditions under which a pair of eigenvalues crosses through the imaginary axis as the model parameters are varied, thus satisfying a Hopf-type criteria. Generically, one expects this to lead to the birth of a branch of periodic orbits. Up to recently however, the Hopf bifurcation has not been rigorously understood in the setting of MFDEs. This situation is remedied in Chapter 2 and in Section 1.4 we use our results to analyze a specific toy economic model that illustrates the points mentioned above.

Recent developments have also led to economic models that lead to MFDEs in a more direct fashion. As an example, we mention the work of d'Albis and Véron [41, 39, 40], who have developed several models describing the dynamical features of an economy featuring only a single commodity, that exhibit oscillations which earlier models could only produce by including multiple commodities. This is accomplished by modelling the population as a continuum of individuals that each live for a finite time and act in such a way that their personal welfare is maximized. Such an approach leads in a natural fashion to a singular version of (1.1), where the derivative $x'(\xi)$ on the left hand side is replaced by zero. Such a model is described and analyzed in detail in Section 1.6, using theory that is developed in Chapter 3.

Chapter Overview

This introductory chapter is organized as follows. In Section 1.1 we discuss the connection between lattice differential equations and mixed type functional differential equations that is provided through the study of travelling waves. The main subject of this thesis is introduced in Section 1.2, where we discuss the important role that invariant manifolds play in the field of dynamical systems. We unfold in an informal manner how the construction of these objects proceeds in the setting of ordinary and delay differential equations. The difficulties that arise when lifting this framework to MFDEs are described in Section 1.3. In addition, we give an overview of the main results concerning invariant manifolds that are obtained in this thesis. These results are illustrated by four worked-out examples, which are presented in Sections 1.4 to 1.7. Throughout these sections, we comment on open problems and possible generalizations of our results. Finally, the numerical methods that were employed to solve the MFDEs arising in the examples are addressed in Section 1.8.

1.1. Travelling Waves in Lattice Systems

In this section we give a brief overview featuring recent developments in the study of lattice differential equations. We will focus particularly on issues related to the search for travelling wave solutions to LDEs, since these waves connect LDEs to functional differential equations of mixed type. It is common practice to distinguish two separate types of LDEs, based on the possibility of defining an energy-type quantity that is conserved over time. Systems that admit such an energy functional are called Hamiltonian, while the other class of LDEs is called dissipative.

Dissipative systems

Many lattice systems that have been considered in the literature can be captured by the following general form, which is posed here on the integer lattice $\mathbb{Z}^2$ for presentation purposes,

$$\dot{u}_{i,j} = \alpha\big((J * u)_{i,j}\big) - f(u_{i,j}, \rho), \quad (i, j) \in \mathbb{Z}^2. \tag{1.4}$$

Here $\alpha \in \mathbb{R}$, while $f : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ typically is a bistable nonlinearity of the form

$$f(u, \rho) = (u - \rho)(u^2 - 1), \tag{1.5}$$

for some parameter $-1 < \rho < 1$. The convolution J, which mixes the different lattice sites, is given by

$$(J * u)_{i,j} = \sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} J(l, m)[u_{i+l,j+m} - u_{i,j}], \tag{1.6}$$

with $\sum_{(l,m)\in\mathbb{Z}^2\setminus\{0\}} J(l,m) = 1$. Typically the support of the discrete kernel J is limited to close neighbours of $0 \in \mathbb{Z}^2$, but we specifically mention here the work of Bates [12], who analyzed a model incorporating infinite range interactions. In many applications, J represents a discrete version of the Laplacian operator. As an example, we introduce the nearest neighbour Laplacian Δ^+ that is defined by

$$(\Delta^+ u)_{i,j} = \frac{1}{4}\big[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\big]. \tag{1.7}$$

With this choice for J, (1.4) turns into the discrete Nagumo equation, given by

$$\dot{u}_{i,j} = \frac{\alpha}{4}\big[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\big] - f(u_{i,j}, \rho). \tag{1.8}$$

This equation is one of the most well-known examples of a lattice differential equation and it has served as a prototype for investigating the properties of dissipative LDEs.

Of course, many other choices for J are possible. In [53], the authors introduce the next-to-nearest neighbour Laplacian $\Delta^\times$ given by

$$(\Delta^\times u)_{i,j} = \frac{1}{4}\big[u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} + u_{i-1,j+1} - 4u_{i,j}\big] \tag{1.9}$$

and numerically study (1.4) with linear combinations of Δ^+ and $\Delta^\times$. In [13] Bates et al. show how an Ising spin model from material science leads to lattice equations (1.4) in which the coefficients of J corresponding to the shifted lattice sites may have both signs. In addition, the kernel J may even lose the natural point symmetry $J(i,j) = J(-i,-j)$.

The discrete Nagumo equation (1.4) with $\alpha = 4h^{-2}$ arises when one discretizes the two dimensional reaction diffusion equation,

$$u_t = \Delta u - f(u, \rho), \tag{1.10}$$

on a rectangular lattice with spacing h. In the analysis of the PDE (1.10), travelling wave solutions of the form $u(x,t) = \phi(k \cdot x - ct)$ have played a crucial role and thus have been studied extensively, starting with the classic work by Fife and McLeod [62]. The unit vector k indicates the propagation direction of the wave and c is the unknown wavespeed, which has to be determined along with the waveprofile ϕ. Following this approach, we can also study travelling wave solutions to equation (1.8). Substituting the travelling wave ansatz $u_{i,j}(t) = \phi(ik_1 + jk_2 - ct)$ into (1.8), we arrive at a mixed type functional differential equation of the form

$$-c\phi'(\xi) = \frac{\alpha}{4}\big(\phi(\xi+k_1)+\phi(\xi-k_1)+\phi(\xi+k_2)+\phi(\xi-k_2)-4\phi(\xi)\big) - f(\phi(\xi), \rho). \tag{1.11}$$

In [35] results are given concerning the asymptotic stability of travelling wave solutions to (1.8), showing that it is indeed worth while to study this class of solutions. The existence of heteroclinic solutions to (1.11) that connect the stable zeroes ± 1 of the nonlinearity f is established in [113].

Many authors have studied the discrete Nagumo equation and other similar LDEs [73, 111, 167, 170, 172]. It is by now well known that away from the continuous limit, i.e., for small positive values of α, the dynamical behaviour of (1.8) is quite different from that of its continuous counterpart (1.10). A feature which is immediately visible from (1.11) is the presence of lattice anisotropy, which means that the wavespeed c of a travelling wave solution to (1.4) depends on the vector of propagation through the lattice k. This is illustrated in Figure 1.2, where we set $k = (\cos\theta, \sin\theta)$ and give a plot of the wavespeed $c(\theta)$ for travelling wave solutions to the system

$$\begin{array}{rcl} \dot{u}_{i,j} & = & \left[\frac{3}{10}u_{i,j+1} + \frac{3}{10}u_{i,j-1} + \frac{1}{5}u_{i+1,j} + \frac{1}{5}u_{i-1,j} - u_{i,j}\right] + [\Delta^{\times} u]_{i,j} \\ & & -\frac{5}{2}(u_{i,j}^2 - 1)(u_{i,j} - \rho), \end{array} \tag{1.12}$$

that satisfy the limits

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1. \tag{1.13}$$

The results were obtained with the numerical method discussed in this thesis by adding a small term $-\gamma\phi''(\xi)$ to the left hand of (1.11), where $\gamma = 10^{-5}$. The polar plots clearly reflect the geometry of the vertically flattened lattice, especially for small values of the detuning parameter ρ. After substituting the travelling wave ansatz into the PDE (1.10), it is clear that this feature of lattice anisotropy vanishes in the continuous limit.

We have already briefly encountered the phenomenon of propagation failure, which also distinguishes lattice differential equations from their continuous counterparts. In the discrete case (1.11), a nontrivial interval of the detuning parameter ρ can exist in which the wavespeed satisfies $c = 0$. This means the waveform $\phi(\xi)$ does not propagate and thus the solution $u_{i,j}(t) = \phi(ik_1 + jk_2 - ct) = \phi(ik_1 + jk_2)$ to (1.4) remains constant in time. This behaviour does not occur for the reaction diffusion equation (1.10). This phenomenon has been studied extensively in [26], where one replaces the cubic nonlinearity f by an idealized nonlinearity to obtain explicit solutions to (1.11). For each propagation angle θ, the quantity $\rho^*(\theta)$ is defined to be the supremum of values $\rho > 0$ for which the wavespeed satisfies $c(\rho, \theta) = 0$. It is proven that this critical value $\rho^*(\theta)$ typically satisfies $\rho^* > 0$, depends continuously on θ when $\tan\theta$ is irrational and is discontinuous when $\tan\theta$ is rational or infinite. Numerical investigations in [53] and the present work suggest that the phenomenon of propagation failure is not just an artifact of the idealized nonlinearity f, but also occurs in the case of a cubic nonlinearity. This has recently been confirmed by Mallet-Paret in [114].

The early work by Chi, Bell and Hassard [32] already contained computations of solutions to lattice differential equations. This numerical work was continued by Elmer and Van Vleck, who have performed extensive calculations on equations of the form (1.11) in [1, 50, 51, 52, 53]. The occurrence of propagation failure presents serious difficulties for numerical schemes to solve (1.11), since solutions may lose their smoothness in the singular perturbation $c \to 0$. This difficulty can be overcome by introducing a term $-\gamma\phi''$ to the left hand side of (1.11) and using numerical continuation techniques to take the positive constant γ as small as possible. In Chapter 5 we shall analyze this approach from a theoretical viewpoint by establishing that this approximation still allows us to uncover part of the behaviour that occurs at $\gamma = 0$.

We conclude our discussion on dissipative LDEs by noting that these equations also occur naturally when studying numerical methods to solve PDEs. We have already seen how LDEs arise when discretizing PDEs. In order to understand the effects of the spatial

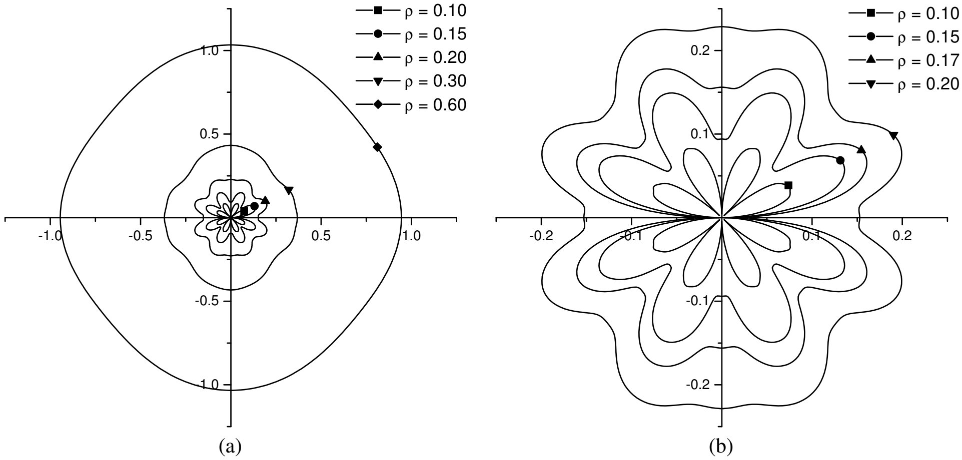


Figure 1.2: *A plot of the wavespeed $c(\theta)$ as a function of the propagation angle θ of travelling waves solutions to (1.12). Figure (b) is just a magnification of (a) to illustrate the behaviour for small values of the wavespeed c in greater detail.*

discretization scheme that any numerical PDE solver must employ one hence has to analyze the resulting LDE [54]. In this context we specially mention the work of Benzoni-Gavage et al. [19, 20, 21], where the numerical computation of shock waves is considered in the setting of LDEs and nonhyperbolic functional differential equations of mixed type are encountered.

Hamiltonian systems

In many physical systems one can define a conserved energy functional in terms of the state variables in a natural fashion. The FPU lattice is a very important example of such a Hamiltonian system. It was introduced by Fermi, Pasta and Ulam in 1955 as a model to describe the behaviour of a string [59]. Their work features the following one dimensional LDE,

$$\ddot{x}_j(t) = V'\big(x_{j+1}(t) - x_j(t)\big) + V'\big(x_j(t) - x_{j-1}(t)\big), \qquad j \in \mathbb{Z}, \tag{1.14}$$

in which the function V describes the interaction potential between neighbouring lattice sites. The harmonic situation where $V(z) \sim z^2$ describes an infinite chain of particles linked together by springs that all behave according to Hooke's law. In this ideal case (1.14) reduces to a linear system which admits a one dimensional family of periodic solutions $x_j(t) = \cos(\omega(k)t - kj)$, parametrized by $k \in \mathbb{R}$. These so-called monochromatic solutions do not interact and hence they can be superpositioned to construct arbitrary solutions of (1.14).

This linear setup is the starting point towards understanding vibrations in crystals. Elementary thermal and elastic properties can already be derived by analyzing the dispersion relation $\omega(k)$ [96]. However, in order to explain more advanced features such as the interplay between vibrations of separate frequencies or the temperature dependance of the elastic constants, it is essential to include higher order terms in the potential V. Models that involve nonlinear versions of the FPU system (1.14) and the very similar Klein-Gordon equation have already been used to describe crystal dislocations [97], localized excitations in ionic crystals [146] and even thermal denaturation of DNA [42].

The Hamiltonian structure of (1.14), together with the evident symmetries present in this equation, have stimulated and facilitated the mathematical analysis of the FPU lattice. To give a recent example, Guo et al. [132] used a Lyapunov-Schmidt reduction to show that generically, a family of small amplitude monochromatic solutions persists for the nonlinear problem (1.14). In addition, under an appropriate resonance condition, two sufficiently small monochromatic solutions that are exactly in or out of phase may be added together to yield a two-parameter family of small bichromatic solutions. These results can be seen in the spirit of the multiscale expansion approach [63, 127, 156], which postulates the existence of solutions to (1.14) of the form

$$x_j(t) = \epsilon A\big(\epsilon^2 t, \epsilon(j - ct)\big)\cos(\omega(k)t - kj) + \mathcal{O}(\epsilon^2). \tag{1.15}$$

It can be easily verified that the envelope function A must now satisfy the nonlinear Schrödinger equation, which has already been widely studied. Formally, the LDE (1.14) has thus been reduced to a PDE. However, this reduction has as yet only been made precise for finite time intervals [70].

Another successful technique that directly uses the Hamiltonian structure of (1.14), relies on the observation that any travelling wave solution $x_j(t) = \phi(j - ct)$ will necessarily be a critical point of the action functional

$$\mathcal{S}(\phi) := \int_{-\infty}^{\infty} \big[\frac{1}{2}c^2\phi'(\xi)^2 - V\big(\phi(\xi+1) - \phi(\xi)\big)\big]d\xi. \tag{1.16}$$

One can use so-called mountain pass methods to characterize the critical points of $\mathcal{S}$ and construct travelling wave solutions to (1.14). Results in this direction for a number of different monotonicity and growth conditions on the potential V can be found in [67, 123, 137, 150]. It is interesting to note that in Section 1.4 we take the exact opposite route, since we will look for the critical points of a similar functional by solving the MFDE that arises from the associated variational problem.

Iooss and Kirchgässner provided an additional important tool in [87], where a center manifold reduction for (1.14) is established. This has allowed for the construction of small amplitude solutions to (1.14) [85, 86, 88, 147, 148]. These results all rely on normal form theory to analyze the reversible system of ODEs that arises after performing the center reduction. We remark here that the techniques that the authors use in [87] to construct the center manifold are tailored specifically for the particular system (1.14) under consideration. By contrast, in Chapter 2 we develop a center manifold framework that holds for arbitrary systems of mixed type. This will allow the normal form computations discussed here to be performed in a far broader setting.

1.2. Classical Construction of Invariant Manifolds

Invariant manifolds have played a fundamental role in the theory of dynamical systems. They can be used to simplify the analysis of complex systems by considerably reducing the relevant dimensions. As we shall see, this may even involve the transformation of infinite dimensional problems into finite dimensional ones. Since invariant manifolds are often robust under modifications of system parameters, they play an important role when analyzing bifurcations and singular perturbations. As an example, Lin described [106] how the existence of multi-hump solutions of large period bifurcating from heteroclinic connections can be established by employing geometric arguments involving intersections between stable and unstable manifolds. In a similar spirit, Fenichel provided three important theorems [58] that facilitate the analysis of dynamics in systems that possess two different natural timescales. In particular, these theorems allow one to robustly link together the dynamics obtained by treating each timescale separately. This is done by exploiting the fact that under suitable conditions the so-called slow manifolds, which are invariant under the slow dynamics, persist when turning on the fast dynamics.

In this section we give a short introduction to the concepts of local stable, unstable and center manifolds. We briefly review how one can prove the existence of these structures when studying ordinary and delay differential equations. This overview will help to identify the issues that need to be resolved if one wishes to consider these manifolds in the context of functional differential equations of mixed type.

Ordinary Differential Equations

For simplicity, we start by considering the following nonlinear ordinary differential equation,

$$x'(\xi) = G(x(\xi)), \tag{1.17}$$

in which x is a $\mathbb{C}^n$-valued function. Let us suppose that the nonlinearity $G : \mathbb{C}^n \to \mathbb{C}^n$ is sufficiently smooth, so that for any initial state $w \in \mathbb{C}^n$ there exist constants $0 < \xi_+ \leq \infty$ and $-\infty \leq \xi_- < 0$ together with a unique solution $x_w : (\xi_-, \xi_+) \to \mathbb{C}^n$ that satisfies (1.17) on the interval (ξ_-, ξ_+), with $x(0) = w$. This allows us to define a flow $\Phi : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ that maps (w, ξ) to $x_w(\xi)$. Care has to be taken that this may not be defined for all $\xi \in \mathbb{R}$, but we will ignore this issue here.

In order to get a grasp of the behaviour of (1.17), an intuitive first step would be to divide the state space into parts that remain invariant under this flow Φ in some sense. In fact, smooth sets that have such a property are precisely the invariant manifolds in which we are interested. Consider first any point $\overline{x} \in \mathbb{C}^n$ for which $G(\overline{x}) = 0$. Such a point is called an equilibrium for (1.17) and obviously it is already an invariant manifold by itself. Two other important examples are given by the stable and unstable manifolds associated to $\overline{x}$, which are defined as

$$\begin{array}{rcl} W^s(\overline{x}) & = & \{w \in \mathbb{C}^n : \Phi(w, \xi) \to \overline{x} \text{ as } \xi \to \infty\}, \\ W^u(\overline{x}) & = & \{w \in \mathbb{C}^n : \Phi(w, \xi) \to \overline{x} \text{ as } \xi \to -\infty\}. \end{array} \tag{1.18}$$

In order to understand the structure of the stable and unstable manifolds near the equilibrium point $\overline{x}$, one needs to linearize (1.17) around this equilibrium. This yields the linear ODE

$$v'(\xi) = Lv(\xi) := DG(\overline{x})v(\xi), \tag{1.19}$$

in which L is an $n \times n$ matrix with complex coefficients. The analysis of (1.19) starts by looking for special solutions of the form

$$v(\xi) = \exp(z\xi)w, \tag{1.20}$$

with $z \in \mathbb{C}$ and $w \in \mathbb{C}^n$. Substitution of this Ansatz into (1.19) shows that z must satisfy the well-known characteristic equation $\Delta(z) = \det(zI - L) = 0$, implying that z is an eigenvalue for the matrix L. Since Δ is a polynomial of degree n, it can be factored as

$$\Delta(z) = \prod_{i=1}^{\ell}(z - \lambda_i)^{\alpha_i}, \tag{1.21}$$

in which each $\lambda_i \in \mathbb{C}$ is a distinct eigenvalue. The integers α_i are called the algebraic multiplicities of the eigenvalues λ_i. The Cayley-Hamilton theorem states that Δ is an annihilating polynomial for the matrix L, which means $\Delta(L) = 0$. One may show that there exists an annihilating polynomial $\mathcal{P} \neq 0$ that divides every other such polynomial. It is obvious that this minimal polynomial $\mathcal{P}$ must admit the following factorization,

$$\mathcal{P}(z) = \prod_{i=1}^{\ell}(z - \lambda_i)^{\nu_i}. \tag{1.22}$$

The integers ν_i are called the ascents of the eigenvalues λ_i and necessarily satisfy $1 \leq \nu_i \leq \alpha_i$. The well-known Jordan decomposition of the square matrix L gives us the following powerful decomposition of the state space,

$$\mathbb{C}^n = \bigoplus_{i=1}^{\ell}\mathcal{N}(\lambda_i I - L)^{\nu_i}. \tag{1.23}$$

The kernel $\mathcal{N}(\lambda_i I - L)^{\nu_i}$ is called the generalized eigenspace associated to λ_i. A useful basis for these eigenspaces can be constructed by computing so-called Jordan chains, which are sequences of n-dimensional vectors $w_0, \ldots, w_k$ that satisfy the relations $Lw_0 = \lambda w_0$ and $Lw_j = \lambda w_j + w_{j-1}$ for $1 \leq j \leq k$. The constituents of a Jordan chain are automatically linearly independent. In fact, the entire generalized eigenspace can be spanned by combining a number of such chains. To appreciate the power of this setup, let us solve (1.19) with the initial condition $v(0) = w_j$. One may easily verify that the unique solution is given by $v = v^j$, with

$$v^j(\xi) = e^{\lambda\xi}[w_j + \xi w_{j-1} + \ldots + \frac{\xi^j}{j!}w_0]. \tag{1.24}$$

The decomposition in (1.23) implies that this observation is sufficient to solve (1.19) with any initial condition. It is easy to check that (1.24) yields an efficient way to compute the matrix exponential $\exp(\xi L)$, using which the general solution of (1.19) can be compactly written as

$$v(\xi) = e^{\xi L} v(0). \tag{1.25}$$

The inhomogeneous linear ODE

$$v'(\xi) = Lv(\xi) + f(\xi) \tag{1.26}$$

can now be solved using the variation-of-constants formula, which yields

$$v(\xi) = \exp(\xi L)v(0) + \int_0^{\xi} e^{(\xi-\xi')L} f(\xi')d\xi'. \tag{1.27}$$

Assume for the moment that the linearization around $\overline{x}$ has no eigenvalues on the imaginary axis. In this case the equilibrium is called hyperbolic. Let us write $\mathbb{C}^n = X_+ \oplus X_-$, in which X_+ is the generalized eigenspace corresponding to the eigenvalues with positive real part and X_- is the generalized eigenspace corresponding to the eigenvalues with negative real part. Write $Q_\pm$ for the associated spectral projections from $\mathbb{C}^n$ onto $X_\pm$ and note that obviously $Q_+ + Q_- = I$.

Any initial condition $w_- \in X_-$ can be extended to an exponentially decaying solution of the linearized homogeneous equation (1.19) on the half-line $[0, \infty)$, namely $x(\xi) = e^{\xi L} w_-$. The analogous statement holds for initial conditions $w_+ \in X_+$, which can be extended to solutions on $(-\infty, 0]$. Such a splitting is called an exponential dichotomy and plays an important role when studying invariant manifolds, as we shall see.

Intuitively speaking, the behaviour of (1.17) near equilibria will be fully dominated by the linear part of the nonlinearity G. In view of this, one may hope that the stable manifold $W^s(\overline{x})$ will locally remain close to the hyperplane $\overline{x} + X_- \subset \mathbb{C}^n$. To shed some light on this issue, we introduce the Banach space $BC(\mathbb{R}_+, \mathbb{C}^n)$ as the set of bounded continuous $\mathbb{C}^n$-valued functions that are defined on the half-line $[0, \infty)$, equipped with the supremum norm. We also introduce a functional $\mathcal{G} : BC(\mathbb{R}_+, \mathbb{C}^n) \times X_- \to BC(\mathbb{R}_+, \mathbb{C}^n)$, defined by

$$\begin{array}{rcl} \mathcal{G}(u, w)(\xi) & = & e^{\xi L} w + \int_0^{\xi} e^{(\xi-\xi')L} Q_-[G\big(\overline{x} + u(\xi')\big) - Lu(\xi')]d\xi' \\ & & + \int_{\infty}^{\xi} e^{(\xi-\xi')L} Q_+[G\big(\overline{x} + u(\xi')\big) - Lu(\xi')]d\xi'. \end{array} \tag{1.28}$$

There are three key features to note regarding this definition. The first is that the splitting of the integral into a part acting on X_- in forward time and a part acting on X_+ in backward time ensures that $\mathcal{G}$ indeed maps into $BC(\mathbb{R}_+, \mathbb{C}^n)$ and not merely into $C(\mathbb{R}_+, \mathbb{C}^n)$. This is where the exponential dichotomy mentioned above comes into play. One may even show that $\mathcal{G}$ maps into the set of exponentially decaying functions. The second observation is that for all arguments $(u, w) \in BC(\mathbb{R}_+, \mathbb{C}^n) \times X_-$, we have $Q_- \mathcal{G}(u, w)(0) = w$. Finally, there is a bijection between solutions $u \in BC(\mathbb{R}_+, \mathbb{C}^n)$ to the fixed point equation $u = \mathcal{G}(u, Q_- u(0))$ and solutions $x \in BC(\mathbb{R}_+, \mathbb{C}^n)$ of the nonlinear equation (1.17), via the correspondance $x(\xi) = u(\xi) + \overline{x}$. To see this, consider any solution $x \in BC(\mathbb{R}_+, \mathbb{C}^n)$ to (1.17). Then $u = x - \overline{x}$ satisfies

$$u'(\xi) = G(x(\xi)) = Lu(\xi) + \big[G\big(u(\xi) + \overline{x}\big) - Lu(\xi)\big]. \tag{1.29}$$

However, by construction also $\mathcal{G}(u,0)$ is a solution to the above equation. This implies that $u = \mathcal{G}(u,0) + y$ for some solution $y \in BC(\mathbb{R}_+, \mathbb{C}^n)$ of the homogeneous equation (1.19). As a consequence, we must have $y(\xi) = e^{\xi L} Q_- y(0) = e^{\xi L} Q_- u(0)$, which implies $u = \mathcal{G}(u, Q_- u(0))$. Conversely, any solution of this fixed point equation satisfies (1.17) by construction.

We have hence reduced the description of the stable manifold $W^s(\overline{x})$ to the task of solving a nonlinear fixed point problem. Unfortunately, this is in general still an intractable procedure. However, notice that the nonlinearity in (1.28) is governed by the expression $G\big(\overline{x} + u(\xi')\big) - DG(\overline{x})u(\xi')$, which is of order $\mathcal{O}(u(\xi')^2)$ for small u. This key fact allows one to at least partially construct the stable manifold $W^s(\overline{x})$. More precisely, we will construct the local stable manifold $W^s_{\text{loc}}(\overline{x}) \subset W^s(\overline{x})$, which contains all $w \in W^s(\overline{x})$ with the additional property that $|\Phi(w,\xi) - \overline{x}| < \epsilon$ for all $\xi \geq 0$. Here $\epsilon > 0$ is a sufficiently small constant.

At the heart of this construction lies the observation that for all sufficiently small $w \in X_-$, the map $\mathcal{G}(\cdot, w)$ is a contraction on a closed and bounded subset of the space $BC(\mathbb{R}_+, \mathbb{C}^n)$. This implies that for all such $w \in X_-$, a unique fixpoint $u^*(w)$ has to exist, which then satisfies $u^*(w) = \mathcal{G}(u^*(w), w)$. This means that $u^*(w)$ solves (1.17) and decays exponentially. Hence $W^s_{\text{loc}}(\overline{x})$ can be written as a graph over the small ball $B_\delta(0) \subset X_-$ of radius $\delta > 0$ around $0 \in X_-$, by means of the map $w \mapsto \overline{x} + [u^*(w)](0)$. One may now easily observe from a Taylor expansion of (1.28) that $W^s_{\text{loc}}(\overline{x})$ is indeed tangent to the hyperplane $\overline{x} + X_-$.

The unstable manifold $W^u(\overline{x})$ can of course be analyzed in a similar fashion. However, when hyperbolicity is lost, i.e., when (1.19) admits eigenvalues on the imaginary axis, somewhat more care needs to be taken. Indeed, in this situation the qualitative large time behaviour of solutions may depend in a subtle fashion on the higher order terms in the Taylor expansion of G. A very powerful tool in this context is the center manifold reduction [30]. To describe this reduction, let us decompose the state space as

$$\mathbb{C}^n = X_- \oplus X_0 \oplus X_+, \tag{1.30}$$

in which $X_\pm$ are defined as before and X_0 is the generalized eigenspace associated to the eigenvalues on the imaginary axis. One may show that there exists a function $h : X_0 \to X_- \oplus X_+$, with $h(0) = 0$ and $Dh(0) = 0$, such that the dynamical behaviour of (1.17) in a sufficiently small neighbourhood of the equilibrium $\overline{x}$ is fully determined by the behaviour of the following ODE

$$y'(\xi) = L_{|X_0} y(\xi) + Q_0\Big[G\big(\overline{x} + y(\xi) + h(y(\xi))\big) - DG(\overline{x})\big(y(\xi) + h(y(\xi))\big)\Big]. \tag{1.31}$$

Notice that this ODE is defined on the subspace $X_0 \subset \mathbb{C}^n$. Stated more precisely, the center manifold theorem guarantees that there exists an $\epsilon > 0$ such that any solution x to (1.17) that has $|x(\xi)| < \epsilon$ for all $\xi \in \mathbb{R}$, yields a solution y to (1.31) via the correspondence $y(\xi) = Q_0[x(\xi) - \overline{x}]$. Conversely, if y satisfies (1.31) on an interval $\mathcal{I} \subset \mathbb{R}$ with $|y(\xi)| < \epsilon$ for all $\xi \in \mathcal{I}$, then the function x defined by $x(\xi) = y(\xi) + h(y(\xi)) + \overline{x}$ satisfies (1.17) on the interval $\mathcal{I}$.

The proof of this center reduction proceeds much along the lines of the procedure outlined above to obtain the local stable and unstable manifolds. There are however a number of additional technical complications that have to be addressed. These are all related to the fact that there may exist initial conditions $w \in X_0$ such that the function $e^{\xi L} w$ grows in a polynomial fashion as $|\xi| \to \infty$. One has to compensate for this possibility by working in exponentially weighted function spaces instead of $BC(\mathbb{R}, \mathbb{C}^n)$, which in turn causes problems when studying the smoothness of the center manifold.

To appreciate the true power of this center manifold reduction, we need to look at parameter dependent versions of (1.17). Let us therefore consider the extended system

$$\begin{cases} x'(\xi) &= G(x(\xi), \mu), \\ \mu' &= 0, \end{cases} \tag{1.32}$$

which should be seen as a version of (1.17) parametrized by a single parameter $\mu \in \mathbb{R}$. Let us suppose for simplicity that $x = 0$ is a parameter independent equilibrium value, which means $G(0, \mu) = 0$ for all $\mu \in \mathbb{R}$. The eigenvalues of the linearization

$$v'(\xi) = DG(0, \mu)v(\xi) \tag{1.33}$$

will now depend on μ. Generically speaking, we expect that eigenvalues will lie on the imaginary axis only at isolated values of μ. Furthermore, it is reasonable to expect that whenever (1.33) does in fact fail to be hyperbolic, there will only be a small number of purely imaginary eigenvalues.

In general, eigenvalues that cross through the imaginary axis as parameters are varied cause a change in the qualitative behaviour of (1.32). Such changes are referred to as bifurcations and their detection and classification play a fundamental role in the theory of dynamical systems [34]. By considerably reducing the dimension of the system that has to be analyzed, the computational and geometric analysis of bifurcations becomes a much more feasible task. In addition, since the physical dimension of the system under investigation becomes almost irrelevant, it becomes worthwhile to isolate commonly occurring bifurcation scenarios and give a standardized treatment for each. Such analyses are covered by the realm of normal form theory.

To give an example, suppose that the hyperbolicity of (1.33) is lost when $\mu = 0$, due to a complex conjugate pair of eigenvalues with algebraic multiplicity one that cross the imaginary axis. We can then construct a $2 + 1$ dimensional center manifold for the extended system (1.32) that captures the behaviour of sufficiently small solutions x to the first component of (1.32), for all small parameters μ. A detailed analysis of this low dimensional system shows that a branch of periodic solutions to (1.31) occurs either for $\mu > 0$ or $\mu < 0$, with amplitudes of order $\mathcal{O}(\sqrt{|\mu|})$ as $\mu \to 0$. These periodic solutions can then be lifted back to the full equation (1.33). A simple sign condition involving the second and third order derivatives of G determines whether these periodic orbits occur for μ positive or negative. This famous result is known as the Hopf bifurcation theorem and by now many generalizations to more complex root crossing scenarios have appeared in the literature.

Delay Differential Equations

Let us now turn our attention to delay equations. For simplicity, we will consider the following nonlinear equation with a single point delay,

$$x'(\xi) = G\big(x(\xi), x(\xi - 1)\big). \tag{1.34}$$

We will always assume that x is a continuous $\mathbb{C}^n$-valued function defined on some interval. In this context the state space X is given by $X = C([-1, 0], \mathbb{C}^n)$. We recall the notation $x_\xi(\theta) = x(\xi + \theta)$ for the state $x_\xi \in X$ of x at ξ. The linearization around an equilibrium $\overline{x}$ is given by

$$v'(\xi) = Lv_\xi := D_1G(\overline{x})v(\xi) + D_2G(\overline{x})v(\xi - 1). \tag{1.35}$$

We again look for solutions to (1.35) of the form $v(\xi) = w \exp(\xi z)$, with $w \in \mathbb{C}^n$. As before, one must have $\Delta(z)w = 0$, but the characteristic matrix Δ is now given by

$$\Delta(z) = zI - D_1G(\overline{x}) - D_2G(\overline{x})e^{-z}. \tag{1.36}$$

Due to the presence of the exponential in (1.36), there will in general be an infinite number of roots to the characteristic equation $\det \Delta(z) = 0$. However, it is still the case that one can capture all the roots to the left of a vertical line in the complex plane, i.e., there exists a number $\gamma_+ \in \mathbb{R}$ such that all roots z have $\operatorname{Re} z < \gamma_+$. Another important observation is that the number of eigenvalues in a vertical strip is finite, i.e., for any pair of reals $\nu_- < \nu_+$, there are only finitely many z with $\nu_- < \operatorname{Re} z < \nu_+$ and $\det \Delta(z) = 0$.

Similarly as in the ODE case, for any root z of the characteristic equation $\det \Delta(z) = 0$ one may compute a Jordan basis for the null space $\mathcal{N}\big(\Delta(z)\big)$ and use the expression (1.24) to construct solutions to the homogeneous equation (1.35). These solutions have the form $v(\xi) = p(\xi)e^{z\xi}$ for polynomials p and are called eigensolutions to (1.35) for the eigenvalue z. Let us write $\mathcal{V} \subset C([-1, 0], \mathbb{C}^n)$ for the span of all these eigensolutions, ranging over all eigenvalues z. There are two important questions that now arise naturally. The first is if any initial condition $\phi \in C([-1, 0], \mathbb{C}^n)$ can be written in terms of these eigensolutions. Stated more precisely, do we have the identity $\overline{\mathcal{V}} = C([-1, 0], \mathbb{C}^n)$. The second question concerns the construction of the natural projections $Q_\pm$, which map initial conditions onto parts that can be extended to bounded solutions of (1.35) on the half-lines $\mathbb{R}_\pm$.

To answer these questions (1.35) needs to be embedded into a more abstract framework. To prepare for this, let us first consider an arbitrary $\phi \in C([-1, 0], \mathbb{C}^n)$ and attempt to solve (1.35) on the interval $[0, 1]$, with the initial condition $x_0 = \phi$. One easily sees that this is equivalent to solving the following initial value problem on $[0, 1]$,

$$\begin{cases} v'(\xi) &= D_1G(\overline{x})v(\xi) + D_2G(\overline{x})\phi(\xi - 1), \\ v(0) &= \phi(0). \end{cases} \tag{1.37}$$

This can be readily solved to yield

$$v_\phi(\xi) = \exp[D_1G(\overline{x})\xi]\phi(0) + \int_0^\xi \exp[D_1G(\overline{x})(\xi - \xi')]D_2G(\overline{x})\phi(\xi' - 1)d\xi' \tag{1.38}$$

for all $0 \leq \xi \leq 1$. This construction can then be repeated on the interval $[1, 2]$. Continuing in this fashion, one can compute a solution v_ϕ to (1.35) on the entire half-line $\mathbb{R}_+$. This procedure has adequately been named the method of steps [45].

We now proceed by considering the setup above from a more abstract point of view. For every $\xi \geq 0$ we have constructed a linear operator $T(\xi) : X \to X$ that maps an initial condition $\phi \in X$ to the state $(v_\phi)_\xi \in X$. Put in other words, we have $v_\phi(\xi) = [T(\xi)\phi](0)$ for all $\xi \geq 0$. One may easily verify that $T(0) = I$ and $T(\xi + \xi') = T(\xi)T(\xi')$ for all positive ξ and ξ', which means that T is in fact a semigroup.

The operators $T(\xi)$ should be seen as the generalization of the matrix exponentials $\exp[L\xi]$ encountered above in the ODE setting. A major difference however is that $T(\xi)$ is only defined for positive ξ. Indeed, any initial condition $\phi \in X$ that is not differentiable can never be extended in backward time. Notice that in the ODE setting, the matrix L can be recovered from the matrix exponential by means of the limit $Lw = \lim_{h \to 0} h^{-1}[\exp(Lh)w - w]$, which exists for all $w \in \mathbb{C}^n$. Inspired by this, the generator A of the semigroup T is defined by setting $A\phi = \lim_{h \downarrow 0} h^{-1}[T(h)\phi - \phi]$, for all $\phi \in X$ for which this limit exists. In an infinite dimensional setting this last existence condition becomes important and in general, the domain $\mathcal{D}(A)$ is indeed a proper subset of the state space X. Using our solution (1.38) it is not hard to explicitly calculate the generator $A : \mathcal{D}(A) \subset X \to X$. It is closed, densely defined and given by

$$\begin{array}{lcl} A\phi & = & \phi', \\ \mathcal{D}(A) & = & \left\{\phi \in C^1([-1, 0], \mathbb{C}^n) \mid \phi'(0) = L\phi = D_1G(\overline{x})\phi(0) + D_2G(\overline{x})\phi(-1)\right\}. \end{array} \tag{1.39}$$

Any $z \in \mathbb{C}$ for which $zI - A : \mathcal{D}(A) \to X$ is bijective is said to be in the resolvent set of A, denoted by $\rho(A)$. Due the closed graph theorem, for any $z \in \rho(A)$ the inverse $(zI - A)^{-1}$, called the resolvent of A at z, is automatically bounded. The spectrum of A is defined as the complement of the resolvent set, i.e., $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

As can be inferred by the ODE case, the properties of the semigroup T are intimately connected to those of the generator A. An important example of such a connection is related to the splitting of the state space X into parts that are invariant under the action of the semigroup T. Let us suppose that $\lambda \in \sigma(A)$ is an isolated pole of finite order m for the function $z \mapsto (zI - A)^{-1}$. Let us also define $\mathcal{M}_\lambda(A) = \mathcal{N}\big((\lambda I - A)^m\big)$ and $\mathcal{R}_\lambda(A) = \mathcal{R}\big((\lambda I - A)^m\big)$. A standard result [45, Theorem IV.2.5] states that the spaces thus defined are maximal, in the sense that $\mathcal{N}\big((\lambda I - A)^k\big) = \mathcal{M}_\lambda(A)$ and $\mathcal{R}\big((\lambda I - A)^k\big) = \mathcal{R}_\lambda(A)$ hold for the kernel and range of $(\lambda I - A)^k$ for all larger integers $k \geq m$. Furthermore, both $\mathcal{M}_\lambda(A)$ and $\mathcal{R}_\lambda(A)$ are closed and invariant under the action of the semigroup T. Finally, one has the spectral decomposition

$$X = \mathcal{M}_\lambda(A) \oplus \mathcal{R}_\lambda(A), \tag{1.40}$$

with a corresponding spectral projection Q_λ onto $\mathcal{M}_\lambda(A)$ that is given by the Dunford integral

$$Q_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (zI - A)^{-1} dz. \tag{1.41}$$

Here the contour $\Gamma_\lambda \subset \rho(A)$ is a small circle around λ that contains no other points of $\sigma(A)$ in its interior.

For the delay equation (1.35) under consideration, the resolvent $(zI - A)^{-1} : X \to X$ may be explicitly computed to be

$$\begin{array}{rcl} [(zI - A)^{-1}\phi](\theta) & = & e^{z\theta}\big(\int_\theta^0 e^{-z\sigma}\phi(\sigma)d\sigma \\ & & +\Delta(z)^{-1}\big[\phi(0) + D_2G(\overline{x})e^{-z}\int_{-1}^0 e^{-z\sigma}\phi(\sigma)d\sigma\big]\big). \end{array} \tag{1.42}$$

This expression provides the link between the eigensolutions that we constructed in an ad-hoc fashion using the characteristic matrix Δ and the abstract spectral theory outlined above for the generator A. In particularly, using the theory for characteristic matrices developed in [89], one can show that the structure and dimension of the generalized eigenspace $\mathcal{M}_\lambda(A)$ is completely determined by the structure of the nullspace $\mathcal{N}(\Delta(\lambda))$, via the expression (1.24). In addition, the spectral projection operators Q_λ can be expressed in terms of residues at $z = \lambda$ of terms involving the matrix-inverse $\Delta(z)^{-1}$. For example, if $z = \lambda$ is a simple root of the characteristic equation $\det \Delta(z) = 0$, there exists a matrix-valued function H that is analytic at $z = \lambda$, such that

$$\Delta(z)^{-1} = (z - \lambda)^{-1}H(z). \tag{1.43}$$

Using this identity, the spectral projection (1.41) reduces to

$$[Q_\lambda\phi](\theta) = e^{\lambda\theta}H(\lambda)\big[\phi(0) + DG_2e^{-\lambda}\int_{-1}^0 e^{-z\sigma}\phi(\sigma)d\sigma\big]. \tag{1.44}$$

Further examples and a description of a systematic procedure for the explicit computation of spectral projections in a general setting can be found in [65].

It remains to address the question whether the set of all generalized eigensolutions is complete. Results in this direction were first given in [160]. It is possible to establish the following decomposition for the state space,

$$X = \overline{\overline{\mathcal{V}} \oplus \mathcal{S}}, \tag{1.45}$$

in which the subspace $\mathcal{S}$ is related to the resolvent of A, via

$$\mathcal{S} = \left\{\phi \in X \mid z \mapsto (zI - A)^{-1}\phi \text{ is entire}\right\}. \tag{1.46}$$

Initial conditions in $\mathcal{S}$ are related to solutions to (1.35) that become identically zero after a finite time. The existence of these so-called small solutions can be verified directly from (1.35). Indeed, one may show that $\mathcal{S}$ is empty if and only if $\det D_2G(\overline{x}) \neq 0$.

Now that the linear homogeneous equation (1.35) has been addressed, we need to move on to the inhomogeneous problem

$$\left\{ \begin{array}{rcl} v'(\xi) & = & Lv_\xi + f(\xi), \\ v_0 & = & \phi. \end{array} \right. \tag{1.47}$$

We note first that in principle the method of steps can again be used to find a solution. However, we are interested here in the construction of invariant manifolds as described above for ODEs. The task at hand is to generalize the variation-of-constants formula (1.27) that was developed for ODEs, now using the semigroup T instead of the matrix exponential $\exp[\xi L]$. Even at first glance one already runs into serious difficulties. Indeed, one needs to compensate for the fact that the inhomogeneity f no longer maps into the state space X, but merely into $\mathbb{C}^n$.

This difficulty has been overcome by the introduction of sun-star calculus. In particular, instead of looking at X we turn our attention to an extended state space $X^{\odot\star} = \mathbb{C}^n \times L^\infty([r_{\min}, 0], \mathbb{C}^n)$. Note that $X \hookrightarrow X^{\odot\star}$ via the natural inclusion $\phi \mapsto (\phi(0), \phi)$. By the nature of the construction, the semigroup T generalizes to a weak-* continuous semigroup $T^{\odot\star}$ on $X^{\odot\star}$. The inhomogeneity f can be naturally extended to a map $F : \mathbb{R} \to X^{\odot\star}$ via $F(\xi) = (f(\xi), 0)$. This leads to the variation-of-constants formula

$$v_\xi = T(\xi - \xi_0)\phi + \int_{\xi_0}^{\xi} T^{\odot\star}(\xi - \xi')F(\xi')d\xi'. \tag{1.48}$$

One may show that this is well-defined in the sense that the righthand side is again an element in the original state space X and that v thus defined satisfies the initial value problem (1.47).

With the development of this variation-of-constants formula the road towards invariant manifold theory for delay equations was opened up. The construction of the stable and unstable manifolds proceeds quite similarly as in the ODE case outlined above. When studying the center manifold, extra precautions need to be taken due to the fact that smooth cut-off functions are generally unavailable in an infinite dimensional setting.

1.3. Invariant Manifolds for MFDEs

We are now ready to move on to functional differential equations of mixed type. For simplicity, we will consider the following nonlinear equation with two shifted arguments,

$$x'(\xi) = G\big(x(\xi), x(\xi - 1), x(\xi + 1)\big). \tag{1.49}$$

We will again assume that x is a continuous $\mathbb{C}^n$-valued function defined on some appropriate interval. The state space X is now given by $X = C([-1, 1], \mathbb{C}^n)$ and the linearization around an equilibrium $\overline{x}$ can be written as

$$v'(\xi) = Lv_\xi := D_1G(\overline{x})v(\xi) + D_2G(\overline{x})v(\xi - 1) + D_3G(\overline{x})v(\xi + 1). \tag{1.50}$$

To start the discussion, we consider a simple illustrative example, due to Härterich et al. [75]. Consider the linear homogeneous MFDE

$$v'(\xi) = v(\xi + 1) - v(\xi - 1), \tag{1.51}$$

with the initial condition $v_0 = 1$. One easily sees that for all $\xi \in (1, 3)$ one must have $v(\xi) = -1$, which in turn implies that $v(\xi) = 1$ for all $\xi \in (3, 5)$. The continuity of the initial condition is hence lost. In other words, (1.50) with an initial condition $v_0 = \phi$ is ill-posed as an initial value problem.

The construction of the semigroup T for delay equations thus breaks done and cannot be repaired. This fact already exhibits itself when analyzing the characteristic matrix

$$\Delta(z) = zI - D_1G(\overline{x}) - D_2G(\overline{x})e^{-z} - D_3G(\overline{x})e^{z}. \tag{1.52}$$

The presence of the two exponential functions with oppositive signs in their arguments will in general cause the characteristic equation $\det \Delta(z) = 0$ to have an infinite set of roots both to the right and left of the imaginary axis. In contrast, the spectrum of semigroup generators can always be contained in a left half plane. The asymptotic location of eigenvalues for (1.50) was analyzed in early work by Bellman and Cooke [16]. An important observation is that any vertical strip $\{z \in \mathbb{C} : \nu_- < \operatorname{Re} z < \nu_+\}$ contains only finitely many roots.

Although it is now no longer the generator of a semigroup, one can still study the operator A that was defined for delay equations by (1.39). In this fashion one can still obtain spectral decompositions of the state space X and the associated spectral projections. However, the relation between the spectral splittings thus obtained and the dynamics of (1.50) is now no longer immediately clear.

The ill-posedness of the initial value problem (1.50) has long limited our understanding of the full nonlinear system (1.49). It was only during the last decade that significant theoretical progress has been made. An important step in this respect was the construction of exponential dichotomies for (1.50). One can show that there exists a closed subspace $P_{\leftarrow} \subset X$ such that for every $\phi \in P_{\leftarrow}$, there exists a function x_ϕ defined on $(-\infty, 1]$, with $x_0 = \phi$, that satisfies (1.50) on the interval $(-\infty, 0]$. Similarly, there exists a closed subspace $P_{\rightarrow} \subset X$ such that every $\phi \in P_{\rightarrow}$ can be extended to a solution of (1.50) on $[0, \infty)$. More importantly, one has the decomposition

$$X = P_{\rightarrow} \oplus P_{\leftarrow}. \tag{1.53}$$

A first result in this direction was obtained in 1989 by Rustichini [130]. Only very recently however, the existence of exponential dichotomies was established for non-autonomous versions of (1.50). These results were obtained independently and simultaneously by Mallet-Paret et al. [115] and Härterich et al. [75].

Another important result concerns the linearization of (1.54) around heteroclinic orbits. In particular, consider any solution x that satisfy the limits $\lim_{\xi\to\pm\infty} x(\xi) = \overline{x}_\pm$, in which both $\overline{x}_\pm \in \mathbb{C}^n$ are equilibria for the nonlinearity G. Linearizing (1.49) around this solution yields $\Lambda v = 0$, in which the linear operator $\Lambda : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ is given by

$$[\Lambda v](\xi) = v'(\xi) - D_1G(x_\xi)v(\xi) - D_2G(x_\xi)v(\xi - 1) - D_3G(x_\xi)v(\xi + 1). \tag{1.54}$$

Notice that since x is a heteroclinic orbit, one can define the limiting operators $A_j^\pm = \lim_{\xi\to\pm\infty} D_jG(x_\xi)$ for $0 \le j \le 2$ and consider the limiting systems

$$v'(\xi) = L^\pm v_\xi = \sum_{j=0}^{2} A_j^\pm v(\xi + r_j). \tag{1.55}$$

When both limiting systems do not have eigenvalues on the imaginary axis, we say that Λ is asymptotically hyperbolic. Assuming this condition, Mallet-Paret was able to establish Fredholm properties for the operator Λ [112]. In particular, the kernel $\mathcal{K}(\Lambda) \subset W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ is finite dimensional, while the range $\mathcal{R}(\Lambda) \subset L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ is closed and has finite codimension. In addition, one has the following useful characterization for the range of Λ,

$$\mathcal{R}(\Lambda) = \left\{ f \in L^{\infty}(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} x^*(\xi) f(\xi) = 0 \text{ for all } x \in \mathcal{K}(\Lambda^*) \right\}, \tag{1.56}$$

in which the adjoint $\Lambda^* : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ is given by

$$[\Lambda^* v](\xi) = v'(\xi) + D_1 G(x_\xi)^* v(\xi) + D_2 G(x_{\xi+1}) v(\xi + 1) + D_3 G(x_{\xi-1}) v(\xi - 1). \tag{1.57}$$

In the special case that x is constant, the operator Λ is an isomorphism and a Greens function G exists such that for any $f \in L^{\infty}(\mathbb{R}, \mathbb{C}^n)$, the equation $\Lambda v = f$ is solved by the function

$$v(\xi) = \int_{-\infty}^{\infty} G(\xi - \xi') f(\xi') d\xi'. \tag{1.58}$$

The linear theory outlined above can be seen as the appropriate generalization of the concepts discussed previously for ordinary and delay differential equations. However, it has not been firmly established how these results can be lifted to a nonlinear setting. In particular, it is unclear whether a version of the sun-star calculus can be developed that allows for the natural embedding of $\mathbb{C}^n$-valued nonlinearities into the state space X.

The approach followed in this thesis circumvents this difficulty by rewriting the relevant fixed point equations discussed in the previous section in such a way, that only the inverse Λ^{-1} is needed. Of course, care must be taken when choosing the correct function spaces on which to define the problem, in order to ensure that this inverse does indeed exist. In addition, we use Laplace transform techniques to obtain the connection between the spectral properties of the operator A discussed above and solutions to the homogeneous equation $\Lambda v = 0$.

In Chapter 2 we combine this technique with a fibred contraction argument due to Vanderbauwhede and Van Gils [159]. In particular, we show how a finite dimensional center manifold can be constructed that captures the dynamics of (1.49) near an equilibrium $\overline{x}$ for G. We illustrate how one can explicitly calculate a Taylor expansion of the resulting ODE (1.31) up to any desired order. In particular, we provide explicit conditions, directly in terms of derivatives of G, under which super and subcritical Hopf bifurcations arise. In Section 1.4 we show how easily these conditions can be explicitly verified in practice. This yields the first result for the occurrence of Hopf bifurcations for general systems of the form (1.1).

In view of the economic modelling problems that are described in the sequel, it is important to extend these results to a slight variant of (1.49). In particular, we need to understand the following algebraic problem of mixed type,

$$0 = F(x_\xi), \tag{1.59}$$

under a smoothness condition that ensures that any solution x to (1.59) automatically satisfies an accompanying differential equation (1.49). We attack this problem in Chapter 3.

In particular, we show that the smoothness condition is sufficient to ensure that (1.59) can still be reduced to an ODE, upon restricting the equation to a small neighbourhood around an equilibrium. Again, one can explicitly compute the Taylor expansion of this ODE up to arbitrary order and we give an example of such a computation in Section 1.6.

One can use the results discussed above to establish the existence of periodic solutions to (1.49) under suitable conditions. Of course, the next interesting question that arises is whether the dynamics around these periodic solutions can be analyzed in a similar fashion. This brings us into the realm of Floquet theory. To set the stage, let us return briefly to the ODE (1.17), under the assumption that this equation admits a 2π-periodic solution p. The Poincaré return map is the key mathematical structure that is used to analyze the dynamics of (1.17) in a neighbourhood of p. It is constructed by fixing a hyperplane $\mathcal{H}$ that contains $p(0)$ and is transversal to $Dp(0)$. Any $w \in \mathcal{H}$ is then mapped to the first subsequent intersection of the orbit through w with this hyperplane $\mathcal{H}$, which is well-defined for all w sufficiently close to $p(0)$. In this fashion, the dynamics in a neighbourhood of the orbit p can be captured by a discrete dynamical system.

The analysis of the Poincaré return-map proceeds by considering the linearization of the ODE (1.17) around p, yielding

$$v'(\xi) = L(\xi)v(\xi) := DG(p_\xi)v(\xi) \tag{1.60}$$

in which the periodic matrix-valued operator $L : \mathbb{R} \to \mathbb{C}^{n\times n}$ shares the period of p. One now analyzes the monodromy map $T_{2\pi} : X \to X$ associated to (1.60). This operator maps initial conditions $w \in \mathbb{C}^n$ onto the translate $v_w(2\pi) \in \mathbb{C}^n$, where v_w solves (1.60) with $v(0) = w$. Eigenvalues of the monodromy map are called Floquet multipliers, while the corresponding solutions to (1.60) are called Floquet solutions. As can be expected, the stability properties of the Poincaré return map depend heavily on the location of this Floquet spectrum. The presence of Floquet multipliers on the unit circle is of particular interest when one is looking for bifurcations that affect a branch of periodic solutions.

The ill-posedness of (1.49) implies that we need to develop a different line of attack when studying MFDEs, since both the Poincaré map and the monodromy operator cannot be directly defined. These issues are explored in Chapter 4, where we provide a center manifold framework that can be used to capture the dynamics of (1.49) near periodic solutions. The techniques used in this chapter were chiefly inspired by those used by Mielke [119], who was also faced with the absence of a semiflow in his work on elliptic PDEs.

The key point is that one can no longer search directly for Floquet multipliers by studying the monodromy operator. However, it is still possible to look for solutions to the linearization

$$v'(\xi) = L(\xi)v_\xi := DG_1(p_\xi)v(\xi) + DG_2(p_\xi)v(\xi - 1) + DG_3(p_\xi)v(\xi + 1), \tag{1.61}$$

that have the special Floquet form $v(\xi) = e^{\lambda\xi}q(\xi)$. Here p is a periodic solution to (1.49), $\lambda \in \mathbb{C}$ is the Floquet exponent and q a continuous periodic function that has the same period as p. In Sections 1.7 and 1.8 we shall see how this feature affects the numerical computations of Floquet multipliers.

Finally, in Chapter 6 we discuss a second mechanism via which periodic orbits can appear. In particular, we use exponential dichotomies to lift Lin's method [106] to the setting of MFDEs. This allows us to study bifurcations from homoclinic solutions to nonlinear MFDEs that depend upon parameters. The so-called ‘blue sky catastrophe’ is an example of such a bifurcation. It describes how a branch of periodic solutions, that have increasingly large period, may suddenly disappear and turn into a homoclinic solution, as the parameters of the system are varied. Many interesting results have been obtained concerning homoclinic bifurcations in ODE settings. In Chapter 6 we show how our framework can be used to translate some of these results to our infinite dimensional system. We explicitly describe the orbit-flip bifurcation, which can lead to very complicated behaviour in the neighbourhood of a homoclinic orbit [134]. In particular, we give conditions for the existence of symbolic dynamics for MFDEs.

1.4. Optimal Control Capital Market Dynamics

Following Rustichini [131] and Benhabib et al. [17], we consider a simple model for the dynamics of a capital market. The economy starts at time $t = 0$ and has a fixed population of unit size. Exactly $n + 1$ distinct types of products are produced. One of these can only be used for consumption purposes while the other n are capital goods that facilitate the manufacture of the consumption good.

We will write $k(t) \in \mathbb{R}^n$ for the total amount of capital goods that are available at time t. By $u(t) \in \mathbb{R}^n$ we will denote the total investments in capital. In order to model effects such as decay and maintenance costs, we introduce a positive constant g as the capital decay rate and obtain the following relation between investments and capital,

$$\dot{k}(t) = u(t) - gk(t). \tag{1.62}$$

At any given moment, the total consumption C naturally depends upon the available production facilities, viz the total amount of capital goods k. Since any investment in capital goods will necessarily divert resources from the production of the consumption good, the consumption C also depends upon the investments u. We thus write $C = C(u, k)$. The welfare $W(t)$ of the society in our model at any given time t is related to the total amount of consumption via $W(t) = \ln C(t)$.

The dynamic behaviour of the capital market can be found by maximizing the functional

$$J(k, \dot{k}) = \int_0^\infty e^{-\rho t} \ln C\big(u(t), k(t)\big) = \int_0^\infty e^{-\rho t} \ln C\big(\dot{k}(t) + gk(t), k(t)\big). \tag{1.63}$$

The parameter $\rho > 0$ represents the discount rate, quantifying the fact that welfare in the present is rated higher than its counterpart in the future.

The Euler-Lagrange equations provide the classic tool for solving (1.63) and lead to the second order ODE

$$\begin{aligned} &e^{-\rho t}[C\big(\dot{k}(t) + gk(t), k(t)\big)]^{-1}[gD_1C\big(\dot{k}(t) + gk(t), k(t)\big) + D_2C\big(\dot{k}(t) + gk(t), k(t)\big)] \\ &= \tfrac{d}{dt}\big[e^{-\rho t}[C\big(\dot{k}(t) + gk(t), k(t)\big)]^{-1} D_1C\big(\dot{k}(t) + gk(t), k(t)\big)\big]. \end{aligned} \tag{1.64}$$

In [17, 18, 131] it is argued that, under some weak assumptions on the function C, periodic solutions to (1.64) are in fact optimal solutions to (1.63). This type of solution is hence particularly interesting from an economic point of view.

It still remains to specify the consumer function $C(u, k)$. Following [17], it can be found by solving the optimization problem

$$
\begin{aligned}
C(u,k) = \max \big\{ Y_c(L^c, K^c) \quad | \quad & Y_u(L^u, K^u) = u, \\
& L^c + \textstyle\sum_{i=1}^n L_i^u = 1, \quad L^c \geq 0, \\
& L_i^u \geq 0, \qquad\qquad\quad 1 \leq i \leq n, \\
& K^c + \textstyle\sum_{i=1}^n K_i^u = k, \quad K^c \geq 0, \\
& K_i^u \geq 0, \qquad\qquad\quad 1 \leq i \leq n \big\}.
\end{aligned} \tag{1.65}
$$

We take Y_c and Y_u to be standard Cobb-Douglas production functions [47], which have the form

$$Y(L, K) = \beta L^{\alpha_L} K^{\alpha_K} \tag{1.66}$$

for positive parameters α_L, α_K and β. The conditions in the maximization problem (1.65) reflect the fact that the total available labour $l = 1$ and capital k have to be divided over the production of the consumption good and capital goods in such a way, that the output of the consumption good is maximal, while the capital good production matches the investments u.

Benhabib et al. found periodic solutions to (1.63) in markets that contain more than one production good, i.e., where $n > 1$ [17]. Rustichini however fixed $n = 1$, but introduced time delays into the problem to account for the fact that both the investment activity and the production of the consumption good take time [131]. In particular, (1.63) is modified to take the form

$$
\begin{array}{rcl}
J(k, \dot{k}) & = & \int_0^\infty e^{-\rho t} \ln C\big(u(t-\tau), k(t-\sigma)\big), \\
\dot{k}(t) & = & u(t-\tau) - gk(t-\tau).
\end{array} \tag{1.67}
$$

In Hughes [81] the Euler-Lagrange equations were generalized to problems that contain delays. In particular, the functional

$$J(y) = \int_a^b f\big(t, y(t-\tau), y(t), \dot{y}(t-\tau), \dot{y}(t)\big) dt \tag{1.68}$$

was considered, in which y ranges over the class of piecewise smooth solutions on the interval $[a - \tau, b]$ that satisfy an initial condition $y(s) = \alpha(s)$ for $s \in [a - \tau, a]$ and a boundary condition $y(b) = \beta$. Solutions that maximize J were shown to satisfy the Euler-Lagrange equation

$$
\begin{array}{rcl}
D_3 f(t, x, y, \dot{x}, \dot{y}) + D_2 f(t+\tau, y, z, \dot{y}, \dot{z}) & = & \frac{d}{dt}[D_5 f(t, x, y, \dot{x}, \dot{y}) \\
& & +D_4 f(t+\tau, y, z, \dot{y}, \dot{z})]
\end{array} \tag{1.69}
$$

on the interval $[a, b - \tau]$, in which $x(\cdot) = y(\cdot - \tau)$ and $z(\cdot) = y(\cdot + \tau)$.

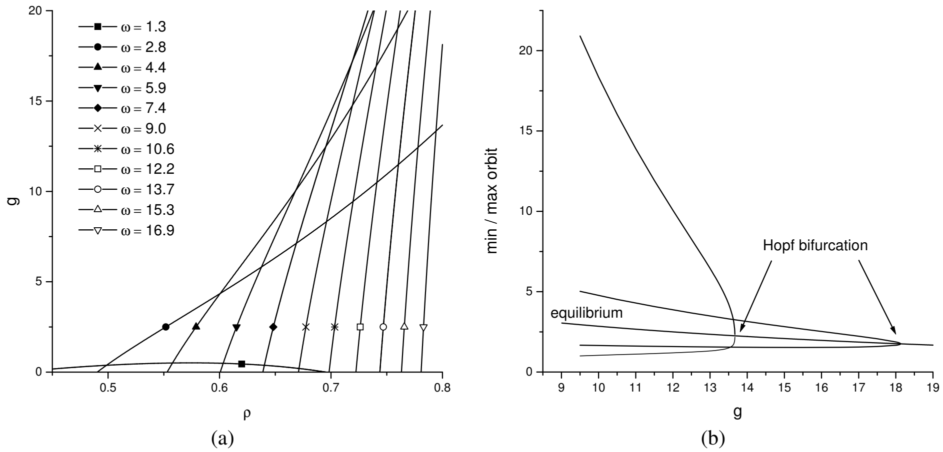


Figure 1.3: *In (a) codimension one families of Hopf bifurcation points have been plotted for (1.70), while (b) is a bifurcation diagram showing two subcritical Hopf bifurcations that the equilibrium undergoes as the parameter g is varied, for fixed* $\rho = 0.80$. *The bifurcating branches were computed using the collocation based MFDE solver discussed in Section 1.8. The remaining parameters were fixed at* $\sigma = \tau = 4.0$.

Generalizing the above equation to cover multiple delays and applying it to (1.67) on the interval $[0, \infty)$, one sees that optimal capital dynamics satisfies the MFDE

$$\begin{array}{l} g e^{-\rho(t+\tau)}[C\big(\dot{k}(t+\tau) + gk(t), k(t+\tau-\sigma)\big)]^{-1} \\ \quad D_1 C\big(\dot{k}(t+\tau) + gk(t), k(t+\tau-\sigma)\big) \\ + e^{-\rho(t+\sigma)}[C\big(\dot{k}(t+\sigma) + gk(t+\sigma-\tau), k(t)\big)]^{-1} \\ \quad D_2 C\big(\dot{k}(t+\sigma) + gk(t+\sigma-\tau), k(t)\big) \\ = \frac{d}{dt}\big[e^{-\rho t}[C\big(\dot{k}(t) + gk(t-\tau), k(t-\sigma)\big)]^{-1} D_1 C\big(\dot{k}(t) + gk(t-\tau), k(t-\sigma)\big)\big]. \end{array} \tag{1.70}$$

We again are interested in periodic solutions to (1.70) [131]. Rustichini showed analytically that under a number of limiting approximations and assumptions the characteristic equation associated to (1.70) will have roots on the imaginary axis, indicating the occurrence of periodic solutions. We will extend this analysis and give an explicit example of a model that exhibits a Hopf bifurcation and hence admits periodic solutions close to an equilibrium. In addition, we will actually compute and exhibit these solutions.

Before we can proceed, we still need to calculate the consumer function C. To simplify our analysis, we choose

$$\begin{array}{rcl} Y_c(L_c, K_c) & = & \sqrt{L_c K_c}, \\ Y_u(L_u, K_u) & = & \sqrt{L_u K_u}. \end{array} \tag{1.71}$$

Notice that the problem (1.65) is only feasible if $u < \sqrt{k}$, so henceforth we will demand that this inequality is satisfied. To solve (1.65), we note that $Y_u(L_u, K_u) = u$ with $K_u = k - K_c$

implies $K_c = k - \frac{u^2}{L_u}$ and hence

$$Y_c = \sqrt{(1 - L_u)(k - \frac{u^2}{L_u})}. \tag{1.72}$$

Maximizing this last expression in the range $\frac{u^2}{k} \leq L_u \leq 1$ shows that the optimal labour distribution is given by $L_u = u/\sqrt{k}$, which implies that the consumer function is given by

$$C(u, k) = \sqrt{k} - u. \tag{1.73}$$

Inserting this into (1.70), we obtain

$$\begin{aligned} \ddot{k}(t) &= \frac{\dot{k}(t-\sigma)}{2\sqrt{k(t-\sigma)}} - g\dot{k}(t-\tau) + [\sqrt{k(t-\sigma)} - \dot{k}(t) - gk(x-\tau)]^2 \times \\ &\quad \Big(\frac{\rho}{\sqrt{k(t-\sigma)} - \dot{k}(t) - gk(t-\tau)} + \frac{ge^{-\rho\tau}}{\sqrt{k(t+\tau-\sigma)} - \dot{k}(t+\tau) - gk(t)} \\ &\quad - \frac{e^{-\rho\sigma}}{2\sqrt{k(t)}(\sqrt{k(t)} - \dot{k}(t+\sigma) - gk(x+\sigma-\tau))}\Big). \end{aligned} \tag{1.74}$$

One immediately finds the equilibrium solution

$$\overline{k} = \frac{e^{-2\rho\sigma}}{4(\rho + ge^{-\rho\tau})^2} \tag{1.75}$$

and upon linearization around this equilibrium, choosing $\sigma = \tau$ and inserting a solution of the form e^{zt}, one obtains the characteristic function

$$\Delta(z) = (z - \rho e^{-(z-\rho)\tau})(z - \rho + \rho e^{z\tau}) - \frac{1}{2}(\rho + ge^{-\rho\tau})(2\rho e^{\rho\tau} + g). \tag{1.76}$$

In Figure 1.3(a) the points in the (ρ, g) plane where the characteristic equation $\Delta(z) = 0$ has roots $z = \pm i\omega$ on the imaginary axis have been plotted, for fixed $\sigma = \tau = 4.0$. At intersection points two distinct pairs of roots $z = \pm i\omega_{1,2}$ cross the imaginary axis simultaneously, but they never do so in a resonant fashion, i.e., $\left\{\omega_2\omega_1^{-1}, \omega_1\omega_2^{-1}\right\} \cap \mathbb{Z} = \emptyset$. Figure 1.5 exhibits the sheets of periodic solutions to (1.74) that emerge from the equilibrium at these Hopf bifurcation points. In order to relate these numerically uncovered bifurcations to the theory in Chapter 2, we will study a one dimensional cross-section of Figures 1.3(a) and 1.5 by fixing $\rho = 0.80$ and treating g as the bifurcation parameter. The characteristic equation $\Delta(z, g) = 0$ has a pair of roots that crosses the imaginary axis at

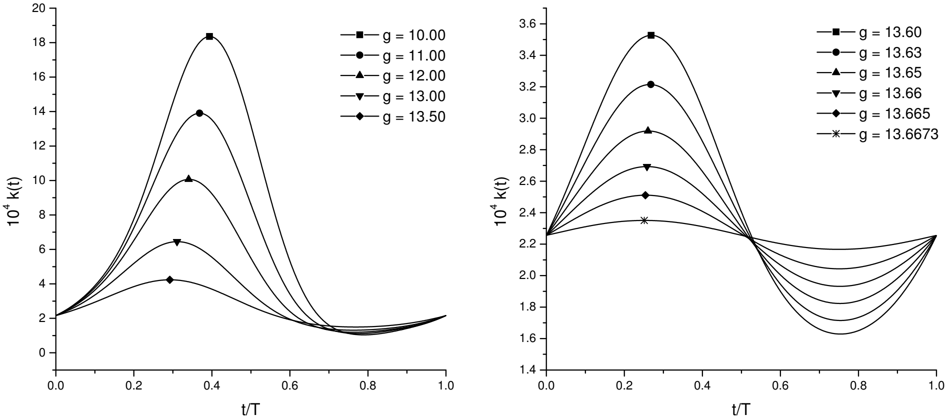
$$(z, g) = (i\omega_0, g_0) \approx (\pm 2.81081i, 13.667698). \tag{1.77}$$

Similarly, another pair of roots crosses at

$$(z, g) = (i\omega_1, g_1) \approx (\pm 16.887783i, 18.128033). \tag{1.78}$$

In order to check the transversality condition (Hζ3) from Theorem 2.2.3, we compute

$$\begin{aligned} D_2\Delta(i\omega_0, g_0) &= -\tfrac{3}{2}\rho - g_0 e^{-\rho\tau} \approx -1.76, \\ D_1\Delta(i\omega_0, g_0) &= (1 + \rho\tau e^{-(i\omega_0-\rho)\tau})(i\omega_0 - \rho + \rho e^{i\omega_0\tau}) \\ &\quad + (z - \rho e^{-(i\omega_0-\rho)\tau})(1 + \rho\tau e^{i\omega_0\tau}) \\ &\approx -2.26 \cdot 10^2 - 18.8i, \end{aligned} \tag{1.79}$$



(a) (b)

Figure 1.4: *Solutions profiles $k(t)$ for periodic solutions to (1.74) at different values of the parameter g. The other model parameters were fixed at $\sigma = \tau = 4$ and $\rho = 0.8$. All solutions are part of the branch bifurcating from the equilibrium at $g = g_0 \approx 13.667698$.*

from which we conclude

$$D_2\Delta(i\omega_0, g_0)/D_1\Delta(i\omega_0, g_0) \approx 7.72 \cdot 10^{-3} - 6.41i \cdot 10^{-4}. \tag{1.80}$$

Together with the fact that there are no other roots on the imaginary axis, this shows that Theorem 2.2.3 can be applied, yielding the existence of periodic solutions for g near g_0 around the equilibrium solution $\overline{k}$, that have period approximately $T = \frac{2\pi}{\omega_0} \approx 2.23536$. To compute the direction of bifurcation, we recall the expansion

$$g^*(\tau) = g_0 + \tau^2[\mathrm{Re}\, D_2\Delta(i\omega_0, g_0)/D_1\Delta(i\omega_0, g_0)]^{-1}\mathrm{Re}\,[c/D_1\Delta(i\omega_0, g_0)] + o(\tau^2), \tag{1.81}$$

with c given by

$$\begin{aligned} c \;&=\; \tfrac{1}{2}D_1^3R(\overline{k}, g_0)(e^{i\omega_0\cdot}, e^{i\omega_0\cdot}, e^{-i\omega_0\cdot}) \\ &\quad + \Delta(0, g_0)^{-1}D_1^2R(\overline{k}, g_0)(e^{i\omega_0\cdot}, \mathbf{1})D_1^2R(\overline{k}, g_0)(e^{i\omega_0\cdot}, e^{-i\omega_0\cdot}) \\ &\quad + \tfrac{1}{2}\Delta(2i\omega_0, g_0)^{-1}D_1^2R(\overline{k}, g_0)(e^{-i\omega_0\cdot}, e^{2i\omega_0\cdot})D_1^2R(\overline{k}, g_0)(e^{i\omega_0\cdot}, e^{i\omega_0\cdot}) \\ &\approx\; [1.51 \cdot 10^{-9} - 4.75i \cdot 10^{-8}] + [1.69 \cdot 10^{-9} + 1.34i \cdot 10^{-8}] \\ &\quad + [-1.05 \cdot 10^{-9} - 2.63i \cdot 10^{-8}] \\ &\approx\; 2.15 \cdot 10^{-9} + 6.00i \cdot 10^{-8}. \end{aligned} \tag{1.82}$$

Since $\mathrm{Re}\,[c/D_1\Delta(i\omega_0, g_0)] \approx \mathrm{Re}\, -9.66 \cdot 10^6 - 1.85i \cdot 10^6 < 0$, we expect, in view of (1.80), that a branch of periodic solutions bifurcates from the equilibrium for $g < g_0$. This is indeed clearly visible from Figures 1.3(b) and 1.5, where the extremal values of periodic orbits for such values of g are exhibited. We refer to Figure 1.4 for examples of the actual profiles of these periodic solutions.

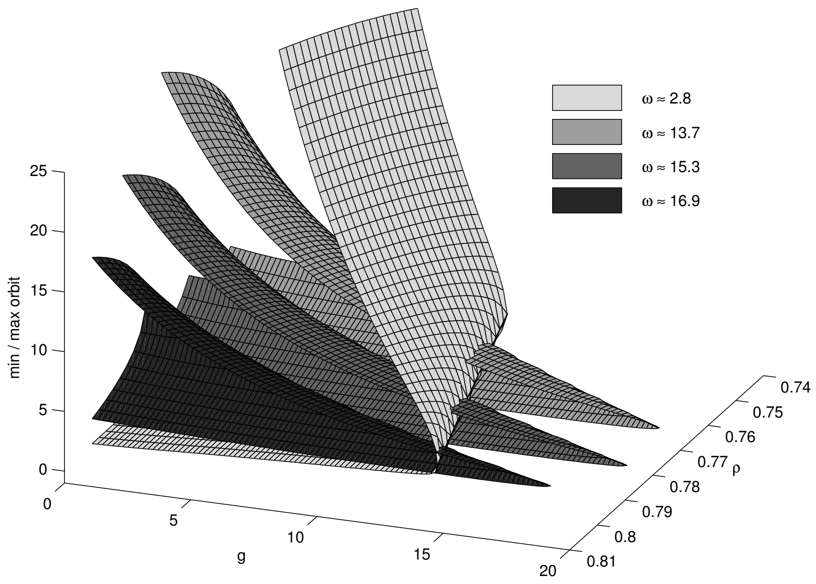


Figure 1.5: *Bifurcation diagram showing the subcritical Hopf bifurcations that the equilibrium undergoes as the parameters ρ and g are varied, while $\sigma = \tau = 4.0$.*

1.5. Economic Life-Cycle Model

As a second example, we discuss here in some detail the work of Albis et al. In [40], they analyze the dynamic behaviour of the capital growth rate in a market economy by using a continuous overlapping-generations model. In particular, they consider a population that consists of individuals that all live for a fixed time of unit length and work during their entire life. The consumption at time t of an individual born at time s is denoted by $c(s,t)$. Similarly, the quantity $a(s,t)$ stands for the assets that such a person owns, on which he receives interest at the rate $r(t)$. The wages that are received are given by $w(t)$ and thus do not depend upon the age of the labourer. Putting this together leads to the following budget constraint,

$$\frac{\partial a(s,t)}{\partial t} = r(t)a(s,t) + w(t) - c(s,t). \tag{1.83}$$

The population is assumed to possess perfect foresight, which in this context means that every member can accurately predict the future behaviour of the interest rate r and the wages w. The economy is driven by the fact that every individual acts in such a way that his total life-time welfare is maximized. This welfare is naturally related to his consumption and is quantified by the expression $\int_s^{s+1} \ln(c(s,\tau))d\tau$, in which s again stands for the date of birth. Every individual except those that already exist at the start of the economy at $t = 0$, is born penniless and may not die in debt, i.e., $a(s, s+1) \geq a(s,s) = 0$ for all $s \geq 0$. Upon solving the above optimization problem, one sees that for any $s \geq 0$ and $t \in [s, s+1]$

the optimal amount of assets $a^*(s,t)$ is a function of the interest rates r_{s+} and wages w_{s+} during the lifetime of an individual. Here $r_{s+} \in C([0,1],\mathbb{R})$ is given by $r_{s+}(\tau) = r(s+\tau)$ and w_{s+} is similarly defined. In particular, in [40] the following expression is derived for the optimal assets,

$$a^*(s,t) = (s+1-t)\int_s^{s+1} w(\sigma)e^{-\int_t^\sigma r(\tau)d\tau}d\sigma - \int_t^{s+1} w(\sigma)e^{-\int_t^\sigma r(\tau)d\tau}d\sigma. \quad (1.84)$$

The height of the interest rate $r(t)$ and wages $w(t)$ depend on the state of the capital and labour markets. We write $l(t)$ for the amount of labour available at any time t and observe that $l(t) = 1$ since the population has fixed unit size. Similarly, we write $k(t)$ for the amount of available capital, which is given by the sum of the assets of all individuals alive at time t, i.e.,

$$k(t) = \int_{t-1}^{t} a^*(\sigma,t)d\sigma. \quad (1.85)$$

There is a unique material good of unit price, which can be used for both consumption and investments. It is produced at the rate Q given by

$$Q(k(t),e(t),l(t)) = Ak(t)^\alpha(e(t)l(t))^\beta, \quad (1.86)$$

for some $A > 0$ and exponents $\alpha > 0$ and $\beta > 0$. Here $e(t)$ is a factor to correct for the increase in labour efficiency over time, which is taken to be $e(t) = k(t)$. The prices for capital and employment equal their respective marginal products, so we can calculate the interest rate $r(t)$ and wages $w(t)$ by partial differentiation of Q, yielding

$$\begin{array}{rclcl} r(t) & = & \alpha Ak(t)^{\alpha-1}(e(t)l(t))^\beta & = & \alpha Ak(t)^{\alpha+\beta-1}, \\ w(t) & = & \beta Ak(t)^\alpha e(t)^\beta l(t)^{\beta-1} & = & \beta Ak(t)^{\alpha+\beta}. \end{array} \quad (1.87)$$

Inserting (1.87) into (1.84) and using the definition (1.85) for $k(t)$, one obtains the market equilibrium condition

$$\begin{array}{rcl} k(t) & = & \beta A\int_{t-1}^{t}(s+1-t)\int_s^{s+1}k(\sigma)^{\alpha+\beta}e^{-\alpha A\int_t^\sigma k(\tau)^{\alpha+\beta-1}d\tau}d\sigma ds \\ & & -\beta A\int_{t-1}^{t}\int_t^{s+1}k(\sigma)^{\alpha+\beta}e^{-\alpha A\int_t^\sigma k(\tau)^{\alpha+\beta-1}d\tau}d\sigma ds. \end{array} \quad (1.88)$$

This integral equation can be transformed into a MFDE by threefold differentiation. An involved computation leads to

$$\begin{array}{rcl} \dddot{k}(t) & = & [(\alpha+\beta)^2A + 2\alpha A]k(t)^{\alpha+\beta-1}\ddot{k}(t) - \alpha A^2(3(\alpha+\beta)^2-\beta)k(t)^{2(\alpha+\beta-1)}\dot{k}(t) \\ & & +(\alpha+\beta-1)A((\alpha+\beta)^2+\alpha)k(t)^{\alpha+\beta-2}\dot{k}(t)^2 \\ & & +2\beta Ak(t)^{\alpha+\beta} + (\alpha A)^2(\alpha+\beta)Ak(t)^{3(\alpha+\beta)-2} \\ & & -\beta Ak(t+1)^{\alpha+\beta}e^{-\alpha A\int_t^{t+1}k(\tau)^{\alpha+\beta-1}d\tau} \\ & & -\beta Ak(t-1)^{\alpha+\beta}e^{-\alpha A\int_t^{t-1}k(\tau)^{\alpha+\beta-1}d\tau}. \end{array} \quad (1.89)$$

In [40] the authors choose $\beta = 1-\alpha$, upon which (1.89) reduces to the linear functional differential equation,

$$\begin{array}{rcl} \dddot{k}(t) & = & A(1+2\alpha)\ddot{k}(t) - \alpha A^2(2+\alpha)\dot{k}(t) - (1-\alpha)Ak(t-1)e^{\alpha A} \\ & & +[2(1-\alpha)+(\alpha A)^2]Ak(t) - (1-\alpha)Ak(t+1)e^{-\alpha A}. \end{array} \quad (1.90)$$

This equation matches the expression derived in [40] by substituting $\alpha + \beta = 1$ directly in (1.87). Since (1.90) is linear, the global behaviour of the capital market can be analyzed by studying the roots of the characteristic equation

$$\Delta(z, \alpha, A) := (z - \alpha A)^3 - (1 - \alpha)A[(z - \alpha A)^2 + 2 - e^{-(z-\alpha A)} - e^{z-\alpha A}] = 0. \quad (1.91)$$

This was performed in [40], where the authors proved that, apart from the triple root at $z = \alpha A$, there is precisely one real root $\overline{g}(A)$ and all other roots satisfy $\operatorname{Re} z \neq \overline{g}(A)$. We remark here that insertion of $k(t) = (A_0 + A_1 t + A_2 t^2)e^{\alpha A t}$ into (1.88) yields $k(t) = 0$, which is why this root needs to be excluded. However, $k(t) \sim e^{\overline{g}t}$ does indeed yield a solution to (1.88). Demanding furthermore that $k(t)$ should remain strictly positive, one sees that the capital dynamics may exhibit oscillations at the start of the economy, but will finally converge to the balanced growth path [40].

We now shift our attention to the case that $\alpha + \beta \neq 1$ and look for non-zero equilibrium solutions to (1.88). It is convenient to introduce the new variable $y = \alpha A e^{(\alpha+\beta-1)\ln k}$. Substitution into (1.88) and setting all derivatives to zero yields the equilibrium condition $f(\overline{y}) = 0$, in which the function f is given by

$$f(y) = (\alpha + \beta)y^2 + 2\beta(1 - \cosh y). \quad (1.92)$$

Lemma 1.5.1. *For all parameters $\alpha > 0$ and $\beta > 0$, the equation $f(y) = 0$ has a unique strictly positive solution $\overline{y} = \overline{y}(A, \alpha, \beta) > 0$.*

Proof. Notice that $f(0) = f'(0) = 0$. In addition, we calculate

$$\begin{aligned} f''(y) &= 2(\alpha + \beta(1 - \cosh y)), \\ f'''(y) &= -2\beta \sinh y, \end{aligned} \quad (1.93)$$

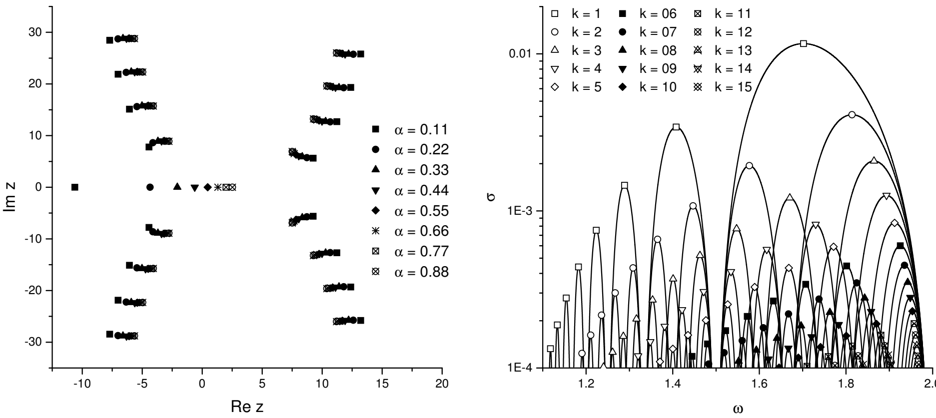
which implies that $f''(0) = 2\alpha > 0$ and $f'''(y) < 0$ for all $y > 0$. The claim now immediately follows upon observing that $\lim_{y\to\infty} f(y) = -\infty$. □

We emphasize here that the equilibrium $\overline{y}$ found above does not translate into a valid equilibrium $\overline{k}$ for the capital when $\alpha + \beta = 1$, since the corresponding variable transformation is ill-defined.

Linearizing (1.89) around an equilibrium $\overline{y}$ and using the condition $f(\overline{y}) = 0$ yields the following characteristic function,

$$\begin{aligned} \alpha\Delta(z) = {} & \alpha(z - \overline{y})^3 + \overline{y}\big(\alpha - (\alpha + \beta)^2\big)(z - \overline{y})^2 \\ & - (\alpha + \beta)\overline{y}^2\big(1 - (\alpha + \beta)\big)(z - \overline{y}) - (\alpha + \beta)\overline{y}\big((\alpha + \beta - 1)\overline{y}^2 + 2\beta\big) \\ & + z^{-1}\big[2\beta\overline{y}\big((\alpha + \beta)(z - \overline{y}) + \overline{y}\big)\cosh(z - \overline{y}) \\ & + \overline{y}^2(\alpha + \beta - 1)\big((\alpha + \beta)\overline{y}^2 + 2\beta\big)\big]. \end{aligned} \quad (1.94)$$

Invoking the equilibrium condition $f(\overline{y}) = 0$, one can easily see that the apparent singularity at $z = 0$ in the above expression is in fact removable. Furthermore, a short calculation shows that $\Delta(\overline{y}) = 0$, but for similar reasons as in the linear case this root needs to be excluded.



(a) (b)

Figure 1.6: *Part (a) depicts the roots of the characteristic equation* $\Delta(z) = 0$*, with* Δ *as in (1.94), for different values of the parameter* $\alpha = \beta$*. Observe the crossing through the imaginary axis of a real root at* $\alpha + \beta = 1$*. Part (b) exhibits curves of parameter values at which Hopf bifurcations occur for (1.98). The integers k that label the branches correspond to the indices in Proposition 1.6.1.*

In Figure 1.6(a) the roots of (1.94) in the complex rectangle $-15 < \operatorname{Re} z < 15$, $-30 < \operatorname{Im} z < 30$ have been plotted for various values of the parameter α, with $\beta = \alpha$. It is clearly visible that a branch of real roots crosses zero at $\alpha = \beta = 0.5$. However, the branches of roots with nonzero imaginary part typically stay away from the imaginary axis. Theorem 2.2.2 implies that for any root z that has negative real part $\operatorname{Re} z = -\lambda < 0$, there are solutions $k(t)$ to (1.89) that exist for all $t > 0$ and satisfy $k(t) = \overline{k} + O(e^{(-\lambda+\epsilon)t})$ as $t \to \infty$, for all sufficiently small $\epsilon > 0$. These solutions will generally exhibit damped oscillations around the equilibrium if in addition $\operatorname{Im} \lambda \neq 0$.

The question concerning the stability of such solutions is at the moment an entirely open one. Due to the presence of infinitely many roots to the left and right of the imaginary axis, one can only hope to obtain satisfactory answers if the analysis is restricted to special classes of solutions. In the current context for example we demand $k \geq 0$, since the total amount of capital may not become negative. This in a way might exclude the oscillating contributions of eigenvalues λ with $\operatorname{Re} \lambda > 0$ and $\operatorname{Im} \lambda \neq 0$, allowing us to focus on the presence of positive real roots for the stability analysis. It is however unclear how to make this precise at the moment.

1.6. Monetary Cycles with Endogenous Retirement

The model that is discussed in this section was developed in [39] and should be seen as an extension of the overlapping generations model discussed above. In particular, the authors again consider a fixed size population of individuals, that now live for the longer time $\omega > 1$, with the restriction that every labourer retires at unit age. In terms of the model components, this means that the wages $w(s,t)$ now do depend upon the time of birth and satisfy $w(s,t) = 1$ for $t \in [s, s+1]$ and $w(s,t) = 0$ otherwise. As before, everybody receives interest at the rate $r(t)$ on their assets $a(s,t)$ while consuming $c(s,t)$. These observations lead to the modified budget constraint

$$\frac{\partial a(s,t)}{\partial t} = r(t)a(s,t) + w(s,t) - c(s,t). \tag{1.95}$$

A second variant in this model is that the utility $u(s)$ as perceived by the generation born at time s is now given by

$$u(s) = \int_s^{s+\omega} \frac{c(s,t)^{1-\sigma^{-1}}}{1-\sigma^{-1}} dt. \tag{1.96}$$

As usual, every individual acts in such a way that his utility is maximized, subject to both (1.95) and the natural budget constraints $a(s,s) = 0$ and $a(s, s+\omega) \geq 0$. In (1.96) the parameter σ stands for the elasticity of intertemporal substitution and is required to satisfy $\sigma > 0$. The economy features a single, nonstorable consumption good, which we will assume to be produced at exactly the rate required by the consumer market. In terms of our model variables, this means that for all time t the following identity must hold,

$$\int_{t-\omega}^{t} w(\sigma,t)d\sigma = \int_{t-\omega}^{t} c(\sigma,t)d\sigma. \tag{1.97}$$

The rules above are sufficient to fix the dynamical behaviour of the economy and following [39], one easily derives the difference equation $\mathcal{F}(r_t) = 0$ for the interest rate $r(t)$, with $\mathcal{F}$ given by

$$\mathcal{F}(r_t) = 1 - \int_{t-\omega}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+\omega} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds. \tag{1.98}$$

Notice that $r = 0$ is an equilibrium solution of (1.98). The linearization around this equilibrium is given by

$$0 = -\frac{1}{\omega}\int_{t-\omega}^{t}\int_s^{s+1}\int_t^v x(u)du\,dv\,ds + \frac{1-\sigma}{\omega^2}\int_{t-\omega}^{t}\int_s^{s+\omega}\int_t^v x(u)du\,dv\,ds. \tag{1.99}$$

Inserting $x(u) = \exp(zu)$ yields the characteristic function

$$\begin{aligned}\Delta(z,\sigma,\omega) \quad = \quad & -\tfrac{1}{\omega^2 z^3}[-\omega e^z + (1-\sigma)e^{z\omega} + (\omega e^z - \omega + 1 - \sigma)e^{-z\omega} \\ & +(\omega - 2 + 2\sigma) + \sigma\omega^2 z^2].\end{aligned} \tag{1.100}$$

The following result, which was partially proven in [39], shows that the characteristic equation $\Delta(z) = 0$ admits simple roots on the imaginary axis that satisfy the conditions associated with the Hopf bifurcation theorem. The proof is deferred to the end of this section.

Proposition 1.6.1. *Consider any $\omega > 1$ such that $(\omega - 1)^{-1} \notin \mathbb{N}$. There exists an infinite sequence of pairs (σ_k, q_k) parametrized by $k \in \mathbb{N}$, with $\sigma_k > 0$ and $q_k > 0$, such that the following properties are satisfied.*

(i) One has the limits $\sigma_k \to 0$ *and* $q_k \to \infty$ *as* $k \to \infty$.

(ii) The characteristic equation $\Delta(z, \sigma_k, \omega) = 0$ *has two simple roots at* $z = \pm i q_k$.

(iii) For all $k \in \mathbb{N}$ *and* $m \in \mathbb{Z} \setminus \{\pm 1\}$, *the inequality* $\Delta(imq_k, \sigma_k, \omega) \neq 0$ *holds.*

(iv) For every $k \in \mathbb{N}$, *the branch of roots* $z(\sigma)$ *of the characteristic equation* $\Delta(z, \sigma, \omega) = 0$ *through* $z = iq_k$ *at* $\sigma = \sigma_k$ *crosses the imaginary axis with positive speed, i.e.,*

$$\mathrm{Re}\, \frac{D_2 \Delta(iq_k, \sigma_k, \omega)}{D_1 \Delta(iq_k, \sigma_k, \omega)} \neq 0. \tag{1.101}$$

Conversely, if $\omega = 1 + n^{-1}$ *for some* $n \in \mathbb{N}$, *then for all* $\sigma > 0$ *the characteristic equation* $\Delta(z, \sigma, \omega) = 0$ *admits no roots with* $\mathrm{Re}\, z = 0$.

Fixing any suitable $\omega > 1$ and treating σ as a bifurcation parameter, the result above will allow us to conclude that the algebraic equation (1.98) admits a branch of periodic solutions bifurcating from the equilibrium $r = 0$ at $\sigma = \sigma_k$, for all $k \in \mathbb{N}$. Similarly, any sufficiently small solution of (1.98) with σ near σ_k can be captured on such a branch. To validate this claim using our main theorem in Section 3.2, we note that twofold differentiation of (1.98) and simplification using the identity (1.98), yields the following mixed type functional differential equation,

$$\begin{array}{rcl} \dot{r}(t) & = & \mathcal{G}(r_t), \quad \text{with} \\ -\sigma \mathcal{G}(r_t) & = & -\sigma^2 r(t)^2 + [\int_t^{t+\omega} e_\sigma(v) dv]^{-1} [e(t+1) - 1] \\ & & -[\int_t^{t+\omega} e_\sigma(v) dv]^{-2} [e_\sigma(t+\omega) - 1] \int_t^{t+1} e(v) dv \\ & & +[\int_{t-\omega}^{t} e_\sigma(v) dv]^{-2} [1 - e_\sigma(t-\omega)] \int_{t-\omega}^{t-\omega+1} e(v) dv \\ & & -[\int_{t-\omega}^{t} e_\sigma(v) dv]^{-1} [e(t-\omega+1) - e(t-\omega)], \end{array} \tag{1.102}$$

in which we have made the abbreviations

$$\begin{array}{rcl} e(w) & = & \exp\big(-\int_t^w r(u) du\big), \\ e_\sigma(w) & = & \exp\big(-(1-\sigma)\int_t^w r(u) du\big). \end{array} \tag{1.103}$$

Linearizing (1.102) around $r = 0$ yields

$$\begin{array}{rcl} -\sigma \dot{x}(t) & = & -\frac{1}{\omega} \int_t^{t+1} x(u) du + \frac{1}{\omega} \int_{t-\omega}^{t-\omega+1} x(u) du \\ & & + \frac{1-\sigma}{\omega^2} \int_t^{t+\omega} x(u) du + \frac{1-\sigma}{\omega^2} \int_t^{t-\omega} x(u) du. \end{array} \tag{1.104}$$

Inserting $x(u) = \exp(zu)$ and normalizing, we find the characteristic function

$$\begin{array}{rcl} \Delta_M(z, \sigma, \omega) & = & \frac{1}{\sigma \omega^2 z} [\sigma \omega^2 z^2 - \omega(e^z - 1) + \omega(e^z - 1) e^{-\omega z} \\ & & +(1-\sigma)(e^{\omega z} + e^{-\omega z} - 2)] \\ & = & -\frac{z^2}{\sigma} \Delta(z, \sigma, \omega), \end{array} \tag{1.105}$$

which immediately implies that condition (HL) in Section 3.2 is satisfied. Using the expressions above, all the other conditions of Theorem 3.2.2 can easily be verified as well.

Hence upon fixing an appropriate $\omega > 1$ and considering any pair (σ_0, q_0) generated by Proposition 1.6.1, we can establish the existence of a $2+1$ dimensional center manifold $u^* : X_0 \times \mathbb{R} \to \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ directly for the difference equation (1.98). Here $X_0 = \mathrm{span}(e^{iq_0\cdot}, e^{-iq_0\cdot})$ and the extra dimension arises by including the bifurcation parameter $\widetilde{\sigma} = \sigma - \sigma_0$ in the center space. The dynamical behaviour of $\widetilde{\sigma}$ on the center manifold is trivial, while the evolution of $\psi(\xi) = x(\xi)e^{iq_0\cdot} + y(\xi)e^{-iq_0\cdot} \subset X_0$ is governed by the ODE

$$\begin{array}{rcl} x' &=& iq_0 x + f_1(x, y, \widetilde{\sigma}) + O\big((|x| + |y|)^3 + |\widetilde{\sigma}|\,(|x| + |y|)(|\widetilde{\sigma}| + |x| + |y|)\big), \\ y' &=& -iq_0 y + f_2(x, y, \widetilde{\sigma}) + O\big((|x| + |y|)^3 + |\widetilde{\sigma}|\,(|x| + |y|)(|\widetilde{\sigma}| + |x| + |y|)\big), \end{array} \tag{1.106}$$

in which the second order terms are given by

$$\begin{array}{rcl} f_1(x, y, \widetilde{\sigma}) &=& -D_2\Delta(iq_0, \sigma_0)D_1\Delta(iq_0, \sigma_0)^{-1}\widetilde{\sigma}x \\ && +\frac{1}{2}D_1\Delta(iq_0, \sigma_0)^{-1}\big(\alpha_{xx}x^2 + 2\alpha_{xy}xy + \alpha_{yy}y^2\big) \\ && +\frac{1}{2iq_0}D_1\Delta(iq_0, \sigma_0)^{-1}\Big(\Delta(2iq_0, \sigma_0)(\beta_{xx} - 4\frac{q_0^2}{\sigma_0}\alpha_{xx})x^2 \\ && \quad -2\Delta(0, \sigma_0)\beta_{xy}xy - \frac{1}{3}\Delta(-2iq_0, \sigma_0)(\beta_{yy} - 4\frac{q_0^2}{\sigma_0}\alpha_{yy})y^2\Big), \\ f_2(x, y, \widetilde{\sigma}) &=& -D_2\Delta(-iq_0, \sigma_0)D_1\Delta(-iq_0, \sigma_0)^{-1}\widetilde{\sigma}y \\ && +\frac{1}{2}D_1\Delta(-iq_0, \sigma_0)^{-1}\big(\alpha_{xx}x^2 + 2\alpha_{xy}xy + \alpha_{yy}y^2\big) \\ && +\frac{1}{2iq_0}D_1\Delta(-iq_0, \sigma_0)^{-1}\Big(\frac{1}{3}\Delta(2iq_0, \sigma_0)(\beta_{xx} - 4\frac{q_0^2}{\sigma_0}\alpha_{xx})x^2 \\ && \quad +2\Delta(0, \sigma_0)\beta_{xy}xy - \Delta(-2iq_0, \sigma_0)(\beta_{yy} - 4\frac{q_0^2}{\sigma_0}\alpha_{yy})y^2\Big) \end{array} \tag{1.107}$$

and the quantities α_{xx} through β_{yy} can be calculated by using

$$\begin{array}{rclcrcl} \alpha_{xx} &=& D_1^2\mathcal{F}(0, \sigma_0)(e^{iq_0\cdot}, e^{iq_0\cdot}), && \beta_{xx} &=& D_1^2\mathcal{G}(0, \sigma_0)(e^{iq_0\cdot}, e^{iq_0\cdot}), \\ \alpha_{xy} &=& D_1^2\mathcal{F}(0, \sigma_0)(e^{iq_0\cdot}, e^{-iq_0\cdot}), && \beta_{xy} &=& D_1^2\mathcal{G}(0, \sigma_0)(e^{iq_0\cdot}, e^{-iq_0\cdot}), \\ \alpha_{yy} &=& D_1^2\mathcal{F}(0, \sigma_0)(e^{-iq_0\cdot}, e^{-iq_0\cdot}), && \beta_{yy} &=& D_1^2\mathcal{G}(0, \sigma_0)(e^{-iq_0\cdot}, e^{-iq_0\cdot}). \end{array} \tag{1.108}$$

Using the transversality condition (iv) from Proposition 1.6.1, it is easily seen that the ODE (1.106) undergoes a Hopf bifurcation at $\sigma = \sigma_0$. This yields a branch of periodic orbits that can be lifted back to periodic solutions of our initial problem (1.98), which establishes our claim above.

Notice that $\Delta_M(z) = 0$ has a double root at $z = 0$ for all valid parameters ω and σ, which arises as an artifact of the differentiation operations needed to derive (1.102). This double root prevents the direct application of the Hopf bifurcation result developed in Chapter 2 to the MFDE (1.102). To give a detailed analysis of the local behaviour of this equation, one would hence have to revert to a complicated normal form reduction. Theorem 3.2.2 has allowed us to circumvent this difficulty by analyzing (1.98) directly.

In Figure 1.6(b) the parameter values at which (1.98) undergoes a Hopf bifurcation have been visualized. The nested structure of the curves and the location of the gaps can be understood from the proof of Proposition 1.6.1, which exploits the rare fact that the roots of (1.100) can be explicitly described. By contrast, the actual computation of the arising periodic orbits is far more challenging due to the presence of the doubly nested integrals in

(1.102). The results in Figure 1.7(a) were computed by using a fine equidistributed grid of 21 thousand mesh points and invoking the non-adaptive Kronrod 21-point rule [43] on each mesh interval to compute the integrals.

Proof of Proposition 1.6.1. For convenience, we write $\widetilde{\Delta}(z,\sigma,\omega) = -\omega^2 z^3 \Delta(z,\sigma,\omega)$. First note that $\widetilde{\Delta}(z,\sigma,\omega) = \frac{1}{2}\omega^2(\omega-1)z^3 + O(z^4)$ around $z = 0$, which implies that $z = 0$ is not a root of $\Delta(z,\sigma_0,\omega)$ for $\omega > 1$. For any $q \in \mathbb{R}$, we write $I(q) = \operatorname{Im} \widetilde{\Delta}(iq,\sigma,\omega)$ and compute

$$I(q) = \omega(\sin\omega q + \sin(1-\omega)q - \sin q) = 4\omega \sin\frac{\omega q}{2} \sin\frac{(1-\omega)q}{2} \sin\frac{q}{2}. \tag{1.109}$$

Similarly, writing $R(q) = \operatorname{Re} \widetilde{\Delta}(iq,\sigma,\omega)$, we compute

$$\begin{aligned} R(q) &= -\omega\cos q + 2(1-\sigma)(\cos\omega q - 1) + \omega(1-\cos\omega q) \\ &\quad + \omega\cos(1-\omega)q - \sigma\omega^2 q^2 \\ &= 2\omega(\sin^2\tfrac{q}{2} + \sin^2\tfrac{\omega q}{2} - \sin^2\tfrac{(1-\omega)q}{2}) - 4\sin^2\tfrac{\omega q}{2} + \sigma(4\sin^2\tfrac{\omega q}{2} - \omega^2 q^2). \end{aligned} \tag{1.110}$$

Notice that for any $l \in \mathbb{N}$ and $q^{(l)} = \frac{2l\pi}{\omega-1}$, we have $\sin\frac{(1-\omega)q^{(l)}}{2} = 0$, while $\sin^2\frac{q^{(l)}}{2} = \sin^2\frac{\omega q^{(l)}}{2}$. This implies that

$$R(q^{(l)}) = 4(\omega-1)\sin^2\frac{q^{(l)}}{2} + \sigma(4\sin^2\frac{q^{(l)}}{2} - (\omega q^{(l)})^2). \tag{1.111}$$

Now assume that $(\omega-1)^{-1} \notin \mathbb{N}$, which implies that $\frac{\pi}{1-\omega} \not\equiv 0 \mod \pi$. There hence exists a strictly increasing sequence of integers $l_k > 0$, parametrized by $k \in \mathbb{N}$, such that $s_k = \sin^2\frac{q^{(l_k)}}{2} > \frac{1}{4}$. Choose $q_k = q^{(l_k)}$ and write

$$\sigma_k = \frac{\omega-1}{\omega^2\frac{q_k^2}{4s_k} - 1} > 0, \tag{1.112}$$

where the last inequality follows from $\omega > 1$ and the fact that $|\sin\theta| < |\theta|$ for $\theta \neq 0$. By construction, we have $\Delta(iq_k,\sigma_k,\omega) = 0$. Suppose that for any $m \in \mathbb{Z} \setminus \{0,\pm 1\}$ we have $\Delta(imq_k,\sigma_k,\omega) = 0$, then using $mq_k = q^{(ml_k)}$ and setting $R(q^{(ml_k)}) = 0$, we find that $\sin^2\frac{q^{(ml_k)}}{2} = s_k m^2 > \frac{1}{4}m^2 > 1$, which is impossible. To prove the claim (iv) involving the derivatives of Δ, note that

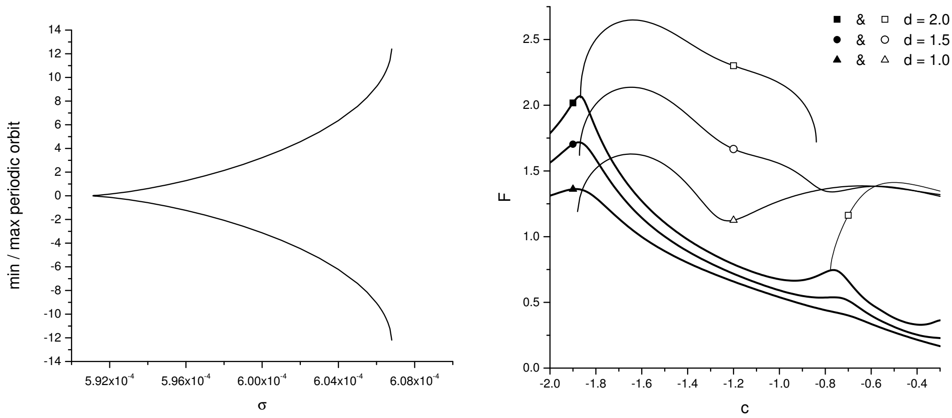
$$D_s\Delta(iq_k,\sigma_k,\omega) = \frac{i}{\omega^2 q_k^3} D_s\widetilde{\Delta}(iq_k,\sigma_k,\omega) \tag{1.113}$$

for $s = 1, 2$. In addition,

$$D_2\widetilde{\Delta}(iq_k,\sigma_k,\omega) = 2(1-\cos\omega q_k) - \omega^2 q_k^2 = 4s_k - \omega^2 q_k^2 < 0. \tag{1.114}$$

It hence suffices to compute

$$\begin{aligned} \operatorname{Re} D_1\widetilde{\Delta}(iq_k,\sigma_k,\omega) &= -\omega\cos q_k + \omega(1-\omega)\cos(1-\omega)q_k + \omega^2\cos\omega q_k \\ &= 2\omega(1-\omega)s_k \neq 0. \end{aligned} \tag{1.115}$$



(a) (b)

Figure 1.7: *Part (a) is a bifurcation diagram illustrating the Hopf bifurcation for (1.98) at* $\sigma = \sigma_0 \approx 5.91 \cdot 10^{-4}$ *and* $\omega = 1.77$. *Please note that the characteristic* $\mathcal{O}(\sqrt{\sigma - \sigma_0})$ *growth of the amplitude is not visible here. However, this growth rate can be recovered upon zooming in around* $\sigma \approx \sigma_0$. *The closed markers and thick lines in part (b) indicate the force versus velocity characteristics for uniform sliding state solutions of the Frenkel-Kontorova model (1.118). For these curves* $\gamma = 0.5$. *The open markers and thin lines in (b) represent curves of computed flip-bifurcations around the sliding state solutions. The parameter* γ *varies along these latter curves, but at the intersection points with the corresponding* F*-*c *characteristic the identity* $\gamma = 0.5$ *holds.*

We conclude the proof by assuming that $\omega = 1 + n^{-1}$ for some $n \in \mathbb{N}$. Substituting $q = 2l\pi$ into $R(q) = 0$ forces $\sigma < 0$, while the choice $q = \frac{2l\pi}{\omega}$ implies $\sigma = 0$. □

1.7. Frenkel-Kontorova models

In this section we return to the setting of lattice differential equations and discuss a Frenkel-Kontorova type model. This model was originally developed to describe the motion of dislocations in a crystal [151, 152], but has numerous other applications in the literature at present. It describes the dynamical behaviour of a chain of particles, which we will index by $j \in \mathbb{Z}$. The positions x_j of these particles evolve according to the following LDE,

$$\ddot{x}_j(t) + \gamma \dot{x}_j(t) = x_{j-1}(t) - 2x_j(t) + x_{j+1}(t) - d \sin x_j(t) + F, \tag{1.116}$$

in which γ and d are parameters and F is an external applied force. In the literature a special class of travelling wave solutions, which have been named uniform sliding states, has been constructed for (1.116). Solutions of this type can be written in the form $x_j(t) = \phi(j - ct)$

for some waveprofile ϕ and wavespeed c. In addition, they must satisfy the special condition $x_{j+N} = x_j + 2\pi M$, in which N and M are fixed integers.

Writing $\sigma = \frac{2\pi M}{N}$, we will search here for travelling wave solutions to (1.116) of the form

$$x_j(t) = \phi(\sigma j - ct), \tag{1.117}$$

with the additional constraint that $\phi(\xi + 2\pi) = \phi(\xi) + 2\pi$. Introducing the new function $\psi(\xi) = \xi + \phi(\xi)$, we find that ψ satisfies the following MFDE,

$$c^2\psi''(\xi) - \gamma c(1+\psi'(\xi)) = \psi(\xi+\sigma) + \psi(\xi-\sigma) - 2\psi(\xi) - d\sin(\xi + \psi(\xi)) + F, \tag{1.118}$$

with the periodic boundary condition $\psi(-\pi) = \psi(\pi)$. In the physics literature this function ψ is called the dynamic hull function.

In order to counter the trivial translation symmetry present in (1.118), it is convenient to impose a normalization condition $\psi(0) = 0$ when numerically constructing periodic solutions to (1.118). In addition, the calculations are simplified considerably by treating c as a constant in (1.118) and finding the corresponding value for F by adding an additional equation $F' = 0$. These calculations have been performed previously in [1] and [151], where F versus c characteristics were obtained for different values of d, using two different numerical approaches. The thick curves in Figure 1.7(b) are F-c characteristics that were obtained using our numerical method described in Section 1.8 and where these characteristics overlap, they agree with the existing computations.

Of considerable interest are the peaks that occur in the F-c characteristics. In [151] these peaks were partially explained in terms of different physical mechanisms leading to resonances, which cause an increasing in the driving force to lead to a relatively small increase in the kinetic energy. Since the kinetic energy is directly related to the wavespeed c, this may serve to explain the peaks. We attempt to approach this problem from a bifurcational point of view. By freeing up the parameter γ we are able to compute curves in the (F, c, γ) plane for which the corresponding periodic orbits undergo flip-bifurcations. The thin lines in Figure 1.7(b) represent the projections of these curves onto the (F, c) plane. The results suggest that the peaks in the F-c characteristics may be related to the nearby flip-bifurcations. Explained physically, near the peaks an increase in the driving force F may serve to excite the bifurcating orbits instead of the branches of sliding states shown in Figure 1.7(b).

1.8. Numerical Methods

In order to apply the main theorems derived in this thesis, it is clear that we need a tool to analyze the location of the roots of the characteristic equation $\det \Delta(z) = 0$. If one finds that a pair of roots crosses the imaginary axis, one can use the Hopf bifurcation theorem to conclude the existence of periodic solutions, but naturally we also wish to be able to explicitly compute these periodic orbits. In addition, in view of the Floquet theory for MFDEs constructed in Chapter 4, a method is needed that allows for the computation of Floquet exponents associated to any period orbit. In this section, we discuss the numerical methods that we use for these purposes.

Solving the characteristic equation

The first tool is a method to compute roots to the transcendental characteristic equation $\det \Delta(z) = 0$. Since these equations in general have an infinite number of roots, we are typically interested in computing all the roots in a vertical strip $\alpha < \operatorname{Re} z < \beta$. The number of roots in such a strip is typically finite and for any vertical strip we can explicitly bound the norm of the imaginary part of any roots, which allows us to restrict the root finding process to bounded rectangles in the complex plane. In the context of delay equations, Engelborghs et al. [56] employed a discretized initial value problem to approximate the rightmost roots of the corresponding characteristic equations. However, for MFDE we no longer have this option, since the associated initial value problem is ill defined and the spectrum extends infinitely to both sides of the imaginary axis. We therefore employ a complex bisection method combined with Newton iterations. The method is based upon the argument principle [98], which states that for any holomorphic function f that does not have any zeroes on the positively oriented simple closed curve Γ, the number of zeroes N of f in the interior of Γ, counted by multiplicity, is given by

$$N = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz. \tag{1.119}$$

One can hence search for all zeroes of a holomorphic function defined on a rectangle R by subdividing the rectangle into four smaller rectangles and discarding the parts for which (1.119) is zero. Since N should always be integer valued, it is possible to accurately monitor the errors involved with the numerical evaluation of the integral in (1.119) and hence this bisection method is very robust. One can continue the subdivision process until a prescribed accuracy has been reached and refine the resulting estimates using a small number of Newton iteration steps. Apart from robustness, a second major advantage of this method compared to the discretization approach of Engelborghs [56] or mapping based algorithms [163], is that the orders of the discovered roots are available. This is quite useful when computing the structure of the spectral subspaces on which the center manifold is defined.

Solving MFDEs

The second numerical tool we discuss here is a collocation solver for MFDEs on finite intervals. The code is able to solve n dimensional problems of the form

$$\tau(\xi)\phi'(\xi) = f\Big(\phi(\xi), \phi\big(\xi + \sigma_1(\xi, \phi(\xi))\big), \ldots, \phi\big(\xi + \sigma_N(\xi, \phi(\xi))\big)\Big), \tag{1.120}$$

for given functions $f : \mathbb{R}^{n(N+1)} \to \mathbb{R}^n$, $\tau : \mathbb{R} \to \mathbb{R}^n$, and shifts $\sigma_i : \mathbb{R}^{1+n} \to \mathbb{R}$. Various types of boundary conditions are allowed. The interesting feature in (1.120) is that the shifts σ_i may depend on the spatial variable ξ as well as on the function value $\phi(\xi)$ itself. This allows us to compute periodic solutions to MFDEs even when the period T is unknown. In particular, we consider T to be a variable and solve the following system on the interval

[0, 1],

$$
\begin{aligned}
\phi'(\xi) &= Tf\Big(\phi(\xi), \phi(\xi + r_1/T \mod 1), \ldots, \phi(\xi + r_N/T \mod 1)\Big), \\
T'(\xi) &= 0, \\
\phi(0) &= \phi(1), \\
P(\phi) &= 0.
\end{aligned}
\tag{1.121}
$$

Here $P(\phi)$ is a phase condition to counter the shift invariance of periodic solutions. We remark here that the computations of periodic solutions with unknown periods that can be found in this chapter are actually novel to the field of MFDEs.

The system (1.121) is solved by dividing the interval [0, 1] into L subintervals $[t_i, t_{i+1}]$ and representing $\phi(\xi)$ on each subinterval in terms of a standard Runge-Kutta monomial basis. The representation is required to be continuous at the boundary points between subintervals and in addition to satisfy (1.121) at the Ld collocation points $t_i + (t_{i+1} - t_i)c_j$, for $i = 1 \ldots L$ and $j = 1 \ldots d$, where $0 < c_1 < \ldots < c_d < 1$ are the Gaussian collocation points of degree d for some $d \geq 3$. Using Newton iterations such a piecewise polynomial ϕ can be found, given a sufficiently close initial estimate. For further details and a convergence analysis of this method, we refer to Chapter 5.

Floquet exponents

In [56] and [153] Floquet exponents associated to periodic solutions were computed by constructing the monodromy matrix, otherwise known as the $T_{2\pi}$-map, for the linearization around the periodic solutions. However, in the case of MFDEs this monodromy matrix is no longer available, since the equation is ill-defined and hence a different technique is needed.

Our approach will be to look for Floquet multipliers on the unit circle by direct construction of the associated Floquet eigenfunction. In particular, given a periodic solution p we will look for functions of the form $v(\xi) = e^{\lambda\xi}q(\xi)$ with $\lambda \in i\mathbb{R}$ and periodic $q \in C(\mathbb{R}, \mathbb{R}^n)$ with $q(\xi) = q(\xi + 2\pi)$, that solve the linearized equation

$$
v'(\xi) = Df(p_\xi)v_\xi. \tag{1.122}
$$

This can be solved within our framework of periodic solutions, by solving (1.122) on the interval $[0, 2\pi]$ with the boundary condition $v(2\pi) = e^{2\pi\lambda}v(0)$ and an appropriate rule $v(\xi + \sigma) = e^{2k\pi\lambda}v(\xi + \sigma - 2k\pi)$, where $k \in \mathbb{Z}$ needs to be chosen such that $\xi + \sigma - 2k\pi \in [0, 2\pi]$.

Generalized Floquet functions can be found in a similar fashion. From Section 4.6, we know that for any Jordan block associated to a Floquet exponent λ, a basis of generalized Floquet eigenfunctions $\{y_k\}$ with $0 \leq k \leq d$ can be constructed, with

$$
y^k(\xi) = e^{\lambda\xi}q^k(\xi) - \sum_{\ell=0}^{k-1} \frac{(-\xi)^{k-\ell}}{(k-\ell)!} y^\ell(\xi). \tag{1.123}
$$

Here $d + 1$ is the dimension of the Jordan block and the functions $q^k \in C(\mathbb{R}, \mathbb{R}^n)$ are periodic with $q^k(\xi + 2\pi) = q^k(\xi)$ for all $\xi \in \mathbb{R}$.

Proposition 1.8.1. *Consider a floquet exponent λ and let a basis for the generalized eigenspace be given by (1.123). Then we have*

$$y^k(\xi + 2\pi) = e^{2\pi\lambda} \sum_{\ell=0}^{k} y^\ell(\xi) \frac{(2\pi)^{k-\ell}}{(k-\ell)!}. \tag{1.124}$$

Proof. We use induction on the integer k. For $k = 0$ the statement is clear directly from (1.123). For $k > 0$, we assume that (1.124) holds for all integers $0 \leq \ell < k$ and compute

$$\begin{array}{rcl} y^k(\xi+2\pi) & = & e^{2\pi\lambda} e^{\lambda\xi} q^k(\xi+2\pi) - e^{2\pi\lambda} \sum_{\ell=0}^{k-1} \frac{(-\xi-2\pi)^{k-\ell}}{(k-\ell)!} \sum_{j=0}^{\ell} y^j(\xi) \frac{(2\pi)^{\ell-j}}{(\ell-j)!} \\ & = & e^{2\pi\lambda}\big[e^{\lambda\xi} q^k(\xi) - \sum_{j=0}^{k-1} y^j(\xi) \sum_{\ell=j}^{k-1} \frac{(-\xi-2\pi)^{k-\ell}}{(k-\ell)!} \frac{(2\pi)^{\ell-j}}{(\ell-j)!}\big] \\ & = & e^{2\pi\lambda}\big[e^{\lambda\xi} q^k(\xi) - \sum_{j=0}^{k-1} y^j(\xi) \frac{1}{(k-j)!}[(-\xi)^{k-j} - (2\pi)^{k-j}]\big] \\ & = & e^{2\pi\lambda}\big[y^k(\xi) + \sum_{j=0}^{k-1} y^j(\xi) \frac{(2\pi)^{k-j}}{(k-j)!}\big], \end{array} \tag{1.125}$$

which completes the proof. □

In practice, it is convenient to compute the periodic solution p simultaneously with the associated Floquet exponents. In order to detect the presence of a Jordan block of dimension $d + 1$ associated to a Floquet exponent λ, one would hence solve the following system on the interval $[0, 2\pi]$,

$$\begin{array}{rcl} p'(\xi) & = & \frac{T}{2\pi} f\Big(p(\xi), p(\xi + 2\pi r_1/T \bmod 2\pi), \ldots, p(\xi + 2\pi r_N/T \bmod 2\pi)\Big), \\ T'(\xi) & = & 0, \\ y^{k\prime}(\xi) & = & \frac{T}{2\pi} Df\Big(p(\xi), p(\xi + 2\pi r_1/T \bmod 2\pi), \ldots, \\ & & \quad p(\xi + 2\pi r_N/T \bmod 2\pi)\Big) \widetilde{\mathrm{ev}}_\xi y^k, \\ p(2\pi) & = & p(0), \\ P(p) & = & 0, \\ y^k(2\pi) & = & e^{2\pi\lambda} \sum_{\ell=0}^{k} y^\ell(0) \frac{(2\pi)^{k-\ell}}{(k-\ell)!}, \\ Y(y^0) & = & 0, \end{array} \tag{1.126}$$

in which $k = 0 \ldots d$. The last condition is a normalization on y^0. The expression $\widetilde{\mathrm{ev}}_\xi y^k$ should be interpreted as the vector $\big(y^k(\xi), y^k(\xi + 2\pi r_1/T), \ldots, y^k(\xi + 2\pi r_N/T)\big)$, which needs to be evaluated using repeated application of (1.124) to translate the arguments into the interval $[0, 2\pi]$.

Chapter 2

Center Manifolds Near Equilibria

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Abstract. We study the behaviour of solutions to nonlinear autonomous functional differential equations of mixed type in the neighbourhood of an equilibrium. We show that all solutions that remain sufficiently close to an equilibrium can be captured on a finite dimensional invariant center manifold, that inherits the smoothness of the nonlinearity. In addition, we provide a Hopf bifurcation theorem for such equations.

2.1. Introduction

The purpose of this chapter is to provide a tool to analyze the behaviour of solutions to a nonlinear functional differential equation of mixed type

$$x'(\xi) = G(x_\xi), \tag{2.1}$$

in the neighbourhood of an equilibrium $\overline{x}$. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \le 0$ and $r_{\max} \ge 0$, hence the nonlinearity G may depend on advanced and retarded arguments simultaneously.

We establish a center manifold theorem for solutions to (2.1) close to $\overline{x}$, that is, we show that all sufficiently small solutions to the equation

$$u'(\xi) = DG(\overline{x})u_\xi + (G(\overline{x} + u_\xi) - DG(\overline{x})u_\xi) \tag{2.2}$$

can be captured on a finite dimensional invariant manifold and we explicitly describe the dynamics on this manifold. This reduction allows us to establish a Hopf bifurcation theorem for (2.1), yielding a very powerful tool to perform a bifurcation analysis on parameter dependent versions of this equation. If the linearization $u'(\xi) = DG(\overline{x})u_\xi$ has no bounded solutions on the line, we say that the equilibrium $\overline{x}$ is hyperbolic and in this case the center manifold contains only the zero function. We will thus be particularly interested in situations where the linear operator $DG(\overline{x})$ has eigenvalues on the imaginary axis, implying that $\overline{x}$ is a nonhyperbolic equilibrium.

The study of center manifolds in infinite dimensions forms one of the cornerstones of the theory of dynamical systems. During the last two decades, many authors have contributed towards developing the general theory. We mention specially the comprehensive overview by Iooss and Vanderbauwhede [158] and the work of Mielke on elliptic partial differential equations [118, 119], in which linear unbounded operators that have infinite spectrum to the right and left of the imaginary axis were analyzed. This type of operator also arises when studying (2.1), but our approach in this chapter is more closely related to the ideas developed in [45], where the theory of semigroups was used to succesfully construct center manifolds for delay equations.

When studying the nonlinear mixed-type functional differential equation (2.1), it is essential to have results for linear systems

$$x'(\xi) = L(\xi)x_{\xi} + f(\xi). \tag{2.3}$$

Mallet-Paret provided the basic theory in [112], showing that a Fredholm alternative theorem holds for hyperbolic systems (2.3) and providing exponential estimates for solutions to such equations. Later, the existence of exponential dichotomies for (2.3) was established independently by Mallet-Paret and Verduyn Lunel [115] and Härterich et al. [75]. In the present work, we extend the framework developed in [112] to nonhyperbolic but autonomous versions of (2.3), which allows us to generalize the center manifold theory developed for delay equations in [45] to equations of mixed type.

Our main results are stated in Section 2.2 and proved in Sections 2.3 through 2.10, where the necessary theory is developed. In particular, in Section 2.3 we discuss and apply the results of Mallet Paret to linear systems (2.3) that violate the hyperbolicity condition needed in [112]. In Section 2.4 we introduce an operator associated with (2.3) on the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$, that in the case of delay equations reduces to the generator of the semigroup associated with the homogeneous version of (2.3). Laplace transform techniques are used in Section 2.5 to combine the results from the previous two sections in order to define a pseudo-inverse $\mathcal{K}$ for (2.3), in the sense that inhomogeneities f are mapped to their corresponding solutions $x = \mathcal{K}f$ modulo a finite dimensional set of solutions to (2.3) with $f = 0$. This set is isomorphic to a finite dimensional subspace $X_0 \subset X$ and the operator $\mathcal{K}$ is used in Section 2.6 in a fixed point argument to construct small solutions $u^*\phi$ to the nonlinear equation (2.2) for any small $\phi \in X_0$. This map u^* is shown to be of class C^k in Section 2.7, while Section 2.8 shows that these small solutions can in fact be described as solutions to a finite dimensional ordinary differential equation. This reduction is used in Section 2.10 to establish a Hopf bifurcation theorem for parameter dependent versions of (2.2). Finally, in Section 2.11 we discuss some examples and explicitly describe the dynamics on the center manifold for a functional differential equation of mixed type that admits a double eigenvalue at zero after linearization. In particular, we exhibit a Takens-Burganov bifurcation and show that for delay equations the results from [45] can be recovered from our framework.

2.2. Main Results

Consider for some $N \geq 0$ the functional differential equation of mixed type

$$x'(\xi) = \sum_{j=0}^{N} A_j x(\xi + r_j) + R(x(\xi + r_0), \ldots, x(\xi + r_N)), \tag{2.4}$$

in which x is a mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$ and each A_j is a $n \times n$ matrix with complex entries. The shifts $r_j \in \mathbb{R}$ may be both positive and negative and for convenience we assume that they are ordered and distinct, i.e., $r_0 < r_1 < \ldots < r_N$. Defining $r_{\min} = r_0$ and $r_{\max} = r_N$, we require $r_{\min} \leq 0 \leq r_{\max}$.

The space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ of continuous $\mathbb{C}^n$-valued functions defined on the interval $[r_{\min}, r_{\max}]$ will serve as a state space when analyzing (2.4). In particular, for any $x \in C(\mathbb{R}, \mathbb{C}^n)$ and any $\xi \in \mathbb{R}$, we define the state $x_\xi \in X$ as the function $x_\xi(\theta) = x(\xi + \theta)$ for any $r_{\min} \leq \theta \leq r_{\max}$. Introducing the bounded linear operator $L : X \to \mathbb{C}^n$ given by

$$L\phi = \sum_{j=0}^{N} A_j \phi(r_j), \tag{2.5}$$

one can rewrite (2.4) as

$$x'(\xi) = Lx_\xi + R(x_\xi). \tag{2.6}$$

In our analysis of (2.6) we will be particularly interested in the scale of Banach spaces

$$BC_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in C(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |x(\xi)| < \infty \right\}, \tag{2.7}$$

parametrized by $\eta \in \mathbb{R}$. The corresponding norm is given by $\|x\|_\eta = \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} |x(\xi)|$. We also need the Banach spaces

$$BC^1_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n) \mid x' \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \right\}, \tag{2.8}$$

with corresponding norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \|x'\|_\eta$. Notice that we have continuous inclusions $BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ and $BC^1_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ for any pair $\eta_2 \geq \eta_1$. We will write $\mathcal{J}_{\eta_2\eta_1}$ and $\mathcal{J}^1_{\eta_2\eta_1}$ respectively for the corresponding embedding operators.

In the analysis of (2.6), it is essential to study the behaviour of the homogeneous linear equation

$$x'(\xi) = Lx_\xi. \tag{2.9}$$

Associated with this system (2.9) one has the characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$, given by

$$\Delta(z) = zI - \sum_{j=0}^{N} A_j e^{zr_j}. \tag{2.10}$$

A value of z such that $\det \Delta(z) = 0$ is called an eigenvalue for the system (2.9). In order to formulate our main results, we need the following proposition.

Proposition 2.2.1. *For any homogeneous linear equation of the form (2.9), there exists a finite dimensional linear subspace $X_0 \subset X$ with the following properties.*

(i) Suppose $x \in \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (2.9). Then for any $\xi \in \mathbb{R}$ we have $x_\xi \in X_0$.

(ii) For any $\phi \in X_0$, we have $\phi' \in X_0$.

(iii) For any $\phi \in X_0$, there is a solution $x = E\phi \in \bigcap_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ of (2.9) that has $x_0 = \phi$. This solution is unique in the set $\bigcup_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$.

We will write Q_0 for the projection operator from X onto X_0, which will be defined precisely in the sequel. The following two assumptions on the nonlinearity $R : X \to \mathbb{C}^n$ will be needed in our results.

(HR1) The nonlinearity R is C^k-smooth for some $k \geq 1$.

(HR2) We have $R(0) = 0$ and $DR(0) = 0$.

We remark here that the smoothness requirement in condition (HR1) in fact refers to the Fréchet differentiability of R, since this operator is defined on the infinite dimensional space X. This technicality should be implicitly understood throughout the remainder of this chapter. However, one should note that this issue becomes irrelevant when considering nonlinearities R as in (2.4), which have a finite dimensional domain.

Theorem 2.2.2. *Consider the nonlinear equation (2.6) and assume that (HR1) and (HR2) are satisfied. Then there exists $\gamma > 0$ such that the characteristic equation* $\det \Delta(z) = 0$ *has no roots with $0 < |\mathrm{Re}\, z| < \gamma$. Fix an interval $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ such that $\eta_{\max} > k\eta_{\min}$, with k as introduced in (HR1). Then there exists a mapping $u^* : X_0 \to \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ and constants $\epsilon > 0$, $\epsilon^* > 0$ such that the following statements hold.*

(i) For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, the function u^* viewed as a map from X_0 into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is C^k-smooth.

(ii) Suppose for some $\zeta > 0$ that $x \in BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (2.6) with $\sup_{\xi\in\mathbb{R}} |x(\xi)| < \epsilon^*$. Then we have $x = u^*(Q_0 x_0)$. In addition, the function $\Phi : \mathbb{R} \to X_0$ defined by $\Phi(\xi) = Q_0 x_\xi \in X_0$ is of class C^{k+1} and satisfies the ordinary differential equation

$$\Phi'(\xi) = A\Phi(\xi) + f(\Phi(\xi)), \tag{2.11}$$

in which $A : X_0 \to X_0$ is the linear operator $\phi \to \phi'$ for $\phi \in X_0$. The function $f : X_0 \to X_0$ is C^k-smooth with $f(0) = 0$ and $Df(0) = 0$ and is explicitly given by

$$f(\psi) = Q_0\big(L(u^*\psi - E\psi)_\theta + R((u^*\psi)_\theta)\big), \tag{2.12}$$

in which the projection Q_0 is taken with respect to the variable θ. Finally, we have $x_\xi = (u^*\Phi(\xi))_0$ for all $\xi \in \mathbb{R}$.

(iii) *For any* $\phi \in X_0$ *such that* $\sup_{\xi \in \mathbb{R}} |(u^*\phi)(\xi)| < \epsilon^*$, *the function* $u^*\phi$ *satisfies (2.6).*

(iv) *For any continuous function* $\Phi : \mathbb{R} \to X_0$ *that satisfies (2.11) and has* $\|\Phi(\xi)\| < \epsilon$ *for all* $\xi \in \mathbb{R}$, *we have that* $x = u^*\Phi(0)$ *is a solution of (2.6). In addition, we have* $x_\xi = (u^*\Phi(\xi))_0$ *for any* $\xi \in \mathbb{R}$.

(v) *Consider the interval* $I = (\xi_-, \xi_+)$, *where* $\xi_- = -\infty$ *and* $\xi_+ = \infty$ *are allowed. Let* $\Phi : I \to X_0$ *be a continuous function that satisfies (2.11) for every* $\xi \in I$ *and in addition has* $\|\Phi(\xi)\| < \epsilon$ *for all such* ξ. *Then for any* $\zeta \in (\xi_-, \xi_+)$ *we have that* $x(\xi) = (u^*\Phi(\zeta))(\xi - \zeta)$ *satisfies (2.6) for all* $\xi \in I$. *In addition, we have* $x_\xi = (u^*\Phi(\xi))_0$ *for all* $\xi \in I$.

The results above should be compared to similar results for delay differential equations, see e.g. [45, Chp. VIII and IX]. When considering delay equations, it is possible to capture all sufficiently small solutions defined only on the half lines $\mathbb{R}_\pm$ on invariant manifolds. This feature is absent when considering mixed type equations, due to the fact that (2.9) is ill-posed as an initial value problem. We believe that the same ill-posedness can be used to explain the fact that the nonlinearity (2.12) on the center manifold cannot immediately be simplified to its delay equation counterpart [45, (IX.8.3)].

An interesting application of statement (v) above arises when one considers functional differential equations of mixed type on finite intervals. This situation arises for example when studying numerical methods to solve such equations on the line, as in Chapter 5. These methods typically truncate the problem to a finite interval, possibly introducing extra solutions. The center manifold reduction will allow us to at least partially analyze the presence of such solutions. Other preliminary research in this area can be found in [115].

In order to state the Hopf bifurcation theorem, it is necessary to include parameter dependance into our framework. In particular, we introduce an open parameter set $\Omega \subset \mathbb{R}^d$ for some integer $d \geq 1$ and consider for $\mu \in \Omega$ the equation

$$x'(\xi) = L(\mu)x_\xi + R(x_\xi, \mu), \tag{2.13}$$

in which R is a nonlinear mapping from $X \times \Omega$ into $\mathbb{C}^n$ and

$$L(\mu)\phi = \sum_{j=0}^{N} A_j(\mu)\phi(r_j). \tag{2.14}$$

We will need the following assumptions on the system (2.13).

(HLμ) The mapping $(\mu, \phi) \to L(\mu)\phi$ from $\Omega \times X$ into $\mathbb{C}^n$ is C^k-smooth for some $k \geq 1$.

(HRμ1) The nonlinearity $R : X \times \Omega \to \mathbb{C}^n$ is C^k-smooth for some $k \geq 1$.

(HRμ2) We have $R(0, \mu) = 0$ and $D_1 R(0, \mu) = 0$ for all $\mu \in \Omega$.

These assumptions are sufficient in order to rewrite the parameter dependent equation (2.13) as an equation of the form (2.6) that satisfies the assumptions of Theorem 2.2.2. This implies that for fixed $\mu_0 \in \Omega$ and corresponding subspace $X_0 = X_0(\mu_0) \subset X$, it is possible to define a mapping $u^* : X_0 \times \Omega \to \bigcap_{\zeta > 0} BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ that is C^k-smooth when considered as a map into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ for suitable values of η. To establish the Hopf bifurcation theorem, we also need the following.

(Hζ1) The parameter space is one-dimensional, i.e., $d = 1$. In addition, the matrices $A_j(\mu)$ have real valued coefficients and the nonlinearity R maps into $\mathbb{R}^n$. Finally, in (HLμ) and (HRμ1) we have $k \geq 2$.

(Hζ2) For some $\mu_0 \in \Omega$ and $\omega_0 > 0$, the characteristic equation $\det \Delta(z, \mu_0) = 0$ has simple roots at $z = \pm i\omega_0$ and no other root belongs to $i\omega_0\mathbb{Z}$.

(Hζ3) Letting $p, q \in \mathbb{C}^n$ be non-zero vectors such that $\Delta(i\omega_0, \mu_0)p = 0$ and $\Delta(i\omega_0, \mu_0)^T q = 0$, normalized such that $q^T D_1\Delta(i\omega_0, \mu_0)p = 1$, we have that $\operatorname{Re} q^T D_2\Delta(i\omega_0, \mu_0)p \neq 0$.

With p as in (Hζ3), we can define the functions $\phi = pe^{i\omega_0\cdot}$ and $\overline{\phi} = \overline{p}e^{-i\omega_0\cdot}$ and it is easy to see that both functions are solutions to the homogeneous equation $x' = L(\mu_0)x_\xi$.

Theorem 2.2.3. *Consider the nonlinear equation (2.13) and assume that (HLμ), (HRμ1), (HRμ2) and (Hζ1) -(Hζ3) all hold. There exist C^{k-1}-smooth functions $\tau \to \mu^*(\tau)$, $\tau \to \rho^*(\tau)$ and $\tau \to \omega^*(\tau)$ taking values in $\mathbb{R}$ and a mapping $\tau \to \psi^*(\tau)$ taking values in X_0, all defined for τ sufficiently small, such that $x^*(\tau) = u^*(\rho^*(\tau)(\phi + \overline{\phi} + \psi^*(\tau)), \mu^*(\tau))$ is a periodic solution of (2.13) at $\mu = \mu^*(\tau)$ with period $\frac{2\pi}{\omega^*(\tau)}$. Moreover, $\mu^*(\tau)$ and $\omega^*(\tau)$ are even in τ, $\mu^*(0) = \mu_0$ and if x is any sufficiently small periodic solution of (2.13) with μ close to μ_0 and period close to $\frac{2\pi}{\omega_0}$, then $\mu = \mu^*(\tau)$ for some τ and there exists $\xi_0 \in [0, \frac{2\pi}{\omega^*(\tau)})$ such that $x(\cdot + \xi_0) = x^*(\tau)(\cdot)$. Finally, we have $\rho^*(\tau) = \tau + o(\tau)$ and $\psi^*(\tau) = o(1)$ as $\tau \to 0$.*

We wish to emphasize here that the corresponding result for delay equations [45, Chp X] can be recovered almost verbatim from the conditions and statement above by making the appropriate restrictions. Our last main theorem establishes a result on the direction of the Hopf bifurcation.

Theorem 2.2.4. *Consider the nonlinear equation (2.13) and assume that (HLμ), (HRμ1), (HRμ2) and (Hζ1) -(Hζ3) all hold, but with $k \geq 3$ in (Hζ1). Let $\mu^*(\tau)$ be as defined in Theorem 2.2.3. Then we have $\mu^*(\tau) = \mu_0 + \mu_2\tau^2 + o(\tau^2)$, with*

$$\mu_2 = \frac{\operatorname{Re} c}{\operatorname{Re} q^T D_2\Delta(i\omega_0, \mu_0)p}, \tag{2.15}$$

in which

$$\begin{aligned} c \;=\; & \tfrac{1}{2}q^T D_1^3 R(0, \mu_0)(\phi, \phi, \overline{\phi}) \\ & + q^T D_1^2 R(0, \mu_0)(\phi, \mathbf{1}\Delta(0, \mu_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \overline{\phi})) \\ & + \tfrac{1}{2}q^T D_1^2 R(0, \mu_0)(\overline{\phi}, e^{2i\omega_0\cdot}\Delta(2i\omega_0, \mu_0)^{-1} D_1^2 R(0, \mu_0)(\phi, \phi)). \end{aligned} \tag{2.16}$$

We conclude this section by remarking that the restriction to point delays in (2.4) is merely a notational convenience to improve the readability of our arguments. In fact, all results carry over almost verbatim to the more general system (2.6) with arbitrary linear $L : X \to \mathbb{C}^n$ and nonlinear $R : X \to \mathbb{C}^n$.

2.3. Linear Inhomogeneous Equations

In this section we develop some results for linear inhomogeneous functional differential equations of mixed type,

$$x'(\xi) = Lx_\xi + f(\xi). \tag{2.17}$$

The techniques used here should be compared to similar ones employed in the context of delay equations, see e.g. [11, 90].

For the moment we take $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, with the bounded linear operator L as defined in (2.5). Associated to the system (2.17) we define a linear operator $\Lambda : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda x)(\xi) = x'(\xi) - Lx_\xi. \tag{2.18}$$

We recall the characteristic matrix $\Delta(z)$ associated to (2.17) as defined in (2.10),

$$\Delta(z) = zI - \sum_{j=0}^{N} A_j e^{zr_j}. \tag{2.19}$$

The following result establishes some elementary properties concerning the behaviour of $\Delta(z)$ on vertical strips in the complex plane.

Lemma 2.3.1. *Consider any closed vertical strip $S = \{z \in \mathbb{C} \mid \gamma_- \le \operatorname{Re} z \le \gamma_+\}$ and for any $\rho > 0$ define $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. Then there exist $K, \rho > 0$ such that* $\det \Delta(z) \neq 0$ *for all $z \in S_\rho$ and in addition $\left|\Delta(z)^{-1}\right| < \frac{K}{|\operatorname{Im} z|}$ for each such z. In particular, there are only finitely many zeroes of* $\det \Delta(z)$ *in S. Furthermore, if* $\det \Delta(z) \neq 0$ *for all $z \in S$, then for any $\alpha \notin S$ the function $R_\alpha(z) = \Delta(z)^{-1} - (z - \alpha)^{-1} I$ is holomorphic in an open neighbourhood of S and in addition there exists $K > 0$ such that $|R_\alpha(z)| \le \frac{K}{1+|\operatorname{Im} z|^2}$ for all $z \in S$.*

Proof. For any $z \in S$, define $A(z) = \sum_{j=0}^{N} A_j e^{zr_j}$ and $\overline{A} = \sup_{z \in S} |A(z)| < \infty$. For any $z \in S$ with $|z| > 2\overline{A}$, we have that $\Delta(z) = z(I - \frac{A(z)}{z})$ is invertible. The inverse is given by

$$\Delta(z)^{-1} = \frac{1}{z} \sum_{j=0}^{\infty} \frac{A(z)^j}{z^j}, \tag{2.20}$$

and satisfies the bound

$$\left|\Delta(z)^{-1}\right| \le \frac{1}{|z|\,(1 - \frac{1}{|z|}\,|A(z)|)} \le \frac{2}{|z|}. \tag{2.21}$$

Now consider the case that $\det \Delta(z) \neq 0$ for all $z \in S$. Since all zeroes of $\det \Delta(z)$ are isolated, there exists an open neighbourhood of S on which $\Delta(z)^{-1}$ and hence $R_\alpha(z)$ is holomorphic. Note that for $|z| > 2\overline{A}$ we have

$$\begin{aligned} |R_\alpha(z)| = \left| \tfrac{\alpha}{z(\alpha - z)} I + \tfrac{1}{z} \textstyle\sum_{j=1}^{\infty} \tfrac{A(z)^j}{z^j} \right| &\le \tfrac{|\alpha|}{|z(z-\alpha)|} + \tfrac{|A(z)|}{|z|^2} \tfrac{1}{1 - \frac{|A(z)|}{|z|}} \\ &\le \tfrac{|\alpha|}{|z(z-\alpha)|} + \tfrac{2\overline{A}}{|z|^2}, \end{aligned} \tag{2.22}$$

which yields the final estimate using the fact that $R_\alpha(z)$ is bounded on the set $\{z \in S \mid |z| < 2\overline{A}\}$. □

The inhomogeneous system (2.17) has been analyzed with respect to the space

$$W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) = \{x \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid x \text{ is absolutely continuous and } x' \in L^\infty(\mathbb{R}, \mathbb{C}^n)\} \tag{2.23}$$

by Mallet-Paret in [112], where he obtained the following result.

Theorem 2.3.2 (Mallet-Paret)**.** *Consider the operator L in (2.5) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has no roots on the imaginary axis. Then the operator* Λ *defined in (2.18) is a bounded linear isomorphism from* $W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ *onto* $L^\infty(\mathbb{R}, \mathbb{C}^n)$*. In particular, there exists a Green's function* $G : \mathbb{R} \to \mathbb{C}^{n\times n}$ *such that the equation* $\Lambda x = f$ *has the unique solution*

$$x(\xi) = \int_{-\infty}^{\infty} G(\xi - s) f(s) ds. \tag{2.24}$$

In addition, we have $G \in L^p(\mathbb{R}, \mathbb{C}^{n\times n})$ *for any* $1 \le p \le \infty$ *and the following identity holds for the Fourier transform (B.1) of G,*

$$\widehat{G}(\eta) = \Delta(i\eta)^{-1}. \tag{2.25}$$

Corollary 2.3.3. *Fix an* $a_- < 0$ *and* $a_+ > 0$ *such that* $\det \Delta(z) \neq 0$ *for all* $a_- \le \operatorname{Re} z \le a_+$ *and choose an* $\alpha < a_-$*. Then we have*

$$|G(\xi)| \le \begin{cases} (1 + K(a_-))e^{a_-\xi} & \text{for all } \xi \ge 0, \\ K(a_+)e^{a_+\xi} & \text{for all } \xi < 0, \end{cases} \tag{2.26}$$

in which

$$K(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |R_\alpha(a + i\omega)| \, d\omega. \tag{2.27}$$

In particular, we have the estimate

$$\left\|\Lambda^{-1}\right\| \le 1 + (\frac{1 + K(a_-)}{-a_-} + \frac{K(a_+)}{a_+})(1 + \sum_{j=0}^{N} |A_j|). \tag{2.28}$$

Finally, suppose f satisfies a growth condition $f(\xi) = O(e^{-\lambda\xi})$ *as* $\xi \to \infty$ *for some* $0 < \lambda < -a_-$*. Then also* $x = \Lambda^{-1} f$ *satisfies* $x(\xi) = O(e^{-\lambda\xi})$ *as* $\xi \to \infty$*. The analogous statement also holds for* $\xi \to -\infty$*.*

Proof. Write $\Delta(z)^{-1} = (z - \alpha)^{-1} I + R_\alpha(z)$. Writing $E(\xi)$ for the inverse transform of $(z - \alpha)^{-1}$, we have that $E(\xi) = e^{\alpha\xi}$ for $\xi > 0$ while $E(\xi) = 0$ for $\xi < 0$. We thus obtain for $\xi > 0$

$$G(\xi) = e^{\alpha\xi} I + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\omega} R_\alpha(i\omega) d\omega = e^{\alpha\xi} I + \frac{e^{a_-\xi}}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\omega} R_\alpha(a_- + i\omega) d\omega, \tag{2.29}$$

where the integration contour was shifted to the line $\operatorname{Re} z = a_-$ in the last step. A similar estimate can be obtained for $\xi < 0$ by shifting the integration contour to $\operatorname{Re} z = a_+$. Lemma 2.3.1 ensures that both $R_\alpha(a_- + i\omega)$ and $R_\alpha(a_+ + i\omega)$ are integrable and this concludes the proof of the exponential decay of G.

Consider the equation $\Lambda x = f$ and notice that $\|x\|_{L^\infty} \le \|G\|_{L^1} \|f\|_{L^\infty}$. Using the estimates above we compute $\|G\|_{L^1} \le \frac{1+K(a_-)}{-a_-} + \frac{K(a_+)}{a_+}$. The differential equation (2.17) now implies

$$\|x\|_{W^{1,\infty}} = \|x\|_{L^\infty} + \left\|x'\right\|_{L^\infty} \le \|x\|_{L^\infty} + \|f\|_{L^\infty} + \sum_{j=0}^{N} \left|A_j\right| \|x\|_{L^\infty}, \tag{2.30}$$

from which the bound for $\left\|\Lambda^{-1}\right\|$ follows.

Finally, if $f(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$, there exists $M > 0$ such that $|f(\xi)| \le Me^{-\lambda\xi}$ for all $\xi > 0$. Hence for all such ξ we compute

$$\begin{array}{rcl} x(\xi) & = & \int_{-\infty}^{\infty} G(\xi - s) f(s) ds \le \frac{1+K(a_-)}{-a_-} e^{a_- \xi} \|f\|_\infty + \int_0^\infty G(\xi - s) f(s) ds \\ & \le & \frac{1+K(a_-)}{-a_-} e^{a_- \xi} \|f\|_\infty + (1 + K(a_-)) e^{a_- \xi} \frac{M}{-a_- - \lambda} \left(e^{(-a_- - \lambda)\xi} - 1\right) \\ & & + \frac{M}{\lambda + a_+} K(a_+) e^{-\lambda \xi}, \end{array} \tag{2.31}$$

which concludes the proof. □

In order to proceed, we need to generalize the results above to the situation where the characteristic equation does have roots on the imaginary axis. The key observation which we shall use is that one can shift the roots of the characteristic equation by multiplying the functions in (2.17) by a suitable exponential. In order to make this precise, we introduce the notation $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$.

Taking any $y \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$, one can compute

$$\left(e_{-\eta} \Lambda e_\eta y\right)(\xi) = y'(\xi) + \eta y(\xi) - \sum_{j=0}^{N} A_j e^{\eta r_j} y(\xi + r_j). \tag{2.32}$$

Upon defining the linear operator $\Lambda_\eta : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda_\eta x)(\xi) = x'(\xi) - \eta x(\xi) - \sum_{j=0}^{N} A_j e^{-\eta r_j} x(\xi + r_j) \tag{2.33}$$

and writing $\Delta_\eta(z)$ for the corresponding characteristic matrix, we see that for any $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ we have

$$\Lambda_\eta e_\eta x = e_\eta \Lambda x \quad \text{and} \quad \Delta_\eta(z) = (z - \eta)I - \textstyle\sum_{j=0}^{N} A_j e^{(z-\eta) r_j} = \Delta(z - \eta). \tag{2.34}$$

In view of these observations we introduce for any $\eta \in \mathbb{R}$ the Banach spaces

$$\begin{array}{rcl} L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in L^\infty(\mathbb{R}, \mathbb{C}^n)\right\}, \\ W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)\right\}, \end{array} \tag{2.35}$$

with norms given by $\|x\|_{L^\infty_\eta} = \left\|e_{-\eta}x\right\|_{L^\infty}$ and similarly $\|x\|_{W^{1,\infty}_\eta} = \left\|e_{-\eta}x\right\|_{W^{1,\infty}}$. The next proposition provides the appropriate generalization of Theorem 2.3.2.

Proposition 2.3.4. *Fix $\eta \in \mathbb{R}$. Consider the operator L in (2.5) and suppose that the characteristic function $\Delta(z)$ has no eigenvalues with $\operatorname{Re} z = \eta$. The operator Λ is a bounded linear isomorphism from $W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$, with inverse given by $\Lambda^{-1} f = e_\eta \Lambda^{-1}_{-\eta} e_{-\eta} f$. In particular, we have $\left\|\Lambda^{-1}\right\| = \left\|\Lambda^{-1}_{-\eta}\right\|$. In addition, there exists $\epsilon_0 > 0$ such that $\Delta(z)$ has no eigenvalues in the strip $\eta - \epsilon_0 < \operatorname{Re} z < \eta + \epsilon_0$. Finally, for any $0 < \epsilon < \epsilon_0$ and $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$, we have the following explicit expression for $x = \Lambda^{-1} f$,*

$$x(\xi) = \frac{1}{2\pi i} \int_{\eta+\epsilon-i\infty}^{\eta+\epsilon+i\infty} e^{\xi z} \Delta(z)^{-1} \widetilde{f}_+(z)dz + \frac{1}{2\pi i} \int_{\eta-\epsilon-i\infty}^{\eta-\epsilon+i\infty} e^{\xi z} \Delta(z)^{-1} \widetilde{f}_-(z)dz, \quad (2.36)$$

where the Laplace transforms $\widetilde{f}_+$ and $\widetilde{f}_-$ are as defined in Appendix B.

Proof. Note that $\Delta_{-\eta}$ has no eigenvalues on the imaginary axis and hence $\Lambda_{-\eta}$ is an isomorphism from $W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ onto $L^\infty(\mathbb{R}, \mathbb{C}^n)$. Since e_η is an isometric isomorphism between $W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and $W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$ and also between $L^\infty(\mathbb{R}, \mathbb{C}^n)$ and $L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$, this proves that Λ is an isomorphism and yields the supplied bound for the norm of the inverse.

Now let $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ and consider $x = \Lambda^{-1} f \in W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$. Write $f = f_+ + f_-$ with $f_+(\xi) = 0$ for $\xi < 0$ and $f_-(\xi) = 0$ for $\xi \geq 0$. Let $x_\pm = \Lambda^{-1} f_\pm = e_\eta \Lambda^{-1}_{-\eta} e_{-\eta} f_\pm$. Using the exponential decay (2.26) of G for $a = \frac{\epsilon_0+\epsilon}{2}$, we easily see that $x_+(\xi) = O(e^{(\eta+a)\xi})$ as $\xi \to -\infty$, and similarly $x_-(\xi) = O(e^{(\eta-a)\xi})$ as $\xi \to \infty$. Using the differential equation (2.17) one sees that similar asymptotic estimates apply for $x'_\pm$. This implies that both $\overline{x}_\pm = e_{-(\eta\pm\epsilon)} x_\pm$ and their first derivatives have exponential decay at both $\pm\infty$ and in particular satisfy $\overline{x}_\pm \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W^{1,2}(\mathbb{R}, \mathbb{C}^n) \cap W^{1,1}(\mathbb{R}, \mathbb{C}^n)$. Similarly, upon defining $\overline{f}_\pm = e_{-(\eta\pm\epsilon)} f_\pm$, we easily see $\overline{f}_\pm \in L^\infty(\mathbb{R}, \mathbb{C}^n) \cap L^1(\mathbb{R}, \mathbb{C}^n) \cap L^2(\mathbb{R}, \mathbb{C}^n)$. Using the identity (2.34) and the fact that both $\Lambda_{-(\eta\pm\epsilon)}$ are isomorphisms from $W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ onto $L^\infty(\mathbb{R}, \mathbb{C}^n)$, we have $\overline{x}_\pm = \Lambda^{-1}_{-(\eta\pm\epsilon)} \overline{f}_\pm$. Since $\overline{x}_\pm, \overline{f}_\pm \in L^2(\mathbb{R}, \mathbb{C}^n) \cap L^1(\mathbb{R}, \mathbb{C}^n)$ we may take the Fourier transform and obtain

$$\widehat{\overline{x}}_\pm(k) = \Delta^{-1}_{-(\eta\pm\epsilon)}(ik) \widehat{\overline{f}}_\pm(k). \quad (2.37)$$

Inversion yields

$$\overline{x}_\pm(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\xi} \Delta^{-1}_{-(\eta\pm\epsilon)}(ik) \widehat{\overline{f}}_\pm(k)dk. \quad (2.38)$$

Writing $z = \eta \pm \epsilon + ik$ and noting $\Delta_{-(\eta\pm\epsilon)}(ik) = \Delta(z)$ together with

$$\begin{array}{rclcl} \widehat{\overline{f}}_+(k) &=& \int_0^\infty e^{-ik\xi} e^{-(\eta+\epsilon)\xi} f_+(\xi)d\xi &=& \widetilde{f}_+(z), \\ \widehat{\overline{f}}_-(k) &=& \int_{-\infty}^0 e^{-ik\xi} e^{-(\eta-\epsilon)\xi} f_-(\xi)d\xi &=& \widetilde{f}_-(z), \end{array} \quad (2.39)$$

we obtain

$$x_\pm(\xi) = \frac{1}{2\pi i} \int_{\eta\pm\epsilon-i\infty}^{\eta\pm\epsilon+i\infty} e^{z\xi} \Delta(z)^{-1} \widetilde{f}_\pm(z)dz. \quad (2.40)$$

□

2.4. The State Space

In this section we focus our attention on the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$. We define a closed and densely defined operator $A : \mathcal{D}(A) \subset X \to X$, via

$$\begin{array}{lcl} \mathcal{D}(A) & = & \left\{\phi \in X \cap C^1([r_{\min}, r_{\max}], \mathbb{C}^n) \mid \phi'(0) = L\phi = \sum_{j=0}^{N} A_j \phi(r_j)\right\}, \\ A\phi & = & \phi'. \end{array} \tag{2.41}$$

Note that the closedness of A can be easily established using the fact that differentiation is a closed operation, together with the continuity of L. The density of the domain $\mathcal{D}(A)$ follows from the density of C^1-smooth functions in X, together with the fact that for any $\epsilon > 0$ and any neighbourhood of zero, one can modify an arbitrary C^1 function ϕ in such a way that $\phi'(0)$ can be set at will, while $\phi(0)$ remains unchanged and $\|\phi\|_X$ changes by at most ϵ. The first lemma of this section shows that X is indeed a state space for the homogeneous equation $\Lambda x = 0$ in some sense, even though one cannot view this equation as an initial value problem.

Lemma 2.4.1. *Suppose that for some $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n)$ we have the identity $\Lambda x = 0$ with $x_{\xi_0} = 0$ for some $\xi_0 \in \mathbb{R}$. If x satisfies the growth condition $x(\xi) = O(e^{b\xi})$ as $\xi \to \infty$ for any $b \in \mathbb{R}$, then $x(\xi) = 0$ for all $\xi \geq \xi_0 + r_{\min}$. Similarly, if $x(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then $x(\xi) = 0$ for all $\xi \leq \xi_0 + r_{\max}$.*

Proof. Without loss of generality take $\xi_0 = 0$ and assume that the growth condition at $+\infty$ holds. Introducing the function y with $y(\xi) = 0$ for all $\xi \leq 0$ and $y(\xi) = x(\xi)$ for all $\xi > 0$, we see that $\Lambda y = 0$. Consider any $\eta > b$ such that $\det \Delta(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z = \eta$ and notice that $y \in W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. It now follows from Proposition 2.3.4 that $y = 0$. □

The next lemma establishes the relationship between the characteristic equation $\det \Delta(z) = 0$ and the spectrum of A.

Lemma 2.4.2. *The operator A has only point spectrum, with $\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}$. In addition, for $z \in \rho(A)$, the resolvent of A is given by*

$$(zI - A)^{-1}\psi = e^{\cdot z} K(\cdot, z, \psi), \tag{2.42}$$

in which $K : [r_{\min}, r_{\max}] \times \mathbb{C} \times X \to \mathbb{C}^n$ is given by

$$K(\theta, z, \psi) = \int_{\theta}^{0} e^{-z\sigma}\psi(\sigma)d\sigma + \Delta(z)^{-1}\big(\psi(0) + \sum_{j=0}^{N} A_j e^{zr_j} \int_{r_j}^{0} e^{-z\sigma}\psi(\sigma)d\sigma\big). \tag{2.43}$$

Proof. Fix $\psi \in X$ and consider the equation $(zI - A)\phi = \psi$ for $\phi \in \mathcal{D}(A)$, which is equivalent to the system

$$\begin{array}{lcl} \phi' & = & z\phi - \psi, \\ \phi'(0) & = & \sum_{j=0}^{N} A_j \phi(r_j). \end{array} \tag{2.44}$$

Suppose $\det \Delta(z) \neq 0$. Solving the first equation yields

$$\phi(\theta) = e^{\theta z}\phi(0) + e^{\theta z}\int_{\theta}^{0} e^{-z\sigma}\,\psi(\sigma)d\sigma \tag{2.45}$$

and hence using the second equation

$$\phi'(0) = z\phi(0) - \psi(0) = \sum_{j=0}^{N} A_j e^{zr_j}\big(\phi(0) + \int_{r_j}^{0} e^{-z\sigma}\,\psi(\sigma)d\sigma\big). \tag{2.46}$$

Thus if we set

$$\phi(0) = \Delta(z)^{-1}\big(\psi(0) + \sum_{j=0}^{N} A_j e^{zr_j}\int_{r_j}^{0} e^{-z\sigma}\,\psi(\sigma)d\sigma\big), \tag{2.47}$$

we see that (2.45) yields a solution to (2.44), showing that indeed $z \in \rho(A)$. On the other hand, consider any $z \in \mathbb{C}$ such that $\det \Delta(z) = 0$. Choosing a non-zero $v \in \mathbb{R}^n$ such that $\Delta(z)v = 0$, one sees that the function $\phi(\theta) = e^{z\theta}v$ satisfies $\phi \in \mathcal{D}(A)$ and $A\phi = z\phi$. This shows that $z \in \sigma_p(A)$, completing the proof. □

The next lemma enables us to compute spectral projections corresponding to sets of eigenvalues in vertical strips in the complex plane. We will particularly be interested in the projection operator corresponding to all eigenvalues on the imaginary axis.

Lemma 2.4.3. *For any pair $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$, set $\Sigma = \Sigma_{\mu,\nu} = \{z \in \sigma(A) \mid \mu < \operatorname{Re} z < \nu\}$. Then Σ is a finite set, consisting of poles of $(zI - A)^{-1}$ that all have finite order. Furthermore, we have the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$, where $\mathcal{M}_\Sigma$ is the generalized eigenspace corresponding to the eigenvalues in Σ. For any $\mu < \gamma_- < \gamma_+ < \nu$ such that $\Sigma_{\gamma_-,\gamma_+} = \Sigma$, the spectral projection Q_Σ onto $\mathcal{M}_\Sigma$ along $\mathcal{R}_\Sigma$ is given by*

$$(Q_\Sigma\phi)(\theta) = \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\theta z}K(\theta, z, \phi)dz + \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\theta z}K(\theta, z, \phi)dz. \tag{2.48}$$

If there are no $z \in \sigma(A)$ with $\operatorname{Re} z = \mu$, then $\gamma_- = \mu$ is allowed. Similary, one may choose $\gamma_+ = \nu$ if there are no $z \in \sigma(A)$ with $\operatorname{Re} z = \nu$.

Proof. Lemma 2.3.1 shows that Σ is finite. Since $\det \Delta(z)$ is a non-zero entire function all zeroes are of finite order, hence the representation (2.42) implies that $(zI - A)^{-1}$ has a pole of order $k \leq k_0$ at λ_0 if λ_0 is a zero of $\det \Delta(z)$ of order k_0. It now follows from standard spectral theory (see e.g. [45, Theorem IV.2.5]) that we have the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$, for some closed linear subspace $\mathcal{M}_\Sigma$. Using Dunford calculus, it follows that for any Jordan path $\Gamma \subset \rho(A)$ with $\operatorname{int}(\Gamma) \cap \sigma(A) = \Sigma$, we have

$$Q_\Sigma = \frac{1}{2\pi i}\int_{\Gamma}(zI - A)^{-1}dz. \tag{2.49}$$

For any $\rho > 0$ such that $|\mathrm{Im}\,\lambda| < \rho$ for any $\lambda \in \Sigma$, we introduce the path $\Gamma_\rho = \Gamma_\rho^{\uparrow} \cup \Gamma_\rho^{\leftarrow} \cup \Gamma_\rho^{\downarrow} \cup \Gamma_\rho^{\rightarrow}$, in which we have introduced the line segments

$$\begin{array}{rclrcl} \Gamma_\rho^{\uparrow} & = & \mathrm{seg}[\gamma_+ - i\rho, \gamma_+ + i\rho], & \Gamma_\rho^{\downarrow} & = & \mathrm{seg}[\gamma_- + i\rho, \gamma_- - i\rho], \\ \Gamma_\rho^{\leftarrow} & = & \mathrm{seg}[\gamma_+ + i\rho, \gamma_- + i\rho], & \Gamma_\rho^{\rightarrow} & = & \mathrm{seg}[\gamma_- - i\rho, \gamma_+ - i\rho]. \end{array} \tag{2.50}$$

Note that the proof is completed if we show that for every $\theta \in [r_{\min}, r_{\max}]$, we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho^{\rightleftarrows}} e^{\theta z}\Big(\int_\theta^0 e^{-z\sigma}\phi(\sigma)d\sigma + \Delta(z)^{-1}\big(\phi(0) + \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\phi(\sigma)d\sigma\big)\Big)dz = 0. \tag{2.51}$$

We treat the case for $\Gamma^{\leftarrow}$, as the other case is analogous. First note that for some $K > 0$ we have the uniform bound

$$\left| e^{\theta z}\big(\phi(0) + \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\phi(\sigma)d\sigma\big)\right| \le K \tag{2.52}$$

in the strip $\gamma_- \le \mathrm{Re}\,z \le \gamma_+$, while by Lemma 2.3.1 $\Delta(z)^{-1} = O(|\mathrm{Im}\,z|^{-1})$ uniformly in this strip. In addition, using Fubini to change the order of integration and applying Lemma B.1, we compute

$$\lim_{\rho\to\infty}\int_{\gamma_+}^{\gamma_-}\int_\theta^0 e^{(i\rho+w)(\theta-\sigma)}\phi(\sigma)d\sigma dw = \lim_{\rho\to\infty}\int_\theta^0 e^{i\rho v}\frac{e^{\gamma_- v} - e^{\gamma_+ v}}{v}\phi(\theta - v)dv = 0, \tag{2.53}$$

which concludes the proof. □

In order to show that $\mathcal{M}_\Sigma$ is finite dimensional, we introduce a new operator $\widehat{A}$ on the larger space $\widehat{X} = \mathbb{C}^n \times X$,

$$\begin{array}{rcl} \mathcal{D}(\widehat{A}) & = & \left\{(c,\phi) \in \widehat{X} \mid \phi' \in X, c = \phi(0)\right\}, \\ \widehat{A}(c,\phi) & = & (L\phi, \phi'). \end{array} \tag{2.54}$$

Writing $j : X \to \widehat{X}$ for the continuous embedding $\phi \to (\phi(0), \phi)$, we see that the part of $\widehat{A}$ in jX is equivalent to A and that the closure of $\mathcal{D}(\widehat{A})$ is given by jX. Hence the spectral analysis of A and $\widehat{A}$ is one and the same. The next lemma shows that $\Delta(z)$ is a characteristic matrix for $\widehat{A}$, in the sense of [45, Def. IV.4.17].

Lemma 2.4.4. *Consider the holomorphic functions $E : \mathbb{C} \to \mathcal{L}(\widehat{X}, \mathcal{D}(\widehat{A}))$ and $F : \mathbb{C} \to \mathcal{L}(\widehat{X}, \widehat{X})$ given by*

$$\begin{array}{rcl} E(z)(c,\psi)(\theta) & = & (c, e^{\theta z}c + e^{\theta z}\int_\theta^0 e^{-z\sigma}\psi(\sigma)d\sigma), \\ F(z)(c,\psi)(\theta) & = & (c + \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\psi(\sigma)d\sigma, \psi(\theta)), \end{array} \tag{2.55}$$

in which $\mathcal{D}(\widehat{A})$ is considered as a Banach space with the graph norm. Then $E(z)$ and $F(z)$ are bijective for every $z \in \mathbb{C}$ and we have the identity

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I \end{pmatrix} = F(z)(zI - \widehat{A})E(z). \tag{2.56}$$

Proof. Writing $E_2(z)$ for the projection of $E(z)$ onto the X component of $\widehat{X}$, we compute

$$\psi(\theta) = zE_2(z)(c, \psi)(\theta) - D(E_2(z)(c, \psi))(\theta). \tag{2.57}$$

On the other hand, using partial integration we find

$$E_2(z)(\psi(0), (zI - D)\psi) = e^{\theta z}\psi(0) + e^{\theta z}\int_\theta^0 e^{-z\sigma}(z\psi(\sigma) - \psi'(\sigma))d\sigma = \psi(\theta), \tag{2.58}$$

from which it easily follows that $E(z)$ is bijective for all $z \in \mathbb{C}$. The bijectivity of $F(z)$ is almost immediate. The last identity in the statement of the lemma follows easily by using the definition of $\Delta(z)$ and computing

$$(zI - \widehat{A})E(z)(c, \psi) = ((z - \sum_{j=0}^N A_j e^{zr_j})c - \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}\psi(\sigma)d\sigma, \psi). \tag{2.59}$$

□

Using the theory of characteristic matrices (see e.g. [45, Theorem IV.4.18], one now obtains the following result.

Corollary 2.4.5. *For any Σ as in the statement of Lemma 2.4.3, the generalized eigenspace $\mathcal{M}_\Sigma$ is finite dimensional.*

We conclude this section by referring the reader to [11, 66], where similar results are obtained in the framework of delay equations.

2.5. Pseudo-Inverse for Linear Inhomogeneous Equations

The goal of this section is to define a pseudo-inverse $\mathcal{K} : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ for the linear inhomogeneous equation (2.17) in the spirit of Theorem 2.3.2, that however can still be defined when the system has eigenvalues on the imaginary axis.

We first need to introduce two families of Banach spaces, parametrized by $\mu, \nu \in \mathbb{R}$, that describe distributions that have controlled exponential growth at $\pm\infty$.

$$\begin{array}{rcl} BX_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) & = & \Big\{x \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX_{\mu,\nu}} := \sup_{\xi<0} e^{-\mu\xi}|x(\xi)| \\ & & \quad + \sup_{\xi\geq 0} e^{-\nu\xi}|x(\xi)| < \infty\Big\}, \\ BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) & = & \Big\{x \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX^1_{\mu,\nu}} := \|x\|_{BX_{\mu,\nu}} \\ & & \quad + \|x'\|_{BX_{\mu,\nu}} < \infty\Big\}. \end{array} \tag{2.60}$$

For any $\eta > 0$, we have continuous inclusions

$$i_{\pm\eta} : W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BX^1_{-\eta,\eta}(\mathbb{R}, \mathbb{C}^n), \tag{2.61}$$

with $\|i_{\pm\eta}\| \leq 2 + |\eta|$. Indeed, this can be seen by considering $x \in W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$, defining $y = e_{\mp\eta}x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and noting that

$$\left|e^{-\eta|\xi|}x'(\xi)\right| = \left|e^{-\eta|\xi|}D(e^{\pm\eta\xi}y(\xi))\right| = \left|e^{\pm\eta\xi-\eta|\xi|}(y'(\xi) \pm \eta y(\xi))\right| \leq (1+\eta)\,\|y\|_{W^{1,\infty}}\,. \tag{2.62}$$

The following important result allows us to relate the projection operators Q_Σ as defined in (2.48) to the solution operator (2.36).

Proposition 2.5.1. *Consider any $x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and write $\Lambda x = f \in BX_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$. Then for any $\gamma_+ > \nu$ and $\gamma_- < \mu$ such that the characteristic equation* $\det \Delta(z) = 0$ *has no roots with* $\operatorname{Re} z = \gamma_\pm$*, and for any $\xi \in \mathbb{R}$, we have*

$$\begin{array}{rcl} x(\xi) & = & \frac{1}{2\pi i}\int_{\gamma_+-i\infty}^{\gamma_++i\infty} e^{\xi z}\big(K(\xi,z,x) + \Delta(z)^{-1}\widetilde{f}_+(z)\big)dz \\ & & +\, \frac{1}{2\pi i}\int_{\gamma_-+i\infty}^{\gamma_--i\infty} e^{\xi z}\big(K(\xi,z,x) - \Delta(z)^{-1}\widetilde{f}_-(z)\big)dz, \end{array} \tag{2.63}$$

in which the operator K defined in (2.43) has been canonically extended to $\mathbb{R}\times\mathbb{C}\times BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$. The Laplace transforms $\widetilde{f}_+$ and $\widetilde{f}_-$ are again as defined in Appendix B.

Proof. An application of Lemma B.2 shows that

$$\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{\gamma_+-i\infty}^{\gamma_++i\infty} e^{\xi z}\Big(\int_\xi^0 e^{-z\sigma}x(\sigma)d\sigma + \widetilde{x}_+(z)\Big)dz. \tag{2.64}$$

Taking the Laplace transform of (2.17) yields

$$\begin{array}{rcl} z\widetilde{x}_+(z) - x(0) & = & \sum_{j=0}^N A_j \int_0^\infty e^{-zu}x(u+r_j)du + \widetilde{f}_+(z) \\ & = & \sum_{j=0}^N A_j e^{zr_j}\big(\widetilde{x}_+(z) + \int_{r_j}^0 e^{-z\sigma}x(\sigma)d\sigma\big) + \widetilde{f}_+(z) \end{array} \tag{2.65}$$

and thus after rearrangement we have

$$\widetilde{x}_+(z) = \Delta(z)^{-1}\Big(x(0) + \sum_{j=0}^N A_j e^{zr_j}\int_{r_j}^0 e^{-z\sigma}x(\sigma)d\sigma + \widetilde{f}_+(z)\Big). \tag{2.66}$$

Now define $y(\xi) = x(-\xi)$ and notice that y satisfies the following equation on $[0, \infty)$,

$$y'(\xi) = -f(-\xi) - \sum_{j=0}^N A_j y(\xi - r_j). \tag{2.67}$$

Taking the Laplace transform of this identity yields

$$z\widetilde{y}_+(z) - y(0) = -\sum_{j=0}^N A_j e^{-zr_j}\big(\widetilde{y}_+(z) + \int_{-r_j}^0 e^{-z\sigma}y(\sigma)d\sigma\big) - \widetilde{f}_-(-z) \tag{2.68}$$

and thus after rearrangement

$$\widetilde{y}_{+}(z) = \Delta(-z)^{-1}\Big(- y(0) + \sum_{j=0}^{N} A_j e^{-zr_j} \int_{-r_j}^{0} e^{-z\sigma} y(\sigma) d\sigma + \widetilde{f}_{-}(-z)\Big). \tag{2.69}$$

Reasoning as in the derivation of (2.64) we obtain the identity

$$\frac{1}{2} y(\xi) = \frac{1}{2\pi i} \int_{-\gamma_- - i\infty}^{-\gamma_- + i\infty} e^{\xi z} \Big(\int_{\xi}^{0} e^{-z\sigma} y(\sigma) d\sigma + \widetilde{y}_{+}(z) \Big) dz \tag{2.70}$$

and thus $\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{-\gamma_- - i\infty}^{-\gamma_- + i\infty} e^{-\xi z}(\Psi(\xi, z) + \Delta(-z)^{-1}\widetilde{f}_{-}(-z))dz$, with

$$\Psi(\xi, z) = \int_{-\xi}^{0} e^{-z\sigma} x(-\sigma) d\sigma - \Delta(-z)^{-1}\big(x(0) - \sum_{j=0}^{N} A_j e^{-zr_j} \int_{-r_j}^{0} e^{-z\sigma} x(-\sigma) d\sigma\big). \tag{2.71}$$

Substituting $z \to -z$, we obtain $\frac{1}{2}x(\xi) = \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z}(\Psi(\xi, -z) - \Delta(z)^{-1}\widetilde{f}_{-}(z))dz$ with

$$\Psi(\xi, -z) = \int_{\xi}^{0} e^{-z\sigma} x(\sigma) d\sigma + \Delta(z)^{-1}\big(x(0) + \sum_{j=0}^{N} A_j e^{zr_j} \int_{r_j}^{0} e^{-z\sigma} x(\sigma) d\sigma\big), \tag{2.72}$$

which follows from (2.71) after the substitution $\sigma \to -\sigma$ and concludes the proof. □

Using Lemma 2.3.1 one sees that there exists $\gamma > 0$ such that (2.17) has no eigenvalues z with $0 < |\mathrm{Re}\, z| < \gamma$. Throughout the rest of this section we fix an arbitrary $\eta \in (0, \gamma)$. We introduce $L^{\infty}(\mathbb{R}, \mathbb{C}^{n\times n})$ functions $\chi_{\pm}$ such that $\chi_{+}(\xi) = I$ for $\xi \geq 0$, $\chi_{-}(\xi) = I$ for $\xi < 0$ and $\chi_{+} + \chi_{-} = I$. Associated with these functions we define bounded linear cutoff operators $\Phi_{\pm} : BC_{\eta}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ by $\Phi_{\pm}x(\xi) = \chi_{\pm}(\xi)x(\xi)$ and notice that $\Phi_{+} + \Phi_{-} = I_{BC_{\eta}(\mathbb{R},\mathbb{C}^n)}$.

Using Proposition 2.3.4 we can define the isomorphisms $\Lambda_{\pm} = \Lambda^{(\eta)}_{\pm} : W^{1,\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ and a linear operator $P_{\eta} : BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n) \to BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ by

$$P_{\eta}x = \Lambda_{+}^{-1}\Phi_{+}\Lambda x + \Lambda_{-}^{-1}\Phi_{-}\Lambda x. \tag{2.73}$$

Notice that P_{η} is well defined, since $\Lambda P_{\eta}x = \Phi_{+}\Lambda x + \Phi_{-}\Lambda x = \Lambda x \in BC_{\eta}(\mathbb{R}, \mathbb{C}^n)$, which together with the differential equation (2.17) implies that the derivative of $P_{\eta}x$ is continous, yielding $P_{\eta}x \in BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ instead of merely $P_{\eta}x \in BX^1_{-\eta,\eta}(\mathbb{R}, \mathbb{C}^n)$. Define the space $\mathcal{R}_{\eta} \subset BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ as the range of P_{η} and the space $\mathcal{N}_0 \subset BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ as the kernel of P_{η}. Notice that the set of eigenvalues $\Sigma = \Sigma_{-\zeta,\zeta}$ is independent of ζ for $0 < \zeta < \gamma$. We introduce the projection $Q_0 : X \to X$ with $Q_0 = Q_{\Sigma}$ and define the finite dimensional linear subspace $X_0 = \mathcal{M}_{\Sigma} \subset X$.

Proposition 2.5.2. *The operator P_η defined above is bounded and in addition is a projection, i.e., it satisfies $P_\eta^2 = P_\eta$. The range $\mathcal{R}_\eta$ is a closed linear subspace of $BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$ and for any $x \in \mathcal{R}_\eta$ we have $Q_0 x_0 = 0$. The kernel $\mathcal{N}_0$ is finite dimensional and does not depend on η, with* $\dim \mathcal{N}_0 = \dim X_0$. *In particular, for any $x \in \mathcal{N}_0$ we have $x_\xi \in X_0$ for all $\xi \in \mathbb{R}$ and conversely, for any $\phi \in X_0$ there exists a unique $x = E\phi$ in $\mathcal{N}_0$ with $x_0 = \phi$. For any $\zeta_0 > 0$, we have that E viewed as a linear operator from X_0 into $BC_{\zeta_0}^1(\mathbb{R}, \mathbb{C}^n)$ is bounded with norm $\|E\|_{\zeta_0}$ that satisfies $\|E\|_{\zeta_1} \le \|E\|_{\zeta_0}$ for $\zeta_1 \ge \zeta_0$.*

Proof. The boundedness of P_η follows from the boundedness of Λ, $\Phi_\pm$ and $\Lambda_\pm^{-1}$, together with the continuous embeddings $i_{\pm\eta} : W_{\pm\eta}^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BX_{-\eta,\eta}^1(\mathbb{R}, \mathbb{C}^n)$. For all $x \in BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$, we notice

$$\Lambda P_\eta x = \Phi_+ \Lambda x + \Phi_- \Lambda x = \Lambda x, \tag{2.74}$$

which yields

$$\begin{array}{rcl} P_\eta^2 x & = & \Lambda_+^{-1} \Phi_+ \Lambda P_\eta x + \Lambda_-^{-1} \Phi_- \Lambda P_\eta x \\ & = & \Lambda_+^{-1} \Phi_+ \Lambda x + \Lambda_-^{-1} \Phi_- \Lambda x = P_\eta x. \end{array} \tag{2.75}$$

The range $\mathcal{R}_\eta$ can now immediately be seen to be closed, since if $P_\eta x_n \to z$, then $P_\eta^2 x_n = P_\eta x_n \to z$, but also $P_\eta x_n \to P_\eta z$, yielding $P_\eta z = z$ and thus $z \in \mathcal{R}_\eta$. Consider any $x \in \mathcal{R}_\eta$ and write $f = \Phi_+ \Lambda x$ and $g = \Phi_- \Lambda x$. It is clear that $\widetilde{f}_-(z) = 0$ and similarly $\widetilde{g}_+(z) = 0$. Combining Propositions 2.3.4 and 2.5.1, we conclude that $Q_0 x_0 = 0$.

Now consider any $x \in \mathcal{N}_0$. It follows from Proposition 2.5.1 that $x_0 = Q_0 x_0$ and since $\mathcal{N}_0$ is invariant under translation, we see $x_\xi = Q_0 x_\xi$ for any $\xi \in \mathbb{R}$. Let $y_0 \in \mathcal{N}_0$ be such that $y_0 = x_0$, then $x - y \in \mathcal{N}_0$ with $(x - y)_0 = 0$, but then Lemma 2.4.1 implies that $x = y$. We thus have $\dim \mathcal{N}_0 \le \dim X_0$. On the other hand, any $\phi \in X_0$ has the form $\phi(\theta) = \sum_{j=0}^{M} p_j(\theta) e^{\lambda_j \theta}$ with $\operatorname{Re} \lambda_j = 0$ and polynomials p_j and can thus be extended to a function $x = E\phi$ on the line, with $x \in \mathcal{N}_0$ and $x_0 = \phi$. Thus $\dim \mathcal{N}_0 = \dim X_0$ and the properties of E easily follow from the specific form of $\phi(\xi)$. This completes the proof. □

We remark here that all the statements in Proposition 2.2.1 have now been proved. Furthermore, we currently have all the ingredients we need to define a bounded pseudo-inverse for Λ. We thus introduce the operator $\mathcal{K}_\eta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to \mathcal{R}_\eta$, given by

$$\mathcal{K}_\eta x = \Lambda_+^{-1} \Phi_+ x + \Lambda_-^{-1} \Phi_- x. \tag{2.76}$$

Notice that the range of $\mathcal{K}_\eta$ is indeed contained in $\mathcal{R}_\eta$, since $x = \Lambda \mathcal{K}_\eta x$ and hence $\mathcal{K}_\eta x = P_\eta \mathcal{K}_\eta x$. This also immediately shows the injectivity of $\mathcal{K}_\eta$, since if $\mathcal{K}_\eta x = 0$, we have $x = \Lambda(0) = 0$. The surjectivity of $\mathcal{K}_\eta$ follows from the identity $y = P_\eta y = \mathcal{K}_\eta \Lambda y$ for any $y \in \mathcal{R}_\eta$. The following result shows that $\mathcal{K}_\eta$ behaves nicely on the scale of Banach spaces $BC_\zeta(\mathbb{R}, \mathbb{C}^n)$.

Lemma 2.5.3. *Consider any pair $0 < \eta_1 < \eta_2 < \gamma$. Then for any $f \in BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n)$ we have*

$$\mathcal{K}_{\eta_1} f = \mathcal{K}_{\eta_2} f. \tag{2.77}$$

Proof. Note that $\overline{f}_+ = e_{-\eta_2}\Phi_+ f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ satisfies a growth condition $\overline{f}_+(\xi) = O(e^{-(\eta_2-\eta_1)\xi})$ as $\xi \to \infty$ and hence $\overline{x}_+ = \Lambda_{-\eta_2}^{-1}\overline{f}_+$ shares this growth condition by Corollary 2.3.3. This implies that the function $x_+ = e_{\eta_2}\overline{x}_+$ satisfies $x_+ = O(e^{\eta_1\xi})$ as $\xi \to \infty$ and since $\overline{x}_+$ is bounded on $\mathbb{R}$, we have $x_+ = O(e^{-\eta_2|\xi|})$ as $\xi \to -\infty$. Using the differential equation (2.17) it follows that $x_+ \in W^{1,\infty}_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \cap W^{1,\infty}_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$. Since $\Lambda x_+ = \Phi_+ f \in L^1_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \cap L^1_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$, we see that $x_+ = (\Lambda_+^{(\eta_1)})^{-1}\Phi_+ f = (\Lambda_+^{(\eta_2)})^{-1}\Phi_+ f$. A similar argument for $\Phi_- f$ completes the proof. □

The next lemma shows that $\mathcal{K}_\eta$ and the translation operator do not commute.

Lemma 2.5.4. *For any $f \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$ and $\xi_0 \in \mathbb{R}$, define the function $y \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ by*

$$y(\xi) = (\mathcal{K}_\eta f)(\xi_0 + \xi) - (\mathcal{K}_\eta f(\xi_0 + \cdot))(\xi). \tag{2.78}$$

Then we have $y \in \mathcal{N}_0$. In particular, we have the identity

$$(I - Q_0)(\mathcal{K}_\eta f)_{\xi_0} = (\mathcal{K}_\eta f(\xi_0 + \cdot))_0. \tag{2.79}$$

Proof. Define functions $x_0(\xi) = (\mathcal{K}_\eta f)(\xi_0 + \xi)$ and $x_1(\xi) = (\mathcal{K}_\eta f(\xi_0 + \cdot))(\xi)$. Notice that for all $\xi \in \mathbb{R}$ we have $(\Lambda x_0)(\xi) = f(\xi_0 + \xi)$ but also $(\Lambda x_1)(\xi) = f(\xi_0 + \xi)$. This implies $\Lambda(x_0 - x_1) = 0$ and hence $y = x_0 - x_1 \in \mathcal{N}_0$. The final statement follows from the fact that $Q_0 y_\xi = y_\xi$ for any $y \in \mathcal{N}_0$ together with the identity $Q_0(\mathcal{K}_\eta f)_0 = 0$ for any $f \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$. □

For notational convenience, we introduce the quantity

$$w = \max(e^{-r_{\min}}, e^{r_{\max}}) \geq 1 \tag{2.80}$$

and note that for any $\eta > 0$, $\xi \in \mathbb{R}$ and $r_{\min} \leq \theta \leq r_{\max}$, we have $e^{-\eta|\xi|}e^{\eta|\xi+\theta|} \leq w^\eta$. This in turn implies that for any $x \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$ and any $\xi \in \mathbb{R}$, we have

$$\|x_\xi\| = \sup_{r_{\min} \leq \theta \leq r_{\max}} e^{\eta|\xi+\theta|}e^{-\eta|\xi+\theta|}\,|x(\xi + \theta)| \leq e^{\eta|\xi|}w^\eta \|x\|_\eta . \tag{2.81}$$

The following corollary to Lemma 2.5.4 shows that the hyperbolic component of $\mathcal{K}f$ remains bounded whenever f is bounded, which in the sequel will allow us to restrict our attention to the growth rate on the center component.

Corollary 2.5.5. *Suppose that $f \in BC_0(\mathbb{R}, \mathbb{C}^n)$. Then for any $\xi \in \mathbb{R}$ we have*

$$\|(I - Q_0)(\mathcal{K}_\eta f)_\xi\| \leq w^\eta \|K_\eta\| \|f\|_0 . \tag{2.82}$$

Proof. Using Lemma 2.5.4 we compute

$$\|(I - Q_0)(\mathcal{K}_\eta f)_\xi\| = \|(\mathcal{K}_\eta f(\xi + \cdot))_0\| \leq w^\eta \|\mathcal{K}_\eta\| \|f(\xi + \cdot)\|_\eta . \tag{2.83}$$

The statement now follows from the observation $\|f(\xi + \cdot)\|_\eta \leq \|f(\xi + \cdot)\|_0 = \|f\|_0$. □

Finally, we show that we can bound the norm of $\mathcal{K}_\eta$ uniformly on closed intervals.

Lemma 2.5.6. *Consider any interval $I = [\eta_-, \eta_+] \subset (0, \gamma)$. Then $\|\mathcal{K}_\eta\|$ is uniformly bounded for $\eta \in I$.*

Proof. In view of the bounds $\|i_{\pm\eta}\| \leq 2 + |\eta|$ for the embedding operators introduced in (2.61), it is enough to show that we can uniformly bound $B_{\pm\eta} = \left\|\Lambda_\pm^{-1}\right\|_{\mathcal{L}(L^\infty_{\pm\eta}(\mathbb{R},\mathbb{C}^n), W^{1,\infty}_{\pm\eta})(\mathbb{R},\mathbb{C}^n)}$. We here concentate on the $+$ case, as the remaining case follows analogously. From Proposition 2.3.4 we know $B_\eta = \|\Lambda_{-\eta}\|$. Fix $a = \min(\frac{1}{2}\eta_-, \frac{1}{2}(\gamma - \eta_+))$ and choose any $\alpha < -a$. Using Corollary 2.3.3 and the fact that the norms $|A_j e^{\eta r_j}|$ are uniformly bounded, we see it is enough to show that the quantities $K^\pm_\eta$ are uniformly bounded for $\eta \in I$, where

$$\begin{aligned} K^\pm_\eta &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Delta_{-\eta}(\pm a + iz)^{-1} - \frac{1}{z-\alpha}I\right| dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Delta(\pm a + \eta + iz)^{-1} - \frac{1}{z-\alpha}I\right| dz. \end{aligned} \tag{2.84}$$

This however follows immediately from Lemma 2.3.1. □

2.6. A Lipschitz Smooth Center Manifold

Using the pseudo-inverse $\mathcal{K}$ defined in the previous section for the inhomogeneous linear equation (2.17), we are now in a position to construct a Lipschitz smooth center manifold for the nonlinear equation (2.6). Throughout this section we consider a fixed nonlinearity $R : X \to \mathbb{C}^n$ that satisfies the assumptions (HR1) and (HR2). In order to employ the Banach contraction theorem, we need to modify the nonlinearity R so that it becomes globally Lipschitz continuous with a sufficiently small Lipschitz constant. To this end, we let $\chi : [0, \infty) \to \mathbb{R}$ be any C^∞-smooth function that satisfies $\chi(\xi) = 0$ for $\xi \geq 2$, $\chi(\xi) = 1$ for $\xi \leq 1$ and $0 \leq \chi(\xi) \leq 1$ for all $1 \leq \xi \leq 2$. For any $\delta > 0$, we define $\chi_\delta : [0, \infty) \to \mathbb{R}$ by $\chi_\delta(\xi) = \chi(\frac{\xi}{\delta})$. Following the approach in [45], we modify the nonlinearity separately in the hyperbolic and nonhyperbolic directions and define $R_\delta : X \to \mathbb{C}^n$ by

$$R_\delta(\phi) = \chi_\delta(\|Q_0\phi\|)\chi_\delta(\|(I - Q_0)\phi\|)R(\phi). \tag{2.85}$$

Lemma 2.6.1. *Let E and F be Banach spaces and let $f : E \to F$ with $f(0) = 0$ be a Lipschitz continuous mapping with Lipschitz constant $L(\delta)$ on on the ball of radius δ. Let $V, W \in \mathcal{L}(E, E)$ with $V + W = I$. Then there exists $C > 0$ such that for all $\delta > 0$ the mapping $x \to \chi_\delta(\|Vx\|)\chi_\delta(\|Wx\|)f(x)$ is globally Lipschitz continuous with Lipschitz constant $(4C\|V\| + 4C\|W\| + 1)L(4\delta)$.*

Proof. There exists $C > 0$ such that χ_δ is globally Lipschitz continuous with Lipschitz constant C/δ. Introduce the shorthands $f_x = f(x)$, $\chi^V_x = \chi_\delta(\|Vx\|)$ and $\chi^W_x = \chi_\delta(\|Wx\|)$

and the corresponding notations for y. We obtain the following estimate,

$$\begin{aligned} \Delta &= \|f(x)\chi_\delta(\|Vx\|)\chi_\delta(\|Wx\|) - f(y)\chi_\delta(\|Vy\|)\chi_\delta(\|Wy\|)\| \\ &= \left\| f_x\chi_x^V\chi_x^W - f_y\chi_y^V\chi_y^W \right\| \\ &\leq \left\| f_x - f_y \right\| \chi_y^V\chi_y^W + \|f_x\| \left|\chi_x^V - \chi_y^V\right| \chi_y^W + \|f_x\| \chi_x^V \left|\chi_x^W - \chi_y^W\right|. \end{aligned} \tag{2.86}$$

We now treat the three different cases. Suppose that both $\chi_x^V\chi_x^W = 0$ and $\chi_y^V\chi_y^W = 0$, then it immediately follows that $\Delta = 0$. Now suppose that both $\chi_x^V\chi_x^W \neq 0$ and $\chi_y^V\chi_y^W \neq 0$, which implies $\|x\|, \|y\| \leq 4\delta$. This means $\|f_x\|, \left\|f_y\right\| \leq 4\delta L(4\delta)$ and hence

$$\begin{aligned} \Delta &\leq L(4\delta)\|x-y\| + 4\delta L(4\delta)\tfrac{C}{\delta}\|V\|\|x-y\| + 4\delta L(4\delta)\tfrac{C}{\delta}\|W\|\|x-y\| \\ &= (4C\|V\| + 4C\|W\| + 1)L(4\delta)\|x-y\|. \end{aligned} \tag{2.87}$$

Notice the only case left to consider is the situation where $\chi_x^V\chi_x^W \neq 0$ but $\chi_y^V\chi_y^W = 0$, since x and y are interchangeable. We obtain

$$\begin{aligned} \Delta &\leq 4\delta L(4\delta)\tfrac{C}{\delta}\|V\|\|x-y\| + 4\delta L(4\delta)\tfrac{C}{\delta}\|W\|\|x-y\| \\ &= (4C\|V\| + 4C\|W\|)L(4\delta)\|x-y\|. \end{aligned} \tag{2.88}$$

□

Corollary 2.6.2. *The mappings $R_\delta : X \to \mathbb{C}^n$ are globally Lipschitz continuous with Lipschitz constants L_{R_δ} that go to zero as δ goes to zero. In addition, $\|R_\delta(\phi)\| \leq 4\delta L_{R_\delta}$ for all $\phi \in X$.*

Proof. The first statement follows from assumption (HR2). The second statement follows by noting that if $R_\delta(\phi) \neq 0$, then $\|\phi\| \leq \|Q_0\phi\| + \|(I - Q_0)\phi\| \leq 2\delta + 2\delta = 4\delta$. □

We observe here that the nonlinearity R_δ induces a map $\widetilde{R}_\delta : C(\mathbb{R}, \mathbb{C}^n) \to C(\mathbb{R}, \mathbb{C}^n)$ via substitution, i.e.,

$$\widetilde{R}_\delta x(\xi) = R_\delta x_\xi. \tag{2.89}$$

Notice that $\widetilde{R}_\delta$ is well-defined, since $i_x : \mathbb{R} \to X$ which sends $\xi \to x_\xi$ is a continuous mapping for any continuous x and hence the same holds for $\widetilde{R}_\delta x = R_\delta \circ i_x$. The next lemma shows that $\widetilde{R}_\delta$ inherits the global Lipschitz continuity of R_δ.

Lemma 2.6.3. *For any $\eta \in \mathbb{R}$, the substitution operator $\widetilde{R}_\delta$ viewed as an operator from $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ into $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ is globally Lipschitz continuous with Lipschitz constant $w^\eta L_{R_\delta}$.*

Proof. Write $x = \widetilde{R}_\delta u$, $y = \widetilde{R}_\delta v$ and compute

$$\begin{aligned} e^{-\eta|\xi|}|y(\xi) - x(\xi)| &= e^{-\eta|\xi|}\left|R_\delta u_\xi - R_\delta v_\xi\right| \leq e^{-\eta|\xi|}L_{R_\delta}\left\|u_\xi - v_\xi\right\| \\ &\leq w^\eta L_{R_\delta}\|u - v\|_\eta. \end{aligned} \tag{2.90}$$

□

We are now ready to construct solutions to the system (2.6) with the modified nonlinearity R_δ substituted for R. This will be done by employing a fixed point argument. To this end, we recall the extension operator $E : X_0 \to \bigcap_{\zeta>0} BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ introduced in Proposition 2.5.2 and define an operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ via

$$\mathcal{G}(u, \phi) = E\phi + \mathcal{K}_\eta \widetilde{R}_\delta(u). \tag{2.91}$$

Choose $\delta > 0$ small enough to guarantee

$$w^\eta L_{R_\delta} \left\|\mathcal{K}_\eta\right\| < \frac{1}{2}. \tag{2.92}$$

Note that if $\|E\|_\eta \|\phi\| < \frac{\rho}{2}$, then $\mathcal{G}(\cdot, \phi)$ leaves the ball with radius ρ in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ invariant. Notice in addition that $\mathcal{G}(\cdot, \phi)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. Since ρ can be chosen arbitrarily, the following theorem can be established using standard arguments.

Theorem 2.6.4. *Consider the system (2.6) and suppose that the conditions (HR1) and (HR2) are satisfied. Fix $\gamma > 0$ such that the characteristic equation* $\det \Delta(z) = 0$ *has no roots with* $0 < |\mathrm{Re}\, z| < \gamma$*. Fix any $\eta \in (0, \gamma)$ and choose $\delta > 0$ such that (2.92) is satisfied. Then there exists a globally Lipschitz continuous mapping u^*_η from X_0 into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ such that $u = u^*_\eta\phi$ is the unique solution in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ of the equation*

$$u = \mathcal{G}(u, \phi). \tag{2.93}$$

The following results show that the family of mappings u^*_ζ defined above behaves appropriately under translations and under shifts of the parameter ζ.

Lemma 2.6.5. *Consider the setting of Theorem 2.6.4 and let $\phi \in X_0$. Then for any $\xi_0 \in \mathbb{R}$ we have the identity*

$$(u^*_\eta\phi)(\xi_0 + \cdot) = (u^*_\eta Q_0(u^*_\eta\phi)_{\xi_0})(\cdot). \tag{2.94}$$

Proof. Using Lemma 2.5.4 we compute

$$\psi := Q_0(u^*_\eta\phi)_{\xi_0} = (E\phi)_{\xi_0} + (\mathcal{K}_\eta \widetilde{R}_\delta(u^*_\eta\phi))_{\xi_0} - (\mathcal{K}_\eta \widetilde{R}_\delta((u^*_\eta\phi)(\xi_0 + \cdot)))_0, \tag{2.95}$$

hence upon defining

$$y(\xi) = E\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{R}_\delta(u^*_\eta\phi)(\xi_0 + \xi) - \mathcal{K}_\eta \widetilde{R}_\delta((u^*_\eta\phi)(\xi_0 + \cdot))(\xi), \tag{2.96}$$

we conclude that $y \in \mathcal{N}_0$ by Lemma 2.5.4 and in addition that $y = E\psi$. Upon calculating

$$\begin{array}{rcl} \mathcal{G}((u^*_\eta\phi)(\xi_0 + \cdot), \psi)(\xi) & = & y(\xi) + \mathcal{K}_\eta \widetilde{R}_\delta((u^*_\eta\phi)(\xi_0 + \cdot))(\xi) \\ & = & E\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{R}_\delta(u^*_\eta\phi)(\xi_0 + \xi) = (u^*_\eta\phi)(\xi_0 + \xi), \end{array} \tag{2.97}$$

we see that due to uniquess of solutions we must have

$$(u^*_\eta\psi)(\xi) = (u^*_\eta\phi)(\xi_0 + \xi), \tag{2.98}$$

from which the claim follows. □

Combining Lemma 2.5.3 and Corollary 2.6.2 immediately yields the final result of this section.

Lemma 2.6.6. *Consider any pair* $0 < \eta_1 < \eta_2 < \gamma$ *and suppose that (2.92) holds for both* η_1 *and* η_2*. Then we have* $u^*_{\eta_2} = \mathcal{J}^1_{\eta_2\eta_1} u^*_{\eta_1}$.

2.7. Smoothness of the center manifold

In the previous section we saw that the mapping $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is Lipschitz continuous. In this section we will extend this result and show that u^*_η inherits the C^k-smoothness of the nonlinearity R. More precisely, we shall establish the following theorem.

Theorem 2.7.1. *Consider the system (2.6) and suppose that the conditions (HR1) and (HR2) are satisfied. Fix* $\gamma > 0$ *such that the characteristic equation* $\det \Delta(z) = 0$ *has no roots with* $0 < |\mathrm{Re}\, z| < \gamma$ *and consider any interval* $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ *with* $k\eta_{\min} < \eta_{\max}$*, where k is as defined in (HR1). Then there exists* $\delta > 0$ *such that the following statements hold.*

(i) For any $\eta \in [\eta_{\min}, \eta_{\max}]$*, we have the inequality*

$$w^\eta L_{R_\delta} \left\| \mathcal{K}_\eta \right\| < \frac{1}{4}. \tag{2.99}$$

(ii) For each integer $1 \leq p \leq k$ *and for each* $\eta \in (p\eta_{\min}, \eta_{\max}]$*, the mapping* $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ *is of class* C^p*, where* u^*_ζ *for* $\zeta \in [\eta_{\min}, \eta_{\max}]$ *is as defined in Theorem 2.6.4 with the above value for* δ*.*

We remark here that the arguments in this section follow closely the lines of [45, Section IX.7]. Throughout this entire section we consider a fixed system (2.6) that satisfies the conditions (HR1) and (HR2), i.e., we shall use the corresponding integer k and C^k-smooth nonlinearity R without further comment in our results.

As a first step towards proving the above theorem, we need to find a suitable domain of definition for $\widetilde{R}_\delta$ to ensure that this operator becomes sufficiently smooth. Due to the presence of the cutoff function on the infinite dimensional complement of X_0, the nonlinearity R_δ loses the C^k-smoothness on X and becomes merely Lipschitz continuous. In view of these observations, we introduce for any $\eta > 0$ the space

$$V^1_\eta(\mathbb{R}, \mathbb{C}^n) = \left\{ u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi\in\mathbb{R}} \left\| Q_h u_\xi \right\| < \infty \right\}, \tag{2.100}$$

in which $Q_h = (I - Q_0)$ is the projection onto the hyperbolic part of X. We provide the above space with the norm

$$\|u\|_{V^1_\eta} = \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|} \left\| Q_0 u_\xi \right\| + \sup_{\xi\in\mathbb{R}} \left\| Q_h u_\xi \right\| + \left\| u' \right\|_\eta, \tag{2.101}$$

with which $V^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is a Banach space that has continuous inclusions $V^1_\eta(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$. In addition, for any $\delta > 0$ we define the open set

$$V^{1,\delta}_\eta = \left\{ u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi \in \mathbb{R}} \left\| Q_h u_\xi \right\| < \delta \right\} \subset V^1_\eta(\mathbb{R}, \mathbb{C}^n). \tag{2.102}$$

Since X_0 is finite dimensional, we have that R_δ is of class C^k on the set $B^h_\delta = \{\phi \in X \mid \|Q_h \phi\| < \delta\}$. In addition, the norms $\|D^p R_\delta \phi\|$ are uniformly bounded on B^h_δ for all $0 \le p \le k$. Thus, for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ for which $\sup_{\xi \in \mathbb{R}} \left\| Q_h u_\xi \right\| < \delta$ and any $0 \le p \le k$, we can define a map $\widetilde{R}^{(p)}_\delta(u) \in \mathcal{L}^{(p)}(C(\mathbb{R}, \mathbb{C}^n), C(\mathbb{R}, \mathbb{C}^n))$ by

$$\widetilde{R}^{(p)}_\delta(u)(v_1, \ldots, v_p)(\xi) = D^p R_\delta(u_\xi)\big((v_1)_\xi, \ldots, (v_p)_\xi\big). \tag{2.103}$$

Here the symbol $\mathcal{L}^{(p)}(Y_1 \times \ldots \times Y_p, Z)$ denotes the space of p-linear mappings from $Y_1 \times \ldots \times Y_p$ into Z. If $Y_1 = \ldots = Y_p = Y$, we use the shorthand $\mathcal{L}^{(p)}(Y, Z)$. Note that the map $\widetilde{R}^{(p)}_\delta(u)$ defined above is well defined, since $D^p R_\delta$ is a continuous map from $B^h_\delta \times X^p$ into $\mathbb{C}^n$, as is the map $i_x : \mathbb{R} \to X$ which sends $\xi \to x_\xi$, for any $x \in C(\mathbb{R}, \mathbb{C}^n)$.

The next lemma shows that for sufficiently small δ, the operator u^*_η maps precisely into the region on which the modification of R in the hyperbolic direction is trivial, which means that R_δ is C^k-smooth on this region.

Lemma 2.7.2. *Let $\delta > 0$ be so small that for some $0 < \eta_0 < \gamma$,*

$$w^{\eta_0} L_{R_\delta} < (4 \left\| \mathcal{K}_{\eta_0} \right\|)^{-1}. \tag{2.104}$$

Then for any $\phi \in X_0$ and $0 < \eta < \gamma$, we have that for all $\xi \in \mathbb{R}$,

$$\left\| Q_h (u^*_\eta \phi)_\xi \right\| < \delta. \tag{2.105}$$

Proof. Note first that the cutoff function ensures that

$$\left\| \widetilde{R}_\delta(u^*_\eta \phi) \right\|_0 \le 4\delta L_{R_\delta}. \tag{2.106}$$

Since Lemma 2.5.3 guarantees that $\mathcal{K}_\eta$ and $\mathcal{K}_{\eta_0}$ agree on $BC_0(\mathbb{R}, \mathbb{C}^n)$, we can use Corollary 2.5.5 to compute

$$\left\| Q_h (u^*_\eta \phi)_\xi \right\| = \left\| Q_h \mathcal{K}_{\eta_0} (\widetilde{R}_\delta(u^*_\eta \phi))_\xi \right\| \le w^{\eta_0} \left\| \mathcal{K}_{\eta_0} \right\| 4\delta L_{R_\delta}. \tag{2.107}$$

□

The next series of results establishes conditions under which the maps $\widetilde{R}_\delta : V^{1,\delta}_\sigma(\mathbb{R}, \mathbb{C}^n) \to BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ are smooth. In the remainder of this section we will for convenience adopt the shorthand $BC^1_\zeta = BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$, together with similar ones for the other function spaces.

Lemma 2.7.3. *Let $1 \le p \le k$, $\zeta_i > 0$ for $1 \le i \le p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta \ge \zeta$. Then for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ such that $\sup_{\xi \in \mathbb{R}} \left\| Q_h u_\xi \right\| < \delta$, we have*

$$\widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta), \tag{2.108}$$

where the norm is bounded by

$$\left\| \widetilde{R}_\delta^{(p)} \right\|_{\mathcal{L}^{(p)}} \le w^\zeta \sup_{\xi \in \mathbb{R}} e^{-(\eta - \zeta)|\xi|} \left\| D^p R_\delta(u_\xi) \right\| < \infty. \tag{2.109}$$

If $\eta > \zeta$ and $\sigma > 0$, then in addition $u \to \widetilde{R}_\delta^{(p)}(u)$ is continuous as a map from $V^{1,\delta}_\sigma$ into $\mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$.

Finally, in the statements above, any subset of the $BC^1_{\zeta_i}$ spaces may be replaced by $V^1_{\zeta_i}$.

Proof. We define $r = \sup_{\xi \in \mathbb{R}} \left\| Q_h u_\xi \right\| < \delta$. The bound for $\left\| \widetilde{R}_\delta^{(p)} \right\|_{\mathcal{L}^{(p)}}$ follows from the estimates $\left\| (v_i)_\xi \right\| \le w^{\zeta_i} e^{\zeta_i |\xi|} \left\| v_i \right\|_{\zeta_i}$ and $\left\| v_i \right\|_{\zeta_i} \le \left\| v_i \right\|_{BC^1_{\zeta_i}}$. Since $\left\| D^p R_\delta \right\|$ is uniformly bounded on B^h_δ, the norm above can be seen to be finite and hence $\widetilde{R}_\delta^{(p)}(u)$ is well defined.

We now consider the case that $\eta > \zeta$ and prove the continuity of $u \to \widetilde{R}_\delta^{(p)}(u)$. Let $\widetilde{B} \subset V^1_\sigma$ be the open ball of radius $\delta - r$ and note that for any $0 < \epsilon < 1$, we have

$$\begin{array}{l} \sup_{g \in \widetilde{B}} \left\| \widetilde{R}_\delta^{(p)}(u + \epsilon g) - \widetilde{R}_\delta^{(p)}(u) \right\|_{\mathcal{L}^{(p)}} \\ \le \sup_{g \in \widetilde{B}} \sup_{\xi \in \mathbb{R}} e^{-(\eta - \zeta)|\xi|} \left\| D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi) \right\| . \end{array} \tag{2.110}$$

Fix an arbitrary $\kappa > 0$. Exploiting the fact that $D^p R_\delta$ is uniformly bounded on B^h_δ, we choose an $A > 0$ such that

$$2e^{-(\eta - \zeta)A} \sup_{\phi \in B^h_\delta} \left\| D^p R_\delta(\phi) \right\| \le \kappa, \tag{2.111}$$

which implies

$$\sup_{g \in \widetilde{B}} \sup_{|\xi| \ge A} e^{-(\eta - \zeta)|\xi|} \left\| D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi) \right\| \le \kappa. \tag{2.112}$$

Due to the compactness of the interval $[-A, A]$, we can choose a finite open covering $\mathrm{Cov} = \bigcup_{j=1}^M B_{\rho_j}(u_{\xi_j}) \subset B^h_\delta \subset X$, with standard open balls $B_\rho(\psi) \subset X$, such that $u_\xi \in \mathrm{Cov}$ for all $\xi \in [-A, A]$ and in addition $\left\| D^p R_\delta(\phi) - D^p R_\delta(u_{\xi_j}) \right\| \le \frac{\kappa}{2}$ for all $\phi \in B_{2\rho_j}(u_{\xi_j})$. Choose any $\epsilon > 0$ such that $\epsilon w^\sigma e^{\sigma A}(\delta - r) < \min \left\{ \rho_j \mid 1 \le j \le M \right\}$. This implies that for every $g \in \widetilde{B}$ and any $1 \le j \le M$ we have $\left\| \epsilon g_\xi \right\| \le \epsilon w^\sigma e^{\sigma |\xi|} \left\| g \right\|_{V^1_\sigma} < \rho_j$ and hence

$$\begin{array}{rcl} \left\| D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_\xi) \right\| & \le & \left\| D^p R_\delta(u_\xi + \epsilon g_\xi) - D^p R_\delta(u_{\xi_{j_0}}) \right\| \\ & + & \left\| D^p R_\delta(u_{\xi_{j_0}}) - D^p R_\delta(u_\xi) \right\| \\ & \le & \frac{\kappa}{2} + \frac{\kappa}{2} = \kappa, \end{array} \tag{2.113}$$

where we have chosen j_0 such that $u_\xi \in B_{\rho_{j_0}}(u_{\xi_{j_0}})$. Since $\xi > 0$ was arbitrary, we have that $u \to \widetilde{R}_\delta^{(p)}(u)$ is indeed continuous as a map from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$. Finally, note that the arguments above carry over upon replacing any subset of the $BC^1_{\zeta_i}$ spaces by their corresponding $V^1_{\zeta_i}$ spaces. □

Lemma 2.7.4. *Let $0 \le p < k$, $\zeta_i > 0$ for $1 \le i \le p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + \sigma$. Then the map $\widetilde{R}_\delta^{(p)} : V_\sigma^{1,\delta} \to \mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$ is of class C^1 with derivative*

$$D\widetilde{R}_\delta^{(p)}(u) = \widetilde{R}_\delta^{(p+1)}(u) \in \mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p} \times V^1_\sigma, BC_\eta). \tag{2.114}$$

In addition, the same statement holds upon replacing any subset of the $BC^1_{\zeta_i}$ spaces with the corresponding $V^1_{\zeta_i}$ spaces.

Proof. Pick an arbitrary $u \in V_\sigma^{1,\delta}$ and write $r = \sup_{\xi\in\mathbb{R}} \left\| Q_h u_\xi \right\| < \delta$. Write $\widetilde{B} \subset V^1_\sigma$ for the open ball with radius $\delta - r$ and note that for any $0 < \epsilon < 1$, we have

$$\begin{aligned}
&\sup_{g\in\widetilde{B}} \tfrac{1}{\epsilon} \left\| \widetilde{R}_\delta^{(p)}(u + \epsilon g) - \widetilde{R}_\delta^{(p)}(u) - \epsilon \widetilde{R}_\delta^{(p+1)}(u) g \right\| \\
&= \sup_{g\in\widetilde{B}} \sup_{\xi\in\mathbb{R}} \sup_{\|v_1\|_{BC^1_{\zeta_1}}=1} \ldots \sup_{\|v_p\|_{BC^1_{\zeta_p}}=1} \tfrac{1}{\epsilon} || D^p R_\delta(u_\xi)((v_1)_\xi, \ldots, (v_m)_\xi) \\
&\quad - D^p R_\delta(u_\xi + \epsilon g_\xi)((v_1)_\xi, \ldots, (v_p)_\xi) + \epsilon D^{p+1} R_\delta(u_\xi)((v_1)_\xi, \ldots, (v_p)_\xi, g_\xi) ||_\eta \\
&\le \sup_{\xi\in\mathbb{R}} \sup_{\phi\in B^h_{(\xi)}} w^{\zeta+\sigma} e^{(-\eta+\zeta+\sigma)|\xi|} \left\| D^{p+1} R_\delta(u_\xi + \epsilon\phi) - D^{p+1} R_\delta(u_\xi) \right\|,
\end{aligned} \tag{2.115}$$

where we have defined $B^h_{(\xi)} = \left\{\phi \in X \mid \|\phi\| < (\delta - r) w^\sigma e^{\sigma|\xi|} \text{ and } \|Q_h\phi\| < \delta - r\right\}$. Since the exponent $-\eta + \zeta + \sigma$ is negative, one can reason as in the proof of Lemma 2.7.3 to conclude that the last expression tends to zero as $\epsilon \to 0$. This implies $D\widetilde{R}_\delta^{(p)}(u) = \widetilde{R}_\delta^{(p+1)}(u)$ as an operator in $\mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times BC^1_{\zeta_p} \times V^1_\sigma, BC_\eta)$. Lemma 2.7.3 ensures that this derivative $u \to \widetilde{R}_\delta^{(p+1)}(u)$ is continuous. Again, the arguments above carry over upon replacing any subset of the $BC^1_{\zeta_i}$ spaces by their corresponding $V^1_{\zeta_i}$ spaces. □

Corollary 2.7.5. *Let $\eta_2 > k\eta_1 > 0$ and $1 \le p \le k$. Then the mapping $\widetilde{R}_\delta : V_{\eta_1}^{1,\delta} \to BC_{\eta_2}$ is of class C^k with*

$$D^p \widetilde{R}_\delta(u) = \widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(V^1_{\eta_1}, BC_{\eta_2}) \cap \mathcal{L}^{(p)}(BC^1_{\eta_1}, BC_{\eta_2}). \tag{2.116}$$

Proof. The fact that $\widetilde{R}_\delta$ is of class C^k follows by repeated application of Lemma 2.7.4. In addition, Lemma 2.7.3 implies that the derivatives $\widetilde{R}_\delta^{(p)}(u) \in \mathcal{L}^{(p)}(V^1_{\eta_1}, BC_{\eta_2})$ can be naturally extended to elements in $\mathcal{L}^{(p)}(BC^1_{\eta_1}, BC_{\eta_2})$.

□

Corollary 2.7.6. *Let $1 \le p \le k$, $\zeta_i > 0$ for $1 \le i \le p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + (k - p)\sigma$ for some $\sigma > 0$. Then the mapping $\widetilde{R}_\delta^{(p)} : V_\sigma^{1,\delta} \to \mathcal{L}^{(p)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p}, BC_\eta)$ is of class C^{k-p}.*

Lemma 2.7.7. *Let $1 \leq p < k$, $\zeta_i > 0$ for $1 \leq i \leq p$, $\zeta = \zeta_1 + \ldots + \zeta_p$ and $\eta > \zeta + \sigma$ for some $\sigma > 0$. Let Φ be a mapping of class C^1 from X_0 into $V_\sigma^{1,\delta}$. Then the mapping $\widetilde{R}_\delta^{(p)} \circ \Phi$ from X_0 into $\mathcal{L}^{(p)}(BC^1_{\zeta_1}, \ldots, BC^1_{\zeta_p}, BC_\eta)$ is of class C^1 with*

$$D(\widetilde{R}_\delta^{(p)} \circ \Phi)(\phi)(v_1, \ldots v_p, \psi) = \widetilde{R}^{(p+1)}(\Phi(\phi))(v_1, \ldots, v_p, \Phi'(\phi)\psi). \tag{2.117}$$

Proof. Let $M = \sup_{\phi \in B_\delta^h} \left\| D^{(p+1)} R_\delta(\phi) \right\|$. Fix $\mathbf{v} = (v_1, \ldots, v_p)$, with $\|v_i\|_{\eta_i} = 1$. Observe that if

$$S(\xi) = \widetilde{R}_\delta^{(p)}(\Phi(\phi))(\mathbf{v})(\xi) - \widetilde{R}_\delta^{(p)}(\Phi(\psi))(\mathbf{v})(\xi) - \widetilde{R}^{(p+1)}(\Phi(\psi))(\mathbf{v}, \Phi'(\psi)(\phi - \psi))(\xi), \tag{2.118}$$

then S can be written as $S(\xi) = S_1(\xi) + S_2(\xi)$, with

$$\begin{array}{rcl} S_1(\xi) & = & \int_0^1 \big(D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) \\ & & \quad - D^{p+1} R_\delta(\Phi(\psi)_\xi)\big)\big(\mathbf{v}_\xi, (\Phi'(\psi)(\phi - \psi))_\xi\big) d\theta, \\ S_2(\xi) & = & \int_0^1 D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) \\ & & \quad \big(\mathbf{v}_\xi, \Phi(\phi)_\xi - \Phi(\psi)_\xi - (\Phi'(\psi)(\phi - \psi))_\xi\big) d\theta. \end{array} \tag{2.119}$$

Define $I(\xi) = \int_0^1 \left\| D^{p+1} R_\delta(\theta \Phi(\phi)_\xi + (1-\theta)\Phi(\psi)_\xi) - D^{p+1} R_\delta(\Phi(\psi)_\xi) \right\| d\theta$ and calculate

$$\begin{array}{rcl} e^{-\eta|\xi|} \left| S_1(\xi) \right| & \leq & w^{\zeta+\sigma} e^{(-\eta+\zeta+\sigma)|\xi|} \left\| \phi - \psi \right\| \left\| \Phi'(\psi) \right\|_{V_\sigma^1} I(\xi) \\ & \leq & w^{\zeta+\sigma} \left\| \phi - \psi \right\| \left\| \Phi'(\psi) \right\|_{V_\sigma^1} \\ & & \max\left\{ 2M e^{(-\eta+\zeta+\sigma)A}, \sup_{\xi \in [-A,A]} I(\xi) \right\}, \\ e^{-\eta|\xi|} \left| S_2(\xi) \right| & \leq & M w^{\zeta+\sigma} e^{(-\eta+\zeta+\sigma)|\xi|} \left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1} \\ & \leq & M w^{\zeta+\sigma} \left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1}. \end{array} \tag{2.120}$$

Fixing some $\epsilon > 0$ and letting $A > 0$ be such that $2Me^{(-\eta+\zeta+\sigma)A} < \epsilon$, we define

$$\Omega = \left\{ \Phi(\psi)_\xi \mid \xi \in [-A, A] \right\} \subset X. \tag{2.121}$$

We can argue as in the proof of Lemma 2.7.3 to show that there exists $\delta_1 > 0$ such that

$$\left\| D^{p+1} R_\delta(\overline{\phi} + \overline{\psi}) - D^{p+1} R_\delta(\overline{\phi}) \right\| < \epsilon \tag{2.122}$$

for any $\overline{\phi} \in \Omega$ and $\left\| \overline{\psi} \right\| < \delta_1$. Since $\sup_{\xi \in [-A,A]} \left\| \Phi(\phi)_\xi - \Phi(\psi)_\xi \right\| \to 0$ as $\phi \to \psi$, there exists $\delta_2 > 0$ such that $\|\phi - \psi\| < \delta_2$ implies $\left\| \Phi(\phi)_\xi - \Phi(\psi)_\xi \right\| < \delta_1$ for all $\xi \in [-A, A]$. In addition, as Φ is differentiable at ψ, there exists $\delta_3 > 0$ such that $\left\| \Phi(\phi) - \Phi(\psi) - \Phi'(\psi)(\phi - \psi) \right\|_{V_\sigma^1} \leq \|\phi - \psi\| \epsilon$ whenever $\|\phi - \psi\| < \delta_3$. Together this implies that if $\|\phi - \psi\| < \min(\delta_2, \delta_3)$, we have

$$\|S(\cdot)\|_\eta \leq \|\phi - \psi\| \, w^{\sigma+\zeta}(M + \left\| \Phi'(\psi) \right\|_{V_\sigma^1})\epsilon, \tag{2.123}$$

which proves that $\widetilde{R}^{(p)} \circ \Phi$ is differentiable. The continuity of this derivative follows from the fact that Φ is of class C^1 together with the continuity of the mapping $u \to \widetilde{R}^{(p+1)}(u)$ from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(p+1)}(BC^1_{\zeta_1} \times \ldots \times BC^1_{\zeta_p} \times V_\sigma^1, BC_\eta)$. □

Corollary 2.7.8. *Consider any pair* $0 < \eta_1 < \eta_2 < \gamma$. *Then the map from* $V^{1,\delta}_{\eta_1}$ *into* $BC^1_{\eta_2}$ *defined by*

$$u \to \mathcal{J}^1_{\eta_2\eta_1}\mathcal{K}_{\eta_1}\widetilde{R}_\delta(u) \tag{2.124}$$

is of class C^1 *with derivative* $u \to \mathcal{J}^1_{\eta_2\eta_1} \circ \mathcal{K}_{\eta_1} \circ \widetilde{R}^{(1)}_\delta(u)$, *which maps into* $\mathcal{L}(V^1_{\eta_1}, BC^1_{\eta_2}) \cap \mathcal{L}(BC^1_{\eta_1}, BC^1_{\eta_2})$.

Proof. Using Lemma 2.5.3 and Corollary 2.6.2 we observe that $\mathcal{J}^1_{\eta_2\eta_1}\mathcal{K}_{\eta_1}\widetilde{R}_\delta(u) = \mathcal{K}_{\eta_2}\widetilde{R}_\delta(u)$. This last map is C^1-smooth by Lemma 2.7.4 and the fact that $\mathcal{K}_{\eta_2}$ is a bounded linear operator from BC_{η_2} into $BC^1_{\eta_2}$, with derivative $\mathcal{K}_{\eta_2} \circ \widetilde{R}^{(1)}_\delta(u)$. The proof is completed upon noting that $\widetilde{R}^{(1)}_\delta(u)$ in fact maps $BC^1_{\eta_1}$ into BC_{η_1} by Lemma 2.7.3. □

Proof of Theorem 2.7.1. In view of Lemma 2.5.6 we can choose the constant $\delta > 0$ in such a way that both (2.99) and (2.104) are satisfied. We start with the case $k = 1$. Let $\eta \in (\eta_{\min}, \eta_{\max}]$. We will apply Lemma A.2 with the Banach spaces $Y_0 = V^1_{\eta_{\min}}$, $Y = BC^1_{\eta_{\min}}$, $Y_1 = BC^1_\eta$ with the corresponding natural inclusions and $\Lambda_0 = \Lambda = X_0$. We fix $\Omega_0 = V^{1,\delta}_{\eta_{\min}} \subset V^1_{\eta_{\min}}$, recall the extension operator $E : X_0 \to \bigcap_{\zeta>0} BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ introduced in Proposition 2.5.2 and choose

$$\begin{array}{lllll} F(u,\phi) & = & E\phi + \mathcal{K}_{\eta_{\min}}\widetilde{R}_\delta(u), & \phi \in X_0, & u \in BC^1_{\eta_{\min}}, \\ F^{(1)}(u,\phi) & = & \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(1)}_\delta(u) \in \mathcal{L}(BC^1_{\eta_{\min}}), & \phi \in X_0, & u \in V^{1,\delta}_{\eta_{\min}}, \\ F^{(1)}_1(u,\phi) & = & \mathcal{K}_\eta \circ \widetilde{R}^{(1)}_\delta(u) \in \mathcal{L}(BC^1_\eta), & \phi \in X_0, & u \in V^{1,\delta}_{\eta_{\min}}. \end{array} \tag{2.125}$$

In the context of Lemma A.2, we have that $G : V^{1,\delta}_{\eta_{\min}} \times X_0 \to BC^1_\eta$ is defined by

$$G(u,\phi) = E\phi + \mathcal{J}^1_{\eta\eta_{\min}}\mathcal{K}_{\eta_{\min}}\widetilde{R}_\delta(u), \tag{2.126}$$

and hence using Corollary 2.7.8 and Lemma 2.7.3 we see that condition (HC1) is satisfied. Since $\sup_{\phi\in B^h_\delta}\|DR_\delta(\phi)\| \le L_{R_\delta}$, we see that (2.99) in combination with Lemma 2.7.3 implies condition (HC2). Condition (HC3) follows from Corollary 2.7.8, (HC4) is evident since $D_2G(u,\phi)\psi = E\psi \in BC^1_{\eta_{\min}}$, (HC5) follows from (2.99) and finally (HC6) follows from Lemma 2.7.2. We conclude that $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^1 and that $D(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}})(\phi) = \mathcal{J}^1_{\eta\eta_{\min}} \circ u^{*(1)}_{\eta_{\min}}(\phi) \in \mathcal{L}(X_0, BC^1_\eta)$, where $u^{*(1)}_{\eta_{\min}}(\phi)$ is the unique solution of the equation

$$u^{(1)} = \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(1)}(u^*_{\eta_{\min}}(\phi))u^{(1)} + E \tag{2.127}$$

in the space $\mathcal{L}(X_0, BC^1_{\eta_{\min}})$.

We now assume that $k \ge 2$ and use induction on p. Let $1 \le p < k$ and suppose that for all $1 \le q \le p$ and all $\eta \in (q\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^q with $D^q(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}) = \mathcal{J}^1_{\eta q\eta_{\min}} u^{*(q)}_{\eta_{\min}}$, for some map $u^{*(q)}_{\eta_{\min}} : X_0 \to \mathcal{L}^{(q)}(X_0, BC^1_{q\eta_{\min}})$. In addition, assume that $u^{*(p)}_{\eta_{\min}}(\phi)$ is the unique solution at $\overline{\eta} = \eta_{\min}$ of an equation of the form

$$u^{(p)} = \mathcal{K}_{p\overline{\eta}} \circ \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}(\phi))u^{(p)} + H^{(p)}_{\overline{\eta}}(\phi) = F^{(p)}_{\overline{\eta}}(u^{(p)},\phi), \tag{2.128}$$

in $\mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}})$. Here we have $H^{(1)}(\phi) = E$ and for $p \geq 2$ we can write $H^{(p)}_{\overline{\eta}}(\phi)$ as a finite sum of terms of the form

$$\mathcal{K}_{p\overline{\eta}} \circ \widetilde{R}^{(q)}_{\delta}(u^*_{\eta_{\min}}(\phi))(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)) \tag{2.129}$$

with $2 \leq q \leq p$ and integers $e_i \geq 1$ such that $e_1 + \ldots + e_q = p$. Notice that these conditions ensure that $F^{(p)}_{\overline{\eta}} : \mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}}) \times X_0 \to \mathcal{L}^{(p)}(X_0, BC^1_{p\overline{\eta}})$ is well-defined for all $\overline{\eta} \in [\eta_{\min}, \frac{1}{p}\eta_{\max}]$ and, in addition, is a uniform contraction for these values of $\overline{\eta}$. We now fix $\eta \in ((p+1)\eta_{\min}, \eta_{\max}]$ and choose σ and ζ such that $\eta_{\min} < \sigma < (p+1)\sigma < \zeta < \eta$. We wish to apply Lemma A.2 in the setting $\Omega_0 = Y_0 = \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma})$, $Y = \mathcal{L}^{(p)}(X_0, BC^1_{\zeta})$, $Y_1 = \mathcal{L}^{(p)}(X_0, BC^1_{\eta})$ with the corresponding natural inclusions and $\Lambda = X_0$. We use the functions

$$\begin{array}{rcl}
F(u^{(p)}, \phi) & = & \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))u^{(p)} + H^{(p)}_{\zeta/p}(\phi), \\
& & \quad \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{\zeta}), \\
F^{(1)}(u^{(p)}, \phi) & = & \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi)) \in \mathcal{L}(\mathcal{L}^{(p)}(X_0, BC^1_{\zeta})), \\
& & \quad \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}), \\
F^{(1)}_1(u^{(p)}, \phi) & = & \mathcal{K}_{\eta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi)) \in \mathcal{L}(\mathcal{L}^{(p)}(X_0, BC^1_{\eta})), \\
& & \quad \phi \in X_0, \quad u^{(p)} \in \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}).
\end{array} \tag{2.130}$$

To check (HC1), we need to show that the map $G : \mathcal{L}^{(p)}(X_0, BC^1_{p\sigma}) \times X_0 \to \mathcal{L}^{(p)}(X_0, BC^1_{\eta})$ given by

$$G(u^{(p)}, \phi) = \mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))u^{(p)} + \mathcal{J}^1_{\eta\zeta} H^{(p)}_{\zeta/p}(\phi) \tag{2.131}$$

is of class C^1. In view of the linearity of this map with respect to $u^{(p)}$, it is sufficient to show that $\phi \to \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}(\phi))$ is of class C^1 as a map from X_0 into $\mathcal{L}(BC^1_{p\sigma}, BC^1_{\zeta})$ and, in addition, that $\phi \to H^{(p)}_{\zeta/p}(\phi)$ is of class C^1 as a map from X_0 into $\mathcal{L}^{(p)}(X_0, BC^1_{\zeta})$. The first fact follows from Lemma 2.7.7 using $\zeta > (p+1)\sigma$ and the C^1-smoothness of the map $\phi \to \mathcal{J}^1_{\sigma\eta_{\min}} u^*_{\eta_{\min}}\phi$. To verify the second fact, we use Lemma 2.7.7 and the chain rule to compute

$$\begin{array}{l}
D_{\phi}\mathcal{K}_{\zeta} \circ \widetilde{R}^{(q)}_{\delta}(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)) \\
\quad = \mathcal{K}_{\zeta} \circ \widetilde{R}^{(q+1)}_{\delta}(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi), u^{*(1)}_{\eta_{\min}}(\phi)) \\
\quad + \sum_{j=1}^{q} \mathcal{K}_{\zeta} \circ \widetilde{R}^{(q)}(u^*_{\eta_{\min}}\phi)(u^{*(e_1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_j+1)}_{\eta_{\min}}(\phi), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi)),
\end{array} \tag{2.132}$$

in which each occurrence of $u^{*(j)}_{\eta_{\min}}$ is understood to map into $BC^1_{j\sigma}$. An application of Lemma 2.7.3 with $\zeta > (p+1)\sigma$, shows that the above map is indeed continuous from X_0 into $\mathcal{L}^{(p+1)}(X_0, BC^1_{\zeta})$. These arguments immediately show that also (HC4) is satisfied. Condition (HC3) can be verified by writing $\mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_{\zeta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}\phi) = \mathcal{K}_{\eta} \circ \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}\phi)$ and applying Lemma 2.7.3 to conclude that $\phi \to \widetilde{R}^{(1)}_{\delta}(u^*_{\eta_{\min}}\phi) \in \mathcal{L}(BC^1_{\zeta}, BC_{\eta})$ is continuous. Conditions (HC2) and (HC5) again follow from (2.99) and Lemma 2.7.3 and

(HC6) follows from the fact that the fixed point of (2.128) lies in $\mathcal{L}^{(p)}(X_0, BC^1_{p\sigma})$ since $p\sigma > p\eta_{\min}$. We thus conclude from Lemma A.2 that $\mathcal{J}^1_{\eta\, p\eta_{\min}} \circ u^{*(p)}_{\eta_{\min}}$ is of class C^1 with $D(\mathcal{J}^1_{\eta\, p\eta_{\min}} \circ u^{*(p)}_{\eta_{\min}})(\phi) = \mathcal{J}^1_{\eta\zeta} \circ u^{*(p+1)}(\phi)$, in which $u^{*(p+1)}(\phi)$ is the unique solution of the equation

$$u^{(p+1)} = \mathcal{K}_\zeta \circ \widetilde{R}^{(1)}_\delta(u^*_{\eta_{\min}}(\phi))u^{(p+1)} + H^{(p+1)}_{\zeta/(p+1)}(\phi) \tag{2.133}$$

in $\mathcal{L}^{(p+1)}(X_0, BC^1_\zeta)$, with

$$H^{(p+1)}_{\zeta/(p+1)}(\phi) = \mathcal{K}_\zeta \circ \widetilde{R}^{(2)}_\delta(u^*_{\eta_{\min}}\phi)(u^{*(p)}(\phi), u^{*(1)}(\phi)) + DH^{(p)}_{\zeta/p}(\phi). \tag{2.134}$$

The arguments in the first part of this proof show that the fixed point $u^{*(p+1)}(\phi)$ is also contained in $\mathcal{L}^{(p+1)}(X_0, BC^1_{(p+1)\eta_{\min}})$. We can hence write $u^{*(p+1)}_{\eta_{\min}} = u^{*(p+1)}(\phi) \in \mathcal{L}^{(p+1)}(X_0, BC^1_{(p+1)\eta_{\min}})$, upon which the proof is completed. □

Corollary 2.7.9. *Consider the setting of Theorem 2.7.1. Then for any $\zeta \in [\eta_{\min}, \eta_{\max}]$ and any $\xi \in \mathbb{R}$, the mapping $\phi \to (u^*_\zeta\phi)_\xi$ from X_0 into X is C^k-smooth.*

Proof. For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, we have $(u^*_\zeta\phi)_\xi = (u^*_\eta\phi)_\xi$. The latter mapping is C^k-smooth as a consequence of Theorem 2.7.1 and the fact that the evaluation at ξ is a bounded linear mapping. □

As a conclusion of this section, we use the explicit expression (2.133) for the derivatives of u^*, together with the fact that $u^*_\eta(0) = 0$, to compute the Taylor expansion of $u^*_\eta\phi$ around $\phi = 0$ up to second order. This can be done if $k \geq 2$ and yields

$$u^*_\eta\phi = E\phi + \frac{1}{2}\mathcal{K}_\eta D^2R_\delta(0)((E\phi)_\xi, (E\phi)_\xi) + o(\|\phi\|^2), \tag{2.135}$$

in which the operator $\mathcal{K}_\eta$ acts with respect to the variable ξ.

2.8. Dynamics on the Center Manifold

In this section we show that the dynamics on the center manifold can be described by an ordinary differential equation. In addition, this reduction will be used to supply the proof of Theorem 2.2.2.

Theorem 2.8.1. *Consider the setting of Theorem 2.7.1 and choose $\eta \in (k\eta_{\min}, \eta_{\max}]$. Then for any $\phi \in X_0$, the function $\Phi : \mathbb{R} \to X_0$ given by $\Phi(\xi) = Q_0(u^*_\eta\phi)_\xi$ is C^{k+1}-smooth and satisfies an ordinary differential equation*

$$\Phi'(\xi) = A\Phi(\xi) + f(\Phi(\xi)). \tag{2.136}$$

Here the function $f : X_0 \to X_0$ is C^k-smooth and is explicitly given by

$$f(\psi) = Q_0(L(u^*_\eta\psi - E\psi)_\theta + R_\delta((u^*_\eta\psi)_\theta)), \tag{2.137}$$

where the projection Q_0 is taken with respect to the variable θ. Finally, we have $f(0) = 0$ and $Df(0) = 0$.

Proof. Notice first that Φ is a continuous function, since $\xi \to (u^*_\eta \phi)_\xi$ is continuous. We calculate

$$\begin{array}{rcl} \Phi'(\xi)(\theta) & = & \lim_{h\to 0} \frac{1}{h}(\Phi(\xi+h)(\theta) - \Phi(\xi)(\theta)) \\ & = & \lim_{h\to 0} \frac{1}{h}(Q_0(u^*_\eta\phi)_{\xi+h}(\theta) - Q_0(u^*_\eta\phi)_\xi(\theta)) \\ & = & Q_0(D(u^*_\eta\phi)(\xi+\cdot))(\theta), \end{array} \tag{2.138}$$

where the continuity of the projection Q_0 together with the fact that $\mathcal{K}_\eta$ maps into $C^1(\mathbb{R}, \mathbb{C}^n)$ was used in the last step. Using the definition of $\mathcal{K}_\eta$ we compute

$$D(u^*_\eta\phi)(\xi+\theta) = L(u^*_\eta\phi)_{\xi+\theta} + R_\delta((u^*_\eta\phi)_{\xi+\theta}). \tag{2.139}$$

For convenience, define $\psi = \Phi(\xi)$. Lemma 2.6.5 implies that $(u^*_\eta\phi)_{\xi+\theta} = (u^*_\eta\psi)_\theta$. The ODE (2.136) now follows upon noting that

$$Q_0(L(E\psi)_\theta) = Q_0(\psi'(\theta)) = Q_0((A\psi)(\theta)) = A\psi. \tag{2.140}$$

The fact that f is C^k-smooth follows from the fact that the C^k-smooth function $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ maps into a region on which $\widetilde{R}_\delta$ is itself C^k-smooth by Corollary 2.7.5. It is easy to see that $f(0) = 0$ and from (HR2) and the Taylor expansion (2.135), it follows that $Df(0) = 0$. The fact that Φ is C^{k+1}-smooth follows from repeated differentiation of (2.136). $\square$

In order to lift solutions of (2.136) back to the original equation (2.6), we need to establish that the nonlinearity in (2.12) agrees with the version in (2.137) in a small neighbourhood of zero. The next lemma shows that this can indeed be realized.

Lemma 2.8.2. *Let $\delta > 0$ and $\epsilon > 0$ be so small that for some $0 < \eta_0 < \gamma$,*

$$\begin{array}{lcl} L_{R_\delta}(w^{2\eta_0} + w^{\eta_0}) & < & (8\left\|\mathcal{K}_{\eta_0}\right\|)^{-1}, \\ \epsilon w^{2\eta_0}\left\|E\right\|_{\eta_0} & < & \frac{1}{2}\delta, \end{array} \tag{2.141}$$

with the Lipschitz constant L_{R_δ} as introduced in Corollary 2.6.2 and the extension operator E as defined in Proposition 2.5.2. Then for any $0 < \eta < \gamma$ and any $\phi \in X_0$ with $\|\phi\| < \epsilon$, we have for all $r_{\min} \le \theta \le r_{\max}$ that

$$\left\| Q_0(u^*_\eta\phi)_\theta \right\| < \delta. \tag{2.142}$$

Proof. Similarly as in the proof of Lemma 2.7.2, we compute

$$Q_0(u^*_\eta\phi)_\theta = (E\phi)_\theta + (\mathcal{K}_{\eta_0}\widetilde{R}_\delta(u^*_\eta\phi))_\theta - (\mathcal{K}_{\eta_0}\widetilde{R}_\delta((u^*_\eta\phi)(\theta+\cdot)))_0 \tag{2.143}$$

and hence using (2.81) we obtain

$$\left\| Q_0(u^*_\eta\phi)_\theta \right\| \le w^{\eta_0}w^{\eta_0}\left\|E\right\|_{\eta_0}\|\phi\| + 4\delta L_{R_\delta}(w^{\eta_0}w^{\eta_0}\left\|\mathcal{K}_{\eta_0}\right\| + w^{\eta_0}\left\|\mathcal{K}_{\eta_0}\right\|) < \delta, \tag{2.144}$$

which completes the proof. $\square$

Proof of Theorem 2.2.2. Choose $\delta > 0$ such that (2.99), (2.104) and (2.141) are all satisfied and fix the constant $\epsilon^* > 0$ such that $\epsilon^* \max(\|Q_0\|, \|I - Q_0\|) < \delta$. Fix $0 < \epsilon < \delta$ such that (2.141) is satisfied, pick any $\eta \in (k\eta_{\min}, \eta_{\max}]$ and write $u^* = u^*_\eta$.

(i) This follows from Theorem 2.7.1 together with $u^* = u^*_\zeta = \mathcal{J}^1_{\zeta\eta_{\min}} u^*_{\eta_{\min}}$ for any $\zeta \in (k\eta_{\min}, \eta_{\max}]$.

(ii) First note that (i) and the conditions (HR1)-(HR2) imply that f is C^k-smooth with $f(0) = 0$ and $Df(0) = 0$. Since $\xi \mapsto x_\xi$ maps into the subset of X on which R and R_δ agree, we have $\Lambda x = \widetilde{R}_\delta(x)$ and hence $Px = \mathcal{K}_\eta \Lambda x = \mathcal{K}_\eta \widetilde{R}_\delta(x)$. Since $Px = x - EQ_0x_0$ we see that $\mathcal{G}(x, Q_0x_0) = x$ and hence due to uniqueness of solutions we indeed have $x = u^* Q_0 x_0$. Note that for all $\xi \in \mathbb{R}$ we have $\|\Phi(\xi)\| < \delta$, which by Lemma 2.6.5 implies that $\|Q_0(u^*\Phi(\xi))_\theta\| < \delta$ for any $\xi \in \mathbb{R}$ and $\theta \in [r_{\min}, r_{\max}]$. Thus the function f defined in (2.137) agrees with (2.12) and hence an application of Theorem 2.8.1 shows that Φ satisfies the ODE (2.11). An application of Lemma 2.6.5 completes the proof.

(iii) This is clear from the fact that $\xi \to (u^*\phi)_\xi$ maps into the subset of X on which R and R_δ agree.

(iv) See (v) with $\xi_- = -\infty$ and $\xi_+ = +\infty$.

(v) Define the function $\Psi(\xi) = Q_0(u^*\Phi(\zeta))_{\xi-\zeta}$ and note that it satisfies (2.136) on $\mathbb{R}$, with $\Psi(\zeta) = \Phi(\zeta)$. Note further that Lemmas 2.7.2 and 2.8.2 imply that the nonlinearities (2.12) and (2.137) agree on the set $\{\phi \in X_0 \mid \|\phi\| < \epsilon\}$. Since both nonlinearities are locally Lipschitz continuous, this implies that in fact $\Psi(\xi) = \Phi(\xi)$ for all $\xi \in (\xi_{\min}, \xi_{\max})$. Thus defining $x(\xi) = (u^*\Phi(\zeta))(\xi - \zeta)$, we see that $Q_0 x_\xi = \Psi(\xi)$ and hence $\left\|Q_0 x_\xi\right\| < \epsilon < \delta$ for all $\xi \in (\xi_{\min}, \xi_{\max})$. Since $(\Lambda x)(\xi) = R_\delta(x_\xi) = R(x_\xi)$ for all such ξ, we see that x indeed satisfies (2.6) on the interval $(\xi_{\min}, \xi_{\max})$. Finally, Lemma 2.6.5 shows that for any ξ in this interval we have $x_\xi = (u^*\Psi(\xi))_0 = (u^*\Phi(\xi))_0$.

□

2.9. Parameter Dependence

We now wish to incorporate parameter dependent equations into our framework. In particular, we will study equations of the form

$$x'(\xi) = L(\mu)x_\xi + R(x_\xi, \mu) \tag{2.145}$$

for parameters $\mu \in \Omega \subset \mathbb{C}^d$ in some open subset Ω and linearities

$$L(\mu)\phi = \sum_{j=0}^{N} A_j(\mu)\phi(r_j). \tag{2.146}$$

We assume here that the conditions (HLμ), (HRμ1) and (HRμ2) all hold. Suppose that for some $\mu_0 \in \Omega$ we have that $\det \Delta_{L(\mu_0)}(z) = 0$ has roots on the imaginary axis. Introducing new coordinates $\nu = \mu - \mu_0$ and defining $\mathbf{y} = (x, \nu)$, we obtain the system

$$\mathbf{y}'(\xi) = \mathbf{L}\mathbf{y}_\xi + \mathbf{R}(\mathbf{y}_\xi), \tag{2.147}$$

in which $\mathbf{L} = (L(\mu_0), 0)$ and $\mathbf{R}((\phi, \nu)) = ((L(\mu_0 + \nu) - L(\mu_0))\phi + R(\phi, \mu_0 + \nu), 0)$. Notice that $\mathbf{R}$ satisfies the assumptions (HR1) and (HR2), which enables the application of the theory developed above. Notice that for any $x \in \mathcal{N}_0$, we have that $\mathbf{y} = (x, \nu)$ satisfies $\mathbf{y}'(\xi) = \mathbf{L}\mathbf{y}_\xi$ and hence we have the identity $\mathbf{X}_0 = X_0 \times \mathbb{C}^d$ for the respective center spaces.

From now on we will simply write $\mathbf{u}^*$ for the function $\mathbf{u}^*_\eta$ defined in Theorem 2.6.4. We split off the part of this operator which acts on the state space for the parameter ν and write $\mathbf{u}^* = (u_1^*, u_2^*)$, with $u_2^*(\phi, \nu) = \nu$. The first component of the differential equation (2.11) on the center manifold in our setting becomes

$$\Phi'(\xi) = A\Phi(\xi) + f(\Phi(\xi), \nu), \tag{2.148}$$

for $\Phi : \mathbb{R} \to X_0$, where $f : X_0 \times \mathbb{C}^d \to X_0$ is given by

$$\begin{aligned} f(\psi, \nu) \quad = \quad & Q_0(L(u_1^*(\psi, \nu) - E\psi)_\theta + (L(\mu_0 + \nu) - L(\mu_0))(u_1^*(\psi, \nu))_\theta \\ & + R((u_1^*(\psi, \nu))_\theta, \mu_0 + \nu)), \end{aligned} \tag{2.149}$$

in which the projection Q_0 is taken with respect to the variable θ. We finish by computing the Taylor expansion of u_1^* to second order, which is possible if $k \geq 2$. We have

$$u_1^*(\phi, \nu) = E\phi + \mathcal{K}\big(L'(\mu_0)\nu(E\phi)_\xi + \frac{1}{2}D_1^2 R(0, \mu_0)((E\phi)_\xi, (E\phi)_\xi)\big) + o((|\nu| + |\phi|)^2), \tag{2.150}$$

in which $\mathcal{K}$ acts with respect to the variable ξ.

2.10. Hopf Bifurcation

In this section we use the projection on the center manifold to apply the finite dimensional Hopf bifurcation theorem to our infinite dimensional setting. In particular, we will consider a system of the form (2.13) that depends on a parameter $\mu \in \mathbb{R}$. We will assume that for some $\mu_0 \in \mathbb{R}$ the linear operator $L = L(\mu_0)$ has simple eigenvalues at $\pm i\omega_0$ for some $\omega_0 > 0$ and we write X_0 for the center subspace at this parameter value μ_0. We will look for small continuous periodic solutions $\Phi : \mathbb{R} \to X_0$ to the equation

$$\Phi'(\xi) = A\Phi(\xi) + f(\Phi(\xi), \nu), \tag{2.151}$$

for small values of ν, with f as in (2.149). Using Theorem 2.2.2 these solutions can be lifted to periodic solutions of the original equation (2.13).

Before we can apply Theorem C.1, we need to study the generalized eigenspace of A for simple eigenvalues.

Lemma 2.10.1. *Consider the system (2.6) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has a simple root at* $z = i\omega_0$. *Then the matrix valued function*

$$H(z) = (z - i\omega_0)\Delta(z)^{-1} \tag{2.152}$$

is analytic in a neighbourhood of $z = i\omega_0$. *In addition, there exist* $p, q \in \mathbb{C}^n$ *such that* $\Delta(i\omega_0)p = \Delta(i\omega_0)^T q = 0$, *while* $q^T \Delta'(i\omega_0)p \neq 0$. *For any such pair the function* $\phi = e^{i\omega_0 \cdot} p$ *is an eigenvector of the operator* A *defined in (2.41) corresponding to the algebraically simple eigenvalue* $i\omega_0$ *and in addition we have the identities*

$$\begin{array}{rcl} H(i\omega_0) & = & pq^T (q^T \Delta'(i\omega_0)p)^{-1}, \\ Q_\phi \psi & = & e^{i\omega_0 \cdot} H(i\omega_0)(\psi(0) + \sum_{j=0}^{N} A_j e^{i\omega_0 r_j} \int_{r_j}^{0} e^{-i\omega_0 \sigma} \psi(\sigma) d\sigma). \end{array} \tag{2.153}$$

Here $Q_\phi : X_0 \to X_0$ *denotes the spectral projection onto the generalized eigenspace of* A *for the eigenvalue* $i\omega_0$.

Proof. Since $\Delta(z)$ is a characteristic matrix for A and $\det \Delta(z) = 0$ has a simple root at $z = i\omega_0$, it follows from the theory of characteristic matrices (see e.g. [45, Theorem IV.4.18]) that $\Delta(z)$ has a pole of order one at $z = i\omega_0$ and A has a simple eigenvalue at $z = i\omega_0$. This proves that $H(z)$ is analytic in a neigbourhood of $z = i\omega_0$. It also follows that the nullspace $\mathcal{N}(\Delta(i\omega_0))$ is the one dimensional span of some $p \in \mathbb{C}^n$. Similarly, we have $\mathcal{N}(\Delta(i\omega_0)^T) = \text{span}\{q\}$ for some $q \in \mathbb{C}^n$. It is easy to check that $\phi = e^{i\omega_0 \cdot} p$ is indeed a corresponding eigenvector for A. Using residue calculus and the formula (2.42) for the resolvent of A to simplify the Dunford integral (2.49), the expression (2.153) for the spectral projection follows easily.

It remains to derive the explicit expression (2.153) for $H(i\omega_0)$. To this end, observe that

$$\Delta(z)H(z) = H(z)\Delta(z) = (z - i\omega_0)I, \tag{2.154}$$

which implies $\mathcal{R}H(i\omega_0) \subset \mathcal{N}\Delta(i\omega_0)$ and $\mathcal{R}H(i\omega_0)^T \subset \mathcal{N}\Delta(i\omega_0)^T$. From this it follows that $H(i\omega_0) = Cpq^T$ for some constant C. Expanding (2.154) in a Taylor series we obtain

$$\begin{array}{rcl} I & = & H'(i\omega_0)\Delta(i\omega_0) + H(i\omega_0)\Delta'(i\omega_0) = \Delta'(i\omega_0)H(i\omega_0) + \Delta(i\omega_0)H'(i\omega_0), \\ 0 & = & H''(i\omega_0)\Delta(i\omega_0) + 2H'(i\omega_0)\Delta'(i\omega_0) + H(i\omega_0)\Delta''(i\omega_0). \end{array} \tag{2.155}$$

Noting that $p = H(i\omega_0)\Delta'(i\omega_0)p = Cpq^T \Delta'(i\omega_0)p$ completes the proof. □

Since we are interested in real valued functions, we need to treat the two complex eigenvalues at $\pm i\omega_0$ together. To this end, we introduce the real valued functions $\psi_\pm \in X_0$ via

$$\begin{array}{rcl} \psi_+(\theta) & = & \frac{1}{2}(\phi(\theta) + \overline{\phi}(\theta)), \\ \psi_-(\theta) & = & -\frac{i}{2}(\phi(\theta) - \overline{\phi}(\theta)) \end{array} \tag{2.156}$$

and we note that the part of A on this basis takes the form $\begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$. On the other hand, we consider the two dimensional real ODE

$$\begin{pmatrix} y_+' \\ y_-' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \tag{2.157}$$

and observe that under the complexification $z = y_+ + iy_-$, this system is transformed into

$$z' = \frac{1}{2}(a_{11} + a_{22} + i(a_{21} - a_{12}))z + \frac{1}{2}(a_{11} + a_{22} + i(a_{12} - a_{21}))\overline{z}. \tag{2.158}$$

The only nontrivial hypothesis we need to check before we can apply Theorem C.1 is the condition (HH3), i.e. $\operatorname{Re} D\sigma(\mu_0) \neq 0$ for the branch $\sigma(\mu)$ of eigenvalues of $D_1 g(0, \mu)$ through $i\omega_0$ at $\mu = \mu_0$. The following lemma indicates how this quantity can be explicitly calculated.

Lemma 2.10.2. *Consider real $m \times m$ matrices M_0 and $M_1(\nu)$ for some integer $m \geq 2$, where each entry $M_1^{(ij)}(\nu)$ of $M_1(\nu)$ is a C^1-smooth function of the real parameter ν with $M_1^{(ij)}(0) = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq m$. Suppose that for some $\omega_0 \in \mathbb{R}$ and $(m-2) \times (m-2)$ matrix B we have $M_0 = \operatorname{diag}(A(\omega_0), B)$ with $A(\omega_0) = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$. Suppose further that the matrices $B \pm i\omega_0 I$ are both invertible, i.e., M_0 has simple eigenvalues $\pm i\omega_0$. Write $\sigma(\nu)$ for the branch of eigenvalues of $M = M_0 + M_1(\nu)$ through $i\omega_0$ at $\nu = 0$. Then we have $\operatorname{Re} D\sigma(0) = \frac{1}{2}(\dot{M}_1^{(11)}(0) + \dot{M}_1^{(22)}(0))$, in which the dot denotes differentiation with respect to ν.*

Proof. We define the function $\Delta(\nu, \lambda) = \det(M_0 + M_1(\nu) - (i\omega_0 + \lambda)I)$ and note that we have the identity $\Delta(\nu, \sigma(\nu) - i\omega_0) = 0$ for small ν. Using implicit differentiation it follows that

$$D\sigma(0) = -D_1\Delta(0, 0)/D_2\Delta(0, 0) \tag{2.159}$$

and hence it suffices to compute

$$\begin{array}{rcl} D_1\Delta(0,0) & = & \big(- i\omega_0 \dot{M}_1^{(11)}(0) - i\omega_0 \dot{M}_1^{(22)}(0) - \omega_0 \dot{M}_1^{(12)}(0) + \omega_0 \dot{M}_1^{(21)}(0)\big) \\ & & \det(B - i\omega_0 I), \\ D_2\Delta(0,0) & = & 2i\omega_0 \det(B - i\omega_0 I), \end{array} \tag{2.160}$$

from which the claim immediately follows. □

Thus in order to calculate $D\sigma(\mu_0)$, it suffices to expand (2.151) up to terms involving $O(\nu\phi)$, i.e.,

$$\Phi' = A\Phi + Q_0 h(\Phi, \nu) + O(|\Phi|^2 + |\nu|^2 + (|\Phi| + |\nu|)^3), \tag{2.161}$$

where $h : X_0 \times \mathbb{R} \to X$ is the bilinear operator

$$h(\psi, \nu)(\theta) = (L\mathrm{ev}_\theta \mathcal{K} + \mathrm{pev}_\theta)\big(\nu L'(\mu_0)\mathrm{ev}_{(\cdot)} E\psi\big), \tag{2.162}$$

in which we have introduced the evaluation function $\mathrm{ev}_{\theta'} f(\cdot) = f_{\theta'}$ and the point evaluation $\mathrm{pev}_{\theta'} f(\cdot) = f(\theta')$. In view of Lemma 2.10.2, the specific form of the transformation of the real ODE (2.157) into (2.158) and the fact that $\phi = \psi_+ + i\psi_-$, it is clear that

$$\operatorname{Re} D\sigma(0) = \operatorname{Re} \widetilde{Q}_\phi Q_0 h(\phi/\nu, \nu), \text{ with } Q_\phi = \phi\widetilde{Q}_\phi. \tag{2.163}$$

In order to evaluate (2.163), we need to calculate $\mathcal{K}e^{i\omega_0 \cdot}v$ for arbitrary $v \in \mathbb{C}^n$. As a preparation, we compute

$$\begin{array}{rcl} Q_\phi e^{i\omega_0\theta}v & = & e^{i\omega_0 \cdot}H(i\omega_0)\Delta'(i\omega_0)v, \\ Q_\phi \theta e^{i\omega_0\theta}v & = & \frac{1}{2}e^{i\omega_0 \cdot}H(i\omega_0)\Delta''(i\omega_0)v, \end{array} \tag{2.164}$$

in which the projections Q_ϕ were taken with respect to the variable θ.

Lemma 2.10.3. *Consider (2.6) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has a simple root at* $z = i\omega_0$*. Let* $H(z)$*,* p *and* q *be as in Lemma 2.10.1. Then for arbitrary* $v \in \mathbb{C}^n$ *we have*

$$(\mathcal{K}e^{i\omega_0 \cdot}v)(\xi) = e^{i\omega_0\xi}(H(i\omega_0)\xi + H'(i\omega_0))v + (E\psi)(\xi), \tag{2.165}$$

for some $\psi \in X_0$ *with* $Q_\phi\psi = 0$*. In addition, we have*

$$Q_\phi((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\omega_0 \cdot}v) = \phi q^T v(q^T\Delta'(i\omega_0)p)^{-1}. \tag{2.166}$$

Proof. For convenience, define $\Psi(\xi) = e^{i\omega_0\xi}(H(i\omega_0)\xi + H'(i\omega_0))v$. We first check that the function above indeed satisfies the differential equation. We compute

$$\Psi'(\xi) = e^{i\omega_0\xi}((i\omega_0\xi + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0))v. \tag{2.167}$$

Similarly, we compute

$$\begin{array}{rcl} L\Psi_\xi & = & e^{i\omega_0\xi}\big((i\omega_0 - \Delta(i\omega_0))H(i\omega_0)\xi + (I - \Delta'(i\omega_0))H(i\omega_0) \\ & & \quad +(i\omega_0 - \Delta(i\omega_0))H'(i\omega_0)\big)v \\ & = & e^{i\omega_0\xi}\big((i\omega_0\xi + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0) - I\big)v, \end{array} \tag{2.168}$$

from which we see that indeed $(\Lambda\Psi)(\xi) = \Psi'(\xi) - L\Psi_\xi = e^{i\omega_0\xi}v$. In addition, using (2.155) we can calculate

$$\begin{array}{rcl} e^{-i\omega_0 \cdot}Q_\phi\Psi_0 & = & (\frac{1}{2}H(i\omega_0)\Delta''(i\omega_0)H(i\omega_0) + H(i\omega_0)\Delta'(i\omega_0)H'(i\omega_0))v \\ & = & -\frac{1}{2}H(i\omega_0)\Delta(i\omega_0)H''(i\omega_0)v = 0, \end{array} \tag{2.169}$$

as required. Finally, we compute

$$\begin{array}{rcl} Q_\phi(L\Psi_\theta + e^{i\omega_0\theta}v) & = & Q_\phi(e^{i\omega_0\theta}((i\omega_0\theta + 1)H(i\omega_0) + i\omega_0 H'(i\omega_0))v) \\ & = & e^{i\omega_0 \cdot}(\frac{1}{2}i\omega_0 H(i\omega_0)\Delta''(i\omega_0)H(i\omega_0) \\ & & \quad +H(i\omega_0)\Delta'(i\omega_0)(H(i\omega_0) + i\omega_0 H'(i\omega_0)))v \\ & = & e^{i\omega_0 \cdot}H(i\omega_0)\Delta'(i\omega_0)H(i\omega_0)v \\ & = & \phi q^T v(q^T\Delta'(i\omega_0)p)^{-1}. \end{array} \tag{2.170}$$

□

Using the above lemma we can now calculate

$$\begin{array}{rcl} \mathrm{Re}\, D\sigma(0) & = & \mathrm{Re}\,\widetilde{Q}_\phi Q_0((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)L'(\mu_0)\phi) \\ & = & -\mathrm{Re}\,\widetilde{Q}_\phi((\mathrm{Lev}_\theta\mathcal{K} + \mathrm{pev}_\theta)D_2\Delta(i\omega_0, \mu_0)e^{i\omega_0 \cdot}p) \\ & = & -\mathrm{Re}\, q^T D_2\Delta(i\omega_0, \mu_0)p(q^T\Delta'(i\omega_0)p)^{-1}. \end{array} \tag{2.171}$$

Proof of Theorem 2.2.3. We apply Theorem C.1 to the ODE (2.151). Conditions (HH1)-(HH2) are immediate from the assumptions on (2.13) and (HH3) follows from (Hζ3) and (2.171). Restricting the allowed values of τ in Theorem C.1 to a small interval I around zero such that $|\mu^*(\tau) - \mu_0| < \frac{\epsilon}{2}$ and $|x^*(\tau)(\xi)| < \frac{\epsilon}{2}$ for all $\xi \in \mathbb{R}$ and $\tau \in I$, with ϵ as in the statement of Theorem 2.2.2, it follows from part (iv) of this theorem that each $x^*(\tau)$ can be lifted to a periodic solution of (2.13). Similarly, every small periodic solution of (2.13) corresponds to a small periodic solution of (2.151), which is captured by Theorem C.1. □

We now set out to compute the direction of bifurcation using Theorem C.2. Notice first that $(E\phi)(\xi) = pe^{i\omega_0\xi}$ and similarly $(E\overline{\phi})(\xi) = \overline{p}e^{-i\omega_0\xi}$. In particular, this implies that $(E\phi)_\xi = e^{i\omega_0\xi}\phi \in X_0$ and similarly $(E\overline{\phi})_\xi = e^{-i\omega_0\xi}\overline{\phi} \in X_0$. In order to evaluate the constant c appearing in Theorem C.2, we need to calculate $\mathcal{K}e^{i\zeta\omega_0\theta}v$ for arbitrary $v \in \mathbb{C}^n$ and $\zeta \in \mathbb{R}$ such that $\det \Delta(i\zeta\omega_0) \neq 0$. We obtain the following result.

Lemma 2.10.4. *Consider (2.6) and suppose that the characteristic equation* $\det \Delta(z) = 0$ *has a simple root at* $z = i\omega_0$. *Let* $H(z)$, p *and* q *be as in Lemma 2.10.1. Then for arbitrary* $v \in \mathbb{C}^n$ *and* $\zeta \in \mathbb{R}$ *such that* $\det \Delta(i\zeta\omega_0) \neq 0$, *we have*

$$(\mathcal{K}e^{i\zeta\omega_0\cdot}v)(\xi) = e^{i\zeta\omega_0\xi}\Delta(i\zeta\omega_0)^{-1}v - Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v). \tag{2.172}$$

In addition, we have the identity

$$Q_0((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\zeta\omega_0\cdot}v) = (i\zeta\omega_0 - A)Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v). \tag{2.173}$$

Proof. For convenience, define $\Psi(\xi) = e^{i\zeta\omega_0\xi}\Delta(i\zeta\omega_0)^{-1}v$. First note that

$$L\Psi_\xi = e^{i\zeta\omega_0\xi}(i\zeta\omega_0 - \Delta(i\zeta\omega_0))\Delta(i\zeta\omega_0)^{-1}v = i\zeta\omega_0\Psi(\xi) - e^{i\zeta\omega_0\xi}v, \tag{2.174}$$

from which it follows that

$$(\Lambda\Psi)(\xi) = i\zeta\omega_0\Psi(\xi) - L\Psi_\xi = e^{i\zeta\omega_0\xi}v, \tag{2.175}$$

which implies the first claim. To substantiate the second claim, note that

$$\begin{array}{rcl} Q_0((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)e^{i\zeta\omega_0\cdot}v) & = & Q_0(L\Psi_\theta + e^{i\zeta\omega_0\theta}v - L(EQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v))_\theta) \\ & = & Q_0(i\zeta\omega_0 e^{i\zeta\omega_0\theta}\Delta(i\zeta\omega_0)^{-1}v \\ & & \quad - AQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v)) \\ & = & (i\zeta\omega_0)Q_0(e^{i\zeta\omega_0\theta}\Delta(i\zeta\omega_0)^{-1}v) \\ & & \quad - AQ_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v) \\ & = & (i\zeta\omega_0 - A)Q_0(e^{i\zeta\omega_0\cdot}\Delta(i\zeta\omega_0)^{-1}v). \end{array} \tag{2.176}$$

□

To explicitly calculate c, we write the nonlinearity $f : X_0 \times \mathbb{R} \to X_0$ in (2.151) in the form

$$f(\psi, \nu) = Q_0((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)\widetilde{\mathbf{R}}_1(u_1^*(\psi, \nu), \nu)), \tag{2.177}$$

in which $\widetilde{\mathbf{R}}_1$ is the substitution operator associated with the first component of the compound operator $\mathbf{R}$ defined in (2.147). We thus need to compute

$$\begin{aligned} & D_1^3(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,0)(\psi_1,\psi_2,\psi_3)(\xi) \\ & \quad = D_1^3 R(0,\mu_0)((E\psi_1)_\xi,(E\psi_2)_\xi,(E\psi_3)_\xi) \\ & \quad\quad + D_1^2 R(0,\mu_0)((E\psi_1)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0,\mu_0)((E\psi_2)_{(\cdot)},(E\psi_3)_{(\cdot)})) \\ & \quad\quad + D_1^2 R(0,\mu_0)((E\psi_2)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0,\mu_0)((E\psi_3)_{(\cdot)},(E\psi_1)_{(\cdot)})) \\ & \quad\quad + D_1^2 R(0,\mu_0)((E\psi_3)_\xi, \mathrm{ev}_\xi \mathcal{K} D_1^2 R(0,\mu_0)((E\psi_1)_{(\cdot)},(E\psi_2)_{(\cdot)})) \end{aligned} \tag{2.178}$$

and hence substituting $\psi_1 = \psi_2 = \phi$ and $\psi_3 = \overline{\phi}$, we obtain

$$\begin{aligned} & D_1^3(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,0)(\phi,\phi,\overline{\phi})(\xi) \\ & \quad = e^{i\omega_0\xi} D_1^3 R(0,\mu_0)(\phi,\phi,\overline{\phi}) \\ & \quad\quad + 2e^{i\omega_0\xi} D_1^2 R(0,\mu_0)\big(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})\big) \\ & \quad\quad - 2D_1^2 R(0,\mu_0)\big((E\phi)_\xi, \mathrm{ev}_\xi E Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi}))\big) \\ & \quad\quad + e^{i\omega_0\xi} D_1^2 R(0,\mu_0)\big(\overline{\phi}, \Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi)\big) \\ & \quad\quad - D_1^2 R(0,\mu_0)\big((E\overline{\phi})_\xi, \mathrm{ev}_\xi E Q_0(e^{2i\omega_0\cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi))\big). \end{aligned} \tag{2.179}$$

In addition, using Lemma 2.10.4 we calculate,

$$\begin{aligned} D_1^2 f(0,\mu_0)(\phi,\phi) &= Q_0((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta) e^{2i\omega_0\cdot} D_1^2 R(0,\mu)(\phi,\phi)) \\ &= (2i\omega_0 - A) Q_0(e^{2i\omega_0\cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu)(\phi,\phi)). \end{aligned} \tag{2.180}$$

A similar computation shows that

$$D_1^2 f(0,\mu_0)(\phi,\overline{\phi}) = -A Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})). \tag{2.181}$$

Using these identities we can write

$$\begin{aligned} & D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,\mu_0)(\phi, -A^{-1} D_1^2 f(0,\mu_0)(\phi,\overline{\phi}))(\xi) \\ & \quad = D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,\mu_0)\big(\phi, Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi}))\big) \\ & \quad = D_1^2 R(0,\mu_0)\big((E\phi)_\xi, \mathrm{ev}_\xi E Q_0(\mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi}))\big) \end{aligned} \tag{2.182}$$

and similarly

$$\begin{aligned} & D_1^2(\widetilde{\mathbf{R}}_1 \circ \mathbf{u}^*)(0,\mu_0)(\overline{\phi}, (2i\omega_0 - A)^{-1} D_1^2 f(0,\mu_0)(\phi,\phi))(\xi) \\ & \quad = D_1^2 R(0,\mu_0)\big((E\overline{\phi})_\xi, \mathrm{ev}_\xi E Q_0(e^{2i\omega_0\cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi}))\big). \end{aligned} \tag{2.183}$$

Putting all our calculations together, we arrive at

$$c\phi = Q_\phi Q_0((L\mathrm{ev}_\theta\mathcal{K} + \mathrm{pev}_\theta)\Psi(\cdot)), \tag{2.184}$$

in which

$$\begin{aligned} \Psi(\xi) \;=\; & \tfrac{1}{2} e^{i\omega_0\xi} D_1^3 R(0,\mu_0)(\phi,\phi,\overline{\phi}) \\ & + e^{i\omega_0\xi} D_1^2 R(0,\mu_0)(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})) \\ & + \tfrac{1}{2} e^{i\omega_0\xi} D_1^2 R(0,\mu_0)(\overline{\phi}, e^{2i\omega_0\cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi)). \end{aligned} \tag{2.185}$$

Finally, an application of Lemma 2.10.3 yields

$$\begin{aligned}(q^T \Delta'(i\omega_0)p)c &= \tfrac{1}{2}q^T D_1^3 R(0,\mu_0)(\phi,\phi,\overline{\phi}) \\ &\quad + q^T D_1^2 R(0,\mu_0)(\phi, \mathbf{1}\Delta(0)^{-1} D_1^2 R(0,\mu_0)(\phi,\overline{\phi})) \\ &\quad + \tfrac{1}{2}q^T D_1^2 N(0,\mu_0)(\overline{\phi}, e^{2i\omega_0 \cdot}\Delta(2i\omega_0)^{-1} D_1^2 R(0,\mu_0)(\phi,\phi)).\end{aligned} \tag{2.186}$$

Proof of Theorem 2.2.4. Using Theorem C.2, the statement follows immediately from the formulas (2.171) and (2.186). □

2.11. Example: Double Eigenvalue At Zero

We here give a concrete example of the power of the finite dimensional reduction by considering a functional differential equation of mixed type that depends on four parameters. For certain values of the parameters the equation reduces to a delay equation, which has already been studied in [45]. This example hence allows us to check that our framework yields reproducable results when restricting to delay equations. The equation we consider has the origin as an equilibrium and in addition has a double eigenvalue at zero with geometric multiplicity one, for certain critical parameter values. This means that the origin is a Takens-Bogdanov point and it is known that for such equilibria only the second order terms are needed to determine the local phase portrait.

In particular, we consider the equation

$$x'(\xi) = \alpha x(\xi) + \beta_- g(x(\xi-1),\mu) + \beta_+ g(x(\xi-1),\mu), \tag{2.187}$$

for some $g \in C^3(\mathbb{R}\times\mathbb{R},\mathbb{R})$. We enforce the conditions $\beta_+ + \beta_- \neq 0$ and $\beta_+ - \beta_- \neq 0$. Suppose that $g(0,\mu) = 0$ for any $\mu \in \mathbb{R}$ and in addition $g'(0,\mu) = \mu$. Linearization around the zero equilibrium yields

$$x'(\xi) = \alpha x(\xi) + \beta_- \mu x(\xi-1) + \beta_+ \mu x(\xi+1) \tag{2.188}$$

and with a short calculation one can verify that this equation has a double eigenvalue zero at $(\alpha,\mu) = (\alpha_0,\mu_0) := (\frac{\beta_-+\beta_+}{\beta_--\beta_+}, \frac{1}{\beta_+-\beta_-})$, with corresponding eigenvectors $\phi_0 = \mathbf{1}$ and $\phi_1 = \{\theta \mapsto \theta\}$. The projection operator $Q_0 : X \to X_0$ onto the span of ϕ_0 and ϕ_1 can be calculated by using residue calculus on the resolvent equation (2.42). We find

$$\begin{aligned}(Q_0\phi)(\theta) &= \frac{2\theta(\beta_--\beta_+)^2 + \frac{2}{3}(\beta_--\beta_+)^2}{(\beta_++\beta_-)^2}\psi(0) \\ &\quad + \frac{\beta_-}{(\beta_++\beta_-)^2}\int_{-1}^0 \big(2(\sigma-\theta)(\beta_++\beta_-) + \tfrac{8}{3}\beta_+ + \tfrac{4}{3}\beta_-\big)\psi(\sigma)d\sigma \\ &\quad + \frac{\beta_+}{(\beta_++\beta_-)^2}\int_0^1 \big(2(\theta-\sigma)(\beta_++\beta_-) + \tfrac{8}{3}\beta_- + \tfrac{4}{3}\beta_+\big)\psi(\sigma)d\sigma.\end{aligned} \tag{2.189}$$

We introduce parameters $\lambda = \alpha - \alpha_0$ and $\nu = \mu - \mu_0$ and investigate (2.187) for small values of λ and ν, keeping β_+ and β_- fixed. Writing

$$\begin{aligned}R(\phi,\lambda,\nu) &= \beta_- g(\phi(-1),\mu_0+\nu) - \beta_-(\mu_0+\nu)\phi(-1) \\ &\quad + \beta_+ g(\phi(+1),\mu_0+\nu) - \beta_+(\mu_0+\nu)\phi(+1) \\ &= \frac{\beta_-}{2}g''(0,\mu_0+\nu)(\phi(-1))^2 + \frac{\beta_+}{2}g''(0,\mu_0+\nu)(\phi(+1))^2 + O(\|\phi\|^3),\end{aligned} \tag{2.190}$$

equation (2.187) transforms into the system

$$\begin{array}{rcl} x'(\xi) &=& \frac{\beta_+ + \beta_-}{\beta_- - \beta_+} x(\xi) + \frac{\beta_-}{\beta_+ - \beta_-} x(\xi - 1) + \frac{\beta_+}{\beta_+ - \beta_-} x(\xi + 1) \\ && + \lambda x(\xi) + \nu \beta_- x(\xi - 1) + \nu \beta_+ x(\xi + 1) + R(x_\xi, \lambda, \nu), \end{array} \tag{2.191}$$

which satisfies the conditions (HRμ1)-(HRμ2) and (HLμ). Using the explicit form of R and the linear part of (2.191), we see that the first component of the second order Taylor expansion (2.150) in our case becomes

$$\begin{array}{rcl} u_1^*(\phi, \lambda, \nu) &=& E\phi + \mathcal{K}\big(\lambda(E\phi)(\cdot) + \beta_- \nu (E\phi)(\cdot - 1) + \beta_+ \nu (E\phi)(\cdot + 1) \\ &+& \frac{\beta_-}{2} g''(0, \mu_0)((E\phi)(\cdot - 1))^2\big) + \frac{\beta_+}{2} g''(0, \mu_0)((E\phi)(\cdot + 1))^2) \\ &+& O(|\phi|^3 + (|\nu| + |\lambda|)\, |\phi|\, (|\nu| + |\lambda| + |\phi|)). \end{array} \tag{2.192}$$

We now set out to calculate the differential equation that is satisfied on the center manifold up to and including second order terms. Using Theorem 2.8.1 we calculate

$$\Phi' = A\Phi + f(\Phi) + O(|\Phi|^3 + (|\lambda| + |\nu|)\, |\Phi|\, (|\lambda| + |\nu| + |\Phi|)), \tag{2.193}$$

in which $A\phi_1 = \phi_0$, $A\phi_0 = 0$ and $f : X_0 \to X_0$ is the function

$$\begin{array}{rcl} f(\psi) &=& Q_0\big(\frac{\beta_- - \beta_+}{\beta_- + \beta_+} \mathrm{pev}_\theta \mathcal{K} + \frac{\beta_-}{\beta_+ - \beta_-} \mathrm{pev}_{\theta - 1} \mathcal{K} + \frac{\beta_+}{\beta_+ - \beta_-} \mathrm{pev}_{\theta + 1} \mathcal{K} + \mathrm{pev}_\theta\big) \\ && \big(\lambda(E\psi)(\cdot) + \beta_- \nu (E\psi)(\cdot - 1) + \beta_+ \nu (E\psi)(\cdot + 1) \\ && + \frac{\beta_-}{2} g''(0, \mu_0)((E\psi)(\cdot - 1))^2 + \frac{\beta_+}{2} g''(0, \mu_0)((E\psi)(\cdot + 1))^2\big), \end{array} \tag{2.194}$$

in which the projection Q_0 is taken with respect to the variable θ. We introduce coordinates $\Phi(\xi)(\theta) = u(\xi) + v(\xi)\theta$ on the center space X_0. Fixing a value of $\xi \in \mathbb{R}$ and writing $\psi = \Phi(\xi)$, $u = u(\xi)$ and $v = v(\xi)$, we compute

$$\begin{array}{l} \lambda(E\psi)(\xi') + \beta_- \nu (E\psi)(\xi' - 1) + \beta_+ \nu (E\psi)(\xi' + 1) \\ \quad + \frac{\beta_-}{2} g''(0, \mu_0)((E\psi)(\xi' - 1))^2 \ + \frac{\beta_+}{2} g''(0, \mu_0)((E\psi)(\xi' + 1))^2 \\ = C_0 + C_1 \xi' + C_2 (\xi')^2, \end{array} \tag{2.195}$$

in which

$$\begin{array}{rcl} C_0 &=& \lambda u + \beta_- \nu (u - v) + \beta_+ \nu (u + v) + \frac{\beta_-}{2} g''(0, \mu_0)(u - v)^2 \\ && + \frac{\beta_+}{2} g''(0, \mu_0)(u + v)^2, \\ C_1 &=& (\lambda + (\beta_- + \beta_+)\nu) v + \beta_- g''(0, \mu_0) v (u - v) + \beta_+ g''(0, \mu_0) v (u + v), \\ C_2 &=& \frac{\beta_- + \beta_+}{2} g''(0, \mu_0) v^2. \end{array} \tag{2.196}$$

In order to proceed, we need to calculate the action of the pseudo-inverse $\mathcal{K}$ on the powers of ξ'. This can be done by using a polynomial ansatz and projecting out the X_0 component at zero. We obtain

$$
\begin{aligned}
(\mathcal{K}\mathbf{1})(\xi) &= \tfrac{\beta_- - \beta_+}{\beta_+ + \beta_-}\xi^2 + \tfrac{2(\beta_+ - \beta_-)^2}{3(\beta_+ + \beta_-)^2}\xi + \tfrac{(\beta_- - \beta_+)(-14\beta_-\beta_+ + \beta_-^2 + \beta_+^2)}{18(\beta_+ + \beta_-)^3}, \\
(\mathcal{K}\xi')(\xi) &= \tfrac{\beta_- - \beta_+}{3(\beta_+ + \beta_-)}\xi^3 + \tfrac{(\beta_- - \beta_+)^2}{3(\beta_+ + \beta_-)^2}\xi^2 + \tfrac{(\beta_- - \beta_+)(-14\beta_-\beta_+ + \beta_-^2 + \beta_+^2)}{18(\beta_+ + \beta_-)^3}\xi \\
&\quad + \tfrac{(\beta_- - \beta_+)(81\beta_-\beta_+^2 - 81\beta_-^2\beta_+ - \beta_-^3 + \beta_+^3)}{270(\beta_+ + \beta_-)^4}, \\
(\mathcal{K}(\xi')^{\mathbf{2}})(\xi) &= \tfrac{\beta_- - \beta_+}{6(\beta_+ + \beta_-)}\xi^4 + \tfrac{2(\beta_- - \beta_+)^2}{9(\beta_+ + \beta_-)^2}\xi^3 + \left(\tfrac{\beta_+ - \beta_-}{6(\beta_+ + \beta_-)} + \tfrac{2(\beta_- - \beta_+)^3}{9(\beta_+ + \beta_-)^3}\right)\xi^2 \\
&\quad + \tfrac{(\beta_- - \beta_+)(81\beta_-\beta_+^2 - 81\beta_-^2\beta_+ - \beta_-^3 + \beta_+^3)}{135(\beta_+ + \beta_-)^4}\xi \\
&\quad + \tfrac{(\beta_+ - \beta_-)(212\beta_-\beta_+^3 - 858\beta_-^2\beta_+^2 + 212\beta_+\beta_-^3 + \beta_-^4 + \beta_+^4)}{1620(\beta_+ + \beta_-)^5}.
\end{aligned}
\tag{2.197}
$$

Inserting (2.195) and (2.197) into (2.193), calculating the relevant projections and performing some extensive formula manipulation now yields the system

$$
\begin{aligned}
u' &= v + \tfrac{2}{3}\tfrac{(\beta_- - \beta_+)^2}{(\beta_- + \beta_+)^2} h(u, v, \lambda, \nu) \\
v' &= 2\tfrac{\beta_- - \beta_+}{\beta_- + \beta_+} h(u, v, \lambda, \nu),
\end{aligned}
\tag{2.198}
$$

with

$$
\begin{aligned}
h(u, v, \lambda, \nu) &= \tfrac{\beta_-}{2} g''(0, \mu_0)(u - v)^2 + \tfrac{\beta_+}{2} g''(0, \mu_0)(u + v)^2 + \lambda u \\
&\quad + \beta_- \nu(u - v) + \beta_+ \nu(u + v) \\
&\quad + O((|u| + |v|)^3 + (|\lambda| + |\nu|)(|u| + |v|)(|\lambda| + |\nu| + |u| + |v|)).
\end{aligned}
\tag{2.199}
$$

If we choose $\beta_- = 1$ and $\beta_+ = 0$, equation (2.187) reduces to a delay equation that has been studied in [45]. The differential equation on the center manifold that was found there using specific delay equation techniques, matches the equation (2.198) derived here.

Chapter 3

Center Manifolds for Smooth Differential-Algebraic Equations

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Abstract. We study a class of mixed type difference equations that enjoy a special smoothening property, in the sense that solutions automatically satisfy an associated functional differential equation of mixed type. Using this connection, a finite dimensional center manifold is constructed that captures all solutions that remain sufficiently close to an equilibrium. The results enable a rigorous analysis of a recently developed model in economic theory, that exhibits periodic oscillations in the interest rates of a simple economy of overlapping generations.

3.1. Introduction

The purpose of this chapter is to develop a center manifold framework that will enable us to analyze the behaviour of near-equilibrium solutions to a class of nonlinear difference equations of mixed type,

$$F(x_\xi) = 0, \tag{3.1}$$

that enjoy a special smoothening property. In particular, we require that any solution to (3.1) automatically satisfies an associated functional differential equation of mixed type (MFDE), which we will denote by

$$x'(\xi) = G(x_\xi). \tag{3.2}$$

In the above x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \le 0$ and $r_{\max} \ge 0$, thus the nonlinearities F and G may depend on advanced and retarded arguments simultaneously.

Our main equation (3.1) should be seen as an infinite dimensional version of a differential-algebraic equation (DAE), i.e., an equation of the form $f(y(\xi), y'(\xi), \xi) = 0$ that yields an ODE after a finite number of differentiations. Such equations have been studied extensively during the last two decades [23, 28, 29], primarily because they have arisen in many scientific disciplines, including chemical engineering [24, 99], mechanics [76, 117, 121], fluid dynamics [166] and electrical circuit theory [27, 122, 139, 155]. We specially emphasize the applications in the latter area, since the incorporation of time delays into the governing model equations turns out to be an important step towards understanding the dynamical behaviour of many circuits [105]. Inclusion of such delayed arguments in a DAE may lead to equations of the form (3.1).

However, at present our primary motivation for the study of (3.1) comes from the area of economic research, where recent developments have led to models involving such equations. In particular, we mention the work of d'Albis and Véron [41, 39, 40], who have developed several models describing the dynamical features of an economy featuring only a single commodity, that exhibit oscillations which earlier models could only produce by including multiple commodities. We refer to Section 1.6 for a detailed discussion of such a model, which describes the effects of retirement on the dynamics of the interest rate. The results from this chapter are used there to establish the existence of periodic cycles for the interest rate. This is accomplished by constructing a smooth local invariant manifold for (3.1) that captures all solutions that remain sufficiently close to an equilibrium and subsequently invoking the Hopf bifurcation theorem.

We remark here that from an economic point of view, periodic cycles are in general considered to be very interesting, since they can be readily observed in actual markets. Traditionally, the Hopf theorem has been widely used to establish the existence of such cycles for economic models involving ODEs. The results developed here allow for the statement of such a theorem in the infinite dimensional setting of (3.1), merely in terms of an explicit finite dimensional characteristic function associated to this equation.

We recall that in Chapter 2 a center manifold was constructed for the MFDE (3.2), based upon earlier work by Mallet-Paret [112], Diekmann et al. [45] and Vanderbauwhede et al. [159]. Writing $\overline{x}$ for any equilibrium $G(\overline{x}) = 0$, this construction allows us to relate the dynamics of any sufficiently small solution to the equation

$$u'(\xi) = DG(\overline{x})u_\xi + (G(\overline{x} + u_\xi) - DG(\overline{x})u_\xi), \tag{3.3}$$

to orbits of a differential equation on a finite dimensional space $\mathcal{N}_G$. This space $\mathcal{N}_G$ contains all the solutions of the linearized equation $u'(\xi) = DG(\overline{x})u_\xi$ that can be bounded by a polynomial. However, if one attempts to analyze the difference equation (3.1) by using the center manifold construction on the associated MFDE, difficulties arise due to the fact that is is unclear how to lift solutions of (3.2) back to solutions of (3.1). In addition, the structure of the space $\mathcal{N}_G$ will in general differ from $\mathcal{N}_F$, the space of polynomially bounded solutions to $0 = DF(\overline{x})u_\xi$. This implies that extra dynamical behaviour may be observed on the center manifold of (3.2) that is not observed in (3.1). For example, in Section 1.6 the parameter dependent characteristic equation associated to the MFDE (3.2) admits a double root at $z = 0$, which is absent when studying (3.1) directly. The presence of this ubiquitous double root is troublesome as it adds a resonance to eigenvalues that cross through the imaginary axis as the parameters are varied. The analysis of (3.2) would hence involve studying complicated zero-Hopf bifurcations [68, 74, 100, 102, 103], a situation one would prefer to avoid.

These issues can be resolved by constructing a center manifold directly for (3.1). We will show that the extra smoothness properties provided by (3.2) enable this reduction to be performed and in addition allow us to describe the dynamics on this center manifold by a differential equation, which of course will be related to the nonlinearity G in (3.2). This procedure is performed systematically in Sections 3.4 and further, mainly in the spirit of Chapter 2. The results that we obtained in this fashion are formulated in Section 3.2.

3.2. Main Results

Consider the following difference equation of mixed type,

$$0 = Lx_\xi + R(x_\xi), \qquad \xi \in \mathbb{R}, \tag{3.4}$$

in which x is a continuous mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$, the operators L and R are a linear respectively nonlinear map from the state space $X = C([-1, 1], \mathbb{C}^n)$ into $\mathbb{C}^n$ and the state $x_\xi \in X$ is defined by $x_\xi(\theta) = x(\xi + \theta)$ for any $-1 \leq \theta \leq 1$. Notice that in terms of the terminology of (3.1), this means that we have fixed $r_{\min} = -1$ and $r_{\max} = 1$. As a consequence of the Riesz representation theorem, there exists a unique $\mathbb{C}^{n\times n}$-valued normalized function of bounded variation $\mu \in \mathrm{NBV}([-1, 1], \mathbb{C}^{n\times n})$, such that for all $\phi \in X$ we have the identity

$$L\phi = \int_{-1}^{1} d\mu(\sigma)\phi(\sigma). \tag{3.5}$$

We recall here that the normalization of μ implies that μ is right-continuous on $(-1, 1)$ and satisfies $\mu(-1) = 0$. Throughout this section, the reader may wish to keep in mind the following typical example equation,

$$x(\xi) = \int_{-1}^{1} x(\xi + \sigma)d\sigma + \Big(\int_{-1}^{1} x(\xi + \sigma)d\sigma\Big)^2. \tag{3.6}$$

As in Chapter 2, we will be particularly interested in the following families of Banach spaces during our analysis of (3.4),

$$\begin{array}{rcl} BC_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in C(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}\, |x(\xi)| < \infty\right\}, \\ BC^1_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n) \mid x' \in BC_\eta(\mathbb{R}, \mathbb{C}^n)\right\}, \end{array} \tag{3.7}$$

parametrized by $\eta \in \mathbb{R}$, with the standard norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \left\|x'\right\|_\eta$. Notice that for any pair $\eta_2 \geq \eta_1$, there exist continuous inclusions $\mathcal{J}_{\eta_2\eta_1} : BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ and $\mathcal{J}^1_{\eta_2\eta_1} : BC^1_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$.

In order to construct a center manifold for (3.4), it is essential to consider the homogeneous linear equation

$$0 = Lx_\xi. \tag{3.8}$$

Associated to this system (3.8) one has the characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n\times n}$, given by

$$\Delta(z) = -\int_{-1}^{1} d\mu(\sigma)e^{z\sigma}. \tag{3.9}$$

The minus sign is included here to ensure notational consistency with the characteristic matrix for MFDEs. A value of z such that $\det \Delta(z) = 0$ is called an eigenvalue for the system (3.8). In order to state our main results, we need to impose the following condition on the operator L and the corresponding characteristic matrix Δ.

(HL) There exists a linear operator $M : X \to \mathbb{C}^n$, an integer $\ell > 0$ and constants $\alpha_M, \beta_M \in \mathbb{C}$ with $\beta_M \neq 0$ such that

$$\Delta(z) = \beta_M^{-1}(z - \alpha_M)^{-\ell}\Delta_M(z), \tag{3.10}$$

where $\Delta_M(z)$ is the characteristic matrix corresponding to the homogeneous linear functional differential equation of mixed type $x'(\xi) = Mx_\xi$.

This condition is related to the fact that we need any solution of the difference equation (3.4) to additionally satisfy a differential equation of mixed type. The operator M should be seen as the linear part of this latter MFDE. For the example equation (3.6) one may conclude that (HL) holds with $\alpha_M = 0$, $\beta_M = 1$ and $\ell = 1$, by computing

$$\Delta(z) = 1 - \frac{1}{z}(e^z - e^{-z}) = \frac{1}{z}(z - e^z + e^{-z}) = \Delta_M(z)/z, \tag{3.11}$$

in which $M\phi = \phi(1) - \phi(-1)$. It is easy to see that this choice for M indeed yields $x'(\xi) = Mx_\xi$ whenever $Lx_\xi = 0$.

Alternatively, the condition (HL) can be verified directly in terms of the measure $d\mu$ associated to L via (3.5). In particular, we will show in Section 3.3 that (HL) is equivalent to the following condition, which roughly states that the first non-smooth derivative of μ may only have a jump at zero.

(HL$'$) There exists an integer $\ell > 0$ such that $\mu \in W^{\ell-1,1}_{\mathrm{loc}}([-1,1], \mathbb{C}^{n\times n})$. In addition, there exist a constant $\kappa \neq 0$ and a normalized function of bounded variation $\zeta \in \mathrm{NBV}([-1,1], \mathbb{C}^{n\times n})$, such that

$$D^{\ell-1}\mu(\sigma) = \kappa I H(\sigma) + \textstyle\int_{-1}^{\sigma}\zeta(\tau)d\tau, \qquad -1 \leq \sigma \leq 1, \tag{3.12}$$

in which H denotes the Heaviside function. Finally, for all $1 \leq s \leq \ell - 1$, we have the identity $D^s\mu(\pm 1) = 0$.

Note that when $\ell \geq 2$ in (HL$'$), it follows directly from (3.12) that $\mu \in C^{\ell-2}([-1,1], \mathbb{C}^{n\times n})$, which ensures that the last condition involving $D^s\mu(\pm 1)$ is well-defined.

The following proposition, which will be proved in Section 3.4, exhibits the finite dimensional space X_0 on which the center manifold will be defined.

Proposition 3.2.1. *For any homogeneous linear equation (3.8) that satisfies the condition (HL), there exists a finite dimensional linear subspace $X_0 \subset X$ with the following properties.*

(i) *Suppose $x \in \bigcap_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (3.8). Then for any $\xi \in \mathbb{R}$ we have $x_\xi \in X_0$.*

(ii) *For any $\phi \in X_0$, we have $D\phi \in X_0$.*

(iii) *For any $\phi \in X_0$, there is a solution $x \in \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ of (3.8) that has $x_0 = \phi$. This solution is unique in the set $\bigcup_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ and will be denoted by $E\phi$.*

We write Q_0 for the projection operator from X onto X_0, which will be defined precisely in the sequel. Before stating our main result, we introduce two conditions on the nonlinearity $R : X \to \mathbb{C}^n$, which again are related to the MFDE that any solution of (3.4) satisfies.

(HR1) For any $x \in C(\mathbb{R}, \mathbb{C}^n)$, the function $f : \xi \mapsto R(x_\xi)$ satisfies $f \in C^\ell(\mathbb{R}, \mathbb{C}^n)$, where ℓ is as introduced in (HL). In addition, there exist operators $R^{(s)} : X \to \mathbb{C}^n$ for $0 \leq s \leq \ell$, with $R^{(0)} = R$, such that

$$D^s f(\xi) \quad = \quad R^{(s)}(x_\xi), \qquad \text{for } 0 \leq s \leq \ell. \tag{3.13}$$

(HR2) The functions $R^{(s)}$ are C^k-smooth for some integer $k \geq 1$ and all $0 \leq s \leq \ell$. In addition, we have $R^{(s)}(0) = DR^{(s)}(0) = 0$ for all $0 \leq s \leq \ell$.

Theorem 3.2.2. *Consider the nonlinear equation (3.4) and assume that (HL), (HR1) and (HR2) are satisfied. Then there exists $\gamma > 0$ such that the characteristic equation* $\det \Delta_M(z) = 0$ *has no roots with $0 < |\mathrm{Re}\, z| < \gamma$. Fix an interval $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ such that $\eta_{\max} > k\eta_{\min}$, with k as introduced in (HR2). Then there exists a mapping $u^* : X_0 \to \bigcap_{\eta>0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, together with constants $\epsilon > 0$ and $\epsilon^* > 0$, such that the following statements hold.*

(i) *For any $\eta \in (k\eta_{\min}, \eta_{\max}]$, the function u^* viewed as a map from X_0 into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is C^k-smooth.*

(ii) *Suppose for some $\zeta > 0$ that $x \in BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (3.4) with* $\sup_{\xi\in\mathbb{R}} |x(\xi)| < \epsilon^*$. *Then we have $x = u^*(Q_0 x_0)$. In addition, the function $\Phi : \mathbb{R} \to X_0$ defined by $\Phi(\xi) = Q_0 x_\xi \in X_0$ is of class C^{k+1} and satisfies the ordinary differential equation*

$$\Phi'(\xi) = A\Phi(\xi) + f(\Phi(\xi)), \tag{3.14}$$

in which $A : X_0 \to X_0$ is the linear operator $\phi \mapsto D\phi$ for $\phi \in X_0$. The function $f : X_0 \to X_0$ is C^k-smooth with $f(0) = 0$ and $Df(0) = 0$ and is explicitly given by

$$f(\psi) = Q_0 \chi_\psi, \tag{3.15}$$

in which the state $\chi_\psi \in X$ is given by

$$\chi_\psi(\sigma) = M(u^*(\psi))_\sigma - M(E\psi)_\sigma + \beta_M(\overline{D} - \alpha_M)^\ell R\big((u^*(\psi))_\sigma\big), \tag{3.16}$$

for $\sigma \in [-1, 1]$. Here the expression $\overline{D}^s R(\cdot)$ should be interpreted as $R^{(s)}(\cdot)$ for $0 \leq s \leq \ell$; see also the remark at the end of this section. Finally, we have $x_\xi = \big(u^*(\Phi(\xi))\big)_0$ for all $\xi \in \mathbb{R}$.

(iii) For any $\phi \in X_0$ such that $\sup_{\xi\in\mathbb{R}} |u^*(\phi)(\xi)| < \epsilon^*$, the function $u^*(\phi)$ satisfies (3.4).

(iv) For any continuous function $\Phi : \mathbb{R} \to X_0$ that satisfies (3.14) with $\|\Phi(\xi)\| < \epsilon$ for all $\xi \in \mathbb{R}$, we have that $x = u^*(\Phi(0))$ is a solution of (3.4). In addition, we have $x_\xi = \big(u^*(\Phi(\xi))\big)_0$ for any $\xi \in \mathbb{R}$.

We conclude this section by noting that (3.16) indeed makes sense, since both $u^*(\psi)$ and $E\psi$ are continuous functions on the line, which ensures that the states $(u^*(\psi))_\sigma$ and $(E\psi)_\sigma$ belong to X and depend continuously on $\sigma \in [-1, 1]$. This allows the operators M and $R^{(s)}$ to be applied, yielding a continuous $\mathbb{C}^n$-valued function χ_ψ on $[-1, 1]$, as required.

3.3. Preliminaries

In this section we provide some preliminary results regarding the linear operators L and M that appeared in Section 3.2. We start by showing that (HL) and (HL′) are equivalent conditions that automatically provide smoothness properties for functions of the form $\xi \mapsto -Lx_\xi$, which will be encountered frequently in the sequel.

Proposition 3.3.1. *Recall the linear operator $L : X \to \mathbb{C}^n$ defined by (3.5). The conditions (HL) and (HL′) on L with equal values of the integer $\ell > 0$ are equivalent. In addition, when these conditions are satisfied the following properties hold.*

(i) For any $x \in C(\mathbb{R}, \mathbb{C}^n)$, the function f defined by $f(\xi) = -Lx_\xi$ satisfies $f \in C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$.

(ii) There exists a constant $C > 0$ such that for any $x \in C(\mathbb{R}, \mathbb{C}^n)$, we have $|D^s f(\xi)| \leq C \|x_\xi\|$ for all $0 \leq s \leq \ell - 1$, where f is again given by $f(\xi) = -Lx_\xi$.

(iii) If the function $f : \xi \mapsto -Lx_\xi$ associated to any $x \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies $f \in C^\ell(\mathbb{R}, \mathbb{C}^n)$, then we must have $x \in C^1(\mathbb{R}, \mathbb{C}^n)$.

Proof. We first show that (HL′) implies (HL) and the properties (i) through (iii) listed in the statement of this result. We proceed by induction on the integer ℓ. Consider therefore an operator $L : X \to \mathbb{C}^n$ with corresponding NBV function μ that satisfies (HL′) with $\ell = 1$. Consider any $x \in C(\mathbb{R}, \mathbb{C}^n)$ and let the function $f : \mathbb{R} \to \mathbb{C}^n$ be defined by $f(\xi) = -Lx_\xi$. The identity (3.12) in (HL′) now implies that we have

$-f(\xi) = \kappa x(\xi) + \int_{\xi-1}^{\xi+1} \zeta(\sigma - \xi)x(\sigma)d\sigma$, from which (i) and (ii) immediately follow. If in fact $f \in C^1(\mathbb{R}, \mathbb{C}^n)$, then differentiation of the above identity yields

$$\kappa Dx(\xi) = -Df(\xi) - \zeta(1)x(\xi+1) + \zeta(-1)x(\xi-1) + \int_{-1}^{1} d\zeta(\sigma)x(\xi+\sigma), \quad (3.17)$$

showing that $x \in C^1(\mathbb{R}, \mathbb{C}^n)$ and hence establishing (iii). An easy calculation involving integration by parts allows us to establish that also condition (HL) holds, with the corresponding operator $M : X \to \mathbb{C}^n$ given by $M\phi = \kappa^{-1}\big(\zeta(-1)\phi(-1) - \zeta(1)\phi(1) + \int_{-1}^{1} d\zeta(\sigma)\phi(\sigma)\big)$. Indeed, choosing $\alpha_M = 0$ and $\beta_M = -\kappa^{-1}$, we may compute

$$\begin{array}{rcl} -\Delta(z) & = & \int_{-1}^{1} d\mu(\sigma)e^{z\sigma} = \kappa I + \int_{-1}^{1} \zeta(\sigma)e^{z\sigma}d\sigma \\ & = & \kappa I + \frac{1}{z}(\zeta(1)e^z - \zeta(-1)e^{-z} - \int_{-1}^{1} d\zeta(\sigma)e^{z\sigma}) \\ & = & \frac{\kappa}{z}(zI - Me^{z\cdot}) = -\beta_M^{-1}z^{-1}\Delta_M(z), \end{array} \quad (3.18)$$

which shows that (3.10) in (HL) is satisfied.

Now let $p > 1$ and consider an operator L with corresponding NBV function μ that satisfies (HL$'$) with $\ell = p$. Observe that (3.12) implies that $D\mu \in L^1_{\text{loc}}([-1, 1], \mathbb{C}^{n\times n})$ is also a NBV function. Therefore it induces the operator $L' : X \to \mathbb{C}^n$ given by

$$L'\phi = \int_{-1}^{1} d[D\mu](\sigma)\phi(\sigma) \quad (3.19)$$

and one may easily verify that L' satisfies the condition (HL$'$) with $\ell = p - 1$. In particular, using our induction hypothesis this means that for some operator $M' : X \to \mathbb{C}^n$, condition (HL) with $\ell = p - 1$ is satisfied by L', together with the properties (i) through (iii) listed above.

Now as before, consider an arbitrary $x \in C(\mathbb{R}, \mathbb{C}^n)$ and its corresponding function f given by $f(\xi) = -Lx_\xi$. We may compute

$$\begin{array}{rcl} -Df(\xi) & = & D[\int_{-1}^{1} d\mu(\sigma)x(\xi+\sigma)] = D[\int_{\xi-1}^{\xi+1} D\mu(\sigma-\xi)x(\sigma)d\sigma] \\ & = & D\mu(1)x(\xi+1) - D\mu(-1)x(\xi-1) - \int_{-1}^{1} d[D\mu](\sigma)x(\xi+\sigma) \\ & = & -\int_{-1}^{1} d[D\mu](\sigma)x(\xi+\sigma) = -L'x_\xi, \end{array} \quad (3.20)$$

where the penultimate equality follows from the conditions in (HL$'$) on $D\mu$. Properties (i) through (iii) with $\ell = p$ now follow immediately from the fact that these properties with $\ell = p - 1$ are satisfied by L'. To show that L also satisfies the condition (HL), one may compute

$$\begin{array}{rcl} -\Delta(z) & = & \int_{-1}^{1} D\mu(\sigma)e^{z\sigma}d\sigma = \frac{1}{z}\big(D\mu(1)e^z - D\mu(-1)e^{-z} - \int_{-1}^{1} d[D\mu](\sigma)e^{z\sigma}\big) \\ & = & -\frac{1}{z}\int_{-1}^{1} d[D\mu](\sigma)e^{z\sigma} = \frac{1}{z}\beta_{M'}^{-1}z^{-p+1}\Delta_{M'}(z). \end{array} \quad (3.21)$$

We now proceed to show that condition (HL) implies (HL$'$). Without loss of generality we will assume $\beta_M = 1$. Using induction on ℓ we will show that if $z \mapsto (z - \alpha)^{-\ell}\Delta_M(z)$

is a holomorphic function, then there exists a NBV function μ that meets the conditions in (HL$'$) and in addition satisfies the identity $-\int_{-1}^{1} d\mu(\sigma)e^{z\sigma} = (z-\alpha)^{-\ell}\Delta_M(z)$. Writing ζ_M for the NBV function associated with M, we introduce the corresponding NBV function $\widetilde{\zeta}$ associated to the operator $\widetilde{M} : X \to \mathbb{C}^n$ given by $\phi \mapsto M\phi - \alpha\phi(0)$. Consider the case $\ell = 1$, write $f(z) = (z-\alpha)^{-1}\Delta_M(z)$ and use repeated integration by parts to compute

$$\begin{array}{rcl} f(z) & = & (z-\alpha)^{-1}(z - \int_{-1}^{1} d\zeta_M(\sigma)e^{z\sigma}) = (z-\alpha)^{-1}(z - \alpha - \int_{-1}^{1} d\widetilde{\zeta}(\sigma)e^{z\sigma}) \\ & = & (z-\alpha)^{-1}e^{z-\alpha}\big(\alpha\int_{-1}^{1} e^{\alpha\sigma}\widetilde{\zeta}(\sigma)d\sigma - \widetilde{\zeta}(1)e^{\alpha}\big) \\ & & +1 - \alpha\int_{-1}^{1} e^{(z-\alpha)\sigma}\int_{-1}^{\sigma} e^{\alpha\tau}\widetilde{\zeta}(\tau)d\tau d\sigma + \int_{-1}^{1} e^{z\sigma}\widetilde{\zeta}(\sigma)d\sigma, \end{array} \tag{3.22}$$

in which we recall the normalization $\widetilde{\zeta}(-1) = 0$. Since f is a holomorphic function, one sees that the following identity must hold,

$$\alpha\int_{-1}^{1} e^{\alpha\sigma}\widetilde{\zeta}(\sigma)d\sigma = \widetilde{\zeta}(1)e^{\alpha}. \tag{3.23}$$

From this it follows that the induction hypothesis is satisfied for the NBV function μ given by

$$\mu(\sigma) = -H(\sigma) - \int_{-1}^{\sigma}\widetilde{\zeta}(\tau)d\tau + \alpha\int_{-1}^{\sigma} e^{-\alpha\tau}\int_{-1}^{\tau} e^{\alpha u}\widetilde{\zeta}(u)du d\tau. \tag{3.24}$$

Now consider an integer $p > 1$ and consider a holomorphic function of the form $f(z) = (z-\alpha)^{-p}\Delta_M(z)$. Assume that our induction hypothesis is satisfied for $\ell = p-1$, which implies that $(z-\alpha)f(z) = -\int_{-1}^{1} d\nu(\sigma)e^{z\sigma}$ for some NBV function ν that satisfies (HL$'$) at $\ell = p-1$. We can thus compute

$$\begin{array}{rcl} f(z) & = & -(z-\alpha)^{-1}\int_{-1}^{1} d\nu(\sigma)e^{z\sigma} \\ & = & (z-\alpha)^{-1}e^{z-\alpha}\big(\alpha\int_{-1}^{1} e^{\alpha\sigma}\nu(\sigma)d\sigma - \nu(1)e^{\alpha}\big) \\ & & +\int_{-1}^{1} e^{z\sigma}\nu(\sigma)d\sigma - \alpha\int_{-1}^{1} e^{(z-\alpha)\sigma}\int_{-1}^{\sigma} e^{\alpha\tau}\nu(\tau)d\tau d\sigma. \end{array} \tag{3.25}$$

Again, since f is holomorphic, (3.23) must hold with ν instead of $\widetilde{\zeta}$ and one may readily verify that the induction hypothesis is satisfied at $\ell = p$, for the NBV function

$$\mu(\sigma) = \alpha\int_{-1}^{\sigma} e^{-\alpha\tau}\int_{-1}^{\tau} e^{\alpha u}\nu(u)du d\tau - \int_{-1}^{\sigma}\nu(\tau)d\tau, \tag{3.26}$$

which concludes the proof. □

We now recall the characteristic matrix Δ_M associated to the homogeneous equation $x'(\xi) = Mx_\xi$ that features in condition (HL) and repeat some useful properties of Δ_M that were established in Chapter 2.

Lemma 3.3.2. *Consider any closed vertical strip* $S = \{z \in \mathbb{C} \mid \gamma_- \le \operatorname{Re} z \le \gamma_+\}$ *and for any* $\rho > 0$ *define* $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. *Then there exist* $C, \rho > 0$ *such that*

$\det \Delta_M(z) \neq 0$ *for all* $z \in S_\rho$ *and in addition* $\left|\Delta_M(z)^{-1}\right| < \frac{C}{|\mathrm{Im}\, z|}$ *for each such* z*. In particular, there are only finitely many zeroes of* $\det \Delta_M(z)$ *in* S*. Furthermore, if* $\det \Delta_M(z) \neq 0$ *for all* $z \in S$*, then for any* $\alpha \notin S$ *the function*

$$R_\alpha(z) = \Delta_M(z)^{-1} - (z - \alpha)^{-1} I \tag{3.27}$$

is holomorphic in an open neighbourhood of S *and in addition there exists* $C' > 0$ *such that* $|R_\alpha(z)| \leq \frac{C'}{1+|\mathrm{Im}\, z|^2}$ *for all* $z \in S$*.*

The final result of this section uses Laplace transform techniques to characterize solutions to $-Lx_\xi = f$ that have controlled exponential growth at $\pm\infty$.

Proposition 3.3.3. *Consider the operator* L *defined by (3.5) and suppose that the condition (HL) is satisfied. Fix constants* $\eta_-, \eta_+ \in \mathbb{R}$ *and consider any* $x \in C(\mathbb{R}, \mathbb{C}^n)$ *that satisfies* $x(\xi) = O(e^{\eta_\pm \xi})$ *as* $\xi \to \pm\infty$*. Define the function* $f : \xi \mapsto -Lx_\xi$*. Then for any* $\gamma_+ > \eta_+$ *and* $\gamma_- < \eta_-$ *such that the characteristic equation* $\det \Delta_M(z) = 0$ *has no roots with* $\mathrm{Re}\, z = \gamma_\pm$ *and for any* $\xi \in \mathbb{R}$*, we have*

$$\begin{array}{rcl} x(\xi) & = & \frac{1}{2\pi i} \int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z} \big(K(\xi, z, x) + \Delta(z)^{-1} \widetilde{f}_+(z)\big) dz \\ & & + \frac{1}{2\pi i} \int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z} \big(K(\xi, z, x) - \Delta(z)^{-1} \widetilde{f}_-(z)\big) dz, \end{array} \tag{3.28}$$

in which $K : \mathbb{R} \times \mathbb{C} \times C(\mathbb{R}, \mathbb{C}^n) \to \mathbb{C}^n$ *is given by*

$$K(\xi, z, x) = \int_\xi^0 e^{-z\tau} x(\tau) d\tau + \Delta(z)^{-1} \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} x(\tau) d\tau. \tag{3.29}$$

The Laplace transforms $\widetilde{f}_+$ *and* $\widetilde{f}_-$ *are as defined in (B.4) and (B.5).*

Proof. Note that Proposition 3.3.1 implies that f shares the growth rate of x at $\pm\infty$. An application of Lemma B.2 hence shows that

$$\frac{1}{2} x(\xi) = \frac{1}{2\pi i} \int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z} \big(\int_\xi^0 e^{-z\tau} x(\tau) d\tau + \widetilde{x}_+(z) \big) dz. \tag{3.30}$$

Taking the Laplace transform of the identity $-Lx_\xi = f(\xi)$ yields

$$0 = \widetilde{f}_+(z) + \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \big(\widetilde{x}_+(z) + \int_\sigma^0 e^{-z\tau} x(\tau) d\tau \big) \tag{3.31}$$

and thus after rearrangement

$$\widetilde{x}_+(z) = \Delta(z)^{-1} \big(\widetilde{f}_+(z) + \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} x(\tau) d\tau \big). \tag{3.32}$$

As in Chapter 2, a similar argument applied to the function $y(\xi) = x(-\xi)$ completes the proof. □

3.4. The state space

In this section we study the state space $X = C([-1, 1], \mathbb{C}^n)$ in the spirit of the corresponding treatment for MFDEs employed in Chapter 2. We recall the linear operator $L : X \to \mathbb{C}^n$ defined by (3.5) and define a closed operator $A : \mathcal{D}(A) \subset X \to X$, via

$$\begin{array}{rcl} \mathcal{D}(A) & = & \left\{\phi \in X \mid \phi \text{ is } C^1\text{-smooth and satisfies } 0 = L\phi = \int_{-1}^{1} d\mu(\sigma)\phi(\sigma)\right\}, \\ A\phi & = & D\phi. \end{array} \tag{3.33}$$

Notice that the domain $\mathcal{D}(A)$ now differs from the corresponding definition in Chapter 2 and in addition, A is no longer densely defined. Nevertheless, it is still possible to relate the resolvent of A to the characteristic matrix Δ. We refer to [89] for a general discussion on characteristic matrices for unbounded operators.

Lemma 3.4.1. *The operator A defined in (3.33) has only point spectrum with $\sigma(A) = \sigma_P(A) = \{z \in \mathbb{C} \mid \det \Delta(z) = 0\}$. In addition, for $z \in \rho(A)$, the resolvent of A is given by*

$$(zI - A)^{-1}\psi = e^{\cdot z} K(\cdot, z, \psi), \tag{3.34}$$

in which $K : [-1, 1] \times \mathbb{C} \times X \to \mathbb{C}^n$ is the appropriate restriction of the operator K defined in (3.29).

Proof. Fix $\psi \in X$ and consider the equation $(zI - A)\phi = \psi$ for $\phi \in \mathcal{D}(A)$, which is equivalent to the system

$$\begin{array}{rcl} D\phi & = & z\phi - \psi, \\ 0 & = & \int_{-1}^{1} d\mu(\sigma)\phi(\sigma). \end{array} \tag{3.35}$$

Suppose that $\det \Delta(z) \neq 0$. Solving the first equation yields

$$\phi(\theta) = e^{\theta z}\phi(0) + e^{\theta z} \int_{\theta}^{0} e^{-z\tau} \psi(\tau) d\tau \tag{3.36}$$

and hence the fullfillment of the second equation requires

$$0 = \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} (\phi(0) + \int_{\sigma}^{0} e^{-z\tau} \psi(\tau) d\tau). \tag{3.37}$$

Thus setting

$$\phi(0) = \Delta(z)^{-1} \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \psi(\tau) d\tau, \tag{3.38}$$

we see that $z \in \rho(A)$. On the other hand, choosing a non-zero $v \in \mathbb{R}^n$ such that $\Delta(z)v = 0$ for some root z of $\det \Delta(z) = 0$, one sees that the function $\phi(\theta) = e^{z\theta} v$ satisfies $\phi \in \mathcal{D}(A)$ and $A\phi = z\phi$. This shows that $z \in \sigma_P(A)$, completing the proof. □

For any pair of reals $\gamma_- < \gamma_+$ such that the characteristic equation $\det \Delta(z) = 0$ has no roots with $\operatorname{Re} z = \gamma_\pm$, define the set $\Sigma = \Sigma_{\gamma_-, \gamma_+} = \{z \in \sigma(A) \mid \gamma_- < \operatorname{Re} z < \gamma_+\}$. Using

Lemma 3.3.2 it is easy to see that Σ is a finite set. Furthermore, the representation (3.34) implies that $(zI - A)^{-1}$ has a pole of finite order at $z = \lambda_0$ for every $\lambda_0 \in \Sigma$. Standard spectral theory [45, Theorem IV.2.5] now yields the decomposition $X = \mathcal{M}_\Sigma \oplus \mathcal{R}_\Sigma$ for some closed linear subspace $\mathcal{M}_\Sigma$, together with a spectral projection $Q_\Sigma : X \to \mathcal{M}_\Sigma$, which is explicitly given by

$$Q_\Sigma = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} dz, \tag{3.39}$$

for any Jordan path $\Gamma \subset \rho(A)$ with $\text{int}(\Gamma) \cap \sigma(A) = \Sigma$. The following result gives conditions under which this Dunford integral can be related to the integral representation in (3.28).

Lemma 3.4.2. *Consider an operator L of the form (3.5) that satisfies (HL). Suppose that $\phi \in C^{\ell-1}([-1, 1], \mathbb{C}^n)$ satisfies $LD^s\phi = 0$ for all $0 \le s \le \ell - 2$, with ℓ as introduced in (HL). Then the spectral projection $Q_\Sigma\phi$ defined above is given by*

$$(Q_\Sigma\phi)(\theta) = \frac{1}{2\pi i} \int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\theta z} K(\theta, z, \phi) dz + \frac{1}{2\pi i} \int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\theta z} K(\theta, z, \phi) dz, \tag{3.40}$$

with K as defined in (3.29).

Proof. For any $\rho > 0$ such that $|\text{Im}\,\lambda| < \rho$ for any $\lambda \in \Sigma$, we introduce the path $\Gamma_\rho = \Gamma_\rho^\uparrow \cup \Gamma_\rho^\leftarrow \cup \Gamma_\rho^\downarrow \cup \Gamma_\rho^\rightarrow$, consisting of the line segments

$$\begin{array}{rclrcl} \Gamma_\rho^\uparrow & = & \text{seg}[\gamma_+ - i\rho, \gamma_+ + i\rho], & \Gamma_\rho^\downarrow & = & \text{seg}[\gamma_- + i\rho, \gamma_- - i\rho], \\ \Gamma_\rho^\leftarrow & = & \text{seg}[\gamma_+ + i\rho, \gamma_- + i\rho], & \Gamma_\rho^\rightarrow & = & \text{seg}[\gamma_- - i\rho, \gamma_+ - i\rho]. \end{array} \tag{3.41}$$

Note that it suffices to show that for every $\theta \in [-1, 1]$, we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho^\rightleftarrows} e^{\theta z} \int_\theta^0 e^{-z\tau}\phi(\tau) d\tau dz + \int_{\Gamma_\rho^\rightleftarrows} e^{\theta z} \Psi(z, \phi) dz = 0, \tag{3.42}$$

with $\Psi(z, \phi)$ given by

$$\Psi(z, \phi) = \beta_M (z - \alpha_M)^\ell \Delta_M(z)^{-1} \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau}\phi(\tau) d\tau. \tag{3.43}$$

The first integral in (3.42) can be shown to converge to zero as in Chapter 2. To treat the second integral, use integration by parts to compute

$$\int_\sigma^0 e^{-z\tau}\phi(\tau) d\tau = \frac{1}{z^{\ell-1}} \int_\sigma^0 e^{-z\tau} D^{\ell-1}\phi(\tau) d\tau + \sum_{k=0}^{\ell-2} \frac{1}{z^{k+1}} \big(e^{-z\sigma} D^k\phi(\sigma) - D^k\phi(0)\big). \tag{3.44}$$

Using the fact that $LD^s\phi = 0$ for $0 \le s \le \ell - 2$, we conclude that $\Psi(z, \phi)$ can be rewritten as

$$\begin{array}{rcl} \Psi(z, \phi) & = & \sum_{k=0}^{\ell-2} \frac{D^k\phi(0)}{z^{k+1}} + \frac{\beta_M (z-\alpha_M)^{\ell-1}}{z^{\ell-1}} \big(\frac{z-\alpha_M}{z-\alpha} + (z - \alpha_M) R_\alpha(z)\big) \\ & & \int_{-1}^1 d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} D^{\ell-1}\phi(\tau) d\tau, \end{array} \tag{3.45}$$

where $R_\alpha(z) = O(1/|z|^2)$ as $\operatorname{Im} z \to \pm\infty$, uniformly in vertical strips. Ignoring the terms in $\Psi(z, \phi)$ that behave as $O(1/z)$ as $\operatorname{Im} z \to \pm\infty$, it remains to show that

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho^{\rightleftarrows}} e^{\theta z} \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} D^{\ell-1}\phi(\tau) d\tau dz = 0. \tag{3.46}$$

This however can also be established using the arguments in Chapter 2. □

In order to show that $\mathcal{M}_\Sigma$ is finite dimensional, we introduce a new operator $\widehat{A}$ on the larger space $\widehat{X} = \mathbb{C}^n \times X$,

$$\begin{array}{rcl} \mathcal{D}(\widehat{A}) & = & \left\{(c, \phi) \in \widehat{X} \mid D\phi \in X \text{ and } c = \phi(0)\right\}, \\ \widehat{A}(c, \phi) & = & (L\phi + D\phi(0), D\phi). \end{array} \tag{3.47}$$

We write $j : X \to \widehat{X}$ for the canonical continuous embedding $\phi \mapsto (\phi(0), \phi)$. The reader should note that the definition of $\widehat{A}$ given here differs from the corresponding definition in Chapter 2. However, this construction ensures that the part of $\widehat{A}$ in jX is equivalent to A and that the closure of $\mathcal{D}(\widehat{A})$ is given by jX. Hence the spectral analysis of A and $\widehat{A}$ is one and the same. The next result shows that $\Delta(z)$ is a characteristic matrix for $\widehat{A}$, in the sense of [45, Def. IV.4.17].

Lemma 3.4.3. *Consider the holomorphic functions $E : \mathbb{C} \to \mathcal{L}(\widehat{X}, \mathcal{D}(\widehat{A}))$ and $F : \mathbb{C} \to \mathcal{L}(\widehat{X}, \widehat{X})$, given by*

$$\begin{array}{rcl} E(z)(c, \psi)(\theta) & = & (c, e^{\theta z} c + e^{\theta z} \int_\theta^0 e^{-z\tau} \psi(\tau) d\tau), \\ F(z)(c, \psi)(\theta) & = & (c - \psi(0) + \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \psi(\tau) d\tau, \psi(\theta)), \end{array} \tag{3.48}$$

in which $\mathcal{D}(\widehat{A})$ is considered as a Banach space with the graph norm. Then $E(z)$ and $F(z)$ are bijective for every $z \in \mathbb{C}$ and we have the identity

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I \end{pmatrix} = F(z)(zI - \widehat{A})E(z). \tag{3.49}$$

Proof. The bijectivity of $E(z)$ follows as in Chapter 2, while the bijectivity of $F(z)$ is almost immediate. The last identity in the statement of the lemma follows easily by using the definition of $\Delta(z)$ and computing

$$(zI - \widehat{A})E(z)(c, \psi) = (\psi(0) - \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} c - \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \psi(\tau) d\tau, \psi). \tag{3.50}$$

□

In Chapter 2 similar results were obtained for the system $x'(\xi) = Mx_\xi$. In particular, writing $\Sigma^M = \Sigma^M_{\gamma_-,\gamma_+} = \{z \in \mathbb{C} \mid \det \Delta_M(z) = 0 \text{ and } \gamma_- < \operatorname{Re} z < \gamma_+\}$, the decomposition $X = \mathcal{M}_{\Sigma^M} \oplus \mathcal{R}_{\Sigma^M}$ was obtained, together with a projection $Q^M_\Sigma : X \to \mathcal{M}_{\Sigma^M}$. Using (HL) it is easy to see that $\Sigma \subset \Sigma^M$. In addition, the next result exhibits how the generalized eigenspaces are related.

Proposition 3.4.4. *Consider the operator L defined in (3.5) and suppose that (HL) holds. Then we have the inclusion $\mathcal{M}_\Sigma \subset \mathcal{M}_{\Sigma^M}$, together with the identity $Q^M_{\Sigma^M} \circ Q_\Sigma = Q_\Sigma$.*

Proof. First recall from Chapter 2 that $\Delta_M(z)$ is a characteristic matrix for the operator $\widehat{A}^M : \mathcal{D}(\widehat{A}) \to X$ given by $\widehat{A}^M(c, \phi) = (M\phi, D\phi)$. As in the proof of [45, Theorem IV.4.18], a basis for $\mathcal{M}_\Sigma$ can be constructed using maximal generalized Jordan chains. It hence suffices to show that every such chain for Δ at $z = \lambda$ is also a Jordan chain for Δ_M at the same value of z. Indeed, for any such chain $v_0, \ldots, v_{m-1}$ of length m we have by definition

$$\Delta(z)(v_0 + (z-\lambda)v_1 + \ldots + (z-\lambda)^{m-1}v_{m-1}) = O((z-\lambda)^m), \tag{3.51}$$

which immediately implies that also

$$\begin{aligned}&\Delta_M(z)(v_0 + (z-\lambda)v_1 + \ldots + (z-\lambda)^{m-1}v_{m-1})\\ &= \beta_M(z-\alpha_M)^\ell \Delta(z)(v_0 + (z-\lambda)v_1 + \ldots + (z-\lambda)^{m-1}v_{m-1}) = O((z-\lambda)^m).\end{aligned} \tag{3.52}$$

The inclusion $\mathcal{M}_\Sigma \subset \mathcal{M}_{\Sigma^M}$ now easily follows, which in turn implies that $Q^M_{\Sigma^M}$ acts as the identity on $\mathcal{M}_\Sigma$, upon which the proof is complete. □

Proof of Proposition 3.2.1. Choose $\gamma > 0$ such that $\det \Delta_M(z) = 0$ has no roots with $0 < |\mathrm{Re}\, z| \leq \gamma$ and write $X_0 = \mathcal{M}_{\Sigma_{-\gamma,\gamma}}$, together with $Q_0 = Q_{\Sigma_{-\gamma,\gamma}}$. Consider any $x \in \bigcap_{\eta>0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ that satisfies $0 = Lx_\xi$. Using Proposition 3.3.1 it follows that $x \in C^1(\mathbb{R}, \mathbb{C}^n)$. However, this implies that also $Lx'_\xi = 0$ for all $\xi \in \mathbb{R}$ and repeated application of this argument shows that in fact $x \in C^\infty(\mathbb{R}, \mathbb{C}^n)$. We can hence combine Proposition 3.3.3 with Lemma 3.4.2 to conclude that $Q_0 x_0 = x_0$ and hence by shifting x along the line, $Q_0 x_\xi = x_\xi$ for all $\xi \in \mathbb{R}$. Due to the fact that a basis for X_0 can be constructed using functions of the form $p(\theta)e^{\lambda\theta}$, in which p is a polynomial and $\det \Delta_M(\lambda) = 0$, one sees that any $\phi \in X_0$ can be extended to a solution $x = E\phi$ of $Lx_\xi = 0$ on the line, with $x_0 = \phi$. To see the uniqueness of this extension, suppose that both x^1 and x^2 satisfy $x^1_0 = x^2_0 = \phi$, with $0 = Lx^1_\xi = Lx^2_\xi$ for all $\xi \in \mathbb{R}$. Write $y(\xi) = x^1(\xi) - x^2(\xi)$ for $\xi \geq 0$ and $y(\xi) = 0$ for $\xi < 0$. Then $y \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies $0 = Ly_\xi$, with $y(\xi) = O(e^{\zeta\xi})$ as $\xi \to \pm\infty$ for some $\zeta > 0$, which can be chosen in such a way that there are no roots of $\det \Delta_M(z) = 0$ in the strip $\zeta - \epsilon \leq \mathrm{Re}\, z \leq \zeta + \epsilon$, for some $\epsilon > 0$. This however implies that for all $\xi \in \mathbb{R}$, we have $y_\xi = Q_{\Sigma_{\zeta-\epsilon,\zeta+\epsilon}} y_\xi = Q_{\{0\}} y_\xi = 0$, i.e., $y = 0$. A similar construction for $\xi \leq 0$ completes the proof. □

3.5. Linear Inhomogeneous Equations

In this section we study the interplay between the linear inhomogeneous equations

$$\begin{array}{rcl} 0 & = & Lx_\xi + f(\xi), \\ y'(\xi) & = & My_\xi + g(\xi), \end{array} \tag{3.53}$$

with L as defined in (3.5) and M as in (HL). Associated to these equations we define the linear operators $\Lambda : C(\mathbb{R}, \mathbb{C}^n) \to C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$ and $\Lambda_M : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ by

$$\begin{array}{rcl} (\Lambda x)(\xi) & = & -Lx_\xi \\ (\Lambda_M x)(\xi) & = & x'(\xi) - Mx_\xi. \end{array} \tag{3.54}$$

The operator Λ_M has been extensively studied in [112] and we will use these results to facilitate our treatment of Λ. We will be particularly interested in the spaces

$$W^{\ell,p}(\mathbb{R}, \mathbb{C}^n) = \left\{ x \in L^p(\mathbb{R}, \mathbb{C}^n) \mid D^s x \in L^p(\mathbb{R}, \mathbb{C}^n) \text{ for all } 1 \le s \le \ell \right\}, \tag{3.55}$$

with $p = 2$ or $p = \infty$. In the first result we choose $p = 2$, which enables us to use Fourier transform techniques to define an inverse for Λ on the space $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$. This inverse will turn out to be closely related to the inverse of Λ_M.

Lemma 3.5.1. *Consider the operator L defined in (3.5) and assume that (HL) is satisfied. Suppose further that the characteristic equation* $\det \Delta_M(z) = 0$ *has no roots with* $\operatorname{Re} z = 0$. *Then Λ is a bounded linear isomorphism from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ onto $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$, with*

$$(D - \alpha_M)^\ell \Lambda x = \beta_M^{-1} \Lambda_M x \tag{3.56}$$

for $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$, in which D denotes the differentiation operator. Conversely, suppose $x = \Lambda^{-1} f$ for $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$, then x is given by

$$\widehat{x}(\eta) = \Delta(i\eta)^{-1} \widehat{f}(\eta). \tag{3.57}$$

In addition, there is a representation

$$x(\xi) = \int_{-\infty}^{\infty} G(\xi - s)((D - \alpha_M)^\ell f)(s) ds = \beta_M \big(\Lambda_M^{-1} (D - \alpha_M)^\ell f\big)(\xi), \tag{3.58}$$

with a Green's function G that satisfies $G \in L^p(\mathbb{R}, \mathbb{C}^{n \times n})$ for all $1 \le p \le \infty$ and whose Fourier tranform is given by

$$\widehat{G}(\eta) = (i\eta - \alpha_M)^{-\ell} \Delta(i\eta)^{-1} = \beta_M \Delta_M(i\eta)^{-1}. \tag{3.59}$$

The function G decays exponentially at both $\pm\infty$. In particular, fixing $a_- < 0$ and $a_+ > 0$ such that $\det \Delta_M(z) \neq 0$ *for all $a_- \le \operatorname{Re} z \le a_+$ and choosing an $\alpha < a_-$, we have the estimate*

$$|G(\xi)| \le \left\{ \begin{array}{ll} \beta_M (1 + K(a_-)) e^{a_- \xi} & \text{for all } \xi \ge 0, \\ \beta_M K(a_+) e^{a_+ \xi} & \text{for all } \xi < 0, \end{array} \right. \tag{3.60}$$

in which

$$K(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |R_\alpha(a + i\omega)| \, d\omega, \tag{3.61}$$

with R_α as introduced in (3.27).

Finally, suppose that f and its derivatives satisfies a growth condition $D^s f(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$ for some $0 < \lambda < -a_-$ and all $0 \le s \le \ell$. Then also $x = \Lambda^{-1} f$ satisfies $x(\xi) = O(e^{-\lambda\xi})$ as $\xi \to \infty$, with the same estimate for x'. The analogous statement also holds for $\xi \to -\infty$.

Proof. Suppose that $\Lambda x = 0$ for some $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$. Due to the Sobolev embedding $W^{1,2}(\mathbb{R}, \mathbb{C}^n) \subset C(\mathbb{R}, \mathbb{C}^n) \cap L^\infty(\mathbb{R}, \mathbb{C}^n)$, we know that x is bounded, hence we can apply Proposition 3.2.1 with $X_0 = \{0\}$ to conclude $x = 0$. Recall the fact that $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ is equivalent to $\eta \mapsto (1 + |\eta| + \ldots + |\eta|^\ell)\widehat{f}(\eta) \in L^2(\mathbb{R}, \mathbb{C}^n)$. The fact that Λ maps $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ into $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ now follows after noting that for some constants C, C' and C'', we have

$$\begin{array}{rcl} (1 + |\eta| + \ldots + |\eta|^\ell)\mathcal{F}^+\Lambda x(\eta) & = & (1 + |\eta| + \ldots + |\eta|^\ell)\beta_M^{-1}(i\eta - \alpha_M)^{-\ell} \\ & & \quad \Delta_M(i\eta)\hat{x}(\eta) \\ & \leq & C\Delta_M(i\eta)\hat{x}(\eta) \leq (C'|\eta| + C'')\hat{x}(\eta). \end{array} \tag{3.62}$$

Observe also that the identity $(D - \alpha_M)^\ell \Lambda x = \beta_M^{-1}\Lambda_M x$ follows immediately from

$$\mathcal{F}^+\big((D - \alpha_M)^\ell \Lambda x\big)(\eta) = (i\eta - \alpha_M)^\ell \Delta(i\eta)\hat{x}(\eta) = \beta_M^{-1}\Delta_M(i\eta)\hat{x}(\eta). \tag{3.63}$$

To show that Λ is invertible, fix any $f \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ and define $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ by

$$\begin{array}{rcl} \hat{x}(\eta) & = & \Delta(i\eta)^{-1}\hat{f}(\eta) = \Delta_M(i\eta)^{-1}\beta_M(i\eta - \alpha_M)^\ell \hat{f}(\eta) \\ & = & \beta_M \Delta_M(i\eta)^{-1}\mathcal{F}^+((D - \alpha_M)^\ell f)(\eta). \end{array} \tag{3.64}$$

It is clear that indeed $\Lambda x = f$ and the remaining statements now follow easily from this identity together with the theory developed in Chapter 2 for the operator Λ_M. □

As in Chapter 2, we need to obtain results on the behaviour of Λ on the exponentially weighted spaces

$$W_\eta^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n) = \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e^{-\eta \cdot}x(\cdot) \in W^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n)\right\}. \tag{3.65}$$

To ease notation, we introduce the function $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ and $\nu \in \mathbb{R}$. Upon defining a transformed operator $\Lambda_\eta : C(\mathbb{R}, \mathbb{C}^n) \to C^{\ell-1}(\mathbb{R}, \mathbb{C}^n)$ by

$$(\Lambda_\eta x)(\xi) = -\int_{-1}^{1} d\mu(\sigma)e^{-\eta\sigma}x(\xi + \sigma), \tag{3.66}$$

one may easily verify the following identity,

$$\Lambda_\eta e_\eta x = e_\eta \Lambda x. \tag{3.67}$$

The corresponding transformation of the characteristic matrix is given by

$$\Delta_\eta(z) = -\int_{-1}^{1} e^{(z-\eta)\sigma} d\mu(\sigma) = \Delta(z - \eta) = (z - \alpha_{\eta,M})^{-\ell}\Delta_{\eta,M}(z), \tag{3.68}$$

with $\alpha_{\eta,M} = \alpha_M + \eta$ and $\Delta_{\eta,M}(z) = \Delta_M(z - \eta)$.

We now wish to use the fact that Λ is invertible as a map from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ into $W^{\ell,2}(\mathbb{R}, \mathbb{C}^n)$ to prove a similar result when considering Λ as a map from $W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$

into $W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. An inverse for Λ will be constructed by writing any $f \in W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ as a sum of functions in $W^{\ell,2}_{\zeta}(\mathbb{R}, \mathbb{C}^n)$ for appropriate values of ζ, on which we can use the inverse of Λ defined in Lemma 3.5.1. In contrast to the situation in Chapter 2, where we merely needed to consider $f \in L^{\infty}(\mathbb{R}, \mathbb{C}^n)$, care has to be taken when splitting the inhomogeneity f to ensure that the components remain sufficiently smooth.

To accomodate this, we choose C^{∞}-smooth basis functions χ_i for $0 \le i \le \ell - 1$ that have compact support contained in $[-1, 1]$, such that $D^i \chi_j(0) = \delta_{ij}$ for $0 \le i, j \le \ell - 1$. We now define the finite dimensional space $BC_{\diamond}(\mathbb{R}, \mathbb{C}^n) = \mathrm{span}\,\{\chi_i \mid 0 \le i \le \ell - 1\} \subset C^{\infty}_c(\mathbb{R}, \mathbb{C}^n)$ and an operator $\Phi_{\diamond} : W^{\ell,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \to BC_{\diamond}(\mathbb{R}, \mathbb{C}^n)$ by

$$\Phi_{\diamond} f = \sum_{i=0}^{\ell-1} \chi_i D^i f(0). \tag{3.69}$$

Notice that $D^i \Phi_{\diamond} f(0) = D^i f(0)$ for all $0 \le i \le \ell - 1$, which ensures that we can define the cutoff operators $\Phi_{\pm} : W^{\ell,1}_{\mathrm{loc}} \to W^{\ell,1}_{\mathrm{loc}}$ via

$$\Phi_{+} f(\xi) = H(\xi)(f - \Phi_{\diamond} f)(\xi) \qquad \Phi_{-} f(\xi) = (1 - H(\xi))(f - \Phi_{\diamond} f)(\xi), \tag{3.70}$$

where $H(\xi)$ denotes the Heaviside function, i.e., $H(\xi) = 1$ for $\xi \ge 0$ and $H(\xi) = 0$ for $\xi < 0$.

Proposition 3.5.2. *Consider any $\eta \in \mathbb{R}$ and $\epsilon_0 > 0$ such that there are no roots of $\det \Delta_M(z) = 0$ in the strip $\eta - \epsilon_0 \le \mathrm{Re}\, z \le \eta + \epsilon_0$. Then the operator Λ is an isomorphism from $W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ onto $W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. In addition, for any $0 < \epsilon < \epsilon_0$ and any $f \in W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ such that $\Phi_{\diamond} f = 0$, we have the following integral expression for $x = \Lambda^{-1} f$,*

$$x(\xi) = \frac{1}{2\pi i} \int_{\eta+\epsilon-i\infty}^{\eta+\epsilon+i\infty} e^{\xi z} \Delta(z)^{-1} \widetilde{f}_{+}(z) dz + \frac{1}{2\pi i} \int_{\eta-\epsilon-i\infty}^{\eta-\epsilon+i\infty} e^{\xi z} \Delta(z)^{-1} \widetilde{f}_{-}(z) dz, \tag{3.71}$$

where the Laplace transforms $\widetilde{f}_{+}$ and $\widetilde{f}_{-}$ are defined as in Section 3.3. Finally, for any $f \in W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$, we have the following Green's formula for $x = \Lambda^{-1} f$,

$$x(\xi) = e^{\eta \xi} \int_{-\infty}^{\infty} G_{-\eta}(\xi - s) e^{-\eta s} ((D - \alpha_M)^{\ell} f)(s) ds = \beta_M \Lambda_M^{-1} (D - \alpha_M)^{\ell} f, \tag{3.72}$$

in which $G_{-\eta}$ has exponential decay at both $\pm\infty$ and is given by

$$\mathcal{F}^{+} G_{-\eta}(k) = \beta_M \Delta_M(ik + \eta)^{-1}. \tag{3.73}$$

Proof. We first show that we can indeed define an inverse for Λ on the space $W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. Pick any $0 < \epsilon < \epsilon_0$ and use the cutoff operators introduced above to define $f_{\pm} \in W^{\ell,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ by $f_{\pm} = \Phi_{\pm} f$ and similarly $f_{\diamond} = \Phi_{\diamond} f$. Note that $f_{\diamond} \in W^{\ell,2}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ and hence we can define $x_{\diamond} = e_{\eta} \overline{x}_{\diamond} \in W^{1,2}_{\eta}(\mathbb{R}, \mathbb{C}^n)$, with $\overline{x}_{\diamond} = \Lambda^{-1}_{-\eta} e_{-\eta} f_{\diamond}$. Since

$e_{-\eta} f_\diamond \in C_c^\infty(\mathbb{R}, \mathbb{C}^n)$, one can use the Green's function representation (3.58) to conclude that also $\overline{x}_\diamond$ and its derivative are uniformly bounded, showing that $x_\diamond \in W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$.

It remains to invert the functions $f_\pm$. To this end, we define

$$\overline{f}_\pm = e_{-(\eta \pm \epsilon)} f_\pm \in W^{\ell,2}(\mathbb{R}, \mathbb{C}^n) \cap W^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W_{\mp\epsilon}^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n), \tag{3.74}$$

which allows us to introduce the functions $x_\pm = e_{\eta \pm \epsilon} \overline{x}_\pm$, in which

$$\overline{x}_\pm = \Lambda_{-\eta\mp\epsilon}^{-1} \overline{f}_\pm \in W^{1,2}(\mathbb{R}, \mathbb{C}^n) \cap W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W_{\mp\epsilon}^{1,\infty}(\mathbb{R}, \mathbb{C}^n), \tag{3.75}$$

where the last two inclusions follow from the Green's function representation of Λ^{-1} in Lemma 3.5.1. This shows that indeed $x_\pm \in W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and hence $x = x_\diamond + x_+ + x_-$ satisfies $\Lambda x = f$. The integral expression (3.71) now follows upon applying the substitution $z = \eta \pm \epsilon + ik$ to the equality

$$\mathcal{F}^+ \overline{x}_\pm(k) = \Delta_{-\eta\mp\epsilon}^{-1}(ik) \mathcal{F}^+ \overline{f}_\pm(k) = \Delta^{-1}(ik + \eta \pm \epsilon) \widetilde{f}_\pm(ik + \eta \pm \epsilon). \tag{3.76}$$

The injectivity of Λ can be shown in exactly the same manner as the corresponding result in Lemma 3.5.1. Notice that Proposition 3.3.1 immediately implies that Λ maps $W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ into $W_\eta^{\ell-1,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W_{\mathrm{loc}}^{\ell,1}(\mathbb{R}, \mathbb{C}^n)$. To show that the mapping is actually into $W_\eta^{\ell,\infty}(\mathbb{R}, \mathbb{C}^n)$, we notice that in a similar fashion as above, any $y \in W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ can be split into $y = y_\diamond + y_+ + y_-$ with $y_\diamond \in C_c^\infty(\mathbb{R}, \mathbb{C}^n)$ and $y_\pm \in W_{\eta\pm\epsilon}^{1,2}(\mathbb{R}, \mathbb{C}^n)$. Applying Lemma 3.5.1 to these individual functions and using (3.67), we find that again $(D - \alpha_M)^\ell \Lambda y = \beta_M^{-1} \Lambda_M y \subset L_\eta^\infty(\mathbb{R}, \mathbb{C}^n)$, which together with Proposition 3.3.1 shows that also $D^\ell \Lambda y \in L_\eta^\infty(\mathbb{R}, \mathbb{C}^n)$.

Finally, we show that the Green's formula representation (3.58) continues to hold. For convenience, we write $\zeta = \eta + \epsilon$ and note that for any $f \in W_{\mathrm{loc}}^{\ell,1}(\mathbb{R}, \mathbb{C}^n)$ the identity $De_\zeta f = e_\zeta(D + \zeta) f$ implies that

$$\begin{array}{rcl} e_\zeta (D - \alpha_{-\zeta,M})^\ell e_{-(\eta+\epsilon)} f & = & e_{\eta+\epsilon}(D - \alpha_M + \eta + \epsilon)^\ell e_{-\zeta} f \\ & = & e_\zeta((D - \zeta) - \alpha_M + \zeta)^\ell f \\ & = & (D - \alpha_M)^\ell f. \end{array} \tag{3.77}$$

This allows us to compute

$$\begin{array}{rcl} x_+(\xi) & = & (e_\zeta \Lambda_{-\zeta}^{-1} e_{-\zeta} f_+)(\xi) \\ & = & e^{\zeta\xi} \int_{-\infty}^\infty G_{-\zeta}(\xi - s) e^{-\zeta s} (D - \alpha_M)^\ell f_+(s) ds \\ & = & e^{\eta\xi} \int_{-\infty}^\infty e^{\epsilon(\xi - s)} G_{-\zeta}(\xi - s) e^{-\eta s} (D - \alpha_M)^\ell f_+(s) ds. \end{array} \tag{3.78}$$

Now noticing that $\mathcal{F}^+ e_\epsilon G_{-\zeta}(k) = \widehat{G}_{-\zeta}(k + i\epsilon)$, we find,

$$\begin{array}{rcl} \mathcal{F}^+ e_\epsilon G_{-\zeta}(k) & = & (ik - \epsilon - \alpha_{-\zeta,M})^{-\ell} \Delta_{-\zeta}(ik - \epsilon)^{-1} \\ & = & (ik - \alpha_M + \eta)^{-\ell} \Delta(ik + \eta)^{-1} = \widehat{G}_{-\eta}(k), \end{array} \tag{3.79}$$

upon which the proof can be completed using similar identities for x_- and $x_\diamond$. □

3.6. The pseudo-inverse

The goal of this section is to define a pseudo-inverse for the linear inhomogeneous equation $\Lambda x = f$ in the spirit of Chapter 2. However, the construction here will differ from the corresponding construction in Chapter 2, due to the fact that we cannot modify the nonlinearities $R^{(s)}$ to become globally Lipschitz continuous in such a way that the differentiation structure in (HR1) is preserved. To bypass this difficulty, we need to decouple the inhomogeneity f from its derivatives, allowing us to replace the vector of functions $(f, Df, \ldots, D^{\ell} f)$ by general vectors $(g_0, \ldots, g_{\ell})$ for which there is no relation between the components. This decoupling should be seen in the context of so-called jet manifolds, which play a role when studying PDEs and DAEs from an algebraic point of view, see e.g. [126].

To formalize this construction, we introduce the product spaces

$$\begin{aligned} L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) &= \underbrace{L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \times \ldots \times L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)}_{\ell+1 \text{ times}}, \\ BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) &= \underbrace{BC_{\eta}(\mathbb{R},\mathbb{C}^n) \times \ldots \times BC_{\eta}(\mathbb{R},\mathbb{C}^n)}_{\ell+1 \text{ times}} \end{aligned} \tag{3.80}$$

and the canonical inclusions $BC^{\ell}_{\eta}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n)$ and $W^{\ell,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \hookrightarrow L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n)$ via $\mathcal{J}x = (x, Dx, \ldots, D^{\ell}x)$. For any $0 \le s \le \ell$ write $\widetilde{D}^s : L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to L^{\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$ for the canonical extensions of the differentiation operators and similarly define $\widetilde{\Phi}_{\diamond} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to C^{\infty}_{c}(\mathbb{R},\mathbb{C}^n)$ and $\widetilde{\Phi}_{\pm} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to L^{\infty,\times(\ell+1)}_{\pm\eta}(\mathbb{R},\mathbb{C}^n)$. Using the explicit representation (3.72) for $\Lambda^{-1} : W^{\ell,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$, we can naturally expand the domain of definition to obtain an operator $\Lambda^{-1} : L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\eta}(\mathbb{R},\mathbb{C}^n)$, given by

$$\Lambda^{-1}\mathbf{f} \;=\; \beta_M \Lambda_M^{-1}(\widetilde{D} - \alpha_M)^{\ell}\mathbf{f}, \qquad \mathbf{f} \in L^{\infty,\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n). \tag{3.81}$$

We will use the longer notation $\Lambda^{-1}_{(\eta)}$ for this operator whenever we wish to emphasize the η-dependence of the underlying exponentially weighted function spaces.

Pick any $\gamma > 0$ such that there are no roots of $\det \Delta_M(z) = 0$ with $0 < |\mathrm{Re}\, z| \le \gamma$ and fix an $\eta \in (0, \gamma)$. Using the construction above we can define the bounded linear operators $\Lambda^{-1}_{\pm} = \Lambda^{-1}_{(\pm\eta)} : L^{\infty,\times(\ell+1)}_{\pm\eta}(\mathbb{R},\mathbb{C}^n) \to W^{1,\infty}_{\pm\eta}(\mathbb{R},\mathbb{C}^n)$. As in the proof of Proposition 3.2.1, we write $X_0 = \mathcal{M}_{\Sigma_{-\gamma,\gamma}}$ for the generalized eigenspace corresponding to roots of $\det \Delta(z) = 0$ on the imaginary axis and $Q_0 = Q_{\Sigma_{-\gamma,\gamma}}$ for the corresponding spectral projection. Similarly, we introduce $X_M = \mathcal{M}_{\Sigma^M_{-\gamma,\gamma}}$ and $Q_M = Q^M_{\Sigma_{-\gamma,\gamma}}$ for the analogues of X_0 and Q_0 associated to the operator Λ_M. Recalling the extension operator $E : X_0 \to BC^1_{\eta}(\mathbb{R},\mathbb{C}^n)$ from Proposition 3.2.1, we have all the ingredients we need to define the pseudo-inverse $\mathcal{K}_{\eta} : BC^{\times(\ell+1)}_{\eta}(\mathbb{R},\mathbb{C}^n) \to BC^1_{\eta}(\mathbb{R},\mathbb{C}^n)$. It is given explicitly by the formula

$$\begin{aligned} \mathcal{K}_{\eta}\mathbf{f} \;=\; & \Lambda_+^{-1}(\tfrac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_+\mathbf{f}) + \Lambda_-^{-1}(\tfrac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_-\mathbf{f}) \\ & - E Q_0 \mathrm{ev}_0\big[\Lambda_+^{-1}(\tfrac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_+\mathbf{f}) + \Lambda_-^{-1}(\tfrac{1}{2}\widetilde{\Phi}_{\diamond}\mathbf{f} + \widetilde{\Phi}_-\mathbf{f})\big], \end{aligned} \tag{3.82}$$

in which we have introduced the evaluation function $\mathrm{ev}_\xi x = x_\xi \in X$. Note that by construction we have the identity $Q_0 \mathrm{ev}_0 \mathcal{K}_\eta \mathbf{f} = 0$. In addition, from (3.81) together with the inclusion $X_0 \subset X_M$, we see that $\Lambda_M \mathcal{K}_\eta \mathbf{f} = \beta_M (\widetilde{D} - \alpha_M)^\ell \mathbf{f}$. The following result shows that $\mathcal{K}$ is well-behaved on the scale of Banach spaces $BC_\zeta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$.

Lemma 3.6.1. *Consider any pair $\eta_1, \eta_2 \in \mathbb{R}$ with $0 < \eta_1 < \eta_2 < \gamma$. Then for any $\mathbf{f} \in BC_{\eta_1}^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$, we have*

$$\mathcal{K}_{\eta_1}\mathbf{f} = \mathcal{K}_{\eta_2}\mathbf{f}. \tag{3.83}$$

Proof. We will merely establish (3.83) under the assumption $\mathbf{f} = \widetilde{\Phi}_+\mathbf{f}$, noting that the remaining components of $\mathbf{f}$ can be treated in a similar fashion. Note that $\overline{h}_+ = e_{-\eta_2}(\widetilde{D} - \alpha_M)^\ell \widetilde{\Phi}_+ \mathbf{f} \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ satisfies a growth condition $\overline{h}_+(\xi) = O(e^{-(\eta_2 - \eta_1)\xi})$ as $\xi \to \infty$ and hence $\overline{x}_+ = \beta_M \Lambda_{-\eta_2, M}^{-1} \overline{h}_+$ shares this growth rate by Corollary 2.3.3. This implies that the function $x_+ = e_{\eta_2}\overline{x}_+$ satisfies $x_+ = O(e^{\eta_1 \xi})$ as $\xi \to \infty$. We can hence argue $x_+ \in W_{\eta_1}^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \cap W_{\eta_2}^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$, from which we conclude

$$\Lambda_{(\eta_1)}^{-1} \widetilde{\Phi}_+ \mathbf{f} = \Lambda_{(\eta_2)}^{-1} \widetilde{\Phi}_+ \mathbf{f}, \tag{3.84}$$

which directly implies that also (3.83) holds. □

The final result of this section should be seen as the analogue of Lemma 2.5.4. The conclusions here are however somewhat weaker, due to the fact that in this more general setting we no longer have an automatic interpretation of $\mathcal{K}_\eta \mathbf{f}$ in terms of the operator Λ. The consequences of this fact shall become clear in Section 3.7, during the analysis of the dynamics on the center manifold.

Lemma 3.6.2. *For any $\mathbf{f} \in BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ and $\xi_0 \in \mathbb{R}$, define the function $y \in BC_\eta^1(\mathbb{R}, \mathbb{C}^n)$ by*

$$y(\xi) = (\mathcal{K}_\eta \mathbf{f})(\xi + \xi_0) - (\mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot))(\xi). \tag{3.85}$$

Then we have $\Lambda_M y = 0$. In particular, we have the identity

$$(I - Q_M)\mathrm{ev}_{\xi_0} \mathcal{K}_\eta \mathbf{f} = (I - Q_M)\mathrm{ev}_0 \mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot). \tag{3.86}$$

In addition, suppose that for all integers s with $0 \le s \le \ell$ we have

$$(\mathbf{f}_s)_{|J} = D^s g \tag{3.87}$$

for some $g \in C^\ell(J, \mathbb{C}^n)$, with $J = [-1 - |\xi_0|, 1 + |\xi_0|]$. Then in fact $\Lambda y = 0$ and

$$(I - Q_0)\mathrm{ev}_{\xi_0} \mathcal{K}_\eta \mathbf{f} = \mathrm{ev}_0 \mathcal{K}_\eta \mathbf{f}(\xi_0 + \cdot). \tag{3.88}$$

Proof. We can no longer as in Chapter 2 apply Λ directly to the definition of $\mathcal{K}$. Instead, we introduce the shift operator T_{ξ_0} that acts as $(T_{\xi_0} f)(\xi) = f(\xi + \xi_0)$ and compute

$$\begin{aligned} y &= x + T_{\xi_0}\Lambda_{(\eta)}^{-1}[\tfrac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]\mathbf{f} - \Lambda_{(\eta)}^{-1}[\tfrac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]T_{\xi_0}\mathbf{f} \\ &\quad + T_{\xi_0}\Lambda_{(-\eta)}^{-1}[\tfrac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]\mathbf{f} - \Lambda_{(-\eta)}^{-1}[\tfrac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]T_{\xi_0}\mathbf{f} \\ &= x + \Lambda_{(\eta)}^{-1}\mathbf{g}_+ + \Lambda_{(-\eta)}^{-1}\mathbf{g}_- \end{aligned} \tag{3.89}$$

for some $x \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$. Using the fact that T_{ξ_0} and $\Lambda_\pm^{-1}$ commute, we can write

$$\begin{array}{rcl} \mathbf{g}_+ & = & T_{\xi_0}[\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]\mathbf{f} - [\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_+]T_{\xi_0}\mathbf{f} \\ \mathbf{g}_- & = & T_{\xi_0}[\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]\mathbf{f} - [\frac{1}{2}\widetilde{\Phi}_\diamond + \widetilde{\Phi}_-]T_{\xi_0}\mathbf{f}. \end{array} \tag{3.90}$$

Now using the identity $I = \widetilde{\Phi}_\diamond + \widetilde{\Phi}_+ + \widetilde{\Phi}_-$, one easily sees that $\mathbf{g}_+ = -\mathbf{g}_-$. Using (3.81) it now easily follows that indeed $\Lambda_M y = 0$.

Now suppose that the differentiability condition (3.87) holds. Since $(T_{\xi_0}\widetilde{\Phi}_\pm \mathbf{f})(\xi) = (\widetilde{\Phi}_\pm T_{\xi_0}\mathbf{f})(\xi) = \mathbf{f}(\xi + \xi_0)$ for all $\xi \geq \max(1, 1 - \xi_0)$ and $\xi \leq \min(-1, -1 - \xi_0)$, it follows that both $\mathbf{g}_\pm$ have compact support and in addition satisfy $\mathbf{g}_\pm = \mathcal{J} g_*$ for some $g_* \in C^\ell(\mathbb{R}, \mathbb{C}^n)$. In this case the conclusion $\Lambda y = 0$ is immediate from (3.89). □

3.7. The Center Manifold

We are now in a position in which we can use the pseudo-inverse defined in the previous section to construct a center manifold for the nonlinear equation (3.4). In order to apply the Banach contraction theorem, we consider the set of nonlinearities $R^{(s)}$ for $0 \leq s \leq \ell$ introduced in condition (HR1) and modify them simultaneously to become globally Lipschitz continuous with a sufficiently small Lipschitz constant. We choose a C^∞-smooth cutoff-function $\chi : [0, \infty) \to \mathbb{R}$ with $\|\chi\|_\infty = 1$ that satisfies $\chi(\xi) = 0$ for $\xi \geq 2$ while $\chi(\xi) = 1$ for $\xi \leq 1$. For any $\delta > 0$ we define $\chi_\delta(\xi) = \chi(\xi/\delta)$. We use the projection Q_M defined in the previous section to modify the nonlinearities separately in the hyperbolic and nonhyperbolic directions and define $\mathbf{R}_\delta : X \to \mathbb{C}^{n\times(\ell+1)}$ componentwise by

$$\mathbf{R}_\delta(\phi)_s = \chi_\delta(\|Q_M\phi\|)\chi_\delta(\|(I - Q_M)\phi\|)R^{(s)}(\phi), \qquad 0 \leq s \leq \ell. \tag{3.91}$$

The fact that we use Q_M instead of Q_0 is motivated by (3.86), which allows us to control the growth of $\xi \mapsto (I - Q_M)\mathrm{ev}_\xi\mathcal{K}$ on the center manifold.

The map $\mathbf{R}_\delta$ induces the map $\widetilde{\mathbf{R}}_\delta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ via $\widetilde{\mathbf{R}}_\delta x(\xi) = \mathbf{R}_\delta x_\xi$. Notice that $\widetilde{\mathbf{R}}_\delta$ is well-defined, since $i_x : \mathbb{R} \to X$ which sends $\xi \mapsto x_\xi$ is a continuous mapping for any continuous x and hence the same holds for $\widetilde{\mathbf{R}}_\delta x = \mathbf{R}_\delta \circ i_x$. The next lemma follows directly from Section 2.6 and shows that the construction above indeed yields a globally Lipschitz smooth substitution operator $\widetilde{\mathbf{R}}_\delta$.

Lemma 3.7.1. *For any $\eta \in \mathbb{R}$, the substitution operator $\widetilde{\mathbf{R}}_\delta$ viewed as an operator from $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ into $BC_\eta^{\times(\ell+1)}(\mathbb{R}, \mathbb{C}^n)$ is globally Lipschitz continuous with Lipschitz constant $\exp(|\eta|)L_\delta$, in which $L_\delta \to 0$ as $\delta \to 0$. In addition, we have $|(\mathbf{R}_\delta\phi)_s| \leq 4\delta L_\delta$ for all $\phi \in X$ and integers s with $0 \leq s \leq \ell$.*

We will apply a fixed point argument to the operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, defined by

$$\mathcal{G}(u, \phi) = E\phi + \mathcal{K}_\eta\widetilde{\mathbf{R}}_\delta(u). \tag{3.92}$$

Theorem 3.7.2. *Consider the system (3.4) and suppose that the conditions (HL), (HR1) and (HR2) are all satisfied. Fix* $\gamma > 0$ *such that the characteristic equation* $\det \Delta_M(z) = 0$ *has no roots with* $0 < |\mathrm{Re}\, z| < \gamma$ *and consider any interval* $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ *with* $k\eta_{\min} < \eta_{\max}$*, where k is as defined in (HR2). Then there exist constants* $0 < \epsilon < \delta$ *such that*

(i) For all $\eta \in [\eta_{\min}, \eta_{\max}]$ *and for any* $\phi \in X_0$*, the fixed point equation* $u = \mathcal{G}(u, \phi)$ *has a unique solution* $u = u^*_\eta(\phi) \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$.

(ii) For any pair $\eta_{\min} \leq \eta_1 < \eta_2 \leq \eta_{\max}$*, we have that* $u^*_{\eta_2} = \mathcal{J}^1_{\eta_2\eta_1} u^*_{\eta_1}$.

(iii) For all $\xi \in \mathbb{R}$ *and all* $\phi \in X_0$*, we have*

$$\left\| (I - Q_M)\mathrm{ev}_\xi u^*_\eta(\phi) \right\| < \delta. \tag{3.93}$$

(iv) For any $\phi \in X_0$ *with* $\|\phi\| < \epsilon$*, we have for all* $-2 \leq \theta \leq 2$ *that*

$$\left\| Q_M \mathrm{ev}_\theta u^*_\eta(\phi) \right\| < \delta. \tag{3.94}$$

(v) For all $\eta \in (k\eta_{\min}, \eta_{\max}]$*, the mapping* $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ *is of class* C^k *and admits the Taylor expansion*

$$u^*_\eta(\phi) = E\phi + \frac{1}{2}\mathcal{K}_\eta D^2\mathbf{R}_\delta(0)(\mathrm{ev}_\xi E\phi, \mathrm{ev}_\xi E\phi) + o(\|\phi\|^2), \tag{3.95}$$

if $k \geq 2$*, in which* $\mathcal{K}_\eta$ *acts on the variable* ξ.

Proof. (i) First note that as in Chapter 2 we can use the Green's function representation (3.72) to uniformly bound $\left\|\mathcal{K}_\eta\right\|$ for $\eta \in [\eta_{\min}, \eta_{\max}]$, hence it is possible to choose δ in such a way that for all such η we have

$$\exp(\eta) L_\delta \left\|\mathcal{K}_\eta\right\| < \frac{1}{4}. \tag{3.96}$$

This ensures that $\mathcal{G}(\cdot, \phi)$ is Lipschitz continuous with constant $\frac{1}{4}$. Since $\mathcal{G}(\cdot, \phi)$ leaves the ball with radius ρ in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ invariant when $\|E\|_\eta \|\phi\| < \frac{\rho}{2}$, the mapping $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ can be defined using the contraction mapping theorem. By computing

$$\begin{aligned} \left\| u^*_\eta(\phi_1) - u^*_\eta(\phi_2) \right\|_{BC^1_\eta} &\leq \|E\|_\eta \|\phi_1 - \phi_2\| \\ &\quad + \left\|\mathcal{K}_\eta\right\| \exp(\eta) L_\delta \left\| u^*_\eta(\phi_1) - u^*_\eta(\phi_2) \right\|_{BC^1_\eta} \\ &\leq \|E\|_\eta \|\phi_1 - \phi_2\| + \tfrac{1}{4} \left\| u^*_\eta(\phi_1) - u^*_\eta(\phi_2) \right\|_{BC^1_\eta}, \end{aligned} \tag{3.97}$$

it is clear that u^*_η is in fact Lipschitz continuous.

(ii) Observing that $\left|\mathbf{R}_\delta\big(\mathrm{ev}_\xi u^*_\eta(\phi)\big)_s\right| \le 4\delta L_\delta$ for all $0 \le s \le \ell$, Lemma 3.6.1 implies that $\mathcal{K}_{\eta_2}\widetilde{\mathbf{R}}_\delta(u^*_{\eta_1}(\phi)) = \mathcal{K}_{\eta_1}\widetilde{\mathbf{R}}_\delta(u^*_{\eta_1}(\phi))$, from which the result follows immediately.

(iii) If $\delta > 0$ is chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$

$$\|I - Q_M\| \exp(\eta_0) L_\delta < (4 \left\|\mathcal{K}_{\eta_0}\right\|)^{-1}, \tag{3.98}$$

then we can use Lemma 3.6.2 and Lemma 3.6.1 to compute

$$\begin{array}{rcl} \left\|(I - Q_M)\mathrm{ev}_\xi u^*_\eta(\phi)\right\| & = & \left\|(I - Q_M)\mathrm{ev}_\xi \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi))\right\| \\ & = & \left\|(I - Q_M)\mathrm{ev}_0 \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi))(\cdot + \xi)\right\| \\ & \le & \|I - Q_M\| \exp(\eta_0) \left\|\mathcal{K}_{\eta_0}\right\| 4\delta L_\delta < \delta. \end{array} \tag{3.99}$$

(iv) If $\delta > 0$ and $\epsilon > 0$ are chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$

$$\begin{array}{rcl} (8 \left\|\mathcal{K}_{\eta_0}\right\|)^{-1} & > & L_\delta(\exp(3\eta_0) + \|I - Q_M\| \exp(\eta_0)) \quad \text{and} \\ \frac{1}{2}\delta & > & \epsilon \exp(3\eta_0) \|E\|_{\eta_0}, \end{array} \tag{3.100}$$

then we can compute

$$\begin{array}{rcl} Q_M \mathrm{ev}_\theta u^*_\eta(\phi) & = & \mathrm{ev}_\theta E\phi + \mathrm{ev}_\theta \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi)) \\ & & \quad -(I - Q_M)\mathrm{ev}_0 \mathcal{K}_{\eta_0}\widetilde{\mathbf{R}}_\delta\big(u^*_\eta(\phi)(\theta + \cdot)\big) \end{array} \tag{3.101}$$

and hence

$$\begin{array}{rcl} Q_M \mathrm{ev}_\theta u^*_\eta(\phi) & \le & \exp(2\eta_0)\exp(\eta_0) \|E\|_{\eta_0} \|\phi\| \\ & & \quad +4\delta L_\delta\big(\exp(2\eta_0)\exp(\eta_0) \|\mathcal{K}\|_{\eta_0} \\ & & \quad + \|I - Q_M\| \exp(\eta_0) \|\mathcal{K}\|_{\eta_0}\big) \\ & < & \delta. \end{array} \tag{3.102}$$

(v) Notice that item (iii) ensures that u^*_η maps precisely into the region on which the modification of R in the infinite dimensional hyperbolic direction is trivial, which means that $\mathbf{R}_\delta$ is C^k-smooth in this region. This fact ensures that we can follow the approach in Chapter 2 to prove that u^* is in fact C^k-smooth, in the sense defined above.

□

In order to show that u^*_η behaves appropriately under translations, we need to be able to control the size of the center part of $u^*_\eta(\phi)$, as is made precise in the next result.

Lemma 3.7.3. *Consider the setting of Theorem 3.7.2 and let $\phi \in X_0$. Consider any $\xi_0 \in \mathbb{R}$ such that $\left\|Q_M \mathrm{ev}_\xi u^*_\eta(\phi)\right\| < \delta$ for all $-1 - |\xi_0| \le \xi \le 1 + |\xi_0|$. Then the following identity holds,*

$$u^*_\eta(\phi)(\xi_0 + \cdot) = \big[u^*_\eta\big(Q_0 \mathrm{ev}_{\xi_0} u^*_\eta(\phi)\big)\big](\cdot). \tag{3.103}$$

Proof. Due to item (iii) of Theorem 3.7.2, we have $\left\|(I - Q_M)\mathrm{ev}_\xi u^*_\eta(\phi)\right\| < \delta$ for all $\xi \in \mathbb{R}$. From (HR1), the definition of $\mathbf{R}_\delta$ in (3.91) and the condition in the statement of the lemma, it now follows that $\widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi))_{|J} = \mathcal{J}g$ for some $g \in C^\ell(J, \mathbb{C}^n)$, where J denotes the interval $J = [-1 - |\xi_0|, 1 + |\xi_0|]$. We can hence apply Lemma 3.6.2 to conclude that the function

$$y(\xi) = E\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi))(\xi_0 + \xi) - \mathcal{K}_\eta \widetilde{\mathbf{R}}_\delta\big(u^*_\eta(\phi)(\xi_0 + \cdot)\big)(\xi) \tag{3.104}$$

satisfies $\Lambda y = 0$, with $y = E\psi$ for $\psi = Q_0 \mathrm{ev}_{\xi_0} u^*_\eta(\phi)$. Upon calculating

$$\begin{array}{rcl} \mathcal{G}\big(u^*_\eta(\phi)(\xi_0 + \cdot), \psi\big)(\xi) & = & y(\xi) + \mathcal{K}_\eta \widetilde{\mathbf{R}}_\delta\big(u^*_\eta(\phi)(\xi_0 + \cdot)\big)(\xi) \\ & = & E\phi(\xi_0 + \xi) + \mathcal{K}_\eta \widetilde{\mathbf{R}}_\delta(u^*_\eta(\phi))(\xi_0 + \xi) \\ & = & u^*_\eta(\phi)(\xi_0 + \xi), \end{array} \tag{3.105}$$

the conclusion follows from the uniqueness of fixed points for $\mathcal{G}$. □

We are now ready to construct the ODE that describes the dynamics on the center manifold. Note that in contrast to the situation in Chapter 2, this is no longer possible globally, but upon combining the results in Lemma 3.7.3 and item (iv) of Theorem 3.7.2 the ODE can at least be defined locally. Nevertheless, the next result will turn out to be strong enough to lift sufficiently small solutions of the ODE (3.106) back to solutions of (3.4).

Proposition 3.7.4. *Consider for any $\phi \in X_0$ the function $\Phi : \mathbb{R} \to X_0$, given by $\Phi(\xi) = Q_0 \mathrm{ev}_\xi u^*_\eta(\phi)$. Suppose that for some $\xi_0 > 0$ we have $\|\Phi(\xi)\| < \epsilon$ for all $\xi \in (-\xi_0, \xi_0)$. Then Φ satisfies the following ODE on the interval $(-\xi_0, \xi_0)$,*

$$\Phi'(\xi) = A\Phi(\xi) + f_\delta(\Phi(\xi)). \tag{3.106}$$

Here the function $f_\delta : X_0 \to X_0$ is C^k-smooth and is explicitly given by

$$f_\delta(\psi) = Q_0\big(M\mathrm{ev}_\theta(u^*_\eta(\psi) - E\psi) + \beta_M(\overline{D} - \alpha_M)^\ell \mathbf{R}_\delta(\mathrm{ev}_\theta u^*_\eta(\psi))\big), \tag{3.107}$$

in which the projection Q_0 is taken with respect to the variable θ and the expression $\overline{D}^s \mathbf{R}_\delta(\cdot)$ should be read as $\mathbf{R}_\delta(\cdot)_s$. Finally, we have $f_\delta(0) = 0$ and $Df_\delta(0) = 0$.

Proof. Notice first that Φ is a continuous function, since $\xi \mapsto \mathrm{ev}_\xi u^*_\eta(\phi)$ is continuous. We calculate

$$\begin{array}{rcl} \Phi'(\xi)(\sigma) & = & \lim_{h\to 0} \frac{1}{h}(\Phi(\xi + h)(\sigma) - \Phi(\xi)(\sigma)) \\ & = & \lim_{h\to 0} \frac{1}{h}\big([Q_0 \mathrm{ev}_{\xi+h} u^*_\eta(\phi)](\sigma) - [Q_0 \mathrm{ev}_\xi u^*_\eta(\phi)](\sigma)\big) \\ & = & \big[Q_0[Du^*_\eta(\phi)](\xi + \cdot)\big](\sigma), \end{array} \tag{3.108}$$

where the continuity of the projection Q_0, together with the fact that $\mathcal{K}_\eta$ maps into $C^1(\mathbb{R}, \mathbb{C}^n)$, was used in the last step. Using the definition of $\mathcal{K}_\eta$, we compute

$$[Du^*_\eta(\phi)](\xi + \theta) = M\mathrm{ev}_{\xi+\theta} u^*_\eta(\phi) + \beta_M(\overline{D} - \alpha_M)^\ell \mathbf{R}_\delta\big(\mathrm{ev}_{\xi+\theta} u^*_\eta(\phi)\big). \tag{3.109}$$

Assume for the moment that for all $\xi \in (-\xi_0, \xi_0)$ and all $-1 \leq \theta \leq 1$ we have that $\mathrm{ev}_{\xi+\theta} u^*_\eta(\phi) = \mathrm{ev}_\theta u^*_\eta(\psi)$, where $\psi = \Phi(\xi)$. Then the ODE (3.106) follows upon noting that

$$Q_0(M\mathrm{ev}_\theta E\psi) = Q_0(D\psi(\theta)) = Q_0((A\psi)(\theta)) = A\psi, \tag{3.110}$$

in which Q_0 acts on the variable θ. The fact that f is C^k-smooth follows from the fact that the C^k-smooth function $u^*_\eta : X_0 \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ maps into a region on which $\widetilde{\mathbf{R}}_\delta$ is itself C^k-smooth by part (iii) of Theorem 3.7.2. It is easy to see that $f_\delta(0) = 0$ and from (HR2) and the Taylor expansion (3.95), it follows that $Df_\delta(0) = 0$. The fact that Φ is C^{k+1}-smooth follows from repeated differentiation of (3.106).

To conclude the proof, write $\phi = \Phi(0)$ and notice that $\|\phi\| < \epsilon$, which by (iv) of Theorem 3.7.2 implies that $\left\| Q_M \mathrm{ev}_\xi u^*_\eta(\phi) \right\| < \delta$ for all $-2 \leq \xi \leq 2$. This allows us to apply Lemma 3.7.3 to conclude that for all $-1 \leq \xi' \leq 1$ and all $\theta \in \mathbb{R}$, we have

$$\mathrm{ev}_{\xi'+\theta} u^*_\eta(\phi) = \mathrm{ev}_\theta u^*_\eta\big(Q_0 \mathrm{ev}_{\xi'} u^*_\eta(\phi)\big) = \mathrm{ev}_\theta u^*_\eta(\Phi(\xi')). \tag{3.111}$$

Since also $\|\Phi(\xi')\| < \epsilon$ for all $-\xi_0 \leq \xi' \leq \xi_0$, the above identity implies that also $\left\| Q_M \mathrm{ev}_\xi u^*_\eta(\phi) \right\| < \delta$ for all $-\min(\xi_0, 1) - 2 \leq \xi \leq \min(\xi_0, 1) + 2$, implying (3.111) for all $-\min(\xi_0, 1) - 1 \leq \xi' \leq 1 + \min(\xi_0, 1)$. Repeating this procedure a sufficient number of times ensures that in fact (3.111) holds for all $\xi' \in (-\xi_0, \xi_0)$, as required. □

Proof of Theorem 3.2.2. We choose $\delta > \epsilon > 0$ as in the statement of Theorem 3.7.2 and fix the constant $\epsilon^* > 0$ such that $\epsilon^* \max(\|Q_M\|, \|Q_0\|, \|I - Q_M\|) < \epsilon$. Pick any $\eta \in (k\eta_{\min}, \eta_{\max}]$ and write $u^* = u^*_\eta$.

(i) This follows from Theorem 3.7.2 together with $u^* = u^*_\zeta = \mathcal{J}^1_{\zeta\eta_{\min}} u^*_{\eta_{\min}}$ for any $\zeta \in (k\eta_{\min}, \eta_{\max}]$.

(ii) The conditions (HR1) and (HR2) together with (i) imply that f is C^k-smooth with $f(0) = Df(0) = 0$. Since $\xi \mapsto x_\xi$ maps into the subset of X on which $\mathbf{R}$ and $\mathbf{R}_\delta$ agree, it is easy to see that $\mathcal{G}(x, Q_0 x_0) = x$ which due to the uniqueness of fixed points immediately implies $x = u^*(Q_0 x_0)$. An application of Lemma 3.7.3 shows that indeed $x_\xi = \mathrm{ev}_0 u^*(\Phi(\xi))$. Note that for all $\xi \in \mathbb{R}$ we have $\|\Phi(\xi)\| < \epsilon$, which implies that Φ satisfies the ODE (3.106) on the line. It hence suffices to show f and f_δ agree on all $\Phi(\xi)$. This however follows immediately from the fact that $\|Q_M \mathrm{ev}_\theta u^*(\Phi(\xi))\| = \left\| Q_M x_{\xi+\theta} \right\| < \epsilon < \delta$.

(iii) This is clear from the fact that $\xi \mapsto \mathrm{ev}_\xi u^*(\phi)$ maps into the subset of x on which $\mathbf{R}$ and $\mathbf{R}_\delta$ agree.

(iv) Define the function $\Psi(\xi) = Q_0 \mathrm{ev}_\xi u^*(\Phi(0))$. Since $\|\Psi(0)\| = \|\Phi(0)\| < \epsilon$, there exists an interval $(-\xi_0, \xi_0)$ with $\xi_0 > 0$, on which the ODE (3.106) is satisfied for Ψ. However, since f and f_δ agree on the set $\{\phi \in X_0 \mid \|\phi\| < \epsilon\}$ and both nonlinearities are Lipschitz continuous, we can conclude that in fact (3.106) is satisfied on the line, with $\Psi(\xi) = \Phi(\xi)$ for all $\xi \in \mathbb{R}$. Thus defining $x = u^*(\Phi(0))$, we have by construction that $\Phi(\xi) = Q_0 x_\xi$. It remains to show that $\| Q_M x_\xi \| < \delta$ for all $\xi \in \mathbb{R}$ and $x_\xi = \mathrm{ev}_0 u^*(\Phi(\xi))$. Writing $\phi = \Phi(0)$, note that $\|\phi\| < \epsilon$ which implies that $\| Q_M \mathrm{ev}_\xi u^*(\phi) \| < \delta$ for all $-2 \le \xi \le 2$. This allows us to apply Lemma 3.7.3 to conclude that for all $-1 \le \xi' \le 1$ and all $\theta \in \mathbb{R}$, we have

$$\mathrm{ev}_{\xi'+\theta} u^*_\eta(\phi) = \mathrm{ev}_\theta u^*_\eta\big(Q_0 \mathrm{ev}_{\xi'} u^*_\eta(\phi)\big) = \mathrm{ev}_\theta u^*_\eta(\Psi(\xi')) = \mathrm{ev}_\theta u^*_\eta(\Phi(\xi')). \quad (3.112)$$

Arguing as in Proposition 3.7.4 we can extend the conclusions above to $\xi \in \mathbb{R}$ and $\xi' \in \mathbb{R}$, which concludes the proof.

□

Chapter 4

Center Manifolds near Periodic Orbits

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Abstract. We study the behaviour of solutions to nonlinear functional differential equations of mixed type (MFDEs), that remain sufficiently close to a prescribed periodic solution. Under a discreteness condition on the Floquet spectrum, we show that all such solutions can be captured on a finite dimensional invariant center manifold, that inherits the smoothness of the nonlinearity. This generalizes the results that were obtained in Chapter 2 for bifurcations around equilibrium solutions to MFDEs.

4.1. Introduction

In this chapter we provide a tool to analyze the behaviour of solutions to a functional differential equation of mixed type (MFDE),

$$x'(\xi) = G(x_\xi), \tag{4.1}$$

that lie in the vicinity of a prescribed periodic solution. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$, the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \leq 0$ and $r_{\max} \geq 0$, hence the operator G may depend on advanced and retarded arguments simultaneously.

In Chapter 1 we have seen how lattice differential equations present a strong motivation for the study of MFDEs. As a specific example which is interesting in view of our main equation (4.1), we recall a Frenkel-Kontorova type model that was analyzed numerically in [1]. This model was originally developed to describe the motion of dislocations in a crystal

[151, 152], but now has numerous other applications in the literature. In particular, consider a chain of particles that have positions x_k, with $k \in \mathbb{Z}$. The dynamics are given by the LDE

$$\ddot{x}_k(t) + \gamma \dot{x}_k(t) = x_{k-1}(t) - 2x_k(t) + x_{k+1}(t) - d \sin x_k(t) + F, \tag{4.2}$$

in which γ and d are parameters and F is an external applied force. In the literature a special class of travelling wave solutions, which have been named uniform sliding states, has been constructed for (4.2). Such solutions can be written in the form $x_k(t) = \phi(k - ct)$ for some waveprofile ϕ and wavespeed c and in addition satisfy the special condition $x_{k+N} = x_k + 2\pi M$, in which N and M are fixed integers. It is not hard to see that (4.2) can be restated in such a way that these states become periodic and hence the study of bifurcations from these solutions can be fitted into the framework developed here.

In a previous chapter, a center manifold approach was developed to capture all solutions of (4.1) that remain sufficiently close to a given equilibrium $\overline{x}$, based upon earlier work by several authors [45, 112, 159]. It was shown that the dimension and linear structure on the center manifold are entirely determined by the holomorphic characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$ associated to the linearized system $v'(\xi) = DG(\overline{x})v_\xi$. This matrix is explicitly given by $\Delta(z) = zI - DG(\overline{x})\exp(z\cdot)$ and is thus relatively straightforward to construct and analyze in many practical applications, see e.g. Chapter 1 and [41]. As an illustration of the strength of this reduction, consider a parameter dependent family of MFDEs,

$$x'(\xi) = G(x_\xi, \mu), \tag{4.3}$$

that admit a common equilibrium $\overline{x}$. In addition, suppose that a pair of roots of the characteristic equation $\det \Delta(z, \mu) = 0$ crosses the imaginary axis at a certain parameter value μ_0. Under suitable conditions the Hopf bifurcation theorem can be lifted to the infinite dimensional setting of (4.3) and hence one may conclude the existence of a branch of periodic solutions to (4.3) bifurcating from the equilibrium $\overline{x}$ for $\mu \sim \mu_0$. In Section 1.4 this approach was used to analyze an economic optimal control problem involving delays. This problem was proposed by Rustichini in order to simplify a model describing the dynamics of a capital market [131], whilst still retaining the periodic orbits that are compulsory for any such model. The existence of these periodic orbits was established by numerically analyzing the resulting characteristic equation and looking for root-crossings through the imaginary axis.

The main goal of this chapter is to facilitate a similar bifurcation analysis around periodic solutions p to (4.1). In order to do this, we will set out to capture all sufficiently small solutions to the equation

$$y'(\xi) = DG(p_\xi)y_\xi + \big(G(p_\xi + y_\xi) - DG(p_\xi)y_\xi - G(p_\xi)\big) \tag{4.4}$$

on a finite dimensional center manifold, hence generalizing the approach in Chapter 2 for equilibria $p = \overline{x}$. Our results should be seen in the setting of Floquet theory in infinite dimensions. In particular, the linear dynamics and structure on the center manifold are related to Floquet solutions of the linear part of (4.4), i.e., functions v of the form $v(\xi) = e^{\lambda\xi}q(\xi)$ that satisfy

$$v'(\xi) = DG(p_\xi)v_\xi. \tag{4.5}$$

Here q is a periodic function that has the same period as p and $\lambda \in \mathbb{C}$ is called a Floquet exponent. In particular, we will be interested in linearized equations that admit Floquet exponents on the imaginary axis.

In contrast to the autonomous case, the construction and subsequent analysis of a characteristic matrix for (4.5) in general poses a significant challenge. In the study of delay equations, at least two approaches have been developed to deal with this problem. The first approach uses the fact that a delay equation may be seen as an initial value problem on the state space $C([r_{\min}, 0], \mathbb{C}^n)$, which allows one to define a monodromy map on this space. It is possible to show that this map is compact, which immediately implies that the set of Floquet exponents is discrete [45]. Applying the theory developed in [89] to the monodromy map, Szalai et al. were able to construct a characteristic matrix for general periodic delay equations (4.5), which in addition can be used efficiently for numerical computations [153]. However, an explicit form for this matrix can only be given in very special cases. In addition, this approach fails whenever $r_{\max} > 0$, since in general MFDEs are ill-defined as initial value problems [75].

If the operator $DG(p_\xi) : C([r_{\min}, r_{\max}], \mathbb{C}^n) \to \mathbb{C}^n$ can be written in the form

$$DG(p_\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(\xi + r_j) \tag{4.6}$$

and if the sizes of the shifts r_j in (4.6) are all rationally related to the period of p, the Floquet exponents can be studied in a more direct fashion. This is done by substituting $q(\xi) = e^{-\lambda\xi}v(\xi)$ into (4.5) and looking for periodic solutions q. The resulting equation can be transformed into an ODE by introducing new variables $q_k(\xi) = q(\xi + kr_*)$, for some r_* that divides all the shifts r_j. In [149, 164], the authors use this reduction to analyze a scalar delay differential equation with a single delay,

$$x'(\xi) = -\mu x(\xi) + f(x(\xi - 1)), \tag{4.7}$$

in which f is an odd C^1-smooth nonlinearity. In particular, a characteristic matrix was constructed for the Floquet exponents associated to the special class of periodic orbits p that satisfy $p(\xi+r) = -p(\xi)$, for some $r > 0$ and all $\xi \in \mathbb{R}$. Under some additional restrictions on f and p it was possible to explicitly verify the presence of Floquet exponents on the imaginary axis. In general however, such an approach will quickly become intractable. One will hence have to resort to numerical calculations in the spirit of [108, 153] to detect Floquet exponents that cross through the imaginary axis as the parameters of a system are varied.

To state our results we will need to assume that the Floquet spectrum of (4.5) is discrete in some sense. In Section 4.3 we will use the ODE reduction described above to verify this condition in a number of special cases, but at the moment it is unclear if this holds for general MFDEs. Our main results are formulated in Section 4.2 and the necessary linear machinery is developed in Sections 4.4 through 4.6. We remark here that the approach in Section 4.4 was chiefly motivated by the work of Mielke. In [119], he constructed a center manifold to study bifurcations in the setting of elliptic PDEs and hence also had to cope with the absence of a time evolution map. However, we will need to deviate from his approach considerably, for reasons which should become clear in the sequel. In Section

4.7 we use the Lyapunov–Perron technique to define the center manifold and derive the associated flow, much along the lines of [45]. Finally, in Section 4.8 we use techniques developed by Vanderbauwhede and van Gils [159] to address the smoothness of the center manifold.

4.2. Main Results

Consider the following functional differential equation of mixed type,

$$x'(\xi) \quad = \quad L(\xi)x_\xi + R(\xi, x_\xi), \quad \xi \in \mathbb{R}, \tag{4.8}$$

in which x is a continuous mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$ and the operators $L(\xi)$ and $R(\xi, \cdot)$ are a linear respectively nonlinear map from the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ into $\mathbb{C}^n$. The state $x_\xi \in X$ is defined by $x_\xi(\theta) = x(\xi + \theta)$ for any $r_{\min} \leq \theta \leq r_{\max}$, with $r_{\min} \leq 0 \leq r_{\max}$. Furthermore, we require throughout this chapter that L and R are periodic, in the sense that $L(\xi + 2\pi)\phi = L(\xi)\phi$ and $R(\xi + 2\pi, \phi) = R(\xi, \phi)$ for all $\xi \in \mathbb{R}$ and $\phi \in X$. For ease of notation, we will present our results for (4.8) under the assumption that L acts on point delays only, i.e., we assume that for some integer N the operator $L(\xi) : X \to \mathbb{C}^n$ can be written in the form

$$L(\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(r_j), \tag{4.9}$$

for $\mathbb{C}^{n\times n}$-valued functions A_j and shifts $r_{\min} = r_0 < r_1 \ldots < r_N = r_{\max}$. We remark however that the arguments developed here can easily be extended to arbitrary $L(\xi) : X \to \mathbb{C}^n$.

As in Chapter 2, we will employ the following families of Banach spaces,

$$\begin{array}{rcl} BC_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in C(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}\, |x(\xi)| < \infty\right\}, \\ BC^1_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in BC_\eta(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n) \mid x' \in BC_\eta(\mathbb{R}, \mathbb{C}^n)\right\}, \end{array} \tag{4.10}$$

parametrized by $\eta \in \mathbb{R}$, with the standard norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \left\|x'\right\|_\eta$. Notice that for any pair $\eta_2 \geq \eta_1$, there exist continuous inclusions $\mathcal{J}_{\eta_2\eta_1} : BC_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$ and $\mathcal{J}^1_{\eta_2\eta_1} : BC^1_{\eta_1}(\mathbb{R}, \mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R}, \mathbb{C}^n)$.

An essential step towards understanding the behaviour of (4.8) is the study of the homogeneous linear equation

$$x'(\xi) = L(\xi)x_\xi. \tag{4.11}$$

In particular, we are interested in the special class of solutions to (4.11) that can be written in the form $x(\xi) = e^{\lambda\xi}p(\xi)$ with $p \in C^{\mathrm{per}}_{2\pi}(\mathbb{R}, \mathbb{C}^n)$, i.e., p is a periodic continuous function with $p(\xi + 2\pi) = p(\xi)$ for all $\xi \in \mathbb{R}$. The parameter $\lambda \in \mathbb{C}$ is called a Floquet exponent for (4.11) if and only if any such solution exists. We need to impose the following restrictions on (4.11).

(HL) The map $\mathbb{R} \to \mathcal{L}(X, \mathbb{C}^n)$ given by $\xi \mapsto L(\xi)$ is of class C^r, for some integer $r \geq 3$.

(HF) There exist $\gamma_- < 0$ and $\gamma_+ > 0$ such that (4.11) has no Floquet exponents $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \in \{\gamma_-, \gamma_+\}$.

When studying delay equations, which in our context means $r_{\max} = 0$, one can show that (HF) is always satisfied [72]. However, the proof requires the existence of an evolution map defined on the entire state space X and hence fails to work when $r_{\max} > 0$. At the moment, it is unclear if equations (4.11) exist for which (HF) fails. However, in Section 4.3 we give some criteria which will help establish (HF) in the case where all the shifts r_j appearing in (4.9) are rationally related to the period 2π.

The following proposition, which will be proved throughout Sections 4.5 and 4.6, exhibits the finite dimensional space X_0 on which the center manifold will be defined.

Proposition 4.2.1. *Consider any homogeneous linear equation (4.11) that satisfies the conditions (HL) and (HF) and pick a constant γ with $0 < \gamma < |\gamma_\pm|$, in which $\gamma_\pm$ are as introduced in (HF). Then there exists a finite dimensional linear subspace $X_0 \subset X$, a C^r-smooth operator $\Pi : \mathbb{R} \to \mathcal{L}(X, X_0)$ and a matrix $W \in \mathcal{L}(X_0)$, such that the following properties hold.*

(i) Suppose $x \in \bigcap_{\eta > 0} BC_\eta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (4.11). Then for any $\ell \in \mathbb{Z}$ we have $x_{2\pi\ell} \in X_0$.

(ii) For any $\phi \in X_0$, there is a unique solution $x = E\phi \in BC_\gamma(\mathbb{R}, \mathbb{C}^n)$ of (4.11) such that $x_0 = \phi$. Moreover, we have that $x \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ for any $\eta > 0$.

(iii) For any $\phi \in X_0$ we have $\Pi(\xi)(E\phi)_\xi = e^{\xi W}\phi$.

We also need to impose the following assumptions on the periodic nonlinearity R, after which we are ready to state our main results.

(HR1) The nonlinearity R is C^k-smooth as a function $\mathbb{R} \times X \to \mathbb{C}^n$, for some integer $k \geq 2$.

(HR2) For all $\xi \in \mathbb{R}$ we have $R(\xi, 0) = 0$ and $D_2 R(\xi, 0) = 0$.

Theorem 4.2.2. *Consider the nonlinear equation (4.8) and assume that (HL), (HF), (HR1) and (HR2) are satisfied. Then there exists $\gamma > 0$ such that (4.11) has no Floquet exponents λ with $0 < |\operatorname{Re} \lambda| < \gamma$. Fix an interval $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ such that $\eta_{\max} > \min(r, k)\eta_{\min}$, with r and k as introduced in (HL) and (HR2). Then there exists a mapping $u^* : X_0 \times \mathbb{R} \to \bigcap_{\eta > 0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ and constants $\varepsilon > 0$, $\varepsilon^* > 0$ such that the following statements hold.*

(i) For any $\eta \in (\min(r, k)\eta_{\min}, \eta_{\max}]$, the function u^* viewed as a map from $X_0 \times \mathbb{R}$ into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is $C^{\min(r,k)}$-smooth.

(ii) Suppose for some $\zeta > 0$ that $x \in BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (4.8) with $\sup_{\xi \in \mathbb{R}} |x(\xi)| < \varepsilon^*$. Then we have $x = u^*(\Pi(0)x_0, 0)$. In addition, the function

$\Phi : \mathbb{R} \to X_0$ defined by $\Phi(\xi) = \Pi(\xi)x_\xi \in X_0$ is of class $C^{\min(r,k+1)}$ and satisfies the ordinary differential equation

$$\Phi'(\xi) = W\Phi(\xi) + f(\xi, \Phi(\xi)). \tag{4.12}$$

Here the function $f : \mathbb{R} \times X_0 \to X_0$ is $C^{\min(r-1,k)}$-smooth with $f(\xi, 0) = 0$ and $Df(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$. Furthermore, it is periodic in the first variable, with $f(\xi + 2\pi, \psi) = f(\xi, \psi)$ for all $(\xi, \psi) \in \mathbb{R} \times X_0$ and given explicitly by

$$\begin{array}{rcl} f(\xi, \psi) & = & [D\Pi(\xi)][u^*(\psi, \xi) - Ee^{-\xi W}\psi]_\xi \\ & & +\Pi(\xi)\chi^1(\psi, \xi) \\ & & +\Pi(\xi)\chi^2(\psi, \xi). \end{array} \tag{4.13}$$

Here the states $\chi^i(\psi, \xi) \in X$, for $i = 1, 2$, are defined as

$$\begin{array}{rcl} \chi^1(\psi, \xi)(\sigma) & = & L(\xi + \sigma)[u^*(\psi, \xi) - Ee^{-\xi W}\psi]_{\xi+\sigma} \\ \chi^2(\psi, \xi)(\sigma) & = & R(\xi + \sigma, (u^*(\psi, \xi))_{\xi+\sigma}). \end{array} \tag{4.14}$$

Finally, we have $x_\xi = \left(u^*(\Phi(\xi), \overline{\xi})\right)_{\overline{\xi}}$ for any pair $\xi, \overline{\xi} \in \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

(iii) For any $\phi \in X_0$ such that $\sup_{\xi\in\mathbb{R}} |u^*(\phi, 0)(\xi)| < \varepsilon^*$, the function $u^*(\phi, 0)$ satisfies (4.8).

(iv) For any continuous function $\Phi : \mathbb{R} \to X_0$ that satisfies (4.12) with $\|\Phi(\xi)\| < \varepsilon$ for all $\xi \in \mathbb{R}$, we have that $x = u^*(\Phi(0), 0)$ is a solution of (4.8). In addition, we have $x_\xi = \left(u^*(\Phi(\xi), \overline{\xi})\right)_{\overline{\xi}}$ for any pair $\xi, \overline{\xi} \in \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

(v) Consider the interval $I = (\xi_-, \xi_+)$, where $\xi_- = -\infty$ and $\xi_+ = \infty$ are allowed. Let $\Phi : I \to X_0$ be a continuous function that satisfies (4.12) for every $\xi \in I$ and in addition has $\|\Phi(\xi)\| < \varepsilon$ for all such ξ. Then for any $\zeta \in (\xi_-, \xi_+)$ we have that $x = u^*(\Phi(\zeta), \zeta)$ satisfies (4.8) for all $\xi \in I$. In addition, we have $x_\xi = \left(u^*(\Phi(\xi), \overline{\xi})\right)_{\overline{\xi}}$ for any pair $(\xi, \overline{\xi}) \in I \times \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

4.3. Preliminaries

In addition to the spaces $BC_\eta(\mathbb{R}, \mathbb{C}^n)$ that contain continuous functions, we introduce two extra families of Banach spaces, parametrized by $\mu, \nu \in \mathbb{R}$, that contain distributions that have controlled exponential growth at $\pm\infty$,

$$\begin{array}{rcl} BX_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) & = & \Big\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX_{\mu,\nu}} := \sup_{\xi<0} e^{-\mu\xi} |x(\xi)| \\ & & \quad + \sup_{\xi\geq 0} e^{-\nu\xi} |x(\xi)| < \infty\Big\}, \\ BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) & = & \Big\{x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \mid \|x\|_{BX^1_{\mu,\nu}} := \|x\|_{BX_{\mu,\nu}} \\ & & \quad + \left\|x'\right\|_{BX_{\mu,\nu}} < \infty\Big\}. \end{array} \tag{4.15}$$

In order to improve the readability of our arguments, we also introduce the notation $\mathrm{ev}_\xi x = x_\xi \in X$ for any $x \in C(\mathbb{R}, \mathbb{C}^n)$ and $\xi \in \mathbb{R}$, together with the shift operators T_ξ defined by $T_\xi f(\cdot) = f(\cdot + \xi)$, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$.

Consider the linear operator $L(\xi) : X \to \mathbb{C}^n$ appearing in (4.8). One may split this operator into an autonomous part and a periodic part, i.e., write $L(\xi) = L_{\mathrm{aut}} + L_{\mathrm{per}}(\xi)$ with

$$\begin{array}{rcl} L_{\mathrm{aut}}\phi & = & \sum_{j=0}^N A^j_{\mathrm{aut}}\phi(r_j), \\ L_{\mathrm{per}}(\xi)\phi & = & \sum_{j=0}^N B^j(\xi)\phi(r_j). \end{array} \tag{4.16}$$

We recall the characteristic matrix $\Delta(z) = zI - \sum_{j=0}^N A^j_{\mathrm{aut}} e^{zr_j}$ associated to L_{aut} and repeat some useful properties of Δ that were established in Chapter 2.

Lemma 4.3.1. *Consider any closed vertical strip $S = \{z \in \mathbb{C} \mid a_- \le \operatorname{Re} z \le a_+\}$ and for any $\rho > 0$ define $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. Then there exist $C, \rho > 0$ such that* $\det \Delta(z) \neq 0$ *for all $z \in S_\rho$ and in addition $\left|\Delta(z)^{-1}\right| < \frac{C}{|\operatorname{Im} z|}$ for each such z. In particular, there are only finitely many zeroes of* $\det \Delta(z)$ *in S.*

Notice that the splitting (4.16) is obviously ambiguous, in the sense that L_{aut} can be chosen freely. We will use this freedom to ensure that the characteristic equation $\det \Delta(z) = 0$ has no roots in a small strip around the imaginary axis, which will allow us to solve linear systems of the form

$$x'(\xi) = L(\xi)x_\xi + f(\xi), \tag{4.17}$$

for suitable classes of inhomogeneities f. As a final matter of notation, for any function x we will write Lx to represent the function $\xi \mapsto L(\xi)x_\xi$.

We conclude this section by discussing the assumption (HF) concerning the Floquet exponents for the system (4.11). We provide a number of results with which this criterion can be verified.

Lemma 4.3.2. *Consider any system of the form (4.11) that has the property that all the shifts are rationally related to the period, i.e., we have $r_j \in \pi\mathbb{Q}$ for all $0 \le j \le N$. Then either every $\lambda \in \mathbb{C}$ is a Floquet exponent, or (HF) is satisfied.*

Proof. Choose $r^* = \frac{2\pi}{M} \in \mathbb{R}$ such that for some numbers $m_j \in \mathbb{Z}$ we have $r_j = m_j r^*$ for all $0 \le j \le N$. Suppose that $\lambda \in \mathbb{C}$ is a Floquet exponent and let $p \in C^{\mathrm{per}}_{2\pi}(\mathbb{R}, \mathbb{C}^n)$ be the corresponding nontrivial periodic function, such that $u' = Lu$ for $u : \xi \mapsto e^{\lambda\xi}p(\xi)$. Associated to p we introduce the $(\mathbb{C}^n)^M$-valued function $\mathbf{p}$, the $\mathbb{C}^n$-valued components of which are defined by $\mathbf{p}_i(\xi) = p(\xi + ir^*)$ for $0 \le i \le M-1$. Since p is periodic, it is clear that $\mathbf{p}_i(r_*) = \mathbf{p}_{i+1 \bmod M}(0)$ for all $0 \le i \le M-1$, which can be reformulated as $\mathbf{p}(r_*) = I_n \otimes \mathcal{T}\mathbf{p}(0)$, in which I_n is the $n \times n$ identity matrix and the $M \times M$-matrix $\mathcal{T}$ is defined by $\mathcal{T}_{i,j} = \delta_{j, i+1 \bmod M}$. After an appropriate shift one may assume $\mathbf{p}(0) \neq 0$. Furthermore, a quick calculation shows that $\mathbf{p}$ satisfies the ODE

$$\mathbf{p}'(\xi) = \mathcal{F}(\xi, \lambda)\mathbf{p}(\xi), \tag{4.18}$$

in which $\mathcal{F}$ is given by

$$(\mathcal{F}(\xi, \lambda)\mathbf{v})_i = -\lambda\mathbf{v}_i + \sum_{j=0}^N e^{\lambda r_j} A_j(\xi + ir^*)\mathbf{v}_{i+m_j \bmod M}. \tag{4.19}$$

Writing $\boldsymbol{\Omega}(\xi, \lambda)$ for the fundamental matrix for the ODE (4.18), we have $\boldsymbol{\Omega}(r_*, \lambda)\mathbf{p}(0) = I_n \otimes \mathcal{T}\mathbf{p}(0)$ and hence

$$\det[\boldsymbol{\Omega}(r_*, \lambda) - I_n \otimes \mathcal{T}] = 0. \tag{4.20}$$

Since the coefficients of the ODE (4.18) depend analytically on $\lambda \in \mathbb{C}$, it follows that for any fixed $\xi \in \mathbb{R}$ the matrix $\boldsymbol{\Omega}(\xi, \cdot)$ is an entire function in the second variable [46, Section 10.7]. This however implies that either (4.20) is satisfied for all $\lambda \in \mathbb{C}$, or the set of solutions is discrete. To complete the proof, observe that $\lambda \in \mathbb{C}^n$ is a Floquet exponent if and only if $\lambda + i$ is a Floquet exponent, which means that the set of real parts of Floquet exponents is discrete whenever the set of Floquet exponents is discrete. □

In some special cases we can get extra information on the fundamental matrix $\boldsymbol{\Omega}$ and show that not all $\lambda \in \mathbb{C}$ can be Floquet exponents.

Corollary 4.3.3. *Consider any scalar system of the form (4.11) that has the property that all shifts are integer multiples of the period, i.e., we have $r_j \in 2\pi\mathbb{Z}$ for all $0 \leq j \leq N$. Then (HF) is satisfied.*

Proof. In this case (4.18) is scalar and the fundamental matrix reduces to $\boldsymbol{\Omega}(2\pi, \lambda) = \exp[-2\pi\lambda + \sum_{j=0}^{N} e^{\lambda r_j} \int_0^{2\pi} A_j(\sigma)d\sigma]$, hence the set of roots of (4.20) is discrete. □

Corollary 4.3.4. *Consider any system of the form (4.11) that has the property that all shifts are rationally related to the period, i.e., we have $r_j \in \pi\mathbb{Q}$ for all $0 \leq j \leq N$. Suppose that there exists a vector $\mathbf{v} \in (\mathbb{C}^n)^M$ that is an eigenvector for the adjoint matrix $\mathcal{F}^*(\xi, \lambda)$ for all ξ and all λ, with $\mathcal{F}$ and r_* as given in (4.19). Then (HF) is satisfied.*

Proof. Observe that the complex conjugate of the eigenvalue $\mu = \mu(\xi, \lambda)$ corresponding to the eigenvector $\mathbf{v}$ of $\mathcal{F}^*(\xi, \lambda)$ is given by

$$\mu^* = -\lambda + P(\xi)(\exp(\lambda r_*)) + Q(\xi)(\exp(-\lambda r_*)), \tag{4.21}$$

in which $P(\xi)(\cdot)$ and $Q(\xi)(\cdot)$ are polynomials for every $\xi \in \mathbb{R}$, with $P(\xi + 2\pi) = P(\xi)$ and $Q(\xi + 2\pi) = Q(\xi)$. Introducing the scalar function $q(\xi) = \mathbf{v}^*\mathbf{p}$, we may now calculate

$$\begin{array}{rcl} q'(\xi) & = & \mathbf{v}^*\mathcal{F}(\xi, \lambda)\mathbf{p} = \mu^*\mathbf{v}^*\mathbf{p} \\ & = & -\lambda q(\xi) + \big[P(\xi)(\exp(\lambda r_*)) + Q(\xi)(\exp(-\lambda r_*))\big]q(\xi). \end{array} \tag{4.22}$$

This means that

$$\begin{array}{rcl} q(2\pi) & = & \exp[-2\pi\lambda + \int_0^{2\pi} P(\sigma)(\exp(\lambda r_*))d\sigma + \int_0^{2\pi} Q(\sigma)(\exp(-\lambda r_*))d\sigma]q(0) \\ & = & q(0), \end{array} \tag{4.23}$$

which concludes the proof. □

As an example to illustrate the result above, consider the equation

$$x'(\xi) = \sin(\xi)x(\xi - \pi) + \sin(\xi)x(\xi + \pi). \tag{4.24}$$

If $\lambda \in \mathbb{C}$ is a Floquet exponent for (4.24) with corresponding scalar $p \in C_{2\pi}^{\text{per}}(\mathbb{R}, \mathbb{C}^n)$, then the $\mathbb{R}^2$-valued function $\mathbf{p}(\xi) = (p_0(\xi), p_1(\xi)) = (p(\xi), p(\xi + \pi))$ satisfies the system

$$\begin{array}{rcl} p_0'(\xi) & = & -\lambda p_0(\xi) + \sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]p_1(\xi), \\ p_1'(\xi) & = & -\lambda p_1(\xi) - \sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]p_0(\xi). \end{array} \tag{4.25}$$

Writing $q(\xi) = p_0(\xi) + ip_1(\xi)$, we find that q solves the scalar ODE

$$q'(\xi) = -\lambda q(\xi) - i\sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]q(\xi) \tag{4.26}$$

and satisfies $q(0) = q(2\pi)$. Using the variation-of-constants formula for q it is clear that (HF) must be satisfied.

4.4. Linear inhomogeneous equations

We introduce the linear operator $\Lambda : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$, given by

$$(\Lambda x)(\xi) = x'(\xi) - L(\xi)x_\xi. \tag{4.27}$$

In this section we set out to solve equations of the form $\Lambda x = f$ and hence define an inverse for Λ. Using Fourier transform techniques, we will first show that Λ is invertible when considered as an operator from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ into $L^2(\mathbb{R}, \mathbb{C}^n)$. This result can then be extended to compute $\Lambda^{-1}f$ for $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$.

Due to the periodicity of $L(\xi)$, the transform $\mathcal{F}^+[Lx](\eta_0)$ will only involve $\hat{x}(\eta_1)$ when $\eta_1 - \eta_0 \in \mathbb{Z}$. It will hence be fruitful to follow the approach employed by Mielke [119] and introduce the sequence space

$$\ell_2 = \big\{w = (w_k)_{k\in\mathbb{Z}} \mid w_k \in \mathbb{C}^n \text{ and } \|w\|_2^2 := \sum_{k\in\mathbb{Z}} |w_k|^2 < \infty\big\}. \tag{4.28}$$

Recalling the splitting (4.16), we need to solve

$$x'(\xi) = \sum_{j=0}^N A^j_{\text{aut}} x(\xi + r_j) + \sum_{j=0}^N B^j(\xi)x(\xi + r_j) + f(\xi). \tag{4.29}$$

Since $B^j \in C^r(\mathbb{R}, \mathbb{C}^{n\times n})$ with $B^j(\xi + 2\pi) = B^j(\xi)$, we may write

$$B^j(\xi) \quad = \quad \textstyle\sum_{k=-\infty}^{\infty} B^j_k e^{ik\xi}, \tag{4.30}$$

in which the coefficients satisfy the estimate

$$\left|B^j_k\right| \le C/(1 + |k|)^r, \tag{4.31}$$

for some $C > 0$. For any $0 \le j \le N$, define the convolution operator $\widetilde{B}^j : \ell_2 \to \ell_2$ by

$$(\widetilde{B}^j w)_n \quad = \quad \textstyle\sum_{k\in\mathbb{Z}} e^{i(n-k)r_j} B_k^j w_{n-k}, \tag{4.32}$$

together with $B_z : \ell_2 \to \ell_2$ given by $B_z = \sum_{j=0}^{N} e^{zr_j} \widetilde{B}^j$. To see that $\widetilde{B}^j$ is well-defined and bounded, use the Cauchy-Schwartz inequality and the estimate (4.31) to compute

$$\begin{array}{rcl} \|\widetilde{B}^j w\|_2^2 & = & \sum_{n\in\mathbb{Z}} \left|\sum_{m\in\mathbb{Z}} e^{i(n-m)r_j} B_m^j w_{n-m}\right|^2 \\ & \le & \sum_{n\in\mathbb{Z}} \big[\sum_{m\in\mathbb{Z}} \left|B_m^j\right|^{\frac{1}{2}} \left|B_m^j\right|^{\frac{1}{2}} |w_{n-m}|\big]^2 \\ & \le & \sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} \left|B_m^j\right|)(\sum_{m\in\mathbb{Z}} \left|B_m^j\right| |w_{n-m}|^2) \\ & = & (\sum_{m\in\mathbb{Z}} \left|B_m^j\right|)^2 \|w\|_2^2 . \end{array} \tag{4.33}$$

Note that it is possible to choose L_{aut} in such a way that $\det \Delta(z) = 0$ has no roots in a strip $|\text{Re}\, z| < \varepsilon$. For any such z, we can hence define a multiplication operator $\Delta_z : \ell_2 \to \ell_2$ by

$$(\Delta_z w)_n \quad = \quad \Delta(z + in)^{-1} w_n. \tag{4.34}$$

We claim that Δ_z is compact. To see this, consider any bounded sequence $\{w^n\}_{n\in\mathbb{N}} \subset \ell_2$, write $v^n = \Delta_z w^n$ and use a diagonal argument to pass to a subsequence for which each component v_k^n converges as $n \to \infty$. For any $K > 0$ we find

$$\left\|v^n - v^m\right\|_2^2 \le \sum_{|k|<K} \left|v_k^n - v_k^m\right|^2 + (1+K)^{-1} \sum_{|k|\ge K} (1+|k|) \left|v_k^n - v_k^m\right|^2 . \tag{4.35}$$

Fixing any $\varepsilon > 0$ and noting that the estimate in Lemma 4.3.1 implies that the second sum can be bounded independently of K, n and m, we can choose $K > 0$ sufficiently large to ensure that the entire second term on the righthand side of (4.35) is bounded by $\varepsilon/2$. Similarly, for such a choice of K we can choose a $M > 0$ such that the first term is bounded by $\varepsilon/2$ for any $n \ge M$ and $m \ge M$, which shows that Δ_z is indeed compact.

For any $\tau \in \mathbb{R}$, consider a function $f : \tau + i\mathbb{R} \to \mathbb{C}^n$ such that $\eta \mapsto f(\tau + i\eta) \in L^2(\mathbb{R}, \mathbb{C}^n)$. For any complex z with $\text{Re}\, z = \tau$, define the sequence $(\mathcal{J}_z f)_k = f(z + ik)$. Notice that for almost all such z, we have $\mathcal{J}_z f \in \ell_2$. Finally, for any $w \in \ell_2$, define $\text{ev}_n w = w_n \in \mathbb{C}^n$ and $(T_n w)_k = w_{k+n}$. With these preparations we are ready to provide the inverse $\Lambda^{-1} f$ for $f \in L^2(\mathbb{R}, \mathbb{C}^n)$.

Proposition 4.4.1. *Suppose that (4.29) admits no Floquet exponents λ with* $\text{Re}\,\lambda = 0$. *Then Λ is an isomorphism from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ onto $L^2(\mathbb{R}, \mathbb{C}^n)$, with inverse given by*

$$\Lambda^{-1} f = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{z\xi} \text{ev}_0 [I - \Delta_z B_z]^{-1} \Delta_z \mathcal{J}_z [\widetilde{f}_+(\cdot) + \widetilde{f}_-(\cdot)] dz. \tag{4.36}$$

In addition, there exists a Green's function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{n\times n}$ such that for every $\xi \in \mathbb{R}$, the function $G(\xi, \cdot) \in L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ satisfies (4.29) in the sense of distributions, with $f(\xi') = \delta(\xi' - \xi)I$. In addition, $G(\xi, \cdot)$ is bounded, admits a jump $G(\xi, \xi+) - G(\xi, \xi-) = I$ and is C^1-smooth on $\mathbb{R} \setminus \{\xi\}$.

Proof. First consider any sequence $w \in \ell_2$ such that $w = \Delta_z B_z w$. Then the function $u(\xi) = e^{z\xi} \sum_{n\in\mathbb{Z}} e^{in\xi} w_n$ satisfies $\Lambda u = 0$. In addition, since $(nw_n) \in \ell_2$, we have that the periodic function $p(\xi) = e^{-z\xi} u(\xi)$ satisfies $p \in W^{1,2}([0, 2\pi], \mathbb{C}^n) \subset C([0, 2\pi], \mathbb{C}^n)$. We hence conclude that z must be a Floquet exponent. Due to the absence of Floquet exponents on the imaginary axis, the Fredholm alternative now implies that $\mathbf{1} - \Delta_z B_z$ is invertible as a map from ℓ_2 onto ℓ_2 for all $z \in i\mathbb{R}$. Since both $z \mapsto \Delta_z$ and $z \mapsto B_z$ are continuous, the same holds for $z \mapsto [\mathbf{1} - \Delta_z B_z]^{-1}$. Notice in addition that one has $B_{z+i} = T_1 B_z T_{-1}$, together with a similar identity for Δ_z. This implies that the norm $\left\|[\mathbf{1} - \Delta_z B_z]^{-1}\right\|$ can be bounded independently of z for $z \in i\mathbb{R}$.

Taking the Fourier transform of (4.29), we arrive at

$$\Delta(i\eta)\hat{x}(\eta) = \hat{f}(\eta) + \sum_{j=0}^{N} \sum_{k=-\infty}^{\infty} e^{i(\eta-k)r_j} B_k^j \hat{x}(\eta - k). \tag{4.37}$$

This identity can be inverted by introducing the sequence $\hat{f}^\theta \in \ell_2$ via $\hat{f}^\theta_n = \hat{f}(\theta+n)$ where this is well-defined and choosing, for $\theta \in [0, 1)$ and $n \in \mathbb{Z}$,

$$\hat{x}(\theta + n) = \mathrm{ev}_n[\mathbf{1} - \Delta_{i\theta} B_{i\theta}]^{-1} \Delta_{i\theta} \hat{f}^\theta. \tag{4.38}$$

It remains to show that $\hat{x}$ thus constructed is in fact an L^2 function. We calculate

$$\begin{array}{rcl} \int_{-\infty}^{\infty} \left|\hat{x}(\eta)\right|^2 d\eta & = & \int_0^1 \sum_{n\in\mathbb{Z}} \left|\hat{x}(\theta + n)\right|^2 d\theta \\ & = & \int_0^1 \left\|[\mathbf{1} - \Delta_{i\theta} B_{i\theta}]^{-1} \Delta_{i\theta} \hat{f}^\theta\right\|_2^2 d\theta \\ & \leq & C \int_0^1 \left\|\Delta_{i\theta} \hat{f}^\theta\right\|_2^2 d\theta = C \int_{-\infty}^{\infty} \left|\Delta(i\eta)^{-1} \hat{f}(\eta)\right|^2 d\eta \\ & \leq & C' \left\|\hat{f}\right\|_2 . \end{array} \tag{4.39}$$

In addition, using (4.37) together with the estimate (4.31) it follows that $\eta \mapsto \eta\hat{x}(\eta)$ is an L^2 function, from which we conclude $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$, as desired. To show that Λ is injective, consider any $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$. There exists a $\theta \in \mathbb{R}$ such that $\hat{x}^\theta \in \ell_2$ with $\hat{x}^\theta \neq 0$ and using (4.37) it follows that $i\theta$ must be Floquet exponent, which yields a contradiction.

Without loss of generality, we will prove the statements concerning the Green's function G only for $\xi = 0$. To this end, note that the construction above remains valid if we take $\widehat{f} = 1$ and $G(0, \cdot) = \Lambda^{-1}\delta(\cdot)$. The only modification that is required is the last inequality in (4.39), which can be replaced by

$$\int_{-\infty}^{\infty} \left|\widehat{G}(\eta)\right|^2 d\eta \leq C \int_{-\infty}^{\infty} \left|\Delta^{-1}(i\eta)\right|^2 d\eta = C \left\|\Delta^{-1}(i\cdot)\right\|_{L^2} \leq C'. \tag{4.40}$$

In view of this, we merely have $G(0, \cdot) \in L^2(\mathbb{R}, \mathbb{C}^{n\times n})$, but using the differential equation we find $G(0, \cdot) \in W^{1,2}((0, \infty), \mathbb{C}^{n\times n}) \cup W^{1,2}((-\infty, 0), \mathbb{C}^{n\times n})$ and hence $G(0, \cdot)$ is C^1-smooth on $\mathbb{R} \setminus \{0\}$, as required. The remaining properties also follow from the distributional differential equation that G satisfies. □

Since we are specially interested in situations where (4.29) does admit Floquet exponents λ with $\operatorname{Re}\lambda = 0$, we will need a tool to shift such exponents off the imaginary axis. To this end, we introduce the notation $e_\nu f = e^{\nu\cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$. In addition, for $\eta \in \mathbb{R}$ we define the shifted linear operator $\Lambda_\eta : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, by

$$(\Lambda_\eta x)(\xi) = x'(\xi) - \eta x(\xi) - \sum_{j=0}^{N} [A^j_{\mathrm{aut}} + B^j(\xi)] e^{-\eta r_j} x(\xi + r_j). \tag{4.41}$$

One may check that $e_\eta \Lambda e_{-\eta} x = \Lambda_\eta x$ and hence for any Floquet exponent λ associated to Λ_η, one has that $\lambda + \eta$ is a Floquet exponent associated to Λ.

In view of these observations, we introduce, for any $\eta \in \mathbb{R}$ and $p \in \{2, \infty\}$, the Banach spaces

$$\begin{array}{lcl} L^p_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in L^p(\mathbb{R}, \mathbb{C}^n) \right\}, \\ W^{1,p}_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in W^{1,p}(\mathbb{R}, \mathbb{C}^n) \right\}, \end{array} \tag{4.42}$$

with norms given by $\|x\|_{L^p_\eta} = \left\| e_{-\eta} x \right\|_{L^p}$ and similarly $\|x\|_{W^{1,p}_\eta} = \left\| e_{-\eta} x \right\|_{W^{1,p}}$. The following result now follows immediately from Proposition 4.4.1.

Corollary 4.4.2. *Suppose that (4.29) admits no Floquet exponents λ with* $\operatorname{Re}\lambda = \eta$. *Then Λ is an isomorphism from $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^2_\eta(\mathbb{R}, \mathbb{C}^n)$, with inverse given by*

$$\Lambda^{-1} f = \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} e^{z\xi} \mathrm{ev}_0 [I - \Delta_z B_z]^{-1} \Delta_z \mathcal{J}_z [\widetilde{f}_+(\cdot) + \widetilde{f}_-(\cdot)] dz. \tag{4.43}$$

In addition, there exists a Green's function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{n \times n}$ such that for every $\xi \in \mathbb{R}$, the function $G(\xi, \cdot) \in L^2_\eta(\mathbb{R}, \mathbb{C}^{n\times n})$ satisfies (4.29) in the sense of distributions, with $f(\xi') = \delta(\xi' - \xi) I$. In addition, $e_{-\eta} G(\xi, \cdot)$ is bounded, while $G(\xi, \cdot)$ admits a jump $G(\xi, \xi+) - G(\xi, \xi-) = I$ and is C^1-smooth on $\mathbb{R} \setminus \{\xi\}$.

In order to avoid confusion, we will write $\Lambda^{-1}_{(\eta)}$ for the inverse of Λ when considered as a map from $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^2_\eta(\mathbb{R}, \mathbb{C}^n)$ and similarly $G_{(\eta)}$ for the corresponding Green's function. In the next section we will use these inverses to construct $\Lambda^{-1} f$ for $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$, by writing f as a sum of two functions in $L^2_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ for appropriate $\eta \in \mathbb{R}$. The next result paves the road for this approach, by showing that $\Lambda^{-1} f$ respects the growth rate of f. As a preparation, we emphasize that on the space $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ one can also define a norm $\|x\|^2_{\widetilde{W}^{1,2}_\eta} := \left\| e_{-\eta} x \right\|_2^2 + \left\| e_{-\eta} x' \right\|_2^2$, which is equivalent to the norm $\|x\|_{W^{1,2}_\eta}$ defined above.

Proposition 4.4.3. *Consider any $\eta \in \mathbb{R}$ and $\varepsilon > 0$ such that (4.29) admits no Floquet exponents λ with* $\operatorname{Re}\lambda \in \{\eta - \varepsilon, \eta + \varepsilon\}$ *and assume that $\Lambda^{-1}_{(\eta+\varepsilon)} g = \Lambda^{-1}_{(\eta-\varepsilon)} g$ for all $g \in L^2_{\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$. Then for any $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, we have $\Lambda^{-1}_{(\eta+\varepsilon)} f \in W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$, with a similar result for $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$.*

Proof. Our arguments here are an adaptation of those presented by Mielke in [119] for elliptic PDEs. Without loss of generality, we will assume that $\eta = 0$ and that time has been rescaled to ensure that $L(\xi)$ has period one. Now consider any $f \in L^\infty(\mathbb{R}, \mathbb{C}^n) \cap L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n)$ and define $x = \Lambda^{-1} f \in W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n)$.

For any $n \in \mathbb{Z}$, let χ_n denote the indicator function for the interval $[n, n+1]$. Writing $f_n = \chi_n f$, we see that $f_n \in L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n) \cap L^2_{-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, with $\sum_{n \in \mathbb{Z}} f_n \to f$ in $L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n)$. We can hence define $x_n = \Lambda^{-1} f_n$ and observe that $x_n \in W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n) \cap W^{1,2}_{-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, again with $\sum_{n \in \mathbb{Z}} x_n \to x$ in $W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n)$. The periodicity of the system (4.29) and the rescaling of time ensure that T_n and Λ^{-1} commute, i.e., $T_n \Lambda^{-1}_{(\pm\varepsilon)} = \Lambda^{-1}_{(\pm\varepsilon)} T_n$. We can exploit this fact to compute

$$
\begin{array}{rcl}
\|x_n\|_{W^{1,2}([m,m+1])} & = & [\int_m^{m+1} x_n(\xi)^2 + x_n'(\xi)^2 d\xi]^{1/2} \\
& = & [\int_{m-n}^{m-n+1} x_n(\xi+n)^2 + x_n'(\xi+n)^2 d\xi]^{1/2} \\
& \leq & [\int_{m-n}^{m-n+1} (x_n(\xi+n)e^{\varepsilon\xi})^2 + (e^{\varepsilon\xi} x_n'(\xi+n))^2 d\xi]^{1/2} e^{-\varepsilon(m-n)} \\
& \leq & e^{-\varepsilon(m-n)} \|T_n x_n\|_{\widetilde{W}^{1,2}_{-\varepsilon}} \leq C_\varepsilon e^{-\varepsilon(m-n)} \|T_n x_n\|_{W^{1,2}_{-\varepsilon}} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} \|T_n f_n\|_{L^2_{-\varepsilon}} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|T_n f_n\|_{L^2} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|\chi_n f\|_\infty \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|f\|_\infty .
\end{array}
\tag{4.44}
$$

In a similar fashion, we obtain

$$
\begin{array}{rcl}
\|x_n\|_{W^{1,2}([m,m+1])} & \leq & [\int_{m-n}^{m-n+1} (x_n(\xi+n)e^{-\varepsilon\xi})^2 + (e^{-\varepsilon\xi} x_n'(\xi+n))^2 d\xi]^{1/2} \\
& & e^{+\varepsilon(m-n+1)} \\
& \leq & e^{\varepsilon(m-n+1)} \|T_n x_n\|_{\widetilde{W}^{1,2}_\varepsilon} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(\varepsilon)} \right\| e^{\varepsilon(m-n)} e^\varepsilon \|f\|_\infty .
\end{array}
\tag{4.45}
$$

Using a Sobolev embedding it now follows that there exists a constant $C > 0$, independent of n and m, such that $\|\chi_m x_n\|_\infty \leq C e^{\pm\varepsilon(m-n)} \|f\|_\infty$. Summing this identity over $n \in \mathbb{Z}$, we obtain

$$
\begin{array}{rcl}
\|\chi_m x\|_\infty & \leq & C \|f\|_\infty \left[\sum_{n \geq m} e^{\varepsilon(m-n)} + \sum_{n<m} e^{\varepsilon(n-m)} \right] \\
& \leq & \frac{2C}{1-e^{-\varepsilon}} \|f\|_\infty .
\end{array}
\tag{4.46}
$$

This bound does not depend on m, hence $x \in L^\infty(\mathbb{R}, \mathbb{C}^n)$, as desired. The differential equation now implies that in fact $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$. □

4.5. The state space

The main goal of this section is to analyze solutions to the homogeneous equation $\Lambda x = 0$ and to provide a pseudo-inverse for Λ that projects out these solutions in some sense. We start by using the Laplace transform to characterize any solution x that satisfies $\Lambda x = f$, even though x may no longer be unique. As a preparation, we introduce the cutoff operators $\Phi_\pm : L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, defined via $[\Phi_+ f](\xi) = 0$ for all $\xi < 0$, $[\Phi_- f](\xi) = 0$ for all $\xi \geq 0$ and $\Phi_+ f + \Phi_- f = f$.

Proposition 4.5.1. *Consider a linear equation of the form (4.29) that satisfies the assumption (HL) and admits no Floquet exponents λ on the lines $\operatorname{Re}\lambda = \gamma_\pm$, for some constants $\gamma_\pm$. Fix a pair $\gamma_- < \mu < \nu < \gamma_+$, consider any function $x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and write $\Lambda x = f$. Then the following identity holds,*

$$x = \Lambda^{-1}_{(\gamma_+)} \Phi_+ f + \Lambda^{-1}_{(\gamma_-)} \Phi_- f + \mathcal{P}_{\gamma_-,\gamma_+} x_0, \tag{4.47}$$

in which $\mathcal{P}_{\gamma_-,\gamma_+} : X \to BX^1_{\gamma_-,\gamma_+}(\mathbb{R}, \mathbb{C}^n)$ is given by

$$\begin{array}{rcl}(\mathcal{P}_{\gamma_-,\gamma_+}\phi)(\xi) & = & \frac{1}{2\pi i}\int_{\gamma_+ - i\infty}^{\gamma_- + i\infty} e^{z\xi} \mathrm{ev}_0 (I - \Delta_z B_z)^{-1} \Delta_z \mathcal{J}_z h_\phi(\cdot) dz \\ & & + \frac{1}{2\pi i}\int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{z\xi} \mathrm{ev}_0 (I - \Delta_z B_z)^{-1} \Delta_z \mathcal{J}_z h_\phi(\cdot) dz,\end{array} \tag{4.48}$$

with

$$h_\phi(z') = \phi(0) + \sum_{j=0}^{N} e^{z' r_j} \int_{r_j}^{0} e^{-\sigma z'} \big(A^j_{\mathrm{aut}} + B^j(\sigma - r_j)\big)\phi(\sigma) d\sigma. \tag{4.49}$$

In addition, we have the representation

$$(\mathcal{P}_{\gamma_-,\gamma_+}\phi)(\xi) = [\Lambda^{-1}_{(\gamma_+)} g_\phi - \Lambda^{-1}_{(\gamma_-)} g_\phi](\xi) + [G_{(\gamma_+)}(0,\xi) - G_{(\gamma_-)}(0,\xi)]\phi(0), \tag{4.50}$$

in which $g_\phi \in L^2(\mathbb{R}, \mathbb{C}^n)$ has compact support, is bounded on $[-r_{\max}, -r_{\min}]$ and is given by

$$\begin{array}{rcl} g_\phi(\xi) & = & \sum_{r_j > 0} \big(A^j_{\mathrm{aut}} + B^j(\xi)\big)\phi(\xi + r_j)\chi_{[-r_j, 0]}(\xi) \\ & & + \sum_{r_j < 0} \big(A^j_{\mathrm{aut}} + B^j(\xi)\big)\phi(\xi + r_j)\chi_{[0, -r_j]}(\xi).\end{array} \tag{4.51}$$

Proof. Taking the Laplace transform of $\Lambda x = f$ yields

$$\begin{array}{rcl} z\widetilde{x}_+(z) & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} \int_0^\infty e^{-zu} x(u + r_j) du \\ & & + \sum_{j=0}^N \int_0^\infty e^{-zu} B^j(u) x(u + r_j) du + \widetilde{f}_+(z) \\ & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} e^{z r_j} \big(\widetilde{x}_+(z) + \int_{r_j}^0 e^{-z\sigma} x(\sigma) d\sigma\big) + \widetilde{f}_+(z) \\ & & + \sum_{j=0}^N e^{z r_j} \big(\int_0^\infty e^{-zu} B^j(u - r_j) x(u) du \\ & & + \int_{r_j}^0 e^{-z\sigma} B^j(\sigma - r_j) x(\sigma) d\sigma\big) \\ & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} e^{z r_j} \big(\widetilde{x}_+(z) + \int_{r_j}^0 e^{-z\sigma} x(\sigma) d\sigma\big) + \widetilde{f}_+(z) \\ & & + \sum_{j=0}^N e^{z r_j} \big[\sum_{k \in \mathbb{Z}} e^{-ik r_j} B^j_k \widetilde{x}_+(z - ik) \\ & & + \int_{r_j}^0 e^{-z\sigma} B^j(\sigma - r_j) x(\sigma) d\sigma\big] \end{array} \tag{4.52}$$

and thus after rearrangement we have

$$\begin{array}{rcl} \Delta(z)\widetilde{x}_+(z) & = & x(0) + \sum_{k\in\mathbb{Z}} \sum_{j=0}^N e^{(z - ik) r_j} B^j_k \widetilde{x}_+(z - ik) + \widetilde{f}_+(z) \\ & & + \sum_{j=0}^N e^{z r_j} \int_{r_j}^0 e^{-z\sigma} \big(A^j_{\mathrm{aut}} + B^j(\sigma - r_j)\big) x(\sigma) d\sigma. \end{array} \tag{4.53}$$

Upon defining $y(\xi) = x(-\xi)$ a similar identity may be obtained for $\widetilde{y}_+(z)$. Similarly as in Chapter 2, an application of the inversion formula (B.7) now yields the desired result (4.48), upon observing that

$$\int_{\gamma_+-i\infty}^{\gamma_++i\infty} e^{\xi z} \int_{\xi}^{0} e^{-z\sigma} x(\sigma) d\sigma dz + \int_{\gamma_-+i\infty}^{\gamma_--i\infty} e^{\xi z} \int_{\xi}^{0} e^{-z\sigma} x(\sigma) d\sigma dz = 0. \tag{4.54}$$

We now establish the representation (4.50), by taking an arbitrary $\phi \in X$, writing $g = g_\phi$ and computing $\widetilde{g}_+$ and $\widetilde{g}_-$. This yields

$$\begin{array}{rcl} \widetilde{g}_+(z) & = & \int_0^\infty e^{-z\xi} g(\xi) d\xi = \sum_{r_j<0} \int_0^{-r_j} e^{-z\xi} \big(A^j_{\text{aut}} + B^j(\xi)\big)\phi(\xi + r_j) d\xi \\ & = & \sum_{r_j<0} e^{zr_j} \int_{r_j}^0 e^{-z\xi'} \big(A^j_{\text{aut}} + B^j(\xi' - r_j)\big)\phi(\xi') d\xi', \\ \widetilde{g}_-(z) & = & \int_0^\infty e^{z\xi} g(-\xi) d\xi = \sum_{r_j>0} \int_0^{r_j} e^{z\xi} \big(A^j_{\text{aut}} + B^j(-\xi)\big)\phi(-\xi + r_j) d\xi \\ & = & \sum_{r_j>0} e^{zr_j} \int_{r_j}^0 e^{-z\xi''} \big(A^j_{\text{aut}} + B^j(\xi'' - r_j)\big)\phi(\xi'') d\xi'', \end{array} \tag{4.55}$$

in which we used the substitutions $\xi' = \xi + r_j$ and $\xi'' = -\xi + r_j$. The result follows using Corollary 4.4.2, together with the observation that the bounded function g_ϕ has compact support, which means $g_\phi \in L^2_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n)$. Finally, using (4.50) and the embeddings $W^{1,2}_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n) \subset L^\infty_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n)$, another application of Corollary 4.4.2 shows that $\mathcal{P}_{\gamma_-,\gamma_+}$ indeed maps into $BX^1_{\gamma_-,\gamma_+}$. □

We now study the set of solutions to the homogeneous equation (4.11) that have controlled exponential growth. We will therefore consider the spaces

$$\begin{array}{rcl} \mathcal{N}_{\mu,\nu} & = & \left\{x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) \mid \Lambda x = 0\right\}, \\ X_{\mu,\nu} & = & \left\{\phi \in X \mid \phi = x_0 \text{ for some } x \in \mathcal{N}_{\mu,\nu}\right\}, \end{array} \tag{4.56}$$

with μ and ν as in Proposition 4.5.1. From the representation (4.47) it follows immediately that for every $\phi \in X_{\mu,\nu}$ there is a unique $x \in \mathcal{N}_{\mu,\nu}$ with $x_0 = \phi$, which we will denote as $x = E\phi$. Using a standard shifting argument, it is clear that for all $x \in \mathcal{N}_{\mu,\nu}$ and any $k \in \mathbb{Z}$, we also have $T_{2k\pi}x \in \mathcal{N}_{\mu,\nu}$. We can hence define the monodromy operators $M_{\pm 2\pi} : X_{\mu,\nu} \to X_{\mu,\nu}$ by $\phi \mapsto \text{ev}_{\pm 2\pi} E\phi$, which satisfy $M_{2\pi} M_{-2\pi} = M_{-2\pi} M_{2\pi} = I$.

Lemma 4.5.2. *Consider a homogeneous linear equation (4.11) that satisfies (HL). Suppose further that for two constants $\gamma_- < \gamma_+$, this equation (4.11) admits no Floquet exponents λ on the lines $\operatorname{Re}\lambda = \gamma_\pm$. Then for any pair $\gamma_- < \mu < \nu < \gamma_+$, we have that $M_{2\pi}$ is a compact operator on $X_{\mu,\nu}$ and $\mathcal{N}_{\mu,\nu}$ is finite dimensional.*

Proof. The representation (4.50) implies that for some $C > 0$ we have a bound $\|M_{2\pi}\phi\| \le C\|\phi\|$ for all $\phi \in X_{\mu,\nu}$, which using the differential equation implies that also $\|DM_{2\pi}\phi\| \le C'\|\phi\|$. An application of the Ascoli-Arzela theorem shows that $M_{2\pi}$ is compact. However, since $M_{2\pi}$ has a bounded inverse, the unit ball in $X_{\mu,\nu}$ is compact and hence this space is finite dimensional. □

Since $M_{2\pi}$ is invertible, we can define a matrix W such that $e^{2\pi W} = M_{2\pi}$. Consider any $\psi \in X_{\mu,\nu}$, then the continuous function $P_\psi : \mathbb{R} \to \mathbb{C}^n$ given by $P_\psi = Ee^{-\xi W}\psi$ is periodic, since

$$\begin{aligned} P_\psi(\xi + 2\pi) &= [Ee^{-(\xi+2\pi)W}\psi](\xi + 2\pi) = [E\mathrm{ev}_{2\pi} EM_{-2\pi} e^{-\xi W}\psi](\xi) \\ &= [EM_{2\pi} M_{-2\pi} e^{-\xi W}\psi](\xi) = P_\psi(\xi). \end{aligned} \tag{4.57}$$

Consider a Jordan chain $\phi^0, \ldots, \phi^\ell$ of length $\ell + 1$ for W at some eigenvalue λ, i.e., $W\phi^0 = \lambda\phi^0$ and $W\phi^i = \lambda\phi^i + \phi^{i-1}$ for $1 \le i \le \ell$. Recall that $e^{W\xi}\phi^i = \sum_{j=0}^{i} \frac{1}{j!}\xi^j e^{\lambda\xi}\phi^{i-j}$. Writing $x^i = E\phi^i$, we now obtain that

$$e^{\lambda\xi} P_{\phi^i}(\xi) = \sum_{j=0}^{i} \frac{1}{j!}(-\xi)^j x^{i-j}(\xi). \tag{4.58}$$

This can be inverted, yielding $x^0 = e^{\lambda\xi} P_{\phi^0}$, which implies that λ is a Floquet multiplier. Similarly, we have

$$x^i(\xi) = e^{\lambda\xi} P_{\phi^i}(\xi) - \sum_{j=1}^{i} \frac{1}{j!}(-\xi)^j x^{i-j}(\xi). \tag{4.59}$$

We hence conclude that $\mathcal{N}_{\mu,\nu}$ is spanned by functions that can be written as sums of terms of the form $e^{\lambda\xi}\xi^j p(\xi)$, with $p \in C^{\mathrm{per}}_{2\pi}(\mathbb{R}, \mathbb{C}^n)$ and λ a Floquet exponent with $\mu \le \operatorname{Re}\lambda \le \nu$. This important observation gives a criterion for the existence of an inverse for $\Lambda : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$, merely in terms of Floquet exponents.

Proposition 4.5.3. *Consider an equation of the form (4.29) that satisfies (HL). Consider any $\eta \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that (4.29) has no Floquet exponents λ in the strip $\eta - \varepsilon_0 < \operatorname{Re}\lambda < \eta + \varepsilon_0$. Then the operator Λ is an isomorphism from $W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$. For any $0 < \varepsilon < \varepsilon_0$, the inverse is given by*

$$\Lambda^{-1} f = \Lambda^{-1}_{(\eta+\varepsilon)}\Phi_+ f + \Lambda^{-1}_{(\eta-\varepsilon)}\Phi_- f. \tag{4.60}$$

Proof. Notice first that the assumptions of Proposition 4.4.3 are satisfied. Indeed, for any function $g \in L^2_{\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, write $x = \Lambda^{-1}_{(\eta+\varepsilon)} g - \Lambda^{-1}_{(\eta-\varepsilon)} g$, then $x \in BX^1_{\eta-\varepsilon,\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$, i.e., $x \in \mathcal{N}_{\eta-\varepsilon,\eta+\varepsilon}$. However, the condition on the Floquet exponents implies that $\mathcal{N}_{\eta-\varepsilon,\eta+\varepsilon} = \{0\}$, hence $x = 0$ as desired. Proposition 4.4.3 now shows that Λ^{-1} defined above indeed maps into $W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$. The injectivity of Λ again follows from the condition on the Floquet multipliers. □

The finite dimensionality of $X_{\mu,\nu}$ can be exploited to define a projection from X onto this subspace, using the operator $\mathcal{P}$ appearing in (4.47).

Lemma 4.5.4. *Consider any set of constants $\widetilde{\gamma}_- < \gamma_- < \gamma_+ < \widetilde{\gamma}_+$ such that the equation (4.29) has no Floquet exponents λ with $\operatorname{Re}\lambda \in \{\gamma_\pm, \widetilde{\gamma}_\pm\}$. Suppose further that (HL) is satisfied. Then the operator $P = P_{\gamma_-,\gamma_+} : X \to X$ defined by $P\phi = \mathrm{ev}_0\mathcal{P}_{\gamma_-,\gamma_+}\phi$ is a projection, with $\mathcal{R}(P_{\gamma_-,\gamma_+}) = X_{\gamma_-,\gamma_+}$.*

Proof. Notice first that the set of real parts of Floquet exponents between $\widetilde{\gamma}_-$ and $\widetilde{\gamma}_+$ is discrete, hence there exist $\gamma_- < \mu < \nu < \gamma_+$ such that $X_{\gamma_-,\gamma_+} = X_{\mu,\nu}$. Now (4.50) implies that $\mathcal{R}(\mathcal{P}_{\gamma_-,\gamma_+}) \subset \mathcal{N}_{\gamma_-,\gamma_+} = \mathcal{N}_{\mu,\nu}$, hence $\mathcal{R}(P_{\gamma_-,\gamma_+}) \subset X_{\mu,\nu}$. In addition, for any $\phi \in X_{\mu,\nu}$ write $x = E\phi \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and notice that (4.47) implies $x = \mathcal{P}_{\gamma_-,\gamma_+}\phi$, yielding

$$\phi = x_0 = \mathrm{ev}_0\mathcal{P}_{\gamma_-,\gamma_+}\phi = P_{\gamma_-,\gamma_+}\phi. \tag{4.61}$$

This shows that indeed $\mathcal{R}(P_{\gamma_-,\gamma_+}) = X_{\mu,\nu} = X_{\gamma_-,\gamma_+}$ and hence also $P^2 = P$. □

From now on fix $\gamma > 0$ such that there are no Floquet exponents with $0 < |\mathrm{Re}\,\lambda| < \gamma$. For any $0 < \mu < \gamma$, define $X_0 = X_{-\mu,\mu}$, $\mathcal{N}_0 = \mathcal{N}_{-\mu,\mu}$ and $Q_0 = P_{-\mu,\mu}$. Note that these definitions are independent of the particular choice of μ. In addition, for any $0 < \eta < \gamma$, define the pseudo-inverse $\mathcal{K} = \mathcal{K}_\eta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ by

$$\mathcal{K}_\eta f = \Lambda^{-1}_{(\eta)}\Phi_+ f + \Lambda^{-1}_{(-\eta)}\Phi_- f. \tag{4.62}$$

Notice that if $0 < \eta_0 < \eta_1 < \gamma$, then $(\mathcal{K}_{\eta_1})_{|BC_{\eta_0}(\mathbb{R},\mathbb{C}^n)} = \mathcal{K}_{\eta_0}$. This can be verified by means of the same reasoning used to established Proposition 4.5.3. In combination with (4.47), this allows us to compute

$$\begin{array}{rcl} \mathcal{K}_{\eta_0} f & = & \mathcal{K}_{\eta_1}\Lambda\mathcal{K}_{\eta_0} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f = \mathcal{K}_{\eta_1} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f \\ & = & \mathcal{K}_{\eta_0} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f \end{array} \tag{4.63}$$

for any $f \in BC_{\eta_0}(\mathbb{R}, \mathbb{C}^n)$, which yields the important identity

$$Q_0 ev_0 \mathcal{K}_{\eta_0} f = 0. \tag{4.64}$$

4.6. Time dependence

For any $\tau \in \mathbb{R}$, consider the shifted mixed type functional differential equation

$$x'(\xi) = L^\tau(\xi)x_\xi + f(\xi) = L(\xi + \tau)x_\xi + f(\xi) \tag{4.65}$$

and write $X_0^\tau, \mathcal{N}_0^\tau, \Lambda^\tau, Q_0^\tau$ and $\mathcal{K}^\tau$ for the spaces and operators associated to (4.65) that are the counterparts of those defined for the original unshifted equation (4.29).

Lemma 4.6.1. *Consider a linear homogeneous equation of the form (4.11) that satisfies (HL). Fix two constants $\tau_0, \tau_1 \in \mathbb{R}$ and suppose that there exists a $\gamma > 0$ such that (4.11) admits no Floquet exponents λ in the strip $0 < |\mathrm{Re}\,\lambda| < \gamma$. Then for any $\phi \in X_0^{\tau_0}$, there is a unique extension $x = E\phi \in \mathcal{N}_0^{\tau_0}$ with the property that $x_0 = \phi$. In addition, we have that $T_{\tau_1-\tau_0}x \in \mathcal{N}_0^{\tau_1}$, i.e.,*

$$Q_0^{\tau_1}\mathrm{ev}_{\tau_1-\tau_0}E\phi = \mathrm{ev}_{\tau_1-\tau_0}E\phi. \tag{4.66}$$

Finally, for any $0 < \eta < \gamma$ and any function $f \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$, the following identity holds,

$$\mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f = T_{\tau_1}\mathcal{K}_\eta^{\tau_0} f - EQ_0^{\tau_0+\tau_1}\mathrm{ev}_{\tau_1}\mathcal{K}_\eta^{\tau_0} f. \tag{4.67}$$

Proof. First, consider any $\phi \in X_0^{\tau_0}$ and write $u = E\phi$, which means that $u'(\xi) = L(\xi + \tau_0)u_\xi$ for all $\xi \in \mathbb{R}$. Defining $\psi = u_{\tau_1 - \tau_0}$, notice that the function $v = E\psi$ has $v(\xi) = u(\xi + \tau_1 - \tau_0)$ and hence satisfies $v'(\xi) = L(\xi + \tau_1)v_\xi$, showing that $\psi \in X_0^{\tau_1}$ as required. Now consider the function y defined by

$$y = T_{\tau_1}\mathcal{K}_\eta^{\tau_0} f - \mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f. \tag{4.68}$$

It is easy to compute

$$y'(\xi) = L(\xi + \tau_0 + \tau_1)y_\xi + f(\xi + \tau_1) - f(\xi + \tau_1) = L(\xi + \tau_0 + \tau_1)y_\xi \tag{4.69}$$

and hence $y \in \mathcal{N}_0^{\tau_0+\tau_1}$. The final statement now follows from $y = Ey_0$, together with the computation

$$\begin{array}{rcl} y_0 & = & Q_0^{\tau_0+\tau_1} y_0 = Q_0^{\tau_0+\tau_1}\mathrm{ev}_{\tau_1}\mathcal{K}_\eta^{\tau_0} f - Q_0^{\tau_0+\tau_1}\mathrm{ev}_0\mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f \\ & = & Q_0^{\tau_0+\tau_1}\mathrm{ev}_{\tau_1}\mathcal{K}_\eta^{\tau_0} f, \end{array} \tag{4.70}$$

where (4.64) was used in the last identity. □

An elementary observation that follows from this result and the uniqueness of continuations, is that if $y \in \mathcal{N}_0^\tau$ for any $\tau \in \mathbb{R}$, then

$$\mathrm{ev}_\xi E \mathrm{ev}_{\xi'} y = \mathrm{ev}_{\xi+\xi'} y. \tag{4.71}$$

We will need the ability to relate the different subspaces X_0^τ to one another in a natural fashion. To this end, we recall the matrix $W \in \mathcal{L}(X_0)$ that is related to the monodromy operator $M_{2\pi}$ by $M_{2\pi} = e^{2\pi W}$. For all $\tau \in \mathbb{R}$, we define the bounded linear operators $\Pi_\rightarrow^\tau : X_0 \to X_0^\tau$ and $\Pi_\leftarrow^\tau : X_0^\tau \to X_0$, via

$$\begin{array}{rcl} \Pi_\rightarrow^\tau \phi & = & \mathrm{ev}_\tau E e^{-\tau W}\phi, \\ \Pi_\leftarrow^\tau \psi & = & e^{\tau W}\mathrm{ev}_{-\tau} E\psi. \end{array} \tag{4.72}$$

In addition, we define a mapping $\Pi : \mathbb{R} \to \mathcal{L}(X, X_0)$ by

$$\Pi(\tau) = \Pi_\leftarrow^\tau Q_0^\tau. \tag{4.73}$$

Using the definition of W and the identity (4.71), it is clear that all three operators defined above are periodic, i.e., $\Pi_\leftarrow^{\tau+2\pi} = \Pi_\leftarrow^\tau$ and similarly for $\Pi_\rightarrow$ and Π. Notice also that $\Pi_\rightarrow^\tau \Pi_\leftarrow^\tau = I$ and $\Pi_\leftarrow^\tau \Pi_\rightarrow^\tau = \Pi(\tau)\Pi_\rightarrow^\tau = I$.

In the remainder of this section we will show that the operator Π inherits the C^r-smoothness of the linear operator L. In [119] this was obtained directly, using an equivalence between the Floquet spectrum and the spectrum of an operator Λ'_{per}, that in our setting should be seen as the restriction of Λ to the space $W_{\mathrm{loc}}^{1,1}(\mathbb{R}, \mathbb{C}^n) \cap C_{\mathrm{per}}^{2\pi}(\mathbb{R}, \mathbb{C}^n)$. In particular, any eigensolution $\Lambda'_{\mathrm{per}} u = -\lambda u$ would lead to a Floquet exponent λ via $x(\xi) = e^{\lambda\xi}u(\xi)$. However, this last observation is only valid in the absence of delayed and advanced arguments in (4.9). This fact forces us to pursue an alternate approach.

Lemma 4.6.2. *Consider a linear equation of the form (4.29) that satisfies the assumption (HL) and suppose that this equation admits no Floquet exponents on the imaginary axis. Then the function* $\overline{\Lambda^{-1}} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}, \mathbb{C}^n), W^{1,2}(\mathbb{R}, \mathbb{C}^n))$ *given by* $\tau \mapsto (\Lambda^\tau)^{-1}$ *is* C^1*-smooth. The derivative is given by*

$$D\overline{\Lambda^{-1}}(\tau) = \overline{\Lambda^{-1}}(\tau)[DL^\tau]\overline{\Lambda^{-1}}(\tau). \tag{4.74}$$

In addition, for any $\xi \in \mathbb{R}$*, the function* $\overline{G} : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ *given by* $\tau \mapsto G^\tau(\xi, \cdot)$ *is* C^1*-smooth. The derivative is given by*

$$D\overline{G}(\tau) = \overline{\Lambda^{-1}}(\tau)[DL^\tau]\overline{G}(\tau). \tag{4.75}$$

Proof. Let us first recall the convolution operators $\widetilde{B}^j$ defined in (4.32). For any $0 \le j \le N$ we now introduce the translated operators $\widetilde{B}^{j,\tau} \in \mathcal{L}(\ell_2, \ell_2)$, which are given by

$$(B^{j,\tau} w)_n = \sum_{k\in\mathbb{Z}} e^{i(n-k)r_j} e^{ik\tau} B^j_k w_{n-k}. \tag{4.76}$$

We claim that $\tau \mapsto \widetilde{B}^{j,\tau}$ is differentiable at $\tau = 0$ and that the derivative is generated by the operator $DB^j \in \mathcal{L}(X, \mathbb{C}^n)$. Indeed, a similar estimate as in (4.33) yields

$$\begin{array}{rcl} \left\| [\widetilde{B}^{j,\tau} - \widetilde{B}^j - \tau \widetilde{DB}^j]w \right\|_2^2 & = & \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} e^{i(n-m)r_j} [e^{i\tau m} - im\tau - 1] B^j_m w_{n-m} \right|^2 \\ & \le & (\sum_{m\in\mathbb{Z}} \left| [e^{i\tau m} - im\tau - 1] B^j_m \right|)^2 \|w\|_2^2 . \end{array} \tag{4.77}$$

Now fix $\varepsilon > 0$ and choose $\varepsilon' = \varepsilon[2\sum_{m\in\mathbb{Z}} \left| mB^j_m \right|]^{-1} > 0$. Since the exponential function is differentiable, there exists a $\delta' > 0$ such that

$$\left| e^z - z - 1 \right| < \varepsilon' |z| \tag{4.78}$$

for all $|z| < \delta'$. Now let $M > 0$ be so large that $(\frac{2}{\delta'} + 1) \sum_{|m|>M} \left| mB^j_m \right| < \frac{\varepsilon}{2}$. Finally, fix $\delta = \frac{\delta'}{M}$. For any $0 < |\tau| < \delta$, write $\Delta = \sum_{m\in\mathbb{Z}} \left| [e^{i\tau m} - im\tau - 1]B^j_m \right|$ and compute

$$\begin{array}{rcl} \Delta & = & \sum_{|m|\le \frac{\delta'}{|\tau|}} \left| [e^{i\tau m} - im\tau - 1]B^j_m \right| + \sum_{|m|> \frac{\delta'}{|\tau|}} \left| [e^{i\tau m} - im\tau - 1]B^j_m \right| \\ & \le & \sum_{|m|\le \frac{\delta'}{|\tau|}} \varepsilon' |\tau| \left| mB^j_m \right| + \sum_{|m|> \frac{\delta'}{|\tau|}} (2 + |m|\,|\tau|) \left| B^j_m \right| \\ & \le & |\tau|\, \varepsilon' \sum_{m\in\mathbb{Z}} \left| mB^j_m \right| + \sum_{|m|> \frac{\delta'}{|\tau|}} (\frac{2}{\delta'} + 1)\, |m|\, |\tau| \left| B^j_m \right| \\ & \le & \frac{\varepsilon}{2} |\tau| + |\tau| \sum_{|m|>M} (\frac{2}{\delta'} + 1) \left| mB^j_m \right| < \varepsilon |\tau| . \end{array} \tag{4.79}$$

This proves the differentiability of $\tau \mapsto \widetilde{B}^{j,\tau}$ at $\tau = 0$ and the same argument can be used to establish this fact for all $\tau \in \mathbb{R}$. Since Δ_z does not depend on τ, this shows that the map $\tau \mapsto I - \Delta_z B^\tau_z \in \mathcal{L}(\ell_2)$ and its inverse are differentiable in the variable τ, uniformly for $z \in i\mathbb{R}$. We find

$$D\big[\tau \mapsto (I - \Delta_z B^\tau_z)^{-1}\big] = \tau \mapsto [I - \Delta_z B^\tau_z]^{-1} \Delta_z [DB]^\tau_z [I - \Delta_z B^\tau_z]^{-1}. \tag{4.80}$$

An estimate analogous to (4.39) now completes the proof. □

The explicit forms (4.74) and (4.75) allow repeated differentiation of $\overline{\Lambda^{-1}}$ and $\overline{G}$, up to the point that the differentiability of L is lost. This observation leads to the following result.

Corollary 4.6.3. *Consider a linear equation of the form (4.29) that satisfies the assumption (HL) and suppose that this equation admits no Floquet exponents on the imaginary axis. Then the functions $\overline{\Lambda^{-1}} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}, \mathbb{C}^n), W^{1,2}(\mathbb{R}, \mathbb{C}^n))$ and $\overline{G} : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ are C^r-smooth. For any $1 \le \ell \le r$ there exist constants $c_{(f_1,\ldots,f_q)}$ such that the following identities hold,*

$$\begin{array}{rcl} D^\ell \overline{\Lambda^{-1}}(\tau) & = & \sum_{(f_1,\ldots,f_q)} c_{(f_1,\ldots,f_q)} \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots \\ & & \overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{\Lambda^{-1}}(\tau), \\ D^\ell \overline{G}(\tau) & = & \sum_{(f_1,\ldots,f_q)} c_{(f_1,\ldots,f_q)} \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots \\ & & \overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{G}(\tau). \end{array} \tag{4.81}$$

Here the sums are taken over tuples $(f_1, \ldots, f_q)$ with $f_i \ge 1$ and $f_1 + \ldots + f_q = \ell$.

We will use the representation (4.50) in order to establish the smoothness of Π. We hence need to extend the results above to show the differentiability of $\overline{\Lambda^{-1}}$ when viewed as an operator that maps into the space of C^{r+1}-smooth functions. To do this, let $K' \subset \mathbb{R}$ be a compact interval and consider the set $C_0(K', \mathbb{C}^n)$ of continuous functions f with support contained in K', i.e., $\text{supp}(f) \subset K'$. Fixing any bounded open interval $\Omega \subset \mathbb{R}$, we now define operators $\overline{\Gamma} = \overline{\Gamma}_{(\eta)} : \mathbb{R} \to \mathcal{L}(C_0(K', \mathbb{C}^n), C^{r+1}(\Omega, \mathbb{C}^n))$ and $\overline{H} = \overline{H}_{(\eta)} : \mathbb{R} \to C^{r+1}(\Omega, \mathbb{C}^{n\times n})$ via

$$\begin{array}{rcl} \overline{\Gamma}(\tau) f & = & \overline{\Lambda^{-1}_{(\eta)}}(\tau) f - \overline{\Lambda^{-1}_{(-\eta)}}(\tau) f, \\ \overline{H}(\tau) & = & \overline{G}_{(\eta)}(\tau) - \overline{G}_{(-\eta)}(\tau). \end{array} \tag{4.82}$$

Notice that indeed $\overline{\Gamma}(\tau) f \in C^{r+1}(\Omega, \mathbb{C}^n)$, since $\Lambda^\tau \overline{\Gamma}(\tau) f = 0$. Throughout the remainder of this chapter, we will use the symbol D_ξ to exclusively represent differentiation with respect to a time-like real-valued variable. The details should be clear from the context. We will also write D_τ for the derivative with respect to the variable τ.

For any suitable integer s, a quick calculation shows that $D_\xi^s D_\tau^\ell \overline{\Lambda^{-1}}(\tau)$ can be written as a sum of elements of the form

$$D_{(e_1,\ldots e_p)} \Lambda_{(f_1,\ldots f_q)}, \tag{4.83}$$

for integers $0 \le p \le s$, $e_i \ge 0$ and $f_i \ge 1$ that satisfy $f_1 + \ldots + f_q \le \ell$ and $p + e_1 + \ldots + e_p + f_1 + \ldots + f_q = s + \ell$, in which

$$\begin{array}{rcl} D_{(e_1,\ldots e_p)} & = & [D^{e_1} L^\tau]\ldots[D^{e_p} L^\tau], \\ \Lambda_{(f_1,\ldots f_q)} & = & \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots\overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{\Lambda^{-1}}(\tau), \end{array} \tag{4.84}$$

together with elements of the form

$$[D^{e_1} L^\tau]\ldots[D^{e_p} L^\tau] D_\xi^f, \tag{4.85}$$

with $e_i \geq 0$, $f \geq 0$ and $p + e_1 + \ldots + e_p + f + 1 = s + \ell$. Now for any tuples $(e_1, \ldots, e_p)$ and $(f_1, \ldots, f_q)$, define the sets

$$\begin{array}{rcl} (e_1, \ldots, e_p) \oplus 1 & = & \big\{(e_1 + 1, e_2, \ldots, e_p), (e_1, e_2 + 1, \ldots, e_p), \ldots, \\ & & \quad (e_1, \ldots, e_p + 1)\big\}, \\ (f_1, \ldots, f_q) \odot 1 & = & (f_1, \ldots, f_q) \oplus 1 \\ & & \quad \cup \big\{(1, f_1, \ldots, f_q), (f_1, 1, \ldots, f_q), \ldots, (f_1, \ldots, f_q, 1)\big\}. \end{array} \tag{4.86}$$

If $q \geq 1$, an easy calculation shows that

$$\begin{array}{rcl} D_\xi D_{(e_1,\ldots,e_p)} \Lambda_{(f_1,\ldots,f_q)} & = & D_{(e_1,\ldots,e_p)\oplus 1} \Lambda_{(f_1,\ldots,f_q)} \\ & & + D_{(e_1,\ldots,e_p,0)} \Lambda_{(f_1,\ldots,f_q)} \\ & & + D_{(e_1,\ldots,e_p,f_1)} \Lambda_{(f_2,\ldots,f_q)} \\ D_\tau D_\xi D_{(e_1,\ldots,e_p)} \Lambda_{(f_1,\ldots,f_q)} & = & D_{(e_1,\ldots,e_p)\oplus 1 \oplus 1} \Lambda_{(f_1,\ldots,f_q)} \\ & & + D_{(e_1,\ldots,e_p)\oplus 1} \Lambda_{(f_1,\ldots,f_q)\odot 1} \\ & & + D_{(e_1,\ldots,e_p,0)\oplus 1} \Lambda_{(f_1,\ldots,f_q)} \\ & & + D_{(e_1,\ldots,e_p,0)} \Lambda_{(f_1,\ldots,f_q)\odot 1} \\ & & + D_{(e_1,\ldots,e_p,f_1)\oplus 1} \Lambda_{(f_2,\ldots,f_q)} \\ & & + D_{(e_1,\ldots,e_p,f_1)} \Lambda_{(f_2,\ldots,f_q)\odot 1}, \end{array} \tag{4.87}$$

upon understanding that $\Lambda_\emptyset = \Lambda^{-1}(\tau)$ and noting that for any set $\mathcal{E}$, one should read $D_\mathcal{E} = \sum_{e \in \mathcal{E}} D_e$. If $q = 0$, then the same identity holds if one writes $f_1 = 0$, $\Lambda_{(f_2,\ldots,f_q)} = \mathrm{id}$ and $\Lambda_{(f_2,\ldots,f_q)\odot 1} = 0$. The important observation, which can be verified by a simple calculation, is that D_τ and D_ξ commute on elements of the form (4.83), i.e.,

$$D_\tau D_\xi D_{(e_1,\ldots,e_p)} \Lambda_{(f_1,\ldots,f_q)} = D_\xi D_\tau D_{(e_1,\ldots,e_p)} \Lambda_{(f_1,\ldots,f_q)}. \tag{4.88}$$

Lemma 4.6.4. *Consider a linear equation of the form (4.29) that satisfies the assumption (HL) and suppose that for some $\gamma > 0$ this equation admits no Floquet exponents λ with $0 < |\mathrm{Re}\,\lambda| < \gamma$. Consider an integer $0 \leq \ell \leq r$ and a parameter $\eta \in (0, \gamma)$. Then the maps $\overline{\Gamma}_\ell = \overline{\Gamma}_{(\eta),\ell} : \mathbb{R} \to \mathcal{L}(C_0(K, \mathbb{C}^n), C^{r+1-\ell}(\Omega, \mathbb{C}^n))$ and $\overline{H}_\ell = \overline{H}_{(\eta),\ell} : \mathbb{R} \to C^{r+1-\ell}(\Omega, \mathbb{C}^{n\times n})$ are C^ℓ-smooth.*

Proof. We will only treat the map $\overline{\Gamma}_\ell$, since the differentiability of $\overline{H}_\ell$ follows in a similar fashion. For any $\tau \in \mathbb{R}$, consider the map $\Phi_\ell(\tau) : C_0(K, \mathbb{C}^n) \to C^{r+1-\ell}(\Omega, \mathbb{C}^n)$ given by

$$\Phi_\ell(\tau) f = [D_\tau^\ell \overline{\Lambda_{(+\eta)}^{-1}}](\tau) f - [D_\tau^\ell \overline{\Lambda_{(-\eta)}^{-1}}](\tau) f. \tag{4.89}$$

In order to see that indeed $\Phi_\ell(\tau) f \in C^{r+1-\ell}(\Omega, \mathbb{C}^n)$, notice first that due to the special form of $\Phi_\ell(\tau)$ we can ignore all the terms of the form (4.85) in the expansion of $[D_\xi^{r+1-\ell} D_\tau^\ell \overline{\Lambda_{(\pm\eta)}^{-1}}](\tau)$. We hence only need to consider the terms of the form (4.83) with $s = r + 1 - \ell$. However, since $e_i \leq r$ for all $1 \leq i \leq p$, these terms will yield a continuous function when applied to f, as desired.

For convenience, we will treat each of the $r + 1 - \ell$ components of $\overline{\Gamma}_\ell f$ separately in order to show that Φ_ℓ is indeed the ℓ-th derivative of $\overline{\Gamma}_\ell$. To this end, define for all $0 \leq s \leq r + 1 - \ell$, the map $\Gamma^{(s)}(\tau) : C_0(K, \mathbb{C}^n) \to C(\Omega, \mathbb{C}^n)$, given by $\Gamma^{(s)}(\tau)f = D_\xi^s \overline{\Gamma}_\ell(\tau)f$. Observe first that due to the Sobolev embeddings $W^{1,2}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n) \subset L^\infty_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$, Corollary 4.6.3 implies that $D_\tau^\ell \Gamma^{(0)} = \Phi_\ell$ when viewing Φ_ℓ as a function mapping into $C(\Omega, \mathbb{C}^n)$. Now due to the commutation relation (4.88), one may use a similar argument to show that for all $0 \le s \le r + 1 - \ell$, $\Gamma^{(s)}$ is ℓ-fold differentiable, with

$$[D_\tau^\ell \Gamma^{(s)}](\tau)f = [D_\xi^s D_\tau^\ell \overline{\Lambda^{-1}_{(+\eta)}} - D_\xi^s D_\tau^\ell \overline{\Lambda^{-1}_{(-\eta)}}](\tau)f \in C(\Omega, \mathbb{C}^n). \tag{4.90}$$

The continuity of Φ_ℓ follows from the continuity of $\overline{\Lambda^{-1}_{(\pm\eta)}}$ as maps $\mathbb{R} \to \mathcal{L}(L^2_{\pm\eta}(\mathbb{R}, \mathbb{C}^n), W^{1,2}_{\pm\eta}(\mathbb{R}, \mathbb{C}^n))$. □

Corollary 4.6.5. *Consider the setting of Lemma 4.6.4. The function $\tau \mapsto \Pi(\tau)$ is C^r-smooth as a map from $\mathbb{R}$ into $\mathcal{L}(X, X_0)$.*

Proof. It is sufficient to show that $\tau \mapsto \mathrm{ev}_{-\tau} E Q_0^\tau$ is C^r-smooth as a map from $\mathbb{R} \to \mathcal{L}(X)$. For an appropriate open $\Omega' \subset \mathbb{R}$, notice that the evaluation function $\Omega' \to \mathcal{L}(C^{\ell+1}(\Omega, \mathbb{C}^n), X)$ defined by $\xi \mapsto \mathrm{ev}_\xi$ is C^ℓ-smooth. In view of Lemma 4.6.4 and the representation (4.50), the C^r-smoothness of Π now follows from Lemma D.1. □

4.7. The center manifold

We are now ready to construct the center manifold for the nonlinear equation (4.8). As a preparation, we need to modify the nonlinearity R so that it becomes globally Lipschitz continuous. This can be realized by choosing a C^∞-smooth cutoff function $\chi : [0, \infty) \to \mathbb{R}$ with $\|\chi\|_\infty = 1$, that satisfies $\chi(\xi) = 0$ for $\xi \ge 2$, while $\chi(\xi) = 1$ for $\xi \le 1$. We subsequently define for any $\delta > 0$ the nonlinearity $R_\delta : \mathbb{R} \times X \to \mathbb{C}^n$, given by

$$R_\delta(\xi, \phi) = \chi(\|\Pi(\xi)\phi\| / \delta')\chi(\left\|(I - Q_0^\xi)\phi\right\| / \delta)R(\xi, \phi), \tag{4.91}$$

in which $\delta' = \delta \sup_{\xi \in \mathbb{R}} \left\|\Pi^\xi_{\rightarrow}\right\|$. As in Chapter 2, one can show that this map is bounded and globally Lipschitz continuous in the second variable. In particular, the Lipschitz constant L_δ is independent of $\xi \in \mathbb{R}$ and satisfies $L_\delta \to 0$ as $\delta \to 0$, while one has the estimate $|R_\delta(\xi, \phi)| \le 4\delta L_\delta$ for all $\xi \in \mathbb{R}$ and $\phi \in X$. Associated to R_δ one can define the substitution map $\widetilde{R}_\delta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC_\eta(\mathbb{R}, \mathbb{C}^n)$, given by $[\widetilde{R}_\delta x](\xi) = R_\delta(\xi, x_\xi)$. The Lipschitz constant associated to this substitution map $\widetilde{R}_\delta$ is given by $w^\eta L_\delta$, in which we have introduced the quantity

$$w = \max(e^{-r_{\min}}, e^{r_{\max}}) \ge 1. \tag{4.92}$$

Following these preliminaries, we introduce the operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \times \mathbb{R} \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$\mathcal{G}(u, \phi, \tau) = Ee^{-\tau W}\big[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)\big] + \mathcal{K}\widetilde{R}_\delta(u). \tag{4.93}$$

Notice that any fixed point $u = \mathcal{G}(u, \phi, \tau)$ will satisfy the equation $u'(\xi) = L(\xi)u_\xi + R_\delta(\xi, u_\xi)$, with $Q_0^\tau u_\tau = \Pi^\tau_{\rightarrow}\phi$. For this reason, we set out to show that for any fixed pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, the map $\mathcal{G}(\cdot, \phi, \tau)$ is a contraction on $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, yielding a fixed point $u = u^*_\eta(\phi, \tau)$.

Theorem 4.7.1. *Consider the nonlinear equation (4.8) and assume that the conditions (HL), (HF), (HR1) and (HR2) are all satisfied. Pick any $\gamma > 0$ such that there are no Floquet exponents λ with $0 < |\mathrm{Re}\,\lambda| < \gamma$ and consider any interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ with $\min(k, r)\eta_{\min} < \eta_{\max}$. Then there exist constants $0 < \varepsilon < \delta$ such that the following properties hold.*

(i) For all $\eta \in [\eta_{\min}, \eta_{\max}]$ and for any pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, the fixed point equation $u = \mathcal{G}(u, \phi, \tau)$ has a unique solution $u = u^*_\eta(\phi, \tau) \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$.

(ii) For any pair $\xi, \overline{\xi} \in \mathbb{R}$ with $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$, we have

$$u^*\big(\Pi(\xi)\mathrm{ev}_\xi u^*(\phi, \tau), \overline{\xi}\big) = T_{\xi-\overline{\xi}}u^*(\phi, \tau). \tag{4.94}$$

(iii) For any pair $\eta_{\min} \le \eta_1 < \eta_2 \le \eta_{\max}$, one has the identity $u^*_{\eta_2} = \mathcal{J}^1_{\eta_2\eta_1}u^*_{\eta_1}$.

(iv) For any pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, we have the inequality

$$\Big\|(I - Q_0^\xi)\mathrm{ev}_\xi u^*_\eta(\phi, \tau)\Big\| < \delta, \tag{4.95}$$

for all $\xi \in \mathbb{R}$.

(v) Consider a pair $(\phi, \tau) \in X_0 \times \mathbb{R}$ that has $\|\phi\| < \varepsilon$. Then the following inequality holds for all $r_{\min} \le \theta \le r_{\max}$,

$$\Big\|\Pi(\tau + \theta)\mathrm{ev}_{\tau+\theta}u^*_\eta(\phi, \tau)\Big\| < \delta. \tag{4.96}$$

(vi) For all $\eta \in (\min(k, r)\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \times \mathbb{R} \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is of class $C^{\min(k,r)}$.

We need a preparatory result to prove this theorem, which allows us to restrict the parameter τ to the interval $[0, 2\pi]$. This in turn will enable us to choose the parameters δ and ε independently of $\tau \in \mathbb{R}$, simplifying the analysis considerably.

Proposition 4.7.2. *Let u satisfy $u = \mathcal{G}(u, \phi, \tau)$. Consider any $\overline{\tau}$ with $\tau - \overline{\tau} \in 2\pi\mathbb{Z}$ and let $v = T_{\tau-\overline{\tau}}u$. Then v satisfies the fixed point equation $v = \mathcal{G}(v, \phi, \overline{\tau})$.*

Proof. First note that Lemma 4.6.1 implies

$$\mathcal{K}\widetilde{R}_\delta(v) = T_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) - EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u), \tag{4.97}$$

using which we compute

$$\begin{array}{rcl} T_{\overline{\tau}-\tau}\mathcal{G}(v, \phi, \overline{\tau}) & = & T_{\overline{\tau}-\tau}Ee^{-\overline{\tau}W}\big[\phi - \Pi(\overline{\tau})\mathrm{ev}_{\overline{\tau}}[T_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) - EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u)]\big] \\ & & \quad +\mathcal{K}\widetilde{R}_\delta(u) - T_{\overline{\tau}-\tau}EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) \\ & = & Ee^{-\tau W}\big[\phi - \Pi(\overline{\tau})\mathrm{ev}_\tau\mathcal{K}\widetilde{R}_\delta(u)\big] + \mathcal{K}\widetilde{R}_\delta(u) \\ & & \quad +Ee^{-\tau W}\Pi(\overline{\tau})\mathrm{ev}_{\overline{\tau}}EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) \\ & & \quad -Ee^{(\overline{\tau}-\tau)W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) \\ & = & u + Ee^{-\tau W}\Pi^{\overline{\tau}}_{\leftarrow}Q_0^{\overline{\tau}}\Pi^{\overline{\tau}}_{\rightarrow}e^{\overline{\tau}W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) \\ & & \quad -Ee^{(\overline{\tau}-\tau)W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_\delta(u) \\ & = & u. \end{array} \tag{4.98}$$

□

We are now ready to prove items (i) through (v) of Theorem 4.7.1. The remaining item (vi) will be treated in Section 4.8, where the necessary machinery is developed.

Partial proof of Theorem 4.7.1. In view of Proposition 4.7.2, we may assume throughout the proof that $\tau \in [0, 2\pi]$.

(i) Choose $\delta > 0$ in such a way that for all $\eta \in [\eta_{\min}, \eta_{\max}]$ and all $\sigma \in \mathbb{R}$, we have

$$w^\eta \left\|\mathcal{K}_\eta\right\| L_\delta\big[1 + \|E\|_\eta e^{2\pi|W|} \|\Pi(\sigma)\| w^\eta e^{2\pi\eta}\big] < \frac{1}{4}. \tag{4.99}$$

Then for any pair $(\phi, \tau) \in X_0 \times [0, 2\pi]$ and all $\eta \in [\eta_{\min}, \eta_{\max}]$, we have the inequality

$$\|\mathcal{G}(u_1, \phi, \tau) - \mathcal{G}(u_2, \phi, \tau)\|_{BC^1_\eta} \le \frac{1}{4} \|u_1 - u_2\|_{BC^1_\eta}. \tag{4.100}$$

In addition, if $\rho \ge 2\|E\|_\eta e^{2\pi|W|}\|\phi\|$, then $\mathcal{G}(\cdot, \phi, \tau)$ maps the ball with radius ρ in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ into itself. We can hence use the contraction mapping theorem to define the unique solution $u = u^*_\eta(\phi, \tau)$ of the fixed point equation $u = \mathcal{G}(u, \phi, \tau)$ for $\tau \in [0, 2\pi]$.

(ii) We first write $\psi = \Pi(\xi)\mathrm{ev}_\xi u^*(\phi, \tau)$ and compute

$$\begin{aligned} \psi &= \Pi(\xi)\mathrm{ev}_\xi E e^{-\tau W}\phi - \Pi(\xi)\mathrm{ev}_\xi E e^{-\tau W}\Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u^*(\phi, \tau)) \\ &\quad + \Pi(\xi)\mathrm{ev}_\xi \mathcal{K}\widetilde{R}_\delta(u^*(\phi, \tau)) \\ &= e^{(\xi-\tau)W}\phi - e^{(\xi-\tau)W}\Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u^*(\phi, \tau)) \\ &\quad + \Pi(\xi)\mathrm{ev}_\xi \mathcal{K}\widetilde{R}_\delta(u^*(\phi, \tau)). \end{aligned} \tag{4.101}$$

Now writing $u = u^*(\phi, \tau)$ and $v = T_{\xi-\overline{\xi}}u$, it suffices to show that $u = T_{\overline{\xi}-\xi}\mathcal{G}(v, \psi, \overline{\xi})$. We can closely follow the computation (4.98) in Proposition 4.7.2 and substitute (4.101) to obtain

$$\begin{aligned} T_{\overline{\xi}-\xi}\mathcal{G}(v, \psi, \overline{\xi}) &= E e^{-\xi W}\big[\psi - \Pi(\overline{\xi})\mathrm{ev}_\xi \mathcal{K}\widetilde{R}_\delta(u)\big] + \mathcal{K}\widetilde{R}_\delta(u) \\ &= E e^{-\tau W}[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)] + \mathcal{K}\widetilde{R}_\delta(u) = u. \end{aligned} \tag{4.102}$$

(iii) This follows immediately using the fact that $\mathcal{K}_{\eta_1}$ and $\mathcal{K}_{\eta_2}$ agree on $BC_0(\mathbb{R}, \mathbb{C}^n)$, together with the estimate $\left|R_\delta(\xi, \mathrm{ev}_\xi u^*(\phi, \tau))\right| \le 4\delta L_\delta$, which holds for all $\xi \in \mathbb{R}$.

(iv) If $\delta > 0$ is chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$ and all $\sigma \in \mathbb{R}$ we have

$$w^{\eta_0} L_\delta < (4\left\|\mathcal{K}^\sigma_{\eta_0}\right\|)^{-1}, \tag{4.103}$$

then we may use Lemma 4.6.1 to compute

$$\begin{aligned} (I - Q^\xi_0)\mathrm{ev}_\xi u^*(\phi, \tau) &= (I - Q^\xi_0)\mathrm{ev}_\xi E e^{-\xi W}\big[\phi - \Pi(\xi)\mathrm{ev}_\xi \mathcal{K}_{\eta_0}\widetilde{R}_\delta(u^*(\phi, \tau))\big] \\ &\quad + (I - Q^\xi_0)\mathrm{ev}_\xi \mathcal{K}_{\eta_0}\widetilde{R}_\delta(u^*(\phi, \tau)) \\ &= (I - Q^\xi_0)\mathrm{ev}_\xi \mathcal{K}_{\eta_0}\widetilde{R}_\delta(u^*(\phi, \tau)) \\ &= \mathrm{ev}_0 \mathcal{K}^\xi_{\eta_0} T_\xi \widetilde{R}_\delta(u^*(\phi, \tau)) \end{aligned} \tag{4.104}$$

and hence

$$\begin{array}{rcl}\left\|(I - Q_0^{\xi})\mathrm{ev}_{\xi} u^*(\phi, \tau)\right\| & \leq & w^{\eta_0} \left\|\mathcal{K}_{\eta_0}^{\xi}\right\| \left\|T_{\xi} \widetilde{R}_{\delta}(u^*(\phi, \tau))\right\|_{\eta_0} \\ & \leq & w^{\eta_0} \left\|\mathcal{K}_{\eta_0}^{\xi}\right\| 4\delta L_{\delta} < \delta. \end{array} \tag{4.105}$$

(v) Choose $\delta > 0$ and $\varepsilon > 0$ sufficiently small to ensure that for some $0 < \eta_0 < \gamma$ and all $\tau, \tau' \in \mathbb{R}$,

$$\begin{array}{lcl} w^{2\eta_0} e^{2\pi\eta_0} \|E\|_{\eta_0} e^{2\pi|W|} \left\|\Pi_{\leftarrow}^{\tau}\right\| \varepsilon & < & \frac{1}{2}\delta, \\ 4L_{\delta} \left\|\mathcal{K}_{\eta_0}\right\| w^{2\eta_0} e^{2\pi\eta_0} \big[\|\Pi(\tau)\| & & \\ \quad + e^{2\pi|W|} \left\|\Pi(\tau')\right\| e^{2\pi\eta_0} w^{\eta_0} \|E\|_{\eta_0} \left\|\Pi_{\leftarrow}^{\tau}\right\| \big] & < & \frac{1}{2}\delta. \end{array} \tag{4.106}$$

Recalling that $\tau \in [0, 2\pi]$ and writing $\Delta = \|\Pi(\tau + \theta)\mathrm{ev}_{\tau+\theta} u^*(\phi, \tau)\|$, we compute

$$\begin{array}{rcl} \Delta & = & \left\|\Pi_{\leftarrow}^{\tau+\theta} \mathrm{ev}_{\tau+\theta} E e^{-\tau W}\big[\phi - \Pi(\tau)\mathrm{ev}_{\tau}\mathcal{K}_{\eta_0}\widetilde{R}_{\delta}(u)\big]\right. \\ & & \quad \left. + \Pi(\tau+\theta)\mathrm{ev}_{\tau+\theta}\mathcal{K}_{\eta_0}\widetilde{R}_{\delta}(u)\right\| \\ & \leq & \left\|\Pi_{\leftarrow}^{\tau+\theta}\right\| w^{\eta_0} w^{\eta_0} e^{2\pi\eta_0} \|E\|_{\eta_0} e^{2\pi|W|}\big[\varepsilon + \|\Pi(\tau)\| e^{2\pi\eta_0} w^{\eta_0} \left\|\mathcal{K}_{\eta_0}\right\| 4\delta L_{\delta}\big] \\ & & + \|\Pi(\tau+\theta)\| w^{\eta_0} w^{\eta_0} e^{2\pi\eta_0} \left\|\mathcal{K}_{\eta_0}\right\| 4\delta L_{\delta} \\ & < & \frac{\delta}{2} + \frac{\delta}{2}. \end{array} \tag{4.107}$$

□

In the remainder of this section we will derive an ODE that is satisfied on the finite dimensional center manifold. To this end, we consider an arbitrary pair $(\phi, \tau) \in X_0 \times \mathbb{R}$ and introduce the function $\Phi : \mathbb{R} \to X_0$, given by

$$\Phi(\xi) = \Pi(\xi)\mathrm{ev}_{\xi} u^*(\phi, \tau). \tag{4.108}$$

Notice that we can apply the identity (4.94) to invert this and express $u^*(\phi, \tau)$ in terms of $\Phi(\xi)$. In particular, for any $\overline{\xi}$ for which $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$, we find

$$u^*(\phi, \tau) = T_{\overline{\xi}-\xi} u^*\big(\Phi(\xi), \overline{\xi}\big). \tag{4.109}$$

Setting out to obtain an ODE for Φ, we introduce the shorthand $u = u^*(\phi, \tau)$ and differentiate (4.108) to find

$$\begin{array}{rcl} \Phi'(\xi) & = & [D\Pi(\xi)]\mathrm{ev}_{\xi} u + \Pi(\xi) D[\mathrm{ev}_{\xi} u] \\ & = & [D\Pi(\xi)]\mathrm{ev}_{\xi} u + \Pi(\xi)\mathrm{ev}_{\xi} Du \\ & = & [D\Pi(\xi)]\mathrm{ev}_{\xi} u + \Pi(\xi)\mathrm{ev}_{\xi} Lu + \Pi(\xi)\mathrm{ev}_{\xi}\widetilde{R}_{\delta}(u) \\ & = & [D\Pi(\xi)]\mathrm{ev}_{\xi} u^*(\Phi(\xi), \xi) + \Pi(\xi)\mathrm{ev}_{\xi} L u^*(\Phi(\xi), \xi) \\ & & \quad + \Pi(\xi)\mathrm{ev}_{\xi}\widetilde{R}(u^*(\Phi(\xi), \xi)) \\ & = & [D\Pi(\xi)]\mathrm{ev}_{\xi} E e^{-\xi W}\psi + \Pi(\xi)\mathrm{ev}_{\xi} L E e^{-\xi W}\psi + f(\xi, \Phi(\xi)). \end{array} \tag{4.110}$$

Here the nonlinearity $f(\xi, \psi)$ is of order $O(\|\psi\|^2)$ as $\psi \to 0$ and is explicitly given by

$$\begin{array}{rcl} f(\xi, \psi) & = & [D\Pi(\xi)]\mathrm{ev}_{\xi}[u^*(\psi, \xi) - E e^{-\xi W}\psi] \\ & & + \Pi(\xi)\mathrm{ev}_{\xi} L[u^*(\psi, \xi) - E e^{-\xi W}\psi] \\ & & + \Pi(\xi)\mathrm{ev}_{\xi}\widetilde{R}_{\delta}(u^*(\psi, \xi)). \end{array} \tag{4.111}$$

Using Proposition 4.7.2 one easily sees that f is 2π-periodic in the first variable, i.e., $f(\xi + 2\pi, \psi) = f(\xi, \psi)$ for all $\xi \in \mathbb{R}$ and $\psi \in X_0$. In addition, the C^r-smoothness of Π and the $C^{\min(r,k)}$-smoothness of u^* imply that $f \in C^{\min(r-1,k)}(\mathbb{R} \times X_0, X_0)$.

It remains to treat the linear part of (4.110). Defining $y = Ee^{-\xi W}\psi \in \mathcal{N}_0$, notice that

$$\begin{array}{rcl} [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi)\mathrm{ev}_\xi L y & = & [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi)\mathrm{ev}_\xi D y \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi) D\mathrm{ev}_\xi y \\ & = & D[\Pi(\xi)\mathrm{ev}_\xi y] = D[e^{\xi W}\mathrm{ev}_{-\xi} E \mathrm{ev}_\xi y] \\ & = & D[e^{\xi W} ev_0 y] = We^{\xi W} ev_0 y \\ & = & W\psi. \end{array} \tag{4.112}$$

We have hence established the following result.

Proposition 4.7.3. *Consider the setting of Theorem 4.7.1. For any $(\phi, \tau) \in X_0 \times \mathbb{R}$, define the function $\Phi : \mathbb{R} \to X_0$ given by $\Phi(\xi) = \Pi(\xi)\mathrm{ev}_\xi u^*(\phi, \tau)$. Then Φ is $C^{\min(r,k+1)}$-smooth and satisfies the ordinary differential equation*

$$\Phi'(\xi) = W\Phi(\xi) + f(\xi, \Phi(\xi)). \tag{4.113}$$

Here the function $f : \mathbb{R} \times X_0 \to X_0$, which is explicitly given by (4.111), is $C^{\min(r-1,k)}$-smooth and satisfies $f(\xi + 2\pi, \psi) = f(\xi, \psi)$ for all $(\xi, \psi) \in \mathbb{R} \times X_0$. Finally, we have $f(\xi, 0) = 0$ and $D_2 f(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$.

In a standard fashion, one may now use the ODE derived above in conjunction with the properties of u^* established in Theorem 4.7.1 to prove our main results in Theorem 4.2.2. As a final remark, we observe that in the constant coefficient situation where $L(\xi) = L$, we have $T_\xi u^*(\cdot, \xi) = u^*(\cdot, 0)$ and $\Pi(\xi) = Q_0$ for all $\xi \in \mathbb{R}$, which shows that the definition of f reduces correctly to the form derived in Chapter 2.

4.8. Smoothness of the center manifold

In this section we address the smoothness of the center manifold established above. In particular, we set out to prove item (vi) of Theorem 4.7.1. Throughout this section we consider a fixed system (4.8) that satisfies the conditions (HL), (HF), (HR1) and (HR2) and recall the corresponding integers r and k. In addition, we fix an interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ as in the setting of Theorem 4.7.1. In order to ease notation we will assume that $r \geq k$, but we remark that upon interchanging k and r all our arguments here remain valid when in fact $r < k$. Our arguments here are based on the strategy developed in [45, Section IX.7] and will extend the proof given in Chapter 2 for autonomous versions of (4.8).

Due to the presence of the cutoff function on the infinite dimensional complement of X_0, the nonlinearity R_δ loses the C^k-smoothness on X and becomes merely Lipschitz continuous. To correct for this situation, we introduce for any $\eta > 0$ the Banach space

$$\begin{array}{rcl} V^1_\eta(\mathbb{R}, \mathbb{C}^n) & = & \Big\{ u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \|u\|_{V^1_\eta} := \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} \left\| \Pi(\xi) u_\xi \right\| \\ & & \quad + \sup_{\xi \in \mathbb{R}} \left\| (I - Q^\xi_0) u_\xi \right\| + \left\| u' \right\|_\eta < \infty \Big\}, \end{array} \tag{4.114}$$

which is continuously embedded in $BC^1_\eta(\mathbb{R},\mathbb{C}^n)$, together with the open set

$$V^{1,\delta}_\eta(\mathbb{R},\mathbb{C}^n) = \left\{u \in BC^1_\eta(\mathbb{R},\mathbb{C}^n) \mid \sup_{\xi\in\mathbb{R}} \left\|(I - Q^\xi_0)u_\xi\right\| < \delta\right\} \subset V^1_\eta(\mathbb{R},\mathbb{C}^n). \tag{4.115}$$

We start by establishing conditions under which the substitution maps $\widetilde{R}_\delta : V^{1,\delta}_\sigma(\mathbb{R},\mathbb{C}^n) \to BC^1_\zeta(\mathbb{R},\mathbb{C}^n)$ are smooth. Notice that R_δ is of class C^k on the set B^h_δ, in which

$$B^h_\delta = \left\{(\xi,\phi) \in \mathbb{R}\times X \mid \left\|(I - Q^\xi_0)\phi\right\| < \delta\right\}. \tag{4.116}$$

Considering any pair of integers $p \geq 0$, $q \geq 0$ with $p+q \leq k$, observe that the norms $\left\|D^p_1 D^q_2 R_\delta(\xi,\phi)\right\|$ are uniformly bounded on B^h_δ. Thus, for any $u \in C(\mathbb{R},\mathbb{C}^n)$ for which $\sup_{\xi\in\mathbb{R}}\left\|(I - Q^\xi_0)u_\xi\right\| < \delta$ and for any $0 \leq p \leq k$, we can define a map $\widetilde{R}^{(p,q)}_\delta(u) \in \mathcal{L}^{(q)}(C(\mathbb{R},\mathbb{C}^n), C(\mathbb{R},\mathbb{C}^n))$ by

$$\widetilde{R}^{(p,q)}_\delta(u)(v_1,\ldots,v_q)(\xi) = D^p_1 D^q_2 R_\delta(\xi,u_\xi)\big((v_1)_\xi,\ldots,(v_q)_\xi\big). \tag{4.117}$$

Here the symbol $\mathcal{L}^{(q)}(Y,Z)$ denotes the space of q-linear mappings from $Y\times\ldots\times Y$ into Z. Note that the map $\widetilde{R}^{(p,q)}_\delta(u)$ defined above is well-defined, since $D^p_1 D^q_2 R_\delta$ is a continuous map from $B^h_\delta \times X^q$ into $\mathbb{C}^n$, as is the map $i_x : \mathbb{R} \to X$ which sends $\xi \mapsto x_\xi$, for any $x \in C(\mathbb{R},\mathbb{C}^n)$. Throughout the remainder of this section we will adopt the shorthand $BC^1_\zeta = BC^1_\zeta(\mathbb{R},\mathbb{C}^n)$, together with analogous ones for the other function spaces. In addition, we write BC^ℓ_ζ for the space of C^ℓ-smooth functions f that have $D^j f \in BC_\zeta$ for all $0 \leq j \leq \ell$. The following two results are stated without proof, as they are very similar to their counterparts in Chapter 2.

Proposition 4.8.1. *Let $p \geq 0$ and $q \geq 0$ be positive integers with $p+q \leq k$. Pick $\eta \geq q\zeta > 0$. Then for any $u \in C(\mathbb{R},\mathbb{C}^n)$ such that $\sup_{\xi\in\mathbb{R}}\left\|(I - Q^\xi_0)u_\xi\right\| < \delta$, we have*

$$\widetilde{R}^{(p,q)}_\delta(u) \in \mathcal{L}^{(q)}(BC^1_\zeta, BC_\eta) \cap \mathcal{L}^{(q)}(V^1_\zeta, BC_\eta), \tag{4.118}$$

where the norm is bounded by

$$\left\|\widetilde{R}^{(p,q)}_\delta\right\|_{\mathcal{L}^{(q)}} \leq w^\zeta \sup_{\xi\in\mathbb{R}} e^{-(\eta-q\zeta)|\xi|}\left\|D^p_1 D^q_2 R_\delta(\xi,u_\xi)\right\| < \infty. \tag{4.119}$$

Furthermore, consider any $0 \leq \ell \leq k-(p+q)$ and any $\sigma > 0$. If $\eta > q\zeta + \ell\sigma$, then in addition the map $u \mapsto \widetilde{R}^{(p,q)}_\delta(u)$ from $V^{1,\delta}_\sigma$ into $\mathcal{L}^{(q)}(BC^1_\zeta, BC_\eta)$ is C^ℓ-smooth, with $D^\ell\widetilde{R}^{(p,q)}_\delta = \widetilde{R}^{(p,q+l)}_\delta$. The same holds when considering $u \mapsto \widetilde{R}^{(p,q)}_\delta(u)$ as a map from $V^{1,\delta}_\sigma$ into $\mathcal{L}^{(q)}(V^1_\zeta, BC_\eta)$.

Finally, if $p+q < k$, consider any $u \in V^{1,\delta}_\sigma$. Then for any q-tuple of functions

$v_1, \ldots, v_q \in BC^1_\zeta$, we have $\widetilde{R}^{(p,q)}_\delta(u)(v_1, \ldots, v_q) \in C^1(\mathbb{R}, \mathbb{C}^n)$, with

$$\begin{aligned} D_\xi \widetilde{R}^{(p,q)}_\delta(u)(v_1, \ldots, v_q) \;=\; & \widetilde{R}^{(p+1,q)}_\delta(u)(v_1, \ldots, v_q) + \widetilde{R}^{(p,q+1)}_\delta(u)(u', v_1, \ldots, v_q) \\ & + \widetilde{R}^{(p,q)}_\delta(u)(v_1', v_2, \ldots, v_q) + \ldots \\ & + \widetilde{R}^{(p,q)}_\delta(u)(v_1, v_2, \ldots, v_q'). \end{aligned} \tag{4.120}$$

Proposition 4.8.2. *Consider integers $p \geq 0$ and $q \geq 0$ with $p + q < k$. Let $\eta > q\zeta + \sigma$ for some $\zeta > 0$ and $\sigma > 0$. Let Φ be a mapping of class C^1 from $X_0 \times \mathbb{R}$ into $V^{1,\delta}_\sigma$. Then the mapping $\widetilde{R}^{(p,q)}_\delta \circ \Phi$ from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(q)}(BC^1_\zeta, BC_\eta)$ is of class C^1 with*

$$\begin{aligned} D(\widetilde{R}^{(p,q)}_\delta \circ \Phi)(\phi, \tau)(v_1, \ldots, v_q, (\psi, \xi)) \;=\; & \widetilde{R}^{(p,q+1)}(\Phi(\phi, \tau)) \\ & (v_1, \ldots, v_q, D\Phi(\phi, \tau)(\psi, \xi)). \end{aligned} \tag{4.121}$$

For convenience, we introduce for any $\eta \in [\eta_{\min}, \eta_{\max}]$ the function $\mathcal{E} : V^{1,\delta}_\eta \times (X_0 \times [0, 2\pi]) \to BC^1_\eta$ via

$$\mathcal{E}\big(u, (\phi, \tau)\big) = Ee^{-\tau W}\big[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)\big]. \tag{4.122}$$

One may compute the partial derivatives

$$\begin{aligned} D_1\mathcal{E}\big(u, (\phi, \tau)\big) \;=\; & -Ee^{-\tau W}\Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}^{(0,1)}_\delta(u), \\ D_2\mathcal{E}\big(u, (\phi, \tau)\big) \;=\; & Ee^{-\tau W} \times \Big(- EWe^{-\tau W}[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)] \\ & -Ee^{-\tau W}[D\Pi(\tau)]\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u) \\ & +\Pi(\tau)\mathrm{ev}_\tau[L\mathcal{K}\widetilde{R}_\delta(u) + \widetilde{R}_\delta(u)]\Big) \end{aligned} \tag{4.123}$$

and easily conclude that these are both continuous functions. This means that $\mathcal{E}$ is at least C^1-smooth and in addition this enables us to define the continuous auxiliary functions $\mathcal{F}_1 : X_0 \times [0, 2\pi] \to \mathcal{L}(V^1_\eta, BC^1_\eta) \cap \mathcal{L}(BC^1_\eta, BC^1_\eta)$ and $\mathcal{F}_2 : X_0 \times [0, 2\pi] \to \mathcal{L}(X_0 \times \mathbb{R}, BC^1_\eta)$ by

$$\begin{aligned} \mathcal{F}_1(\phi, \tau) \;=\; & D_1\mathcal{E}\big(u^*(\phi, \tau), (\phi, \tau)\big), \\ \mathcal{F}_2(\phi, \tau) \;=\; & D_2\mathcal{E}\big(u^*(\phi, \tau), (\phi, \tau)\big). \end{aligned} \tag{4.124}$$

Notice that Proposition 4.8.1 implies that $\mathcal{F}_1$ is indeed well-defined as a map into $\mathcal{L}(BC^1_\eta, BC^1_\eta)$.

We will employ an induction approach towards establishing the smoothness of u^*. The next result serves as a starting point by obtaining the C^1-smoothness.

Proposition 4.8.3. *For all $\eta \in (\eta_{\min}, \eta_{\max}]$, the function $\mathcal{J}^1_{\eta\eta_{\min}} u^*_{\eta_{\min}} : X_0 \times [0, 2\pi] \to BC^1_\eta$ is C^1-smooth. In addition, for each $1 \leq p \leq k$ and all $\eta \in (p\eta_{\min}, \eta_{\max}]$, the function*

$$(\phi, \tau) \mapsto \mathcal{J}^1_{\eta\; p\eta_{\min}} D^p_\xi u^*_{\eta_{\min}}(\phi, \tau), \tag{4.125}$$

which maps $X_0 \times [0, 2\pi]$ into BC^1_η, is continuous.

Proof. Consider any $\eta \in (\eta_{\min}, \eta_{\max}]$. We will apply Lemma A.2 in the setting $Y_0 = V^1_{\eta_{\min}}$, $Y = BC^1_{\eta_{\min}}$ and $Y_1 = BC^1_{\eta}$, together with their natural inclusions. Furthermore, we choose $\Omega_0 = V^{1,\delta}_{\eta_{\min}} \subset V^1_{\eta_{\min}}$ and let $\Lambda = X_0 \times \mathbb{R}$ with $\Lambda_0 = X_0 \times [0, 2\pi]$. For any $(\phi, \tau) \in X \times [0, 2\pi]$, the operators featuring in Appendix A are defined by

$$\begin{array}{rcll} F(u, \phi, \tau) & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_{\eta_{\min}} \widetilde{R}_\delta(u), & u \in BC^1_{\eta_{\min}}, \\ F^{(1)}(u, \phi, \tau) & = & D_1\mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(0,1)}_\delta(u) \in \mathcal{L}(BC^1_{\eta_{\min}}), & u \in V^{1,\delta}_{\eta_{\min}}, \\ F^{(1)}_1(u, \phi, \tau) & = & D_1\mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_{\eta} \circ \widetilde{R}^{(0,1)}_\delta(u) \in \mathcal{L}(BC^1_{\eta}), & u \in V^{1,\delta}_{\eta_{\min}}. \end{array} \tag{4.126}$$

In the context of Lemma A.2 this means that $G : V^{1,\delta}_{\eta_{\min}} \times X_0 \times [0, 2\pi] \to BC^1_\eta$ is defined by

$$\begin{array}{rcl} G(u, \phi, \tau) & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{J}^1_{\eta\eta_{\min}} \mathcal{K}_{\eta_{\min}} \widetilde{R}_\delta(u) \\ & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_\eta \widetilde{R}_\delta(u), \end{array} \tag{4.127}$$

in which the final equality follows from the fact that $\mathcal{K}_{\eta_{\min}}$ and $\mathcal{K}_\eta$ agree on BC_0.

Conditions (HC1), (HC3) and (HC4) are satisfied due to the C^1-smoothness of $\mathcal{E}$, together with Proposition 4.8.1. The inequality (4.99) implies (HC2) and (HC5), while (HC6) follows from (4.103). We conclude that $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^1 and that $D(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}})(\phi, \xi) = \mathcal{J}^1_{\eta\eta_{\min}} \circ u^{*(1)}_{\eta_{\min}}(\phi, \xi) \in \mathcal{L}(X_0 \times \mathbb{R}, BC^1_\eta)$, where $u^{*(1)}_{\eta_{\min}}(\phi, \xi)$ is the unique solution of the equation

$$u^{(1)} \;=\; [\mathcal{F}_1(\phi, \tau) + \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(0,1)}(u^*_{\eta_{\min}}(\phi, \tau))]u^{(1)} + \mathcal{F}_2(\phi, \tau) \tag{4.128}$$

in the space $\mathcal{L}(X_0 \times \mathbb{R}, BC^1_{\eta_{\min}})$. We compute

$$\begin{array}{rcl} D_\xi u^*_{\eta_{\min}}(\phi, \tau) & = & L u^*_{\eta_{\min}}(\phi, \tau) + \widetilde{R}_\delta(u^*_{\eta_{\min}}(\phi, \tau)) \\ D^2_\xi u^*_{\eta_{\min}}(\phi, \tau) & = & [DL]u^*_{\eta_{\min}}(\phi, \tau) + L[D_\xi u^*_{\eta_{\min}}(\phi, \tau)] + \\ & & \widetilde{R}^{(1,0)}(u^*_{\eta_{\min}}(\phi, \xi)) + \widetilde{R}^{(0,1)}(u^*_{\eta_{\min}}(\phi, \tau)) D_\xi u^*_{\eta_{\min}}(\phi, \tau) \end{array} \tag{4.129}$$

and hence $(\phi, \tau) \mapsto \mathcal{J}^1_{\eta\eta_{\min}} D_\xi u^*_{\eta_{\min}}(\phi, \tau)$ is continuous. It is easy to see that (4.129) can be differentiated another $k - 1$ times, showing that in general $(\phi, \tau) \mapsto \mathcal{J}^1_{\eta\,\ell\eta_{\min}} D^\ell_\xi u^*_{\eta_{\min}}(\phi, \tau)$ is continuous for $1 \le \ell \le k$. □

In the interest of clarity, we specify in some detail the induction hypothesis that we use prior to performing the induction step, To this end, consider any integer ℓ that satisfies $1 \le \ell < k$ and suppose that for all $1 \le q \le \ell$, there exist mappings

$$u^{*(q)}_{\eta_{\min}} : X_0 \times [0, 2\pi] \to \mathcal{L}^{(q)}(X_0 \times \mathbb{R}, BC^1_{q\eta_{\min}}), \tag{4.130}$$

such that the following properties are satisfied.

(IH1) For all $1 \le q \le \ell$ and for all $\eta \in (q\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^q with

$$D^q(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}) = \mathcal{J}^1_{\eta\, q\eta_{\min}} \circ u^{*(q)}_{\eta_{\min}}. \tag{4.131}$$

(IH2) For all integer pairs (p,q) with $0 \le q \le \ell$ and $1 \le p \le k-q$ and all $\eta \in ((p+q)\eta_{\min}, \eta_{\max}]$, the function $X_0 \times [0,2\pi] \to \mathcal{L}^{(q)}(X_0 \times \mathbb{R}, BC^1_\eta)$, defined by

$$(\phi,\tau) \mapsto \mathcal{J}^1_{\eta\,(p+q)\eta_{\min}} D^p_\xi u^{*(q)}(\phi,\tau), \tag{4.132}$$

is continuous.

(IH3) For any pair $(\phi,\tau) \in X_0 \times [0,2\pi]$, the map $u^{*(\ell)}_{\eta_{\min}}(\phi,\tau)$ is the unique solution at $\overline{\eta} = \eta_{\min}$ of an equation of the form

$$u^{(\ell)} = F^{(\ell)}_{\overline{\eta}}(u^{(\ell)},\phi,\tau) \tag{4.133}$$

in the space $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\overline{\eta}})$, with

$$\begin{aligned} F^{(\ell)}_{\overline{\eta}}(u^{(\ell)},\phi,\tau) \;=\; & \big[\mathcal{F}_1(\phi,\tau) + \mathcal{K}_{\ell\overline{\eta}} \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))\big]u^{(\ell)} \\ & + D^{\ell-1}\mathcal{F}_2(\phi,\xi) + H^{(\ell)}_{\overline{\eta}}(\phi,\tau). \end{aligned} \tag{4.134}$$

Here we have $H^{(1)}(\phi,\tau) = 0$ and for $\ell \ge 2$ we can write $H^{(\ell)}_{\overline{\eta}}(\phi,\tau)$ as a finite sum of terms of two different forms, the first of which is given by

$$\mathcal{K}_{\ell\overline{\eta}} \circ \widetilde{R}^{(0,q)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(e_1)}_{\eta_{\min}}(\phi,\tau),\ldots,u^{*(e_q)}_{\eta_{\min}}(\phi,\tau)), \tag{4.135}$$

with $2 \le q \le \ell$ and integers $e_i \ge 1$ such that $e_1 + \ldots + e_q = \ell$. The second form can be written as

$$D^{f_1}\mathcal{F}_1(\phi,\tau)u^{*(f_2)}_{\eta_{\min}}(\phi,\tau), \tag{4.136}$$

with integers $f_1 \ge 1$ and $f_2 \ge 1$ that satisfy $f_1 + f_2 = \ell$.

Using Proposition 4.8.3 it is easily verified that the assumptions above are satisfied for $\ell = 1$. Before proceeding with the remaining cases, we need to study the smoothness of the operators $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 4.8.4. *Suppose that for some integer $1 \le \ell < k$ the induction assumptions (IH1) through (IH3) all hold. Then for any $\eta \in [\eta_{\min}, \eta_{\max}]$, the functions $\mathcal{F}_1 : X_0 \times [0,2\pi] \to \mathcal{L}(BC^\ell_\eta, BC^1_\eta)$ and $\mathcal{F}_2 : X_0 \times [0,2\pi] \to \mathcal{L}(X_0 \times \mathbb{R}, BC^1_\eta)$ are C^ℓ-smooth.*

Proof. Upon defining $\mathcal{E}'(u,(\phi,\tau)) = \mathcal{E}(u,(\phi,\tau)) - Ee^{-\tau W}\phi$, we remark that it is sufficient to establish the claim for the operators $\mathcal{F}'_1$ and $\mathcal{F}'_2$ associated to $\mathcal{E}'$. Observe first that for $i = 1,2$ we can write $D^\ell\mathcal{F}'_i(\phi,\tau)$ as a sum of terms of the form

$$D_1^{\alpha_1}D_2^{\alpha_2}\mathcal{E}'(u^*(\phi,\tau),(\phi,\tau))u^{*(\beta_1)}(\phi,\tau)\ldots u^{*(\beta_{n_\beta})}(\phi,\tau), \tag{4.137}$$

in which $\beta_j \ge 1$ for $1 \le j \le n_\beta$. If $i = 1$, then we have in addition that $\alpha_1 \ge 1$, $n_\beta = \alpha_1 - 1$ and $\alpha_2 + \beta_1 + \ldots + \beta_{n_\beta} = \ell$. If however $i = 2$, then we have $\alpha_2 \ge 1$, $n_\beta = \alpha_1$ and $\alpha_2 + \beta_1 + \ldots + \beta_{n_\beta} = \ell + 1$.

Now notice that the only nonzero component of $D_1^{\alpha_1} D_2^{\alpha_2} \mathcal{E}'(u, (\phi, \tau))$ can be written as a sum of terms of the form

$$EW^{\gamma_0} e^{-\tau W} (D^{\gamma_1} \Pi)(\tau) \mathrm{ev}_\tau D_\xi^{\gamma_2} \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u), \tag{4.138}$$

in which $\gamma_j \geq 0$ for $0 \leq j \leq 2$ with $\gamma_0 + \gamma_1 + \gamma_2 = \alpha_2$. Setting out to compute the derivatives with respect to ξ appearing in (4.138), notice first that

$$D_\xi \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u) = L \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u) + \widetilde{R}_\delta^{(0,\alpha_1)}(u). \tag{4.139}$$

Generalizing, we obtain that $D_\xi^{\gamma_2} \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u)$ can be written as a sum of terms of two different forms, the first of which is given by

$$[D^{e_1} L] \ldots [D^{e_{n_e}} L] \widetilde{R}^{(p,\alpha_1+q)}(u)(D_\xi^{f_1} u, \ldots, D_\xi^{f_q} u)(D_\xi^{g_1}, \ldots, D_\xi^{g_{\alpha_1}}), \tag{4.140}$$

in which we have $p \geq 0$, $q \geq 0$ and $n_e \geq 0$, together with $e_j \geq 0$ for all $1 \leq j \leq n_e$, $f_j \geq 1$ for all $1 \leq j \leq q$ and $g_j \geq 0$ for all $1 \leq j \leq \alpha_1$. In addition, we must have

$$1 + n_e + e_1 + \ldots + e_{n_e} + p + f_1 + \ldots + f_q + g_1 + \ldots + g_{\alpha_1} = \gamma_2. \tag{4.141}$$

The second form is given by

$$[D^{e_1} L] \ldots [D^{e_{n_e}} L] \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u), \tag{4.142}$$

in which $n_e \geq 0$, $e_j \geq 0$ for all $1 \leq j \leq n_e$ and

$$n_e + e_1 + \ldots + e_{n_e} = \gamma_2. \tag{4.143}$$

Indeed, this can be verified directly for $\gamma_2 = 1$ and differentiation of the terms in (4.140) and (4.142) again gives terms of these forms.

It remains to show that the terms (4.140) and (4.142) are continuous after substituting $u = u^*(\phi, \tau)$. In view of Proposition (IH2), it suffices to check that we have $\alpha_1 + p + q \leq k$, $e_j \leq r$ for $1 \leq j \leq n_e$, $f_j \leq \ell$ for $1 \leq j \leq q$ and $g_j + \beta_j \leq \ell$ for $1 \leq j \leq n_\beta$. If in fact we have $i = 1$, i.e., we are considering $D^\ell \mathcal{F}_1'$, then we in addition need $g_{\alpha_1} \leq \ell - 1$ to ensure that BC_η^ℓ is mapped into BC_η^1 under the operator $D_\xi^{g_{\alpha_1}}$. All these inequalities can easily be verified by using the conditions (4.141) and (4.143). □

Proof of item (vi) of Theorem 4.7.1. Assume that for some $1 \leq \ell < k$, the induction assumptions (IH1) through (IH3) are satisfied. Notice that these conditions ensure that $F_{\overline{\eta}}^{(\ell)} : \mathcal{L}^{(\ell)}(X_0, BC_{\ell\overline{\eta}}^1) \times X_0 \to \mathcal{L}^{(\ell)}(X_0, BC_{\ell\overline{\eta}}^1)$ is well-defined for all $\overline{\eta} \in [\eta_{\min}, \frac{1}{\ell}\eta_{\max}]$ and, in addition, is a uniform contraction for these values of $\overline{\eta}$. We now fix $\eta \in ((\ell + 1)\eta_{\min}, \eta_{\max}]$ and choose σ and ζ such that $\eta_{\min} < \sigma < (\ell + 1)\sigma < \zeta < \eta$. We wish to apply Lemma A.2 in the setting $\Omega_0 = Y_0 = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC_{\ell\sigma}^1)$, $Y = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC_\zeta^1)$, $Y_1 = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC_\eta^1)$ with the corresponding natural inclusions, with the parameter space given by $\Lambda_0 = X_0 \times [0, 2\pi]$ with $\Lambda_0 \subset \Lambda = X_0 \times \mathbb{R}$.

For any $(\phi, \tau) \in X_0 \times [0, 2\pi]$, we define the functions

$$\begin{array}{rcl} F(u^{(\ell)}, \phi, \tau) & = & [\mathcal{F}_1(\phi, \tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))]u^{(\ell)} + D^{\ell-1}\mathcal{F}_2(\phi, \tau) \\ & & +H^{(\ell)}_{\zeta/\ell}(\phi), \\ F^{(1)}(u^{(\ell)}, \phi, \tau) & = & \mathcal{F}_1(\phi, \tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau)) \in \mathcal{L}(\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)), \\ F^{(1)}_1(u^{(\ell)}, \phi, \tau) & = & \mathcal{F}_1(\phi, \tau) + \mathcal{K}_\eta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau)) \in \mathcal{L}(\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\eta)), \end{array} \tag{4.144}$$

in which we take $u^{(\ell)} \in \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)$ in the definition of F and $u^{(\ell)} \in \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma})$ for $F^{(1)}$ and $F^{(1)}_1$. To check (HC1), we need to show that the map $G : \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma}) \times X_0 \times [0, 2\pi] \to \mathcal{L}^{(\ell)}(X_0, BC^1_\eta)$ given by

$$\begin{array}{rcl} G(u^{(\ell)}, \phi, \tau) & = & \big[\mathcal{F}_1(\phi, \tau) + \mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))\big]u^{(\ell)} \\ & & +D^{\ell-1}\mathcal{F}_2(\phi, \tau) + \mathcal{J}^1_{\eta\zeta} H^{(\ell)}_{\zeta/\ell}(\phi, \tau) \end{array} \tag{4.145}$$

is of class C^1. In view of the linearity of this map with respect to $u^{(\ell)}$, together with the smoothness of $\mathcal{F}_1$ and $\mathcal{F}_2$ as established in Proposition 4.8.4, it is sufficient to show that $(\phi, \tau) \mapsto \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))$ is of class C^1 as a map from $X_0 \times \mathbb{R}$ into $\mathcal{L}(BC^1_{\ell\sigma}, BC^1_\zeta)$ and, in addition, that $(\phi, \tau) \mapsto H^{(\ell)}_{\zeta/\ell}(\phi, \tau)$ is of class C^1 as a map from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)$. The first fact follows from Proposition 4.8.2 using $\zeta > (\ell + 1)\sigma$ and the C^1-smoothness of the map $(\phi, \tau) \mapsto \mathcal{J}^1_{\sigma\eta_{\min}} u^*_{\eta_{\min}}(\phi, \tau)$. To verify the second fact, we again use Proposition 4.8.2 to differentiate the components of $H^{(\ell)}$ given in (4.135) and (4.136). The first component yields

$$\begin{array}{l} D\mathcal{K}_\zeta \circ \widetilde{R}^{(0,q)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))(u^{*(e_1)}_{\eta_{\min}}(\phi, \tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi, \tau)) \\ \quad = \mathcal{K}_\zeta \circ \widetilde{R}^{(0,q+1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))(u^{*(e_1)}_{\eta_{\min}}(\phi, \tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi, \tau), u^{*(1)}_{\eta_{\min}}(\phi, \tau)) \\ \quad + \sum_{j=1}^q \mathcal{K}_\zeta \circ \widetilde{R}^{(0,q)}(u^*_{\eta_{\min}}(\phi, \tau))(u^{*(e_1)}_{\eta_{\min}}(\phi, \tau), \ldots, u^{*(e_j+1)}_{\eta_{\min}}(\phi, \tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi, \tau)), \end{array} \tag{4.146}$$

in which each occurrence of $u^{*(j)}_{\eta_{\min}}$ is understood to map into $BC^1_{j\sigma}$. An application of Proposition 4.8.1 with $\zeta > (\ell+1)\sigma$, shows that the above map is indeed continuous from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_\zeta)$. The second component can be treated using similar arguments in conjunction with Proposition 4.8.4. These arguments immediately show that also (HC4) is satisfied. Conditions (HC2), (HC3) and (HC5) can be verified much as before. Finally, (HC6) follows from the fact that $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\eta_{\min}}) \subset \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma})$.

We thus conclude from Lemma A.2 that $\mathcal{J}^1_{\eta\,\ell\eta_{\min}} \circ u^{*(\ell)}_{\eta_{\min}}$ is of class C^1 with $D(\mathcal{J}^1_{\eta\,\ell\eta_{\min}} \circ u^{*(\ell)}_{\eta_{\min}})(\phi, \tau) = \mathcal{J}^1_{\eta\zeta} \circ u^{*(\ell+1)}(\phi, \tau)$, in which $u^{*(\ell+1)}(\phi, \tau)$ is the unique solution of the equation

$$\begin{array}{rcl} u^{(\ell+1)} & = & \big[\mathcal{F}_1(\phi, \tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))\big]u^{(\ell+1)} + D^\ell\mathcal{F}_2(\phi, \tau) \\ & & +H^{(\ell+1)}_{\zeta/(\ell+1)}(\phi, \tau) \end{array} \tag{4.147}$$

in $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_\zeta)$, with

$$H^{(\ell+1)}_{\zeta/(\ell+1)}(\phi, \tau) = \mathcal{K}_\zeta \circ \widetilde{R}^{(0,2)}_\delta(u^*_{\eta_{\min}}(\phi, \tau))(u^{*(\ell)}_{\eta_{\min}}(\phi, \tau), u^{*(1)}_{\eta_{\min}}(\phi, \tau)) + DH^{(\ell)}_{\zeta/\ell}(\phi, \tau). \tag{4.148}$$

However, the definition (4.148) remains valid upon writing $\zeta = (\ell + 1)\eta_{\min}$. This allows one to define $H^{(\ell+1)}_{\eta_{\min}} \in \mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$ in a natural fashion, with $H^{(\ell+1)}_{\zeta/(\ell+1)} = \mathcal{J}^1_{\zeta\ (\ell+1)\eta_{\min}} H^{(\ell+1)}_{\eta_{\min}}$. We hence conclude that the fixed point $u^{*(\ell+1)}(\phi, \tau)$ of (4.147) is also contained in $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$. We can hence define $u^{*(\ell+1)}_{\eta_{\min}} = u^{*(\ell+1)}(\phi, \tau) \in \mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$. In order to complete the proof, it remains only to consider the statements in (IH2) that involve the D_ξ derivatives. However, these follow from inspection, repeatedly using $D_\xi \mathcal{K} f = L\mathcal{K} f + f$ together with (4.120). □

Chapter 5

Travelling Waves Close to Propagation Failure

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Abstract. We analyze a variant of Newton's Method for computing travelling wave solutions to scalar bistable lattice differential equations. We prove that the method converges to a solution, obtain existence and uniqueness of solutions to such equations with a small second order term and study the limiting behaviour of such solutions as this second order term tends to zero. The robustness of the algorithm will be discussed using numerical examples. These results will also be used to illustrate phenomena like propagation failure, which are encountered when studying lattice differential equations. We finish by discussing the broad application range of the method and illustrate that higher dimensional systems exhibit richer behaviour than their scalar counterparts.

5.1. Introduction

The main purpose of this chapter is to analyze a numerical method to solve families of scalar bistable differential difference equations of the form

$$-\gamma\phi''(\xi) - c\phi'(\xi) = F\big(\phi(\xi), \phi(\xi + r_1), \ldots, \phi(\xi + r_N), \rho\big). \tag{5.1}$$

Here $\gamma \geq 0$ is a fixed parameter, c is an unknown wavespeed, ρ can be thought of as a detuning parameter and the diagonal function $-F(x, \ldots, x, \rho)$ is an N-shaped function which depends C^1-smoothly on ρ. The numbers r_i are shifts which may have either sign.

The algorithm we discuss consists of a combination of a Newton-type method with parameter continuation techniques and is based upon ideas proposed in [1, 10, 53]. Our main contribution here is to give a detailed analysis of the method. In particular, we shall show that the algorithm converges to a solution of (5.1) and use numerical examples to discuss some of the issues involved when solving (5.1). In addition, we shall obtain existence and uniqueness of connecting solutions to (5.1) and prove that these solutions depend C^1-smoothly on the detuning parameter ρ. These results extend earlier results obtained by Mallet-Paret in [113], where the $\gamma = 0$ case was treated. To relate this interesting and widely studied case to the numerically feasible situation where $\gamma > 0$, we shall also prove that a sequence of solutions to (5.1) with γ tending to zero converges to a solution with $\gamma = 0$.

The primary motivation for the study of our main equation (5.1) comes from the analysis of travelling wave solutions to lattice differential equations, as explained in detail in Chapter 1. The early work by Chi, Bell and Hassard [32] already contained computations of solutions to LDEs and Elmer and Van Vleck have performed extensive calculations on equations of the form (5.1) in [50, 51, 52, 53]. In their early works [50, 51], the nonlinearity f was replaced by an idealized nonlinearity, but this restriction was lifted in [53], where a larger class of bistable functions f is considered. At present, they are pursuing a collocation approach to solve a class of functional differential equations which includes the family (5.1) [1]. We note here that when applying the methods in [1, 53] to (5.1), one essentially performs a series of Newton iterations of the same type as those studied in this chapter, which means that the theory developed here can be directly applied to this situation. Our results should thus be seen as a first step towards establishing a general theoretical background for the numerical analysis of (5.1).

Notice that the equation (1.11) which governs the behaviour of travelling wave solutions to LDEs does not contain a second derivative term, in contrast to the family (5.1) where γ may be strictly positive. As we have seen in Sectin 1.1 while discussing the phenomenon of propagation failure, very interesting features of lattice differential equations arise when $\gamma = 0$ and the wavespeed c satisfies $c \approx 0$. Unfortunately, the possible lack of continuity properties of the solutions in this regime makes it extremely difficult to numerically solve (5.1) directly, as all known methods would require handling singularly perturbed boundary value problems. However, setting $\gamma > 0$ in (5.1) has a smoothening effect on solutions, ensuring every solution to be at least twice differentiable. This allows the succesfull application of numerical techniques to solve (5.1) even as $c \to 0$, but immediately raises the question if the rich behaviour in the limit $\gamma, c \to 0$ can still be uncovered. In this chapter we give rigorous theoretical and numerical evidence that this is indeed the case. In particular, we prove in Theorem 5.3.12 that solutions to (5.1) with increasingly small γ converge to a solution with $\gamma = 0$. We strengthen the argument in Sections 5.5 and 5.6 by discussing a number of numerical examples which clearly exhibit the phenomenon of propagation failure. These examples also illustrate the important fact that the convergence proved in Theorem 5.3.12 already occurs at numerically feasible values of γ.

In addition to the technical reasons mentioned above, there is also a physical reason to introduce a second order term in (5.1). Such a term arises naturally if we consider systems which have local as well as nonlocal interactions and it allows us to perform continuation from systems with a continuous Laplacian to systems with a discrete Laplacian. As an example in solid-state physics, we mention the Frenkel-Kontorova type equations discussed in [151, 152].

The numerical method discussed in this chapter combines the merits of both the strategies employed in [1, 53]. In particular, we remark here that the direct collocation technique employed in [1] is numerically robuster than the method used in [53], but also requires significantly more computer time and storage space to execute a Newton iteration step. In Section 5.4 we show that away from the continuous limit, i.e., for small values of α in (1.11), the approach in [53] can be expected to work best. In Section 5.5 this information is combined with our continuation techniques to give a more thorough investigation into the phenomenon of propagation failure than previously possible. On the other hand, in Section 5.6.2 we numerically solve a two dimensional periodic diffusion problem, which requires the robustness of the direct collocation technique along with our path following strategies.

This chapter is organized as follows. In Section 5.2 we recall the general Fredholm theory developed in [112] for linear functional equations of mixed type and apply it to scalar second order equations. In Section 5.3, we set out to establish existence and uniqueness of solutions to (5.1). We introduce the operator $\mathcal{G} : W_0^{2,\infty} \times \mathbb{R} \times V \to L^\infty$ associated with (5.1) and given by

$$\mathcal{G}(\phi, c, \rho)(\xi) = -\gamma \phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi), \phi(\xi + r_1), \ldots, \phi(\xi + r_N), \rho\big). \quad (5.2)$$

Solutions to (5.1) correspond to zeroes of $\mathcal{G}$. In the first part of Section 5.3, Theorem 5.2.10 is used to prove that the Frechet derivative $D_{1,2}\mathcal{G}$ of $\mathcal{G}$, evaluated at a solution (ϕ, c) to (5.1) at some parameter ρ_0, is, in fact, an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ to L^∞ (Proposition 5.3.8). This allows us to make a smooth local continuation $\big(\phi(\rho), c(\rho)\big)$ of solutions around $\rho = \rho_0$. In the second part of Section 5.3, we establish the uniqueness of solutions and prove Theorem 5.3.12. This enables us to turn the local continuation from the first part into a global continuation. In order to obtain the existence of solutions, we solve an explicit equation of the form (5.1) and use a homotopy of systems to extend this solution to an arbitrary family (5.1).

Having developed the underlying theory, we discuss the algorithm in Section 5.4 and we prove its convergence to a solution of (5.1). The algorithm is a modified Newton iteration, which uses the inverse of a linear operator $D_{1,2}\mathcal{F}$ that is closely related to the operator $D_{1,2}\mathcal{G}$, but with a relaxation on the shifted terms. Our analysis of the method relies heavily on the isomorphism result in Proposition 5.3.8, which can be extended to the operator $D_{1,2}\mathcal{F}$. In Section 5.5 we use our algorithm to calculate solutions to a specific family (5.1). The results are used to illustrate some of the technical difficulties involved in the application of our method. Considerable attention is devoted to the phenomenon of propagation failure and the issue of approaching the solutions in the singular perturbation limit $\gamma \to 0$ and $c \to 0$.

Finally, in Section 5.6, we address some issues connected to a possible generalization of the theory developed in this chapter. In particular, the numerical method can handle a broader class of equations than those analyzed here. We illustrate this by numerically computing solutions to a differential difference equation that arises when studying Ising models, which are very important for applications in the material sciences [13]. In addition, we discuss higher dimensional systems of the form (5.1) and show numerically that here the uniqueness of solutions breaks down, indicating that higher dimensional systems have a richer structure than their one dimensional counterparts. In future work this will be analyzed in a more theoretical setting.

5.2. Linear Functional Differential Equations of Mixed Type

In this section we apply the results obtained in [112] to second order scalar linear functional differential equations of mixed type

$$-\gamma x''(\xi) - cx'(\xi) = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j) + h(\xi). \tag{5.3}$$

Here x, A_j and h represent real valued functions and the parameter γ is assumed to satisfy $\gamma \neq 0$ throughout this section. In the homogeneous case we have $h = 0$ and (5.3) reduces to

$$-\gamma x''(\xi) - cx'(\xi) = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j). \tag{5.4}$$

Linear equations of the form (5.3) arise when one considers the linearization of (5.1) around a particular solution $\phi(\xi)$. In order to investigate the nonlinear equation (5.1) it will turn out to be crucial to understand the properties of the associated linear differential difference equation. Results in this direction will be given in this section, after we have introduced the terminology we shall need.

Throughout this section we will assume that the coefficients $A_j : J \to \mathbb{R}$ are measurable and uniformly bounded on some (usually infinite) interval J and that the inhomogeneity $h : J \to \mathbb{R}$ is locally integrable. The quantities r_j, the so-called shifts, can have either sign. As a technical restriction we shall assume $r_0 = 0$ and $r_i \neq r_j$ whenever $i \neq j$. For convenience we demand that $N \geq 1$. It should be noted that in this case this is not a restriction on (5.3), as we can always take any coefficient A_j to vanish identically on J.

Following the standard notation for differential difference equations as introduced in [112], we define the quantities

$$\begin{aligned} r_{\min} &= \min \left\{ r_j \mid j = 0 \ldots N \right\}, \\ r_{\max} &= \max \left\{ r_j \mid j = 0 \ldots N \right\} \end{aligned} \tag{5.5}$$

and observe that $r_{\min} \leq 0 \leq r_{\max}$ and $r_{\min} < r_{\max}$. We also define the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{R})$ of a solution by $x_\xi(\theta) = x(\xi + \theta)$ for $\theta \in [r_{\min}, r_{\max}]$. This allows us to rewrite (5.3) as

$$-\gamma x''(\xi) - cx'(\xi) = L(\xi)x_\xi + h(\xi). \tag{5.6}$$

Here $L(\xi)$, for almost every $\xi \in J$, denotes the bounded linear functional

$$L(\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(r_j), \quad \phi \in C([r_{\min}, r_{\max}], \mathbb{R}) \tag{5.7}$$

from $C([r_{\min}, r_{\max}], \mathbb{R})$ into $\mathbb{R}$. When the function h is absent, we have the homogeneous system

$$-\gamma x''(\xi) - cx'(\xi) = L(\xi)x_\xi. \tag{5.8}$$

A special case of (5.7) occurs when all the matrix functions $A_j(\xi)$ are constants, giving rise to the constant coefficient operator

$$L_0(\phi) = \sum_{j=0}^{N} A_{j,0}\phi(r_j) \tag{5.9}$$

and the homogeneous constant coefficient system

$$-\gamma x''(\xi) - cx'(\xi) = L_0 x_\xi. \tag{5.10}$$

Definition 5.2.1. *A solution to equation (5.6) on an interval J is a continuously differentiable function $x : J^\# \to \mathbb{R}$, defined on the larger interval*

$$J^\# = \{\xi + \theta \mid \xi \in J \text{ and } \theta \in [r_{\min}, r_{\max}]\}, \tag{5.11}$$

such that both x and x' are absolutely continuous on J and x satisfies (5.6) for almost every $\xi \in J$. □

From now on we shall assume $J = \mathbb{R}$, unless explicitly stated otherwise. We will be particularly interested in the spaces

$$\begin{aligned} W^{1,\infty} &= \left\{f \in L^\infty \mid f \text{ is absolutely continuous and } f' \in L^\infty\right\}, \\ W^{2,\infty} &= \left\{f \in L^\infty \mid f \text{ is absolutely continuous and } f' \in W^{1,\infty}\right\}, \end{aligned} \tag{5.12}$$

where we have used the shorthand $L^\infty = L^\infty(\mathbb{R}, \mathbb{R})$.

Associated to the homogeneous equation (5.8) we have the bounded linear operator $\Lambda_{c,\gamma,L} : W^{2,\infty} \to L^\infty$ defined by

$$(\Lambda_{c,\gamma,L}x)(\xi) = -\gamma x''(\xi) - cx'(\xi) - L(\xi)x_\xi. \tag{5.13}$$

The adjoint equation of (5.8) is given by

$$-\gamma y''(\xi) + cy'(\xi) = -L^*(\xi)y_\xi, \tag{5.14}$$

in which

$$L^*(\xi)\phi = -\sum_{j=0}^{N} A_j(\xi - r_j)\phi(-r_j), \quad \phi \in C([-r_{\max}, -r_{\min}], \mathbb{R}). \tag{5.15}$$

The corresponding adjoint operator $\Lambda^*_{c,\gamma,L} : W^{2,\infty} \to L^\infty$ is defined by

$$(\Lambda^*_{c,\gamma,L}y)(\xi) = -\gamma y''(\xi) + cy'(\xi) + L^*(\xi)y_\xi \tag{5.16}$$

and one can indeed easily verify that for test functions x and y we have $(x, \Lambda_{c,\gamma,L}y) = (\Lambda^*_{c,\gamma,L}x, y)$, where $(,)$ denotes the standard inner product $(x, y) = \int_{-\infty}^{\infty} x(\xi)y(\xi)d\xi$.

Associated to the constant coefficient system (5.10) is the characteristic equation, given by

$$\Delta_{c,\gamma,L_0}(s) = 0, \tag{5.17}$$

where Δ_{c,γ,L_0}, called the characteristic function, is given by

$$\Delta_{c,\gamma,L_0}(s) = -\gamma s^2 - cs - \sum_{j=0}^{N} A_{j,0} e^{s r_j}. \tag{5.18}$$

We recall that a number $\lambda \in \mathbb{C}$ is an eigenvalue of the constant coefficient system (5.10) if and only if it satisfies the characteristic equation, i.e., $\Delta_{c,\gamma,L_0}(\lambda) = 0$. Elementary solutions $y(\xi)$ of the constant coefficient system (5.10) corresponding to the eigenvector λ can be written as $y(\xi) = \operatorname{Re} e^{\lambda\xi} p(\xi)$, for some complex polynomial p. We will also refer to these solutions as eigensolutions.

Definition 5.2.2. *The constant coefficient system (5.10) is called hyperbolic in case $\Delta_{c,\gamma,L_0}(i\eta) \neq 0$ for all $\eta \in \mathbb{R}$, i.e., there are no eigenvalues on the imaginary axis.* □

We shall often write the operator $L(\xi)$ in (5.7) as a sum

$$L(\xi) = L_0 + M(\xi) \tag{5.19}$$

of a constant coefficient operator L_0 and a perturbation operator $M(\xi) : C([r_{\min}, r_{\max}], \mathbb{R}) \to \mathbb{R}$ and we will be specially interested in cases where $M(\xi)$ vanishes as $\xi \to \pm\infty$.

Definition 5.2.3. *The system (5.8) (or more simply L) is asymptotically autonomous at $\pm\infty$ if there exist L_0 and M as in (5.19), for which*

$$\lim_{\xi \to \pm\infty} \|M(\xi)\| = 0. \tag{5.20}$$

In this case (5.10) is called the limiting equation at $\pm\infty$. If in addition this limiting equation is hyperbolic, then we say that (5.8) is asymptotically hyperbolic at $\pm\infty$. If (5.8) is asymptotically autonomous or hyperbolic at both $\pm\infty$, then we simply drop the suffix "at $\pm\infty$". □

We are now ready to state the main theorem of this section which establishes useful properties of the operator $\Lambda_{c,\gamma,L}$. In addition, two important propositions concerning the asymptotic behaviour of solutions to (5.6) are included. These results can be seen as extensions of the main results from [112] to second order scalar systems and are derived in [82] by embedding the second order equation (5.3) into a first order two-dimensional system which is covered by the results in [112].

Theorem 5.2.4 (The Fredholm Alternative)**.** *Assume the homogeneous equation (5.8) is asymptotically hyperbolic. Then the operator $\Lambda_{c,\gamma,L}$ from $W^{2,\infty}$ to L^∞ is a Fredholm operator and its range $\mathcal{R}(\Lambda_{c,\gamma,L}) \subseteq L^\infty$ is given by*

$$\mathcal{R}(\Lambda_{c,\gamma,L}) = \left\{ h \in L^\infty \mid \int_{-\infty}^{\infty} y(\xi) h(\xi) d\xi = 0 \text{ for all } y \in \mathcal{K}(\Lambda^*_{c,\gamma,L}) \right\}. \tag{5.21}$$

In particular,

$$\begin{array}{lcllcl} \dim \mathcal{K}(\Lambda^*_{c,\gamma,L}) & = & \operatorname{codim}\mathcal{R}(\Lambda_{c,\gamma,L}), & \dim \mathcal{K}(\Lambda_{c,\gamma,L}) & = & \operatorname{codim}\mathcal{R}(\Lambda^*_{c,\gamma,L}), \\ \operatorname{ind}(\Lambda_{c,\gamma,L}) & = & -\operatorname{ind}(\Lambda^*_{c,\gamma,L}), & & & \end{array} \tag{5.22}$$

where ind *denotes the Fredholm index. Furthermore, the Fredholm index of $\Lambda_{c,\gamma,L}$ depends only on the limiting operators $L_\pm$, namely the limits of $L(\xi)$ as $\xi \to \pm\infty$. Finally, if L_ρ for $-1 \le \rho \le 1$ is a continuously varying one-parameter family of hyperbolic constant coefficient operators (5.9) with $L_{\pm 1} = L_\pm$, then* $\operatorname{ind}(\Lambda_{c,\gamma,L}) = 0$.

The next proposition will turn out to be extremely useful when obtaining asymptotic estimates on solutions to (5.1). It enables us to turn the detailed information about the eigenvalues of (5.8) which we shall obtain for our class of differential difference equations into very precise statements concerning the decay rate of the solutions. However, this result does not rule out the existence of solutions which decay superexponentially, as defined below.

Definition 5.2.5. *Let $x : J \to \mathbb{R}$ be a continuous function on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Then we say x decays superexponentially or has superexponential decay at $+\infty$ if*

$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0 \tag{5.23}$$

for every $b \in \mathbb{R}$. We define superexponential decay at $-\infty$ analogously. We will drop the distinction "at $\pm\infty$" if this is clear from the context. □

Proposition 5.2.6. *Let $x : J^\# \to \mathbb{R}$ be a solution to equation (5.8) on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Assume that x does not decay superexponentially and that (5.8) is asymptotically autonomous at $+\infty$, with L written as in (5.19). Also assume for some real number a and some positive number $k > 0$, that*

$$x(\xi) = O(e^{-a\xi}), \;\; x'(\xi) = O(e^{-a\xi}), \;\; \|M(\xi)\| = O(e^{-k\xi}), \;\; \xi \to \infty. \tag{5.24}$$

Then there exist $b \ge a$ and $\epsilon > 0$ such that

$$\begin{array}{lcll} x(\xi) & = & y(\xi) + O(e^{-(b+\epsilon)\xi}), & \xi \to \infty, \\ x'(\xi) & = & y'(\xi) + O(e^{-(b+\epsilon)\xi}), & \xi \to \infty, \end{array} \tag{5.25}$$

where y is a nontrivial eigensolution of the limiting equation (5.10) corresponding to the nonempty set of eigenvalues with $\operatorname{Re}\lambda = -b$.

In light of Proposition 5.2.6, the following lemma will be useful when studying the asymptotic behaviour of solutions to the linear homogeneous equation (5.8).

Lemma 5.2.7. *Consider a real-valued function $x : [\tau, \infty) \to \mathbb{R}$ of the form*

$$x(\xi) = y(\xi) + O(e^{-(b+\epsilon)\xi}), \;\; \xi \to \infty, \tag{5.26}$$

for some $b \in \mathbb{R}$ and $\epsilon > 0$, where y is a nontrivial solution of the constant coefficient system (5.10) with $\gamma \ne 0$, given by a finite sum of eigensolutions corresponding to a set Λ

of eigenvalues λ, all of which satisfy $\operatorname{Re}\lambda = -b$. *If* $\operatorname{Im}\lambda \neq 0$ *for all* $\lambda \in \Lambda$, *then there exist arbitrarily large* ξ *for which* $x(\xi) > 0$ *and arbitrarily large* ξ *for which* $x(\xi) < 0$. *On the other hand, if* $\Lambda = \{-b\}$, *then* $x(\xi) \neq 0$ *for all large* ξ. *The analogous result for* $\xi \to -\infty$ *also holds.*

The next proposition shows that solutions to (5.8) which are in $W^{2,\infty}$ decay exponentially. Note that it is not required here that the coefficients $A_j(\xi)$ approach their limits exponentially fast.

Proposition 5.2.8. *Assume that equation (5.6) is asymptotically hyperbolic at* $+\infty$. *Then there exist positive quantities* K, K' *and* a *such that for all pairs of functions* $x \in W^{2,\infty}$ *and* $h \in L^\infty$ *which satisfy* $\Lambda_L x = h$, *the estimate*

$$\left(x(\xi)^2 + x'(\xi)^2\right)^{\frac{1}{2}} \leq K e^{-a\xi} \left(\|x\|_{L^\infty}^2 + \left\|x'\right\|_{L^\infty}^2 \right)^{\frac{1}{2}} + K' \|h\|_{L^\infty} \tag{5.27}$$

holds for all $\xi \geq 0$.

Due to the conditions we impose on our nonlinear equation (5.1), the linear equations (5.3) encountered in the sequel often satisfy the following conditions.

Assumption 5.2.9. *The parameter* γ *satisfies* $\gamma > 0$ *and the function* $h : J \to \mathbb{R}$ *is a continuous function satisfying* $h(\xi) \geq 0$ *for all* $\xi \in J$. *In addition, for every* $0 \leq j \leq N$, *the function* $A_j(\xi)$ *is continuous on* J *and there exist constants* α_j, β_j *such that*

$$\alpha_j \leq A_j(\xi) \leq \beta_j, \quad \xi \in J. \tag{5.28}$$

In addition, we have $\alpha_j > 0$ *for* $1 \leq j \leq N$. □

The final theorem of this section concerns homogeneous equations (5.4) that satisfy the above conditions and will be the main ingredient for establishing the results in the next section. The proof is deferred to Appendix 5.7, where the necessary machinery is developed.

Theorem 5.2.10. *Consider the homogeneous linear equation (5.4) and suppose that Assumption (5.2.9) is satisfied. Assume that equation (5.4) is asymptotically autonomous and that in addition the limiting equations are approached at an exponential rate, so*

$$\left|A_j(\xi) - A_{j\pm}\right| = O(e^{-k|\xi|}), \quad \xi \to \pm\infty, \quad j = 0 \dots N \tag{5.29}$$

for some $k > 0$. *Also assume that each of the sums* $A_{\Sigma_\pm}$ *given below, of the limiting coefficients at* $\pm\infty$, *is negative, namely*

$$A_{\Sigma_\pm} = \sum_{j=0}^{N} A_{j\pm} < 0. \tag{5.30}$$

Finally, assume that there exists a nontrivial solution $x = p(\xi) \in W^{2,\infty}$ *to (5.4) which satisfies* $p(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}$. *Then equation (5.4) is asymptotically hyperbolic and the associated operator* $\Lambda_{c,\gamma,L} \colon W^{2,\infty} \to L^\infty$ *is a Fredholm operator. In addition, we have*

$$\dim \mathcal{K}(\Lambda_{c,\gamma,L}) = \dim \mathcal{K}(\Lambda_{c,\gamma,L}^*) = \operatorname{codim}\mathcal{R}(\Lambda_{c,\gamma,L}) = 1, \quad \operatorname{ind}(\Lambda_{c,\gamma,L}) = 0. \tag{5.31}$$

The element $p \in \mathcal{K}(\Lambda_{c,\gamma,L})$ is strictly positive,

$$p(\xi) > 0, \quad \xi \in \mathbb{R} \tag{5.32}$$

and there exists an element $p^* \in \mathcal{K}(\Lambda^*_{c,\gamma,L})$ which is strictly positive,

$$p^*(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{5.33}$$

5.3. Global Structure

In this section we study the family of autonomous differential difference equations introduced in the introduction,

$$-cx'(\xi) - \gamma x''(\xi) = F\big(x(\xi + r_0), x(\xi + r_1), x(\xi + r_2), \ldots, x(\xi + r_N), \rho\big), \tag{5.34}$$

in which $\gamma > 0$. As in the previous section, we demand that $r_0 = 0$, $r_i \neq r_j$ if $i \neq j$ and $r_i \neq 0$ for $i = 1 \ldots N$, where $N \geq 1$. Here we take $\rho \in \overline{V}$ to be a parameter, where V is an open subset of $\mathbb{R}$. We shall prove existence and uniqueness of solutions to (5.34) under certain conditions and establish the C^1-dependence of the solutions on the parameter ρ.

We start out by making precise the requirements given in the introduction and give a list of conditions on the function F which we will assume to hold throughout this section.

(b1) The nonlinearity $F : \mathbb{R}^{N+1} \times \overline{V} \to \mathbb{R}$ is C^1-smooth in $\mathbb{R}^{N+1}$ and $\overline{V}$.

(b2) The derivative $D_1 F : \mathbb{R}^{N+1} \times \overline{V} \to \mathbb{R}^{N+1}$ with respect to the first argument $v \in \mathbb{R}^{N+1}$ is locally Lipschitz in v.

(b3) For each $\rho \in \overline{V}$ and for $j = 1, \ldots, N$, we have, writing $v = (v_0, v_1, \ldots, v_N) \in \mathbb{R}^{N+1}$, that either

$$\frac{\partial F(v,\rho)}{\partial v_j} \equiv 0, \quad \text{or} \quad \frac{\partial F(v,\rho)}{\partial v_j} > 0, \tag{5.35}$$

that is, either F is totally independent of v_j or is strictly increasing in v_j. Furthermore, for each $\rho \in \overline{V}$ there is at least one j, satisfying $1 \leq j \leq N$, for which the nonlinearity F is not totally independent of v_j.

(b4) Let $\Phi : \mathbb{R} \times \overline{V} \to \mathbb{R}$ be defined as

$$\Phi(\phi, \rho) = F(\phi, \phi, \ldots, \phi, \rho). \tag{5.36}$$

Then for some quantity $q = q(\rho) \in [-1, 1]$ we have that

$$\begin{aligned} &\Phi(-1, \rho) = \Phi(q(\rho), \rho) = \Phi(1, \rho) = 0, \\ &\Phi(\phi, \rho) > 0, \quad \phi \in (-\infty, -1) \cup (q, 1), \\ &\Phi(\phi, \rho) < 0, \quad \phi \in (-1, q) \cup (1, \infty). \end{aligned} \tag{5.37}$$

In case $\rho \in V$ we demand $q(\rho) \in (-1, 1)$.

(b5) We have for $q = q(\rho)$ that

$$\begin{aligned} &D_1\Phi(-1, \rho) < 0 \text{ if } q \neq -1, \\ &D_1\Phi(q, \rho) > 0 \text{ if } q \in (-1, 1), \\ &D_1\Phi(1, \rho) < 0 \text{ if } q \neq 1, \end{aligned} \tag{5.38}$$

with D_1 denoting the derivative with respect to the first argument $x \in \mathbb{R}$.

Condition (b3) allows us to consider families in which the shifts r_j may vary with ρ, by adding extra shifts r_j which do not affect the value of F for certain values of ρ.

In (5.34) the wavespeed c is an unknown parameter. From the above conditions we see that equation (5.34) has exactly three constant equilibrium solutions, namely $x = \pm 1$ and $x = q(\rho)$. We will be interested in solutions to (5.34) joining the two equilibrium points ± 1. As (5.34) is autonomous, we see that all translates of a solution $x(\xi)$ to (5.34) are also solutions. We can use this freedom to demand that $x(0) = 0$. It will turn out that after this normalization the solution to (5.34) is unique. We thus seek our solutions in the space

$$W_0^{2,\infty} = \left\{ x \in W^{2,\infty} \mid x(0) = 0 \right\}. \tag{5.39}$$

It will be useful to introduce the operator $\mathcal{G} : W_0^{2,\infty} \times \mathbb{R} \times V \to L^\infty$ defined by

$$\mathcal{G}(\phi, c, \rho)(\xi) = -\gamma \phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi + r_0), \phi(\xi + r_1), \ldots, \phi(\xi + r_N), \rho\big). \tag{5.40}$$

We are now ready to define the concept of a connecting solution to (5.34).

Definition 5.3.1. *Given $\rho \in V$, a connecting solution to the nonlinear autonomous differential difference equation (5.34) is a pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ that satisfies (5.34) and joins the two equilibrium solutions ± 1, i.e., for which the limits*

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1 \tag{5.41}$$

hold. □

Please note that we will continue to use the term "solution" to indicate a function $x \in W^{2,\infty}$ satisfying the equation (5.34), but not necessarily joining the two equilibria ± 1 and not necessarily having $x(0) = 0$.

We are now in a position to state the main theorem of this section.

Theorem 5.3.2. *Consider a family of autonomous differential difference equations (5.34) that satisfies the conditions (b1) through (b5). There exist C^1-smooth functions $c : V \to \mathbb{R}$ and $P : V \to W_0^{2,\infty}$ such that for all $\rho_0 \in V$, the pair $\big(P(\rho_0), c(\rho_0)\big)$ is a connecting solution to equation (5.34). Moreover, these are the only connecting solutions to (5.34).*

Before proceeding with the proof of the main theorem, let us consider the differential difference equation (5.34) with fixed parameters c, γ and ρ. If x_1 and x_2 are two bounded solutions of this equation (5.34), then the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear homogeneous equation (5.8) with coefficients given by

$$A_j(\xi) = \int_0^1 \frac{\partial F(u,\rho)}{\partial u_j}\bigg|_{u=t\pi(x_1,\xi)+(1-t)\pi(x_2,\xi)} dt. \tag{5.42}$$

Here π is the state projection

$$\pi(\phi,\xi) = \big(\phi(\xi + r_0), \dots, \phi(\xi + r_N)\big) \in \mathbb{R}^{N+1}. \tag{5.43}$$

This can easily be seen by using the formula

$$\begin{aligned} F(v,\rho) - F(w,\rho) &= \int_0^1 \frac{dF(tv+(1-t)w,\rho)}{dt} dt \\ &= \sum_{j=0}^N \left(\int_0^1 \frac{\partial F(tv+(1-t)w,\rho)}{\partial u_j} dt \right) (v_j - w_j). \end{aligned} \tag{5.44}$$

Similarly, suppose that $x : \mathbb{R} \to \mathbb{R}$ is any solution to (5.34) for some $\rho \in \overline{V}$. Then $x'(\xi)$ is a solution of the linearization around x, that is, the linear equation (5.8) with coefficients

$$A_j(\xi) = \frac{\partial F(u,\rho)}{\partial u_j}\bigg|_{u=\pi(x,\xi)}. \tag{5.45}$$

The linearization around the three equilibrium solutions $x = \pm 1$ and $x = q(\rho)$ are constant coefficient equations given by (5.10). We shall write L_+, L_- and $L_\diamond$ for the associated linear operators (5.9) and shall refer to the corresponding constant coefficients as

$$\begin{aligned} A_{j\pm}(\rho) &= \frac{\partial F(u,\rho)}{\partial u_j}\bigg|_{u=\kappa(\pm 1)}, \\ A_{j\diamond}(\rho) &= \frac{\partial F(u,\rho)}{\partial u_j}\bigg|_{u=\kappa(q(\rho))}, \end{aligned} \tag{5.46}$$

where κ is the diagonal map $\kappa(x) = (x, \dots, x) \in \mathbb{R}^{N+1}$. Writing $A_{\Sigma\pm} = \sum_{j=0}^N A_{j\pm}$, we have the identity

$$A_{\Sigma\pm} = D_1\Phi(\pm 1, \rho). \tag{5.47}$$

Note that when $\rho \in V$, condition (b5) in combination with Lemma 5.7.4 implies that the linearization of (5.34) around $x = \pm 1$ is asymptotically hyperbolic at $\pm\infty$.

The proof of Theorem 5.3.2 will be given in two parts. First we shall concentrate on the existence of functions $P(\rho)$ and $c(\rho)$ as in the statement of Theorem 5.3.2 in a small neighbourhood of the detuning parameter ρ_0, given a connecting solution (P_0, c_0) for $\rho = \rho_0$. After we have established the existence of this local continuation in Proposition 5.3.3, we show that it can be extended to all $\rho \in V$ and thus prove the existence and uniqueness claims in the statement of Theorem 5.3.2.

Proposition 5.3.3. *Let* $(P_0, c_0) \in W_0^{2,\infty} \times \mathbb{R}$ *be a connecting solution to (5.34) for some* $\rho_0 \in V$ *and for some* $c_0 \in \mathbb{R}$. *Then for each* ρ *near* ρ_0 *there exists an unique* $(P, c) = \big(P(\rho), c(\rho)\big) \in W_0^{2,\infty} \times \mathbb{R}$, *that depends* C^1*-smoothly on* ρ, *for which* $\mathcal{G}\big(P(\rho), c(\rho), \rho\big) = 0$, *with* $c(\rho_0) = c_0$ *and* $P(\rho_0) = P_0$. *This function* $P(\rho)$ *satisfies the boundary conditions* $\lim_{\xi\to\pm\infty} P(\rho)(\xi) = \pm 1$ *and thus* $\big(P(\rho), c(\rho)\big)$ *is a connecting solution to (5.34).*

Our approach to proving the result above will be to invoke the implicit function theorem on the operator $\mathcal{G}$ defined by (5.40). Consequently, in Proposition 5.3.8 we study the Frechet derivative of $\mathcal{G}$, which is given by

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0)(\psi, b)(\xi) = -bP_0'(\xi) + (\Lambda_{c_0,\gamma,L}\psi)(\xi), \tag{5.48}$$

where $\Lambda_{c_0,\gamma,L}$ is the linear operator associated to the linearization of (5.34) around the solution P_0. We shall establish that Theorem 5.2.10 applies to the operator $\Lambda_{c_0,\gamma,L}$ and that the derivative P_0' is strictly positive (Lemma 5.3.7). In particular, this means that $P_0' \notin \mathcal{R}(\Lambda_{c_0,\gamma,L})$ and $\mathcal{K}(\Lambda_{c_0,\gamma,L}) \cap W_0^{2,\infty} = \emptyset$. From this it is easy to see that $D_{1,2}\mathcal{G}$ is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞, which legitimizes the use of the implicit function theorem.

We shall need the following technical lemma to prove that solutions to (5.34) which are close to connecting solutions in the $W^{2,\infty}$ norm are in fact also connecting solutions. The proof of this result closely follows the corresponding argument for $\gamma = 0$ and we therefore refer to [82] for the details.

Lemma 5.3.4. *Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to (5.34) for some $\rho \in \overline{V}$ and $c \in \mathbb{R}$. Define*

$$\mu_- = \inf_{\xi\in\mathbb{R}} x(\xi), \quad \mu_+ = \sup_{\xi\in\mathbb{R}} x(\xi), \tag{5.49}$$

and assume that both $\mu_\pm$ are finite. Then

$$\mu_- \in [-1, q(\rho)] \cup \{1\}, \quad \mu_+ \in \{-1\} \cup [q(\rho), 1]. \tag{5.50}$$

The same conclusion (5.50) holds for

$$\mu_- = \liminf_{\xi\to\infty} x(\xi), \quad \mu_+ = \limsup_{\xi\to\infty} x(\xi) \tag{5.51}$$

and similarly for the lim inf *and* lim sup *at $-\infty$.*

Corollary 5.3.5. *If $(P, c) \in W_0^{2,\infty} \times \mathbb{R}$ is a connecting solution to (5.34), then*

$$-1 < P(\xi) < 1, \quad \xi \in \mathbb{R}. \tag{5.52}$$

Proof. Lemma 5.3.4 implies that $-1 \le P(\xi) \le 1$ for all $\xi \in \mathbb{R}$. The strict inequalities now follow from an application of Lemma 5.7.9. □

Lemma 5.3.6. *Let $(P_0, c_0) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.34). Then for some quantities $C_\pm > 0$ and $\epsilon > 0$ we have that*

$$P_0(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u+\epsilon)\xi}), & \xi \to -\infty, \\ 1 - C_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s-\epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{5.53}$$

where $\lambda_-^u \in (0, \infty)$ is the unique positive eigenvalue of the linearization of (5.34) about $x = -1$ and $\lambda_+^s \in (-\infty, 0)$ is the unique negative eigenvalue of the linearization about $x = 1$. The formulae for $P'(\xi)$ obtained by formally differentiating (5.53) also hold.

Proof. We consider only the limit $\xi \to \infty$, as the proofs of the results for $\xi \to -\infty$ are similar. Defining $y(\xi) = 1 - P(\xi)$, we see that y satisfies the linear equation (5.4) with coefficients $A_j(\xi)$ given by (5.42) with $x_1 = 1$ and $x_2 = P$. Note that $\lim_{\xi\to\infty} A_j(\xi) = A_{j+}(\rho)$, thus this linear equation is asymptotically hyperbolic. Proposition 5.2.8 now implies that $y(\xi)$ decays exponentially. Using the expression (5.42) together with the Lipschitz condition (b2) on the derivative of F, it follows that the coefficients $A_j(\xi)$ approach their limits exponentially fast. One can now proceed as in the proof of Theorem 5.2.10 to establish the claim. □

Lemma 5.3.7. *If $(P, c) \in W_0^{2,\infty} \times \mathbb{R}$ is a connecting solution to (5.34), then $P'(\xi) > 0$ for all $\xi \in \mathbb{R}$.*

Proof. We note that it is sufficient to prove that $P'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, since Corollary 5.7.8 then immediately implies the strict positivity $P'(\xi) > 0$.

By (5.53) we see that there exists $\tau > 0$ such that $P'(\xi) > 0$ whenever $|\xi| \geq \tau$ and such that $P(-\tau) < P(\xi) < P(\tau)$ whenever $|\xi| < \tau$. From this we have $P(\xi + k) > P(\xi)$ for all $\xi \in \mathbb{R}$, provided that $k \geq 2\tau$. Now suppose that $P'(\xi) < 0$ for some ξ and set

$$k_0 = \inf\{k > 0 \mid P(\xi + k) > P(\xi) \text{ for all } \xi \in \mathbb{R}\}. \tag{5.54}$$

Certainly $k_0 > 0$. Also, $k_0 \leq 2\tau$ and $P(\xi + k_0) \geq P(\xi)$ for all $\xi \in \mathbb{R}$. If $0 < k < k_0$ then $P(\xi + k) \leq P(\xi)$ for some ξ, where necessarily $|\xi| \leq \tau$. Therefore, there exists some ξ_0, with $|\xi_0| \leq \tau$, for which $P(\xi_0 + k_0) = P(\xi_0)$. We can now define $x_1(\xi) = P(\xi + k_0)$ and $x_2(\xi) = P(\xi)$. Because $x_1(\xi) \geq x_2(\xi)$ for all $\xi \in \mathbb{R}$ and $x_1(\xi_0) = x_2(\xi_0)$, Lemma 5.7.9 implies that $P(\xi + k_0) = P(\xi)$ for all $\xi \in \mathbb{R}$. This is a contradiction, because $P'(\xi) > 0$ for all large $|\xi|$. □

Proposition 5.3.8. *Let $(P_0, c_0) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.34) for some $\rho_0 \in V$ and for some $c_0 \in \mathbb{R}$. Consider the linearization (5.4) of equation (5.34) about P_0 and let $\Lambda_{c_0,\gamma,L}$ denote the associated linear operator from $W^{2,\infty}$ to L^∞. Then the derivative of $\mathcal{G}$,*

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0) : W_0^{2,\infty} \times \mathbb{R} \to L^\infty, \tag{5.55}$$

at the solution (P_0, c_0), with respect to the first two arguments, is given by

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0)(\psi, b)(\xi) = -bP_0'(\xi) + (\Lambda_{c_0,\gamma,L}\psi)(\xi) \tag{5.56}$$

and is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞.

Proof. The fact that $\mathcal{G}$ is C^1-Frechet differentiable follows from the fact that F is a C^1-function and the explicit formula (5.56) follows by direct differentiation of (5.40). The operator $\Lambda_{c_0,\gamma,L}$ can be easily seen to satisfy all the conditions of Theorem 5.2.10. In particular, $x(\xi) = P_0'(\xi)$ satisfies the linear equation (5.4), which by Lemma 5.3.7 gives the strictly positive $p = P_0' \in \mathcal{K}_{c_0,\gamma,L}$ in the statement of Theorem 5.2.10. Thus, by Theorem 5.2.10, the kernel $\mathcal{K}_{c_0,\gamma,L}$ of $\Lambda_{c_0,\gamma,L}$ is precisely the one-dimensional span of P_0'. The strict positivity $P_0'(0) > 0$ implies that $P_0' \notin W_0^{2,\infty}$, hence $\mathcal{K}(\Lambda_{c_0,\gamma,L}) \cap W_0^{2,\infty} = \emptyset$. In addition, the presence of the strictly positive $p^* \in \mathcal{K}(\Lambda^*_{c_0,\gamma,L})$ guarantees $P_0' \notin \mathcal{R}(\Lambda_{c_0,\gamma,L})$ by Theorem 5.2.4, which establishes the claim.

□

Proof of Proposition 5.3.3. The local continuation follows from the implicit function theorem, together with Lemma 5.3.8. The limit at $+\infty$ follows from the observation that the quantity $\mu_-(\rho)$ in (5.51) for $P(\rho)$ varies continuously with ρ, together with $\mu_-(\rho_0) = 1$ and the identity (5.50). The limit at $-\infty$ follows similarly. □

We now set out to give the proof of Theorem 5.3.2. Lemma 5.3.9 establishes the uniqueness claim in Theorem 5.3.2. Theorem 5.3.12 will allow us to extend the local continuation in Proposition 5.3.3 to a global continuation for all $\rho \in V$, by proving that limits of connecting solutions are connecting solutions to the limiting differential difference equation. This means that once we have established the existence of a connecting solution to (5.34) for one value of the detuning parameter, $\rho_0 \in V$, we know that (5.34) has a connecting solution for all values $\rho \in V$. This is why we give an explicit solution to a prototype differential difference equation in Lemma 5.3.13. By constructing a new family (5.34), which mixes the original differential difference equation and the prototype system, we can combine Theorem 5.3.12 and Proposition 5.3.3 to establish the existence of a connecting solution to our original family (5.34) at one value of the detuning parameter ρ, as required.

We merely state the following lemma and refer to [82] for the complete proof, which closely follows the corresponding argument for the $\gamma = 0$ case.

Lemma 5.3.9. *For each $\rho \in V$ there exists at most one value $c \in \mathbb{R}$ such that equation (5.34) possesses a monotone increasing solution $x = P(\xi)$, satisfying the boundary conditions*

$$\lim_{\xi \to \pm\infty} x(\xi) = \pm 1. \tag{5.57}$$

For each $c \in \mathbb{R}$ and $\rho \in V$ there exists at most one solution $x = P(\xi)$ of (5.34), up to translation, satisfying the boundary conditions (5.57).

The following result, concerning the linearization around the (unstable) equilibrium $q(\rho)$, will prove to be useful in establishing the boundary conditions $x(\pm\infty) = \pm 1$ for limits of connecting solutions x_n.

Lemma 5.3.10. *For every $\rho \in V$, $\gamma \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}$ there do not exist two monotone increasing solutions $x_\pm : \mathbb{R} \to \mathbb{R}$ of equation (5.34) such that*

$$\begin{array}{ll} \lim_{\xi \to -\infty} x_-(\xi) = -1, & \lim_{\xi \to \infty} x_-(\xi) = q(\rho), \\ \lim_{\xi \to -\infty} x_+(\xi) = q(\rho), & \lim_{\xi \to \infty} x_+(\xi) = 1. \end{array} \tag{5.58}$$

Proof. The case where $\gamma = 0$ was considered in [113, Lemma 7.1], so we will assume $\gamma > 0$. First notice that

$$\Delta_{c,\gamma,L_\diamond(\rho)}(0) = -A_{\Sigma\diamond}(\rho) = -D_1\Phi(q(\rho), \rho) < 0, \tag{5.59}$$

which by Lemma 5.7.5 implies that there do not simultaneously exist eigenvalues $\lambda^u_\diamond < 0 < \lambda^s_\diamond$ for the constant coefficient system $L_\diamond$ defined in (5.46).

Now assume that there exist monotone increasing x_- and x_+ satisfying conditions (5.58). Consider $y(\xi) = q(\rho) - x_-(\xi)$, which is a monotone decreasing function on the real line, satisfying (5.4) with coefficients given by (5.42), with $x_1 = q(\rho)$ and $x_2 = x_-(\xi)$. This linear equation satisfies all the assumptions of Lemma 5.7.1 and thus reasoning as in the proof of this lemma we see that for all $\xi \in \mathbb{R}$,

$$y'(\xi) \geq -By(\xi) \tag{5.60}$$

for some $B > 0$. Now take any sequence $\xi_n \to \infty$, and let $z_n(\xi) = y(\xi + \xi_n)/y(\xi_n)$. Then each z_n also satisfies $z_n'(\xi) \geq -Bz_n(\xi)$ on $\mathbb{R}$. As $z_n(0) = 1$, we conclude that the sequence of functions z_n is uniformly bounded and equicontinuous on each compact interval and so without loss we have that $z_n(\xi) \to z(\xi)$ uniformly on compact intervals. From the differential equation (5.34) we see that we can use the uniform bound on z_n' to obtain a uniform bound on $z_n''(\xi)$, thus concluding that also $z_n'(\xi)$ is equicontinuous on each compact interval. One now easily sees that z satisfies the autonomous limiting constant coefficient equation associated to $L_\diamond$. Moreover, $-Bz(\xi) \leq z'(\xi) \leq 0$ for all $\xi \in \mathbb{R}$, with $z(0) = 1$, so $z(\xi) > 0$ and z does not decay faster than exponentially. We may now apply Proposition 5.2.6 to the solution z. We conclude that $z(\xi) = w(\xi) + O(e^{-(b+\epsilon)\xi})$ as $\xi \to \infty$, where w is a nontrivial sum of eigensolutions corresponding to a set of eigenvalues with $\operatorname{Re}\lambda = -b \leq 0$. The positivity of z, together with Lemma 5.2.7, implies that the linearization about $x = q(\rho)$ possesses a nonpositive eigenvalue $\lambda_\diamond^s \leq 0$. Since $\Delta_{c,\gamma,L_\diamond(\rho)}(0) < 0$ we have $\lambda_\diamond^s < 0$. We can use similar reasoning applied to $x_+(\xi)$ to conclude that the linearization about $x = q(\rho)$ must also possess a positive eigenvalue $\lambda_\diamond^u > 0$. This yields a contradiction. □

Remark 5.3.11. *In the above proof we could not apply Proposition 5.2.6 directly to the function $y(\xi)$, as it may not be the case that $y(\xi)$ approaches its limits $y(\pm\infty)$ exponentially fast.*

The next theorem enables us to take limits of connecting solutions, which will be crucial in establishing global existence of solutions.

Theorem 5.3.12. *Let $\rho_n \in V$ and $\gamma_n \in \mathbb{R}_{>0}$ be two sequences satisfying $\gamma_n \to \gamma_0$ and $\rho_n \to \rho_0$ as $n \to \infty$, possibly with $\gamma_0 = 0$. Let $(P_n(\xi), c_n)$ denote any connecting solution to (5.34) with $\rho = \rho_n$ and $\gamma = \gamma_n$. Then, after possibly passing to a subsequence, the limit*

$$\lim_{n\to\infty} P_n(\xi) = P_0(\xi) \tag{5.61}$$

exists pointwise and also the limit

$$\lim_{n\to\infty} c_n = c_0 \tag{5.62}$$

exists, with $|c_0| < \infty$. Furthermore, $P_0(\xi)$ satisfies the limiting differential difference equation

$$-\gamma_0 P_0''(\xi) - c_0 P_0'(\xi) = F\big(P_0(\xi), P_0(\xi + r_1), \ldots, P_0(\xi + r_N), \rho_0\big) \tag{5.63}$$

almost everywhere. In addition, we have the limits

$$\lim_{\xi\to\pm\infty} P_0(\xi) = \pm 1. \tag{5.64}$$

Proof. Using the fact that the functions $P_n(\xi)$ satisfy $P_n' > 0$, we may argue in a standard fashion that, after passing to a subsequence, the pointwise limit $P_0(\xi) = \lim_{n\to\infty} P_n(\xi)$ exists for all $\xi \in \mathbb{R}$. Due to the limits $\lim_{n\to\infty} P_n(\xi) = \pm 1$, we have $\int_{-\infty}^{\infty} P_n'(s)ds = 2$. Writing $F(\xi) = \liminf_{n\to\infty} P_n'(\xi)$ we obtain, using Fatou's Lemma,

$$\int_{-\infty}^{\infty} F(s)ds \leq 2. \tag{5.65}$$

In particular, this implies that the measure of the set for which $F(s) = \infty$ is zero. Letting β_n be any sequence with $\beta_n \to 0$ as $n \to \infty$, we have that, if we choose ξ_0 appropriately,

$$\liminf_{n\to\infty} \beta_n(P_n'(\xi) - P_n'(\xi_0)) = 0 \quad \text{almost everywhere .} \tag{5.66}$$

Now suppose that $\liminf_{n\to\infty} |c_n| = \infty$. Without loss assume $c_n > 0$. Write $q_0 = q(\rho_0)$ and fix a point

$$q_* \in (q_0, 1). \tag{5.67}$$

Let $x_n(\xi) = P_n(c_n\xi + \xi_n)$, where $\xi_n \in \mathbb{R}$ is such that $P_n(\xi_n) = q_*$. Then (5.34) in integrated form gives us

$$\begin{aligned} -\gamma_n c_n^{-2}\big(x_n'(\xi) - x_n'(\xi_0)\big) \quad = \quad & \textstyle\int_{\xi_0}^{\xi} F\big(x_n(s), x_n(s + r_1 c_n^{-1}), \ldots, \\ & x_n(s + r_N c_n^{-1}), \rho_n\big)ds + \big(x_n(\xi) - x_n(\xi_0)\big). \end{aligned} \tag{5.68}$$

Again, because the x_n are monotonically increasing functions, we can pass to a subsequence for which the pointwise limit $x(\xi) = \lim_{n\to\infty} x_n(\xi)$ exists and is continuous at all but countably many points. We have seen above that $\liminf_{n\to\infty} \beta_n c_n^{-1} x_n'(\xi) = 0$ almost everywhere, for a sequence $\beta_n \to 0$. After taking the limit $\liminf_{n\to\infty}$ we thus obtain, using $\beta_n = c_n^{-1} \to 0$,

$$-\big(x(\xi) - x(\xi_0)\big) = \int_{\xi_0}^{\xi} F\big(x(s), x(s), \ldots, x(s), \rho_0\big)ds, \tag{5.69}$$

which holds almost everywhere. By redefining x on a set of measure zero, which does not affect the right hand side of (5.69), we can assume this identity to hold everywhere. From this identity we also see that $x(\xi)$ is differentiable and satisfies

$$-x'(\xi) = \Phi(x(\xi), \rho_0). \tag{5.70}$$

Since $x(\xi) \leq q_*$ for almost all $\xi \leq 0$, we cannot have $x(\xi) = 1$ for some ξ, as this would imply $x(\xi) = 1$ for all ξ. Now $x_n(\xi) \geq q_*$ for all $\xi \geq 0$, hence also $1 > x(\xi) \geq q_*$ for $\xi \geq 0$ and thus $x'(\xi) = -\Phi(x(\xi), \rho_0) < 0$ whenever $\xi \geq 0$. On the other hand, $x_n'(\xi) > 0$, hence $x'(\xi) \geq 0$, for all ξ. This contradiction implies that $\liminf_{n\to\infty} |c_n| < \infty$. Thus, after passing to a subsequence, the limit $c_0 = \lim_{n\to\infty} c_n$ exists.

Integration of (5.34) yields

$$\begin{aligned} -\gamma_n\big(P_n'(\xi) - P_n'(\xi_0)\big) \quad = \quad & \textstyle\int_{\xi_0}^{\xi} F\big(P_n(s), P_n(s + r_1), \ldots, \\ & P_n(s + r_N), \rho_n\big)ds + c_n\big(P_n(\xi) - P_n(\xi_0)\big). \end{aligned} \tag{5.71}$$

Consider the case where $\gamma_0 > 0$. Notice that $y_n(\xi) = 1 - P_n(\xi)$ is a monotone decreasing function on the real line, which satisfies the linear equation (5.3) with coefficients given by (5.42), with $x_1 = 1$ and $x_2 = P_n$. Referring to these coefficients as $A_{j,n}(\xi)$, we see that (5.4) satisfies the conditions of Lemma 5.7.1 and we hence obtain from the proof of this lemma

$$y_n'(\xi) \geq -B_n y_n(\xi), \tag{5.72}$$

in which $B_n = \sqrt{\frac{c_n^2}{4\gamma_n^2} - \frac{\alpha_{0,n}}{\gamma_n}} + \frac{c_n}{\gamma_n}$. Now there exists α_0 such that $0 \geq \alpha_{0,n} \geq \alpha_0$, as the functions $y_n(\xi)$ are uniformly bounded and $D_1 F$ is a continuous function, which attains its maxima and minima on compact sets. This means that the constants B_n are bounded, $0 \leq B_n \leq B$ for some B. From (5.72) we now see that y_n' and hence P_n' are uniformly bounded. From the differential equation (5.34) it now also follows that the functions P_n'' are uniformly bounded. Thus P_n' is an equicontinuous family, allowing us to pass to a subsequence for which $P_n'(\xi) \to P_0'(\xi)$ and $P_n(\xi) \to P_0(\xi)$ uniformly on compact intervals.

Thus, taking the limit $\liminf_{n\to\infty}$ in (5.71), we now obtain for all $\gamma_0 \geq 0$

$$\begin{aligned} -\gamma_0\big(P_0'(\xi) - P_0'(\xi_0)\big) \;=\; & \int_{\xi_0}^{\xi} F\big(P_0(s), P_0(s+r_1), \ldots, \\ & P_0(s+r_N), \rho_0\big)ds + c_0\big(P_0(\xi) - P_0(\xi_0)\big), \end{aligned} \tag{5.73}$$

which holds for all $\xi \in \mathbb{R}$ if $\gamma_0 \neq 0$ and almost everywhere if $\gamma_0 = 0$. In case $\gamma_0 = 0$ and $c_0 \neq 0$ we can again by redefining P_0 on a set of measure zero ensure that (5.73) holds for all $\xi \in \mathbb{R}$. After differentiation we see that $P_0(\xi)$ satisfies the differential difference equation stated in the theorem.

We now set out to prove the limits (5.64). Because $P_0(\xi)$ is a bounded monotonically increasing function, the limits $\lim_{\xi\to\pm\infty} P_0(\xi)$ exist. We will refer to these limits as $P_0(\pm\infty)$. When $c_0 \neq 0$, the function $P_0'(\xi)$ decays exponentially, and when $\gamma_0 \neq 0$, the function $P_0''(\xi)$ decays exponentially. Taking the limits $\xi \to \pm\infty$ in equation (5.63) we obtain

$$0 = F\big(P_0(\pm\infty), P_0(\pm\infty), \ldots, P_0(\pm\infty), \rho_0\big) = \Phi\big(P_0(\pm\infty), \rho_0\big), \tag{5.74}$$

which implies that

$$P_0(\pm\infty) \in \{-1, q(\rho_0), 1\}. \tag{5.75}$$

Since we know that $P_n(\xi) < 0$ if $\xi < 0$ and $P_n(\xi) > 0$ if $\xi > 0$, we have that $P_0(\xi) \leq 0$ if $\xi < 0$ and $P_0(\xi) \geq 0$ if $\xi > 0$ almost everywhere. In particular, if $q(\rho_0) = \pm 1$ then the proof is complete as then necessarily $P_0(\pm\infty) = \pm 1$. Thus assume that $q(\rho_0) \in (-1, 1)$. Fix any points q_1 and q_2 satisfying $-1 < q_1 < q(\rho_0) < q_2 < 1$ and let $\xi_n, \zeta_n \in \mathbb{R}$ be such that

$$\begin{array}{lllll} P_n(\xi) & \leq & q_1 & \text{for} & \xi < \zeta_n, \qquad q_1 \leq P_n(\xi) \leq q_2 \quad \text{for} \quad \zeta_n < \xi < \xi_n, \\ P_n(\xi) & \geq & q_2 & \text{for} & \xi > \xi_n. \end{array} \tag{5.76}$$

Without loss (we may always pass to a subsequence) we may assume that the limits $\xi_n \to \xi_0$ and $\zeta_n \to \zeta_0$ both exist, although they may possibly be infinite. It is enough to show that the difference $\xi_n - \zeta_n$ is bounded. Indeed, if this is the case, and if ξ_n and hence also ζ_n are bounded themselves, so that ξ_0 and ζ_0 are both finite, then $P_0(\xi) \leq q_1$ for all

$\xi < \zeta_0$ and $P_0(\xi) \geq q_2$ for all $\xi > \xi_0$, which with (5.75) implies the limits (5.64). The case $\xi_0 = \zeta_0 = \pm\infty$ cannot occur, since then either $P_0(\xi) \leq q_1$ or $P_0(\xi) \geq q_2$, hence $P_0(\xi) = \pm 1$ for all $\xi \in \mathbb{R}$, which is a contradiction.

To prove that $\xi_n - \zeta_n$ is bounded, assume $\xi_n - \zeta_n \to \infty$ and define

$$x_{n+}(\xi) = P_n(\xi + \xi_n), \;\; x_{n-}(\xi) = P_n(\xi + \zeta_n). \tag{5.77}$$

Upon passing to a subsequence and taking limits $x_{n\pm} \to x_{\pm}$ as above, we obtain solutions of (5.63) which satisfy the four boundary conditions in (5.58) with $q(\rho_0)$ replacing $q(\rho)$. However, this is impossible by Lemma 5.3.10. □

Lemma 5.3.13. *Suppose that the function $q : \overline{V} \to \mathbb{R}$ associated to (5.34) satisfies $q(\rho^*) = 0$ for some $\rho^* \in \overline{V}$. Then (5.34) with $\rho = \rho^*$ has a connecting solution $(P(\xi), c)$ for some $c \in \mathbb{R}$.*

Proof. First we consider the specific equation for some $k > 0$,

$$-\gamma x''(\xi) - x'(\xi) = \beta^{-1}\big(x(\xi - k) - x(\xi)\big) - f\big(x(\xi)\big), \tag{5.78}$$

in which f is given by

$$f(x) = \frac{\beta x(x^2 - 1)}{1 - \beta x} + 2\gamma x(x^2 - 1), \;\; \beta = \tanh k, \tag{5.79}$$

for $x \in [-1, 1]$. Outside this interval f is modified to be a nonzero C^1 function on the real line. It is routine to check that $x = \tanh \xi$ satisfies (5.78).

Now let $g : [0, 1] \to [0, 1]$ be any C^1 smooth function satisfying $g(\frac{1}{4}) = 0$ and $g(\frac{3}{4}) = 1$ and consider the family of equations

$$\begin{array}{rcl} -\gamma x''(\xi) - cx'(\xi) & = & (1 - g(\rho))\Big(\beta^{-1}\big(x(\xi - k) - x(\xi)\big) - f\big(x(\xi)\big)\Big) \\ & & +g(\rho)F\big(x(\xi + r_0), \ldots, x(\xi + r_N), \rho^*\big) \end{array} \tag{5.80}$$

for $\rho \in [0, 1]$. It is easy to see that this family satisfies the conditions (b1) through (b5), with $q(\rho) = 0$ for all $\rho \in [0, 1]$. We know that at $\rho = \frac{1}{4}$ the equation (5.80) has a connecting solution, namely $c = 1, x = \tanh \xi$. Due to Proposition 5.3.3, we see that solutions to (5.80) exist in a neighbourhood of $\rho = \frac{1}{4}$ and Theorem 5.3.12 allows us to extend this continuation to the interval $(0, 1)$. This proves the claim, as at $\rho = \frac{3}{4}$ the system reduces to the specified equation (5.34) with $\rho = \rho^*$. □

In case there is no value ρ^* for which $q(\rho^*) = 0$, the following lemma shows that we can choose an arbitrary value $\rho_0 \in V$ and embed the differential difference equation (5.34) with $\rho = \rho_0$ into a new family which does have $q(\rho_*) = 0$ for some ρ^*. We can then apply the same reasoning as in the proof of Lemma 5.3.13 to the new family to obtain a connecting solution to our original family at $\rho = \rho_0$.

Lemma 5.3.14 (see [113, Lemma 8.6])**.** *Consider the system*

$$-\gamma x''(\xi) - cx'(\xi) = F_0\big(x(\xi + r_0), \dots, x(\xi + r_N)\big) \tag{5.81}$$

satisfying the conditions (b1) through (b5) without the parameter ρ. *Assume that* $q = q_0 \in (-1, 1)$ *for the quantity in condition (b5). Then there exists a family (5.34), with* $V = (-1, 1)$ *and* $q(\rho) = \rho$, *satisfying the conditions (b1) through (b5), which reduces to (5.81) at* $\rho = q_0$.

We now have all the ingredients to complete the proof of Theorem 5.3.2.

Proof of Theorem 5.3.2. One can use Lemma's 5.3.13 and 5.3.14 to establish the existence of a solution at some parameter $\rho_* \in V$, after which a global continuation for all $\rho \in V$ of this solution can be constructed using Theorem 5.3.12 and Proposition 5.3.3. Uniqueness follows from Lemma 5.3.9. Here we have assumed V is connected, if not, use this construction for each connected component of V. □

5.4. The Algorithm

In this section we present and analyze a numerical method for solving the nonlinear autonomous differential difference equation

$$-\gamma x''(\xi) - cx'(\xi) = F\big(x(\xi), \overline{x}(\xi)\big), \tag{5.82}$$

where we have introduced the notation $\overline{\phi}(\xi) = \big(\phi(\xi+r_1), \phi(\xi+r_2), \dots, \phi(\xi+r_N)\big) \in \mathbb{R}^N$. As in the previous section, we demand that $\gamma > 0$, $r_i \neq r_j$ if $i \neq j$ and $r_i \neq 0$ for $i = 1 \dots N$, where $N \geq 1$. Throughout this section we will also assume F satisfies the conditions (b1) through (b5) from Section 5.3.

Following Definition 5.3.1, a connecting solution to (5.82) is a pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ that satisfies (5.82) and has the limits

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1. \tag{5.83}$$

Solutions to (5.82) correspond to zeroes of the operator $\mathcal{G}$ defined in (5.84), which in the present notation is given by

$$\mathcal{G}(\phi, c)(\xi) = -\gamma \phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi), \overline{\phi}(\xi)\big). \tag{5.84}$$

The numerical method we use to solve the differential difference equation (5.82) consists of applying a variant of Newton's method to find a zero of the operator $\mathcal{G}$ which satisfies the boundary conditions (5.83). Normally, applying Newton's method to seek a zero of $\mathcal{G}$ would involve an iteration step of the form

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{G}(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n). \tag{5.85}$$

To execute this step one would have to solve the linear differential difference equation

$$D_{1,2}\mathcal{G}(\phi_n, c_n)(\phi_{n+1}, c_{n+1}) = D_{1,2}\mathcal{G}(\phi_n, c_n)(\phi_n, c_n) - \mathcal{G}(\phi_n, c_n). \tag{5.86}$$

Since this is a computationally expensive procedure due to the presence of the shifted arguments [1], we want to reduce their contribution as much as possible. To this end, we fix a relaxation parameter $\mu \in [0, 1]$ and introduce the linear operator $D_{1,2}\mathcal{F}^\mu : W^{2,\infty} \times \mathbb{R} \to L^\infty$, given by

$$\begin{array}{rcl} D_{1,2}\mathcal{F}^\mu(\phi, c)(\psi, b)(\xi) & = & -\gamma\,\psi''(\xi) - c\psi'(\xi) - D_1F(\phi, \overline{\phi})\psi(\xi) \\ & & -\mu D_2F(\phi, \overline{\phi})\overline{\psi}(\xi) - b\phi'(\xi). \end{array} \tag{5.87}$$

Here $D_1F(x, \overline{x})$ denotes the derivative of F with respect to the first unshifted argument and $D_2F(x, \overline{x})$ denotes the derivative with respect to the shifted arguments. This operator $D_{1,2}\mathcal{F}^\mu(\phi, c)$ will play an important role in the variant of Newton's method we employ to solve (5.82). In particular, the iteration step in our method consists of solving the linear differential difference equation

$$D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)(\phi_{n+1}, c_{n+1}) = D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)(\phi_n, c_n) - \mathcal{G}(\phi_n, c_n). \tag{5.88}$$

We note here that when $\mu = 1$, the iteration step (5.88) is equivalent to the Newton iteration defined in (5.85). However, when $\mu = 0$, (5.88) is just an ordinary differential equation, which can be solved efficiently using standard techniques.

It will be useful to rewrite (5.88) in the form

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n). \tag{5.89}$$

At this point it is not yet clear if this iteration step is well-defined. In particular, we will show that for μ close enough to 1, the operator $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$ is invertible for all pairs (ϕ_*, c_*) sufficiently close to the solution (ϕ, c). The main theorem of this section roughly states that the numerical method introduced above converges to a solution of (5.82). In order to make this precise, we need to define what we mean by a point of attraction of the Newton iteration (5.88).

Definition 5.4.1. *A pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ is a point of attraction of the Newton iteration (5.88) if there is an open neighbourhood $S \subseteq W_0^{2,\infty} \times \mathbb{R}$, with $(\phi, c) \in S$, such that for any $(\phi_0, c_0) \in S$, the iterates defined by (5.88) all lie in $W_0^{2,\infty} \times \mathbb{R}$ and converge to (ϕ, c).* □

Theorem 5.4.2. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to the nonlinear autonomous differential difference equation (5.82). Then there exists $\epsilon > 0$ such that (ϕ, c) is a point of attraction for the Newton iteration (5.88) for all μ satisfying $|\mu - 1| < \epsilon$.*

Theorem 5.4.2 will be proved in a number of steps. We first prove that the Newton iteration (5.88) is well-defined for appropriate choices of the parameter μ and the initial condition (ϕ_0, c_0). Then we will consider the linearization of (5.89) around the solution (ϕ, c) and prove that the spectral radius of this linearized operator is smaller than one, which will allow us to complete the proof.

The first two lemma's use the fact that $D_{1,2}\mathcal{G}(\phi, c)$ is an isomorphism to show that this also holds for the operator $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$, for pairs (ϕ_*, c_*) sufficiently close to (ϕ, c).

Lemma 5.4.3. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.82). Then there exists $\epsilon > 0$ such that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism for all μ satisfying $|\mu - 1| < \epsilon$.*

Proof. We start out by noting that $D_{1,2}\mathcal{F}^1(\phi, c) = D_{1,2}\mathcal{G}(\phi, c)$, which is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞. It follows from [128, Theorem 5.10] that $[D_{1,2}\mathcal{G}(\phi, c)]^{-1}$ is a bounded linear operator. We can thus write $\upsilon = \left\|[D_{1,2}\mathcal{G}(\phi, c)]^{-1}\right\|$ and since $D_{1,2}\mathcal{G}(\phi, c)$ is a nontrivial operator, $0 < \upsilon < \infty$ must hold. Noticing that

$$\left\|[D_{1,2}\mathcal{F}^{\mu_1}(\phi, c)] - [D_{1,2}\mathcal{F}^{\mu_2}(\phi, c)]\right\| = |\mu_1 - \mu_2| \left\|D_2 F(\phi, \overline{\phi})\right\| \tag{5.90}$$

and using the fact that $\left\|D_2 F(\phi, \overline{\phi})\right\| < \infty$ as ϕ is bounded, we see that we can choose $\epsilon > 0$ such that

$$\left\|[D_{1,2}\mathcal{F}^{\mu}(\phi, c)] - [D_{1,2}\mathcal{G}(\phi, c)]\right\| < \frac{1}{2\upsilon} \tag{5.91}$$

whenever $|\mu - 1| < \epsilon$. Now fix $\mu \in (1 - \epsilon, 1 + \epsilon)$ and let I be the identity operator on $W_0^{2,\infty} \times \mathbb{R}$. Since

$$\begin{aligned} &\left\|I - [D_{1,2}\mathcal{G}(\phi, c)]^{-1}[D_{1,2}\mathcal{F}^\mu(\phi, c)]\right\| \\ &= \left\|[D_{1,2}\mathcal{G}(\phi, c)]^{-1}([D_{1,2}\mathcal{G}(\phi, c)] - [D_{1,2}\mathcal{F}^\mu(\phi, c)])\right\| \le \tfrac{1}{2\upsilon}\upsilon = \tfrac{1}{2} < 1, \end{aligned} \tag{5.92}$$

Neumann's Lemma implies that $[D_{1,2}\mathcal{G}(\phi, c)]^{-1}[D_{1,2}\mathcal{F}^\mu(\phi, c)]$ is invertible and hence $D_{1,2}\mathcal{F}^\mu(\phi, c)$ has a left inverse. Because $D_{1,2}\mathcal{G}(\phi, c)$ is an isomorphism, it has a left and right inverse and so by an analogous argument involving the identity operator on L^∞ the existence of a right inverse for $D_{1,2}\mathcal{F}^\mu(\phi, c)$ can be established. This completes the proof that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism. □

For convenience, we define open balls $B_{\psi,b,\delta}$ in $W_0^{2,\infty} \times \mathbb{R}$ given by

$$B_{\psi,b,\delta} = \left\{(\phi_*, c_*) \in W^{2,\infty} \times \mathbb{R} \mid \|(\psi, b) - (\phi_*, c_*)\| < \delta\right\}. \tag{5.93}$$

Lemma 5.4.4. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.82). Then there exists $\epsilon > 0$, such that for all $\mu \in \mathbb{R}$ with $|\mu - 1| < \epsilon$, there is an open ball $B = B_{\phi,c,\delta}$, for some $\delta > 0$, with the property that the linear operator $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$ is an isomorphism for all $(\phi_*, c_*) \in B$.*

Proof. The proof is analogous to the proof of Lemma 5.4.3. One uses the fact that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is invertible and the observation that $\left\|D_{1,2}\mathcal{F}^\mu(\widetilde{\phi}, \widetilde{c})\right\|$ is continuous with respect to $(\widetilde{\phi}, \widetilde{c})$ in the norm on $W_0^{2,\infty} \times \mathbb{R}$. To establish this, one needs the local Lipschitz condition (b2) on the derivatives of F, which implies the global Lipschitz continuity of $D_1 F$ on compact subsets of $\mathbb{R}^{N+1}$. Together with the boundedness of all $\phi_* \in W_0^{2,\infty}$, this establishes that for fixed ϕ_* and for all ϕ_{**} with $\|\phi_{**} - \phi_*\| \le C$, we have $\left|D_1 F(\phi_*, \overline{\phi}_*)(\xi) - D_1 F(\phi_{**}, \overline{\phi}_{**})(\xi)\right| \le D \|\phi_* - \phi_{**}\|$ for some $D < \infty$. With this estimate and a similar one for $D_2 F$, the continuity is easily established. □

We remark that Lemma 5.4.4 guarantees that for μ close enough to 1, there exists $\delta > 0$ such that the Newton iteration step given by (5.88) is well-defined whenever $(\phi_n, c_n) \in B_{\phi,c,\delta}$. We can now define the operator $H^\mu : B_{\phi,c,\delta} \to W_0^{2,\infty} \times \mathbb{R}$ given by

$$H^\mu(\phi_*, c_*) = (\phi_*, c_*) - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}\mathcal{G}(\phi_*, c_*). \tag{5.94}$$

Lemma 5.4.5. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.82). Then there exists $\epsilon > 0$ such that for all μ satisfying $|\mu - 1| < \epsilon$, the operator H^μ defined by (5.94) is Frechet differentiable at (ϕ, c). For these values of μ, the corresponding derivative with respect to ϕ_* and c_* at this point is given by*

$$D_{1,2}H^\mu(\phi, c) = I - [D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}D_{1,2}\mathcal{G}(\phi, c). \tag{5.95}$$

Proof. From Lemma 5.4.3 we know that there exists $\epsilon > 0$ such that for all μ satisfying $|\mu - 1| < \epsilon$, $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism. From the proof of Lemma 5.4.3 we also know that for such μ we have $\left\|I - [D_{1,2}\mathcal{G}(\phi, c)]^{-1}D_{1,2}\mathcal{F}^\mu(\phi, c)\right\| < 1$. Now fix μ satisfying $|\mu - 1| < \epsilon$.

Fix $\beta > 0$. We know that G is Frechet-differentiable at (ϕ, c), hence there exists δ_1 such that

$$\left\|\mathcal{G}(\phi_*, c_*) - \mathcal{G}(\phi, c) - D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]\right\| \leq \beta \left\|(\phi_*, c_*) - (\phi, c)\right\| \tag{5.96}$$

for all $(\phi_*, c_*) \in B_{\phi,c,\delta_1}$. From Lemma 5.4.4 we know that there exists δ_2 such that $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$ is an isomorphism for all $(\phi_*, c_*) \in B_{\phi,c,\delta_2}$. In the proof of Lemma 5.4.4 we have seen that $\left\|D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)\right\|$ is continuous in ϕ_* and c_*. Using this and the continuity of the inverse, we see that there exists $\delta_3 > 0$ such that

$$\left\|[[D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1} - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}][D_{1,2}\mathcal{G}(\phi, c)]\right\| \leq \beta \tag{5.97}$$

whenever $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$. From (5.97) it also follows that when $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$ we have

$$\left\|[D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)]\right\| \leq \beta + \left\|[D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)]\right\| = \beta + C \tag{5.98}$$

for some finite constant C. Using the identity

$$[D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1} = [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)][D_{1,2}\mathcal{G}(\phi, c)]^{-1}, \tag{5.99}$$

we see that $\left\|[D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}\right\| \leq D(\beta + C)$ for some finite constant D, whenever $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$.

Now choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. Using the fact that $(\phi, c) = H^\mu(\phi, c)$ we obtain for all $(\phi_*, c_*) \in B_{\phi,c,\delta}$

$$\begin{aligned}
&\left\|H^\mu(\phi_*, c_*) - H^\mu(\phi, c) - [I - [D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}D_{1,2}\mathcal{G}(\phi, c)][(\phi_*, c_*) - (\phi, c)]\right\| \\
&= \left\|[D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)] - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}\mathcal{G}(\phi_*, c_*)\right\| \\
&\leq \left\|-[D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}[\mathcal{G}(\phi_*, c_*) - \mathcal{G}(\phi, c) - D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]]\right\| \\
&\quad + \left\|[[D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1} - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}][D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]]\right\| \\
&\leq (D(\beta + C)\beta + \beta)\left\|(\phi_*, c_*) - (\phi, c)\right\|.
\end{aligned} \tag{5.100}$$

This completes the proof that H^μ is Frechet differentiable. □

We can now use the fact that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism to establish the crucial fact that the spectral radius of the linear operator $D_{1,2}H^\mu(\phi, c)$ is less than one.

Lemma 5.4.6. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.82). Let $\hat{\sigma}^\mu$ denote the spectral radius of $D_{1,2}H^\mu(\phi, c)$. Then there exists $\epsilon > 0$, such that for all μ satisfying $|\mu - 1| < \epsilon$, we have $\hat{\sigma}^\mu < 1$.*

Proof. Writing out the eigenvalue problem for $D_{1,2}H^\mu(\phi, c)$, we obtain the equation

$$(1-\mu)[D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}[D_2F(\phi, \overline{\phi})\overline{\psi}] - \lambda(\psi, b) = (0, 0), \tag{5.101}$$

where λ is the eigenvalue and (ψ, b) are the eigenfunctions. After applying $D_{1,2}\mathcal{F}^\mu(\phi, c)$ and using the explicit form of $D_{1,2}\mathcal{F}^\mu$ this is equivalent to

$$-D_{1,2}\mathcal{F}^{\tilde{\mu}(\lambda)}(\phi, c)(\psi, b) = 0, \tag{5.102}$$

in which

$$\tilde{\mu}(\lambda) = \mu + \frac{1-\mu}{\lambda}. \tag{5.103}$$

We know from Lemma 5.4.3 that there exists $\delta > 0$ such that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism for all μ satisfying $|\mu - 1| < \delta$. If we now choose $\epsilon = \frac{\delta}{2}$, we see that for all μ satisfying $|\mu - 1| < \epsilon$ and for all $|\lambda| \geq 1$,

$$|\tilde{\mu}(\lambda) - 1| \leq \frac{\delta}{2} + \frac{\delta}{2}|\lambda|^{-1} \leq \delta. \tag{5.104}$$

In particular, this means that for these μ and λ equation (5.102) has only the zero solution, as $D_{1,2}\mathcal{F}^{\tilde{\mu}(\lambda)}(\phi, c)$ is an isomorphism. Thus for these μ there cannot be any eigenvalues λ with $|\lambda| \geq 1$, proving that $\hat{\sigma}^\mu < 1$. □

We are now ready to complete the proof of Theorem 5.4.2.

Proof of Theorem 5.4.2. Fix $\beta > 0$ such that for all μ satisfying $|\mu - 1| < \beta$, we have that the operator H^μ is well-defined in a neighbourhood of (ϕ, c) and Frechet differentiable at (ϕ, c), together with the inequality $\hat{\sigma}^\mu < 1$, where $\hat{\sigma}^\mu$ is the spectral radius of $D_{1,2}H^\mu(\phi, c)$. Now fix μ satisfying $|\mu - 1| < \beta$, write $H = H^\mu$ and $\hat{\sigma} = \hat{\sigma}^\mu$ and choose $\epsilon > 0$ such that $\hat{\sigma} + \epsilon < 1$. Let H^p be the p-fold iterate of H. Since H is Frechet-differentiable at (ϕ, c), so is H^p. From the chain rule it follows that $D_{1,2}H^p(\phi, c) = [D_{1,2}H(\phi, c)]^p$.

From the Gelfand and Mazur formula [129, Theorem 10.13] for the spectral radius $\hat{\sigma}$, it follows that we may choose p such that

$$\begin{aligned} &\left\| [D_{1,2}H(\phi, c)]^p \right\| \leq (\hat{\sigma} + \epsilon)^p, \\ &(\hat{\sigma} + \epsilon)^p + \epsilon < 1. \end{aligned} \tag{5.105}$$

Let s be an integer. From the Frechet-differentiability of H^s we know that there exists $\delta > 0$, such that for all $(\phi_*, c_*) \in B_{\phi,c,\delta}$ and for all $1 \leq s \leq p$,

$$\left\| H^s(\phi_*, c_*) - H^s(\phi, c) - [D_{1,2}H(\phi, c)]^s[(\phi_*, c_*) - (\phi, c)]\right\| \leq \epsilon \left\|(\phi_*, c_*) - (\phi, c)\right\|. \tag{5.106}$$

With this we can compute

$$\begin{aligned} \|H^s(\phi_*, c_*) - (\phi, c)\| &\leq \left\| H^s(\phi_*, c_*) - H^s(\phi, c) \right. \\ &\quad \left. -[D_{1,2}H(\phi, c)]^s[(\phi_*, c_*) - (\phi, c)]\right\| \\ &\quad + \left\|[D_{1,2}H(\phi, c)]^s\right\| \|(\phi_*, c_*) - (\phi, c)\| \\ &\leq \left(\left\|[D_{1,2}H(\phi, c)]^s\right\| + \epsilon\right) \|(\phi_*, c_*) - (\phi, c)\|. \end{aligned} \tag{5.107}$$

Writing

$$w = \max\left(\epsilon, \max\left\{\left\|[D_{1,2}H(\phi, c)]^s\right\| \mid s = 1 \ldots p\right\}\right), \tag{5.108}$$

we see that we can ensure $H^s(\phi_0, c_0) \in B_{\phi,c,\delta_*}$ for $s = 1 \ldots p$ by choosing $(\phi_0, c_0) \in B_{\phi,c,\delta_*/2w}$. For $s = p$ equation (5.107) reduces to

$$\left\| H^p(\phi_*, c_*) - (\phi, c)\right\| \leq [(\hat{\sigma} + \epsilon)^p + \epsilon] \|(\phi_*, c_*) - (\phi, c)\|. \tag{5.109}$$

Combining everything, we see that by choosing $(\phi_0, c_0) \in B_{\phi,c,\delta/2w}$ all the Newton iterates lie in the ball $B_{\phi,c,\delta}$. Now choosing $\delta > 0$ so small that H is well-defined on $B_{\phi,c,\delta}$, we see that the Newton process is well-defined and satisfies

$$\lim_{n\to\infty} \|(\phi_n, c_n) - (\phi, c)\| \leq \lim_{n\to\infty} (2w)\left((\hat{\sigma} + \epsilon)^p + \epsilon\right)^{\left\lfloor \frac{n}{p} \right\rfloor} \|(\phi_0, c_0) - (\phi, c)\| = 0. \tag{5.110}$$

This concludes the proof of the theorem. □

Remark 5.4.7. *It is not clear if Theorem 5.4.2 holds for $\mu = 0$. Setting $\mu = 0$ in the Newton iteration step (5.88) is easily seen to be equivalent to making the approximation $\overline{\phi}_{n+1} = \overline{\phi}_n$. Intuitively, this approximation should become increasingly accurate as the iterates ϕ_n converge to the solution of (5.82). In addition, the equations (5.90) and (5.91) from the proof of Lemma 5.4.3 give us information about the values of ϵ which satisfy the claim in Theorem 5.4.2. In particular, smaller values of $\|D_2F\|$ give us larger possible values for ϵ. Referring back to (1.11), we see there that $\|D_2F\|$ is proportional to the parameter α. Since we are interested in solutions to (1.11) far from the continuous limit, i.e., for small values of the parameter α, these observations lead us to believe we can take $\mu = 0$ in many cases of interest. See Section 5.5 for a further discussion and some numerical examples.*

5.5. Examples

In this section we present some numerical results obtained by our algorithm in order to illustrate some of the key phenomena encountered in the qualitative study of lattice differential equations, together with some of the technical difficulties involved with the numerical computation of solutions to such equations. We note here that all the Newton iteration steps

(5.88) which were executed in order to obtain the results in this section were performed with $\mu = 0$.

In the literature it has by now become somewhat classic to study travelling wave solutions to the spatially discretized reaction diffusion equation (1.8). The simplest corresponding differential difference equation is given by

$$-c\phi'(\xi) = \alpha\big(\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big) - (\phi(\xi)^2 - 1)(\phi(\xi) - \rho), \qquad (5.111)$$

where $\alpha > 0$ and $\rho \in (-1, 1)$ is a detuning parameter. It has been widely studied both numerically and theoretically [1, 15, 35, 51, 53, 93, 170, 171]. The relative simplicity of (5.111) and the fact that solutions exhibit many of the interesting features mentioned in the introduction ensure that this equation is an ideal test problem for any differential difference equation solver.

In [82] (5.111) was solved numerically by adding a small diffusion term $-\gamma\phi''(\xi)$ to the left-hand side of (5.111) and our results were compared to previously established solutions in [1, 53]. In particular, we remark that our implementation allows us to choose $\gamma = 3 \times 10^{-10}$, while up to now the smallest possible choice for γ was given by $\gamma = 10^{-6}$ [1]. The phenomenon of propagation failure is clearly visible from the results for $\alpha = 0.1$ and the corresponding wave profiles already reach their limiting forms at $\gamma = 10^{-5}$.

We wish to empasize here that, in contrast to the presentation in [53], the formulation of the algorithm given in the previous section allows us to consider differential difference equations which involve a nonlinear mixing of shifted terms and ordinary terms. In this section we illustrate this feature by numerically studying the differential difference equation

$$-\gamma\phi''(\xi) - c\phi'(\xi) = 2\alpha \tanh\big(\tfrac{1}{2}\phi(\xi+1) + \tfrac{1}{2}\phi(\xi-1) - \phi(\xi)\big) - \tfrac{1}{4} f(\phi(\xi), \rho). \quad (5.112)$$

Here $\gamma, \alpha > 0$ are two positive parameters and f is the cubic nonlinearity given by

$$f(x, \rho) = (x^2 - 1)(x - \rho), \qquad (5.113)$$

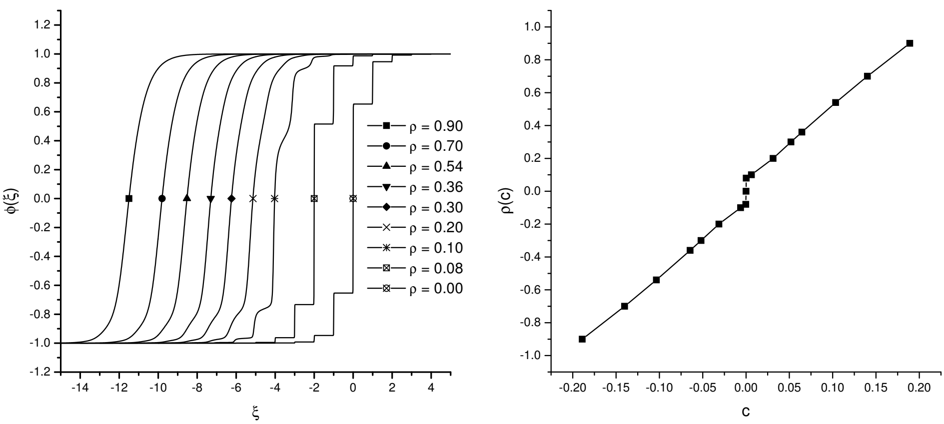
where $\rho \in (-1, 1)$ is a continuation parameter. The solutions of (5.112) were required to satisfy the limits

$$\lim_{\xi\to-\infty} \phi(\xi) = -1, \quad \lim_{\xi\to\infty} \phi(\xi) = 1 \qquad (5.114)$$

and were normalized to have $\phi(0) = 0$. Equations similar to (5.112) play an important role when studying Glauber type Ising models [49] in material science.

It is easy to verify that the family (5.112) satisfies all the requirements (b1) through (b5). Also note that if $\phi(\xi)$ is a solution to the problem (5.112) satisfying the limits (5.114) at some parameter $\rho = \rho_0$ with wavespeed $c = c_0$, then $\psi(\xi) = -\phi(-\xi)$ is a solution to the same problem with $\rho = -\rho_0$ and wavespeed $c = -c_0$ and also satisfies the limits (5.114).

The phenomenon of propagation failure has been studied extensively in [113]. In particular, in Corollary 2.5 of [113] it is shown that for our family (5.112) with $\gamma = 0$, there exist quantities $-1 \le \rho_- \le \rho_+ \le 1$, such that (5.112) only has connecting solutions with wavespeed $c = 0$ for $\rho_- \le \rho \le \rho_+$. It may happen that $\rho_- < \rho_+$, that is, that there is a nontrivial interval of the detuning parameter ρ for which the wavespeed vanishes. In this

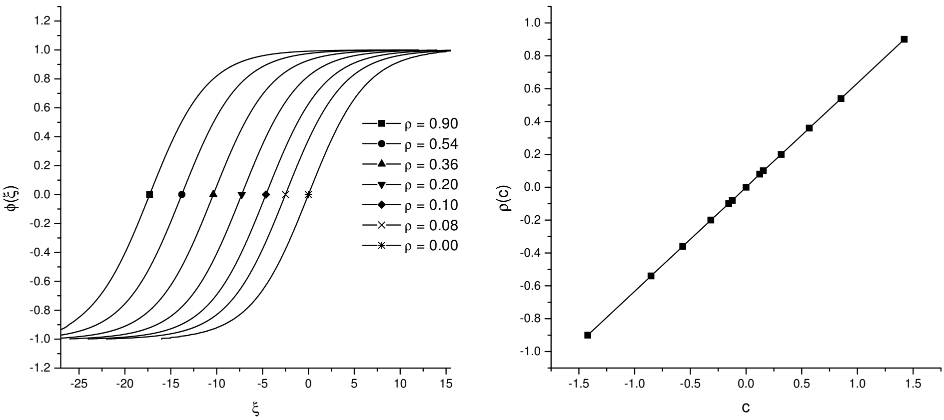


(a) (b)

Figure 5.1: *In (a) the waveprofiles $\phi(\xi)$ have been plotted for solutions to the differential difference equation (5.112) with $\gamma = 10^{-6}$ and $\alpha = 0.1$, at different values of the detuning parameter ρ. For presentation purposes the curves have been shifted by different amounts along the ξ-axis. In (b) the $\rho(c)$ relation has been plotted, i.e., for each value of the detuning parameter ρ the corresponding wavespeed c is given. The solid dots represent the wavespeeds corresponding to the curves in (a), which have been continued to $\rho < 0$, using the observation that $\psi(\xi) = -\phi(-\xi)$ is a connecting solution with wavespeed $-c$ if $\phi(\xi)$ is a connecting solution with wavespeed c. From (b) it is easily seen that there exists a nontrivial interval of ρ in which $c \sim 0$, hence propagation failure occurs.*

region one generally expects the solutions to become discontinuous. However, since all the numerical computations were performed with $\gamma > 0$, which forces the solutions to remain continuous, it is a priori not clear if one can accurately reproduce the solution profiles at $\gamma = 0$ and thus actually uncover the propagation failure. The essential tool here is Theorem 5.3.12, which establishes that if we have a sequence of solutions $P_n(\xi)$ to (5.34) with $\gamma = \gamma_n$, where $\gamma_n \to 0$, a subsequence of the functions P_n will convergence to a solution at $\gamma = 0$. Ideally, this convergence should occur at a value for the parameter γ which can be handled numerically and the solution curves should remain computationally stable below this value. This was the case for the solutions to (5.111) calculated in [82] and we show here that the same property holds for the problem (5.112) currently under investigation.

In Figure 5.1 the calculated solutions to (5.112) are presented, together with their wavespeeds. One sees clearly from Figure 5.1(b) that there is a nontrivial interval of the detuning parameter ρ for which the wavespeed c vanishes. Looking at Figure 5.1(a), one sees that the solutions for these values of ρ exhibit step-like behaviour. In the calculations we used $\gamma = 10^{-6}$, which thus indicates that for γ small enough, one can be confident that the effects of propagation failure will be observed and accurate predictions can be made about the parameter values at which it will occur. Propagation failure does not occur at each value of α, as the $\rho(c)$ curve in Figure 5.2(b) shows. Notice that the solutions in Figure



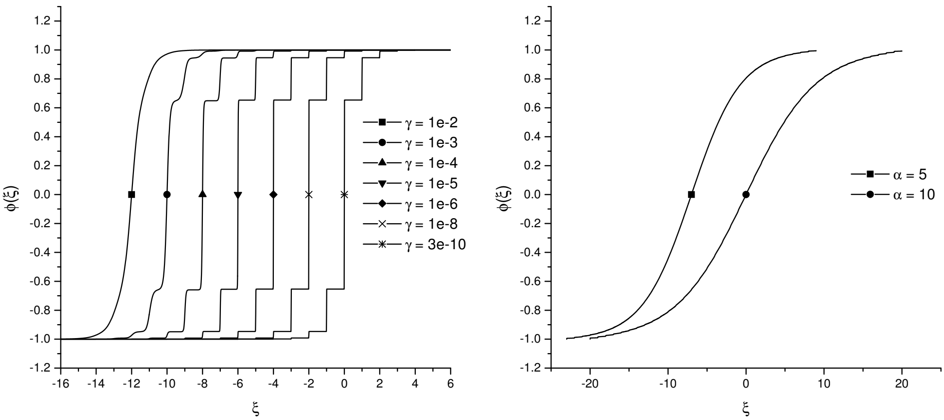
(a) (b)

Figure 5.2: *In (a) the waveprofiles $\phi(\xi)$ have been plotted for solutions to (5.112) with $\gamma = 10^{-4}$ and $\alpha = 5$, at different values of the detuning parameter ρ. The wavespeeds for these solutions are given in (b). The calculations to obtain the solution curves in this figure were performed on the finite interval* $[-20, 20]$. *Notice that in (b) there is no nontrivial interval of ρ for which $c = 0$. Indeed, the solution curves in (a) remain continuous as $\rho \to 0$.*

5.2(a) indeed remain smooth as $\rho \to 0$. We remark here that the wavespeed necessarily satisfies $c = 0$ when $\rho = 0$, but it is clear that for this specific system (5.112), the solutions only exhibit discontinuous behaviour when the wavespeed vanishes for a nontrivial interval of the detuning parameter ρ.

In Figure 5.3(a) the solution curves to (5.112) have been plotted for a number of different values of γ, ranging from $\gamma = 10^{-2}$ to $\gamma = 3 \times 10^{-10}$. The transition from smooth to steplike solutions is clearly visible and already occurs at $\gamma \approx 10^{-3}$. Notice that the solution curves remain stable for $\gamma = 10^{-5}$ to $\gamma = 3 \times 10^{-10}$, while the curve for $\gamma = 10^{-4}$ does not differ too much. One sees here that in this example computations with $\gamma \sim 10^{-5}$ will provide an excellent approximation to the actual solutions with $\gamma = 0$. In particular, the computations indicate that the discontinuous behaviour due to propagation failure, which occurs at $\gamma = 0$ and $c = 0$, is already visible at $\gamma = 10^{-5}$. Indeed, upon recalculation of the curves in Figure 5.1 using $\gamma = 10^{-8}$, the results were observed to remain exactly the same.

When we take $\mu = 0$ in the Newton iteration (5.88), we are neglecting the presence of the shifted terms D_2F. In particular, referring to (5.90) in the proof of Lemma 5.4.3, one expects that when the norm of the shifted term D_2F becomes large, problems will arise with the invertibility of the operator $D_{1,2}\mathcal{F}^{\mu}$ and hence with the convergence of the algorithm. In our case, the importance of the shifted term is given by the parameter α. For large α, the hyperbolic tangent term in (5.112) becomes increasingly important. Nevertheless, by using a suitable continuation scheme, we are able to obtain solutions to (5.112) for $\alpha = 5$ and $\alpha = 10$ at $\gamma = 10^{-4}$ and $\rho = 0$. These solutions have been plotted in Figure 5.3(b). At



(a) (b)

Figure 5.3: *In (a) waveprofiles* $y(\xi)$ *for solutions to (5.112) at different values of* γ *are given, for fixed* $\rho = 0$ *and* $\alpha = 0.1$*, demonstrating the robust convergence in the* $\gamma \to 0$ *limit and showing that already at* $\gamma = 10^{-5}$ *the waveform has attained its limiting profile. In (b) solutions* $y(\xi)$ *to (5.112) at* $\alpha = 5$ *and* $\alpha = 10$ *in the critical case* $\rho = 0$ *are given. The parameter* γ *was fixed at* 10^{-4}.

these levels of α the shifted term has become the dominant term. If one wishes to increase α even further, it no longer suffices to take $\mu = 0$ in (5.88). It is however quite satisfactory that this choice for μ can be used for practical purposes up to these levels of α, which are already far from the interesting case $\alpha \approx 0.1$.

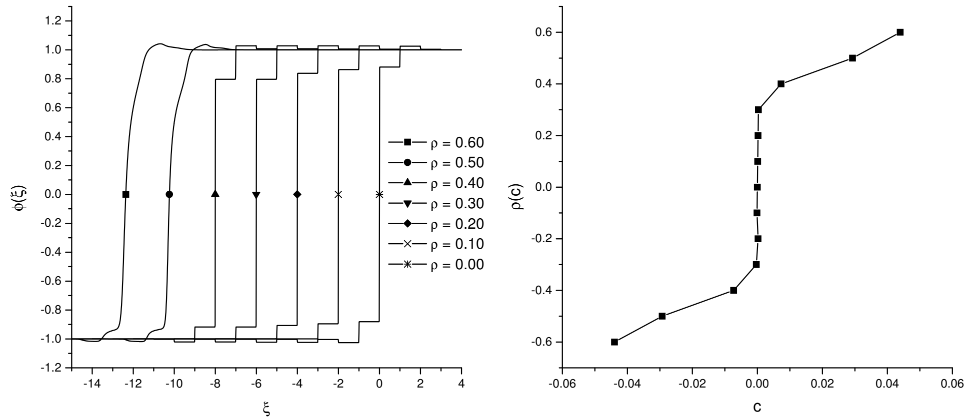
5.6. Extensions

Although all the theory developed in this chapter applies only to one dimensional families (5.1) that satisfy the conditions (b1) through (b5), it turns out that the application range of the numerical method is much broader. In addition, interesting models exist which lead to differential difference equations that violate the above assumptions. To gain some insight into these issues, we numerically study two important systems that are not covered by the theory developed in this chapter, which both give rise to novel dynamical behaviour.

5.6.1. Ising models

In this subsection we numerically study the differential difference equation given by

$$\begin{array}{rcl} -\gamma\phi''(\xi) - c\phi'(\xi) & = & \alpha\big(\phi(\xi-1) + \phi(\xi+1) - \frac{1}{4}\phi(\xi-2) - \frac{1}{4}\phi(\xi+2) - \frac{3}{2}\phi(\xi)\big) \\ & & -\frac{1}{4}f(\phi(\xi),\rho), \end{array} \quad (5.115)$$



(a) (b)

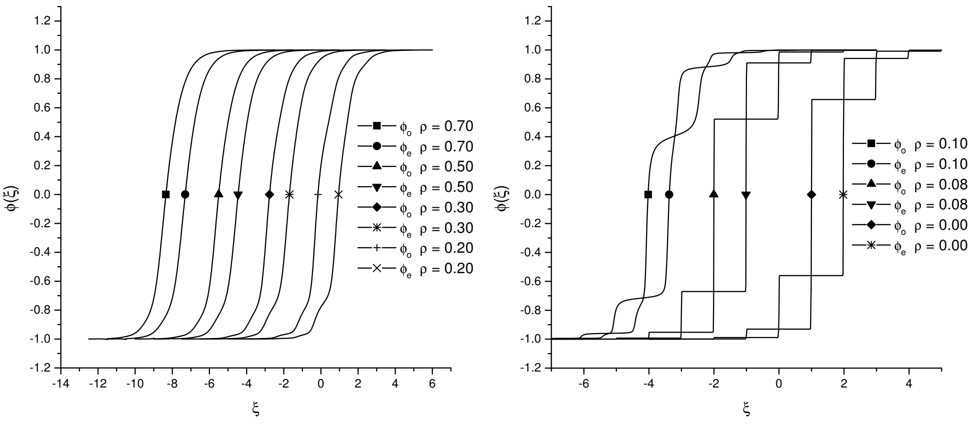
Figure 5.4: *In (a) the waveprofiles $\phi(\xi)$ have been plotted for solutions to (5.115) with $\gamma = 10^{-6}$ and $\alpha = 0.042$, at different values of the detuning parameter ρ. The corresponding wavespeeds for these solutions are given in (b).*

where f again denotes the bistable cubic nonlinearity $f(x, \rho) = (x^2 - 1)(x - \rho)$ for some parameter $\rho \in (-1, 1)$ and $\alpha > 0$ is a strictly positive parameter. We again impose the limits $\phi(\pm\infty) = \pm 1$ and the normalization condition $\phi(0) = 0$ and again take $\mu = 0$ when performing the iteration steps (5.88). The interesting feature in (5.115) is that the coefficients in front of the shifted terms $\phi(\xi \pm 2)$ are now negative, which implies that this equation does not satisfy the assumption (b3) introduced in Section 5.3. In particular, we no longer have any guarantee that (5.115) in fact has a solution or that the numerical method will be able to find it.

Equation (5.115) with $\gamma = 0$ is an example of a class of differential difference equations which was proposed in [13] to provide a discrete convolution model for Ising-like phase transitions. The equation was derived by considering groups of atoms arranged on a lattice and computing the gradient flow of a Helmholtz free energy functional. This energy functional takes into account interactions within each group of atoms together with interactions between groups, thus incorporating both local and non-local effects into the model. Due to the nature of the physical forces involved, the long-range interaction coefficients can be both positive and negative.

Unnormalized solutions to (5.115) with $\gamma = 0$, $c = 0$, fixed ρ and sufficiently small $\alpha > 0$ were analyzed in [13]. In particular, for each sufficiently small $\alpha > 0$ it was shown that there exist three intervals $I_j(\alpha)$ for $1 \le j \le 3$, such that for any two disjoint sets $S_1, S_2 \subset \mathbb{Z}$, there exists a unique (unnormalized) solution u to (5.115) that satisfies $u(x) \in I_i$ whenever $\lfloor x \rfloor \in S_i$, for all $1 \le i \le 3$. Here we have defined $S_3 = \mathbb{Z} \setminus \big(S_1 \cup S_2\big)$. We remark here that $-1 \in \mathrm{int}(I_1)$ and $1 \in \mathrm{int}(I_2)$. From this it is clear that the set of solutions to (5.115) with $\gamma = c = 0$ has a rich structure.

In Figure 5.4 the results of an application of the numerical method to (5.115) with



(a) (b)

Figure 5.5: *Waveprofiles $\phi_o(\xi)$ and $\phi_e(\xi)$ have been plotted for solutions to (5.118) with $\alpha_o = 1.5$, $\alpha_e = 1.7$ and $\gamma = 10^{-5}$. In addition to the global shift for different values of ρ, the curves for ϕ_e in (a) have been shifted by* 1 *along the ξ-axis relative to their accompanying ϕ_o curves. This additional shift has not been applied to the curves in (b).*

$\gamma = 10^{-6}$ and $\alpha = 0.042$ are displayed. The transition from smooth solution curves to discontinuous step functions as ρ approaches the critical value $\rho = 0$ is clearly illustrated. Notice that in contrast to the results from the previous section, the solution curves are no longer monotonic and in addition are no longer restricted to the interval $[-1, 1]$. The values taken by the step functions in Figure 5.4 are in agreement with the predictions from [13] and these results again illustrate that the algorithm is robust enough to be able to uncover some of the behaviour at $\gamma = 0$.

5.6.2. Higher Dimensional Systems

Up to now all the theory has been developed for scalar differential difference equations of the form (5.34). The question of course immediately arises if the results can be extended to higher dimensional systems and if the numerical method is able to handle them as well. In this subsection we briefly discuss some of the issues involved, using a bistable reaction-diffusion equation on a one dimensional lattice with spatially varying diffusion coefficients as an example. Specifically, we will study the system

$$\dot{u}_j(t) = \alpha_j\big(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big) - \frac{15}{4}\big(u_j^2(t) - 1\big)\big(u_j(t) - \rho\big), \quad j \in \mathbb{Z}, \quad (5.116)$$

where $\rho \in (-1, 1)$ is a detuning parameter and the coefficients α_j are periodic with period two, i.e., we have $\alpha_{j+2} = \alpha_j$ for all $j \in \mathbb{Z}$.

Lattice differential equations of the form (5.116) arise naturally when modelling diffusion processes in discrete systems which are spatially periodic. As a specific biological

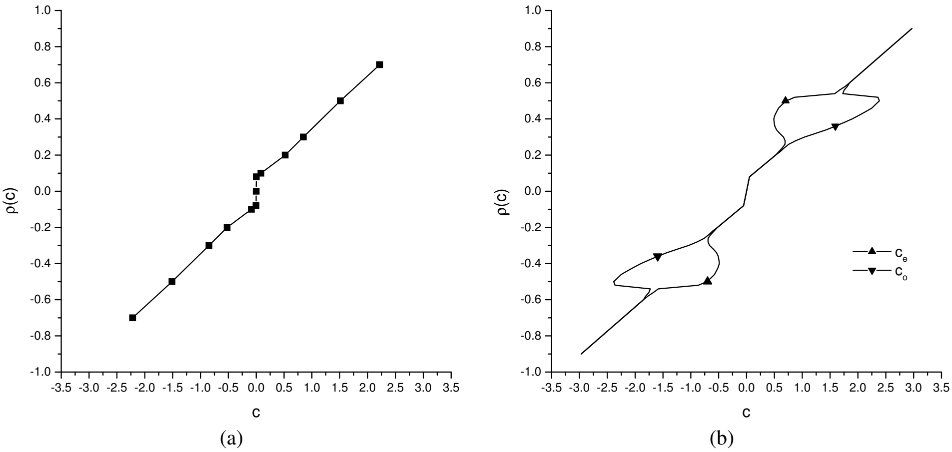


Figure 5.6: *In (a) the wavespeed plot for the solutions to (5.118) with $\alpha_o = 1.5$, $\alpha_e = 1.7$ and $\gamma = 10^{-5}$ is given. Notice the nontrivial interval of the detuning parameter ρ for which $c = 0$, indicating that propagation failure occurs for the periodic diffusion problem (5.116). In (b) the wavespeed plot for the constant diffusion system (5.119) with $\alpha = 1.6$ and $\gamma = 10^{-5}$ is given. Notice the existence of two regions for which $c_e \neq c_o$. These regions are called period two bifurcation regions. The presence of these regions demonstrates that, unlike one dimensional systems, higher dimensional systems do not necessarily have unique solutions.*

example we mention a model that describes the behaviour of nerve fibers by employing an electrical circuit model for the excitable fiber membrane [92, Sec 9.3]. One considers myelinated nerve fibers that have periodic gaps, called nodes, in their coating. Assuming a one dimensional grid of nodes and writing V_j for the voltage at node j, one derives the equations [92]

$$p\big(C\dot{V}_j + I_{\text{ion}}(V_j)\big) = \frac{1}{\mu_j L R}\big(V_{j+1} + V_{j-1} - 2V_j\big), \quad j \in \mathbb{Z}. \tag{5.117}$$

Here p is the perimeter length of the fibre, C is the capacitance, L is the length of myelin sheath between nodes, R is the resistance per unit of length and μ_j is the length of node j. Allowing the node length μ_j to vary periodically among nodes and remarking that a cubic nonlinearity is a natural form for the ionic current $I_{\text{ion}}(V)$, one sees that (5.117) transforms into a system of the form (5.116).

As in previous sections, we numerically analyze the system (5.116) by adding a small artificial diffusion term and looking for travelling wave solutions. In particular, the points on the lattice are split into two groups, called even and odd, which admit their own waveforms ϕ_e, ϕ_o and diffusion coefficients α_e, α_o. Substituting the travelling wave anszatz $u_{2k}(t) = \phi_e(2k - ct)$ and $u_{2k+1} = \phi_o(2k + 1 - ct)$ into (5.116), we arrive at the two dimensional differential difference equation

$$\left\{\begin{array}{rcl} -\gamma\phi_o''(\xi) - c\phi_o'(\xi) & = & \alpha_o\big(\phi_e(\xi+1) + \phi_e(\xi-1) - 2\phi_o(\xi)\big) \\ & & -\frac{15}{4}\big(\phi_o^2(\xi) - 1\big)\big(\phi_o(\xi) - \rho\big), \\ -\gamma\phi_e''(\xi) - c\phi_e'(\xi) & = & \alpha_e\big(\phi_o(\xi+1) + \phi_o(\xi-1) - 2\phi_e(\xi)\big) \\ & & -\frac{15}{4}\big(\phi_e^2(\xi) - 1\big)\big(\phi_e(\xi) - \rho\big). \end{array}\right. \tag{5.118}$$

In addition, we impose the asymptotic limits $\phi_o(\pm\infty) = \pm 1$, $\phi_e(\pm\infty) = \pm 1$ and introduce the phase condition $\phi_o(0) = 0$ in order to control the translational invariance of (5.118).

In contrast to previous sections, it was necessary to take $\mu = 1$ when performing the Newton iterations (5.88) needed to numerically solve (5.118). The results can be found in Figure 5.5, where solution curves to (5.118) with $\alpha_o = 1.5$, $\alpha_e = 1.7$ and $\gamma = 10^{-5}$ have been plotted for various values of the detuning parameter ρ. The corresponding wavespeeds can be found in Figure 5.6(a). It is clear from the latter wavespeed plot and the steplike behaviour exhibited in Figure 5.5(b) that propagation failure can occur for the discrete periodic diffusion system (5.116). We also mention the interesting fact that as the norm of the detuning parameter ρ increases, the two waveprofiles ϕ_e and ϕ_o become increasingly alike, indicating that the significance of the diffusion term in (5.118) decreases as $|\rho| \to 1$.

At the moment it is unclear if we have existence and uniqueness of solutions to equations of the form (5.118) and if the convergence theory established in this chapter for the Newton iteration (5.88) continues to hold. Some of the arguments used to derive the current results are strictly one dimensional in nature and do not generalize trivially to higher dimensions. It will be a topic of future research to address these issues, but for the moment we finish by numerically illustrating that extending the theory to higher dimensions is not simply an exercise in bookkeeping.

Note that when $\alpha_o = \alpha_e$, (5.118) is guaranteed to have at least one solution. This can be seen by choosing $\phi_o = \phi_e$ and applying Theorem 5.3.2 to the resulting equation. However, in [52, 82] the two dimensional system

$$\left\{\begin{array}{rcl} -\gamma\phi_o''(\xi) - c_o\phi_o'(\xi) & = & \alpha\big(\phi_e(\xi+1) + \phi_e(\xi-1) - 2\phi_o(\xi)\big) \\ & & -\frac{15}{4}\big(\phi_o^2(\xi) - 1\big)\big(\phi_o(\xi) - \rho\big), \\ -\gamma\phi_e''(\xi) - c_e\phi_e'(\xi) & = & \alpha\big(\phi_o(\xi+1) + \phi_o(\xi-1) - 2\phi_e(\xi)\big) \\ & & -\frac{15}{4}\big(\phi_e^2(\xi) - 1\big)\big(\phi_e(\xi) - \rho\big). \end{array}\right. \tag{5.119}$$

was analyzed with the boundary conditions $\phi_*(\pm\infty) = \pm 1$ and $\phi_*(0) = 0$ for $* = o, e$. A corresponding wavespeed plot can be found in Figure 5.6(b) and the interesting feature is the presence of solutions with $c_o \neq c_e$, indicating that for the two dimensional system (5.119) uniqueness of solutions is lost.

5.7. Proof of Theorem 5.2.10

The aim of this section is to provide some basic results on the class of scalar differential difference equations encountered when studying (5.1) and to use these results to prove Theorem 5.2.10. We will mainly be concerned with the subclass of linear equations (5.3) that arises when linearizing (5.1) around solutions. However, we shall also provide a number of comparison principles for solutions to the nonlinear equation (5.1) which can directly be applied to the linear equations mentioned above.

The first result gives conditions under which (5.4) admits no positive solutions which decay superexponentially. This is especially useful in combination with Proposition 5.2.6, as in the absence of superexponentially decaying solutions this Proposition allows us to obtain asymptotic descriptions of the solutions to (5.4).

Lemma 5.7.1. *Consider the homogeneous equation (5.4) and let* $x : J^{\#} \to \mathbb{R}$ *be a solution to this equation on* $J = [\tau, \infty)$ *for some* $\tau \in \mathbb{R}$*. Suppose that Assumption 5.2.9 holds, possibly with* $\alpha_j = 0$ *for one or more* $1 \leq j \leq N$*. Assume further that* $x(\xi) \geq 0$ *for all* $\xi \in J^{\#}$*, but that there does not exist an* $R > 0$ *such that* $x(\xi) = 0$ *for all* $\xi \geq R$*. Then* x *does not decay superexponentially. The analogous result for* $J = (-\infty, \tau]$ *also holds.*

Proof. Without loss we shall also assume $J = [\tau, \infty)$, as the case of $J = (-\infty, \tau]$ can be treated by a change of variables $\xi \to -\xi$, which does not change the sign of γ. For convenience, we introduce the quantity $\widetilde{\alpha}_0 = \min(\alpha_0, \frac{c^2}{4\gamma} - \epsilon)$, where $\epsilon > 0$ is an arbitrary number.

We start out by noting that we can rescale equation (5.8) by defining $y(\xi) = e^{\lambda\xi}x(\xi)$, where λ can be chosen appropriately. It is easy to see that $y(\xi)$ satisfies the following differential difference equation

$$y''(\xi) = (2\lambda - \frac{c}{\gamma})y'(\xi) - \lambda(\lambda - \frac{c}{\gamma})y(\xi) - \frac{1}{\gamma}\sum_{j=0}^{N} A_j(\xi)e^{-\lambda r_j}y(\xi + r_j). \qquad (5.120)$$

Since $y(\xi) \geq 0$ for all $\xi \in J^{\#}$, we have the inequality

$$y''(\xi) \leq (2\lambda - \frac{c}{\gamma})y'(\xi) + \Big(-\lambda(\lambda - \frac{c}{\gamma}) - \frac{\widetilde{\alpha}_0}{\gamma}\Big)y(\xi). \qquad (5.121)$$

Now choosing $\lambda = \frac{c}{2\gamma}$ we obtain

$$y''(\xi) \leq By(\xi), \quad \xi \in J, \qquad (5.122)$$

in which $B = \frac{c^2}{4\gamma^2} - \frac{\widetilde{\alpha}_0}{\gamma} > 0$. Using a standard argument for ordinary differential equations which can be found in [82, Lemma A.1], one sees that for arbitrary $\xi_0 \in J$,

$$y(\xi) \leq C_1 e^{\sqrt{B}(\xi - \xi_0)} + C_2 e^{-\sqrt{B}(\xi - \xi_0)} \qquad (5.123)$$

holds for all $\xi \geq \xi_0$. The coefficients C_1 and C_2 in this expression are given by

$$\begin{aligned} C_1 &= \tfrac{1}{2\sqrt{B}}\Big(y'(\xi_0) + \sqrt{B}y(\xi_0)\Big), \\ C_2 &= \tfrac{1}{2\sqrt{B}}\Big(-y'(\xi_0) + \sqrt{B}y(\xi_0)\Big). \end{aligned} \qquad (5.124)$$

From the nonnegativity of $y(\xi)$ we see that we must have $C_1 \geq 0$, as otherwise (5.123) would imply that $y(\xi) < 0$ for sufficiently large ξ. From this we conclude

$$y'(\xi_0) \geq -\sqrt{B}y(\xi_0), \quad \xi_0 \in J, \tag{5.125}$$

which immediately implies that $y(\xi)$ and hence $x(\xi)$ cannot have superexponential decay. □

The following lemma will be crucial to establish comparison principles for solutions to the nonlinear equation (5.1). It can be easily derived by employing the scaling argument introduced in the proof of Lemma 5.7.1.

Lemma 5.7.2. *Let $x : J^{\#} \to \mathbb{R}$ be a solution to (5.3) on $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$ and suppose that Assumption 5.2.9 holds, possibly with $\alpha_j = 0$ for one or more $1 \leq j \leq N$. Assume further that $x(\xi) \geq 0$ for all $\xi \in J^{\#}$. Then if $x(\xi_0) = 0$ for some $\xi_0 \in J$, we have $x(\xi) = 0$ for all $\xi \geq \xi_0$.*

We now restrict ourselves to constant coefficient equations

$$-\gamma x''(\xi) - cx'(\xi) - L_0(x_\xi) = 0 \tag{5.126}$$

that satisfy Assumption 5.2.9. Our goal will be to obtain detailed information about the eigenvalues of such systems. This will allow us to give precise asymptotic descriptions of solutions to the nonautomatous linear equation (5.8) and to provide conditions for which (5.126) is hyperbolic. To this end, we introduce the quantity

$$A_\Sigma = -\Delta_{c,\gamma,L_0}(0) = \sum_{j=0}^{N} A_{j,0}, \tag{5.127}$$

associated to the constant coefficient operator L_0. The following lemma relates the existence of complex eigenvalues of (5.126) to the sign of the characteristic function $\Delta_{c,\gamma,L_0}(s)$ for real values of s.

Lemma 5.7.3. *Consider the constant coefficient equation (5.126), suppose that Assumption 5.2.9 holds and in addition assume that $A_\Sigma < 0$. Consider an arbitrary $a \in \mathbb{R}$. If $\Delta_{c,\gamma,L_0}(a) \geq 0$, then there do not exist any eigenvalues $\lambda \in \mathbb{C}$ such that* $\operatorname{Re}\lambda = a$ *except possibly $\lambda = a$ itself.*

Proof. Note that $A_{0,0} < 0$, since $A_\Sigma < 0$ and $A_{j,0} > 0$ for $1 \leq j \leq N$. Writing $\lambda = a + i\eta$ with $a, \eta \in \mathbb{R}$, we compute

$$\begin{aligned}
\left|c\lambda + \gamma\lambda^2 + A_{0,0}\right|^2 &= \left|ca + \gamma a^2 - \gamma\eta^2 + A_{0,0} + i(2a\gamma\eta + c\eta)\right|^2 \\
&= \left|ca + \gamma a^2 + A_{0,0}\right|^2 \\
&\quad +\eta^2(\eta^2\gamma^2 + 2a\gamma c + 2a^2\gamma^2 + c^2 - 2\gamma A_{0,0}) \\
&= \left|ca + \gamma a^2 + A_{0,0}\right|^2 + \eta^2 p(a),
\end{aligned}$$

where p is a second degree polynomial. It is elementary to see that

$$p(a) \geq \frac{1}{2}c^2 + \eta^2\gamma^2 - 2\gamma A_0 \geq \eta^2\gamma^2 \geq 0.$$

We thus have

$$\left|c\lambda + \gamma\lambda^2 + A_{0,0}\right| \geq \left|ca + \gamma a^2 + A_{0,0}\right|, \tag{5.128}$$

with equality if and only if $\lambda = a$.

Now suppose that $\lambda = a + i\eta$ satisfies $\Delta_{c,\gamma,L_0}(\lambda) = 0$ for some real η and that $\Delta_{c,\gamma,L_0}(a) \geq 0$. Then using (5.128), we arrive at

$$\begin{array}{rcl} \left|ca + \gamma a^2 + A_{0,0}\right| & \leq & \left|c\lambda + \gamma\lambda^2 + A_{0,0}\right| = \left|\sum_{j=1}^N A_{j,0}e^{\lambda r_j}\right| \\ & \leq & \sum_{j=1}^N A_{j,0}e^{ar_j} \\ & \leq & -(ca + \gamma a^2 + A_{0,0}). \end{array} \tag{5.129}$$

By examining the first and last terms in (5.129), we see that the three inequalities have to be equalities. This can only be if $\lambda = a$, from which the claim immediately follows.

□

Notice that under the assumptions of the previous lemma, we have $\Delta_{c,\gamma,L_0}(0) > 0$ and $\Delta''_{c,\gamma,L_0} < 0$. The behaviour of the characteristic function is thus easy to analyze on the real line and we can use the result above to establish the following two claims about the eigenvalues of (5.126).

Lemma 5.7.4. *Consider the equation (5.126) and suppose that $A_\Sigma < 0$ and Assumption 5.2.9 is satisfied. Then equation (5.126) is hyperbolic. Furthermore, there exists precisely one real positive eigenvalue $\lambda^+ \in (0,\infty)$ and precisely one real negative eigenvalue $\lambda^- \in (-\infty, 0)$ and each of these eigenvalues is simple. The eigenvalues λ^- and λ^+ depend C^1 smoothly on c and the coefficients $A_{j,0}$. In addition, we have that*

$$\frac{\partial\lambda^-}{\partial c} < 0 \ \ and \ \ \frac{\partial\lambda^+}{\partial c} < 0. \tag{5.130}$$

All the remaining eigenvalues satisfy

$$\operatorname{Re}\lambda \in (-\infty, \lambda^-) \cup (\lambda^+, \infty), \quad \operatorname{Im}\lambda \neq 0. \tag{5.131}$$

Lemma 5.7.5. *Consider the equation (5.126) and suppose that $A_\Sigma > 0$ and Assumption 5.2.9 is satisfied. Then either all real eigenvalues of (5.126) lie in $(0,\infty)$, or else they all lie in $(-\infty, 0)$.*

We now shift our focus to nonlinear differential difference equations of the form

$$-\gamma x''(\xi) - cx'(\xi) = G\big(\xi, x(\xi), x(\xi + r_1), \ldots, x(\xi + r_N)\big). \tag{5.132}$$

In the automatous case we write

$$-\gamma x''(\xi) - cx'(\xi) = F\big(x(\xi), x(\xi + r_1), \ldots, x(\xi + r_N)\big). \tag{5.133}$$

We will impose the following conditions on (5.132).

Assumption 5.7.6. *The parameter γ satisfies $\gamma > 0$ and the shifts satisfy $r_i \neq r_j$ when $i \neq j$ and $r_i \neq 0$. There is at least one shifted argument, i.e. $N \geq 1$. The function $G : \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}$, written as $G(\xi, u)$, where $u = (u_0, u_1, \ldots, u_N)$, is C^1 smooth and the derivative D_2G of G with respect to the second argument $u \in \mathbb{R}^{N+1}$ is locally Lipshitz in u. In addition, for every $\xi \in \mathbb{R}$ we have that*

$$\frac{\partial G(\xi, u)}{\partial u_j} > 0, \quad u \in \mathbb{R}^{N+1}, \quad 1 \leq j \leq N. \tag{5.134}$$

□

The following lemma roughly states that solutions to (5.133) are uniquely specified by their initial conditions. The proof is almost completely analogous to that in [113], so we omit it.

Lemma 5.7.7. *Let $x_j : J \to \mathbb{R}$ for $j = 1, 2$ be two solutions of equation (5.132) with the same parameters c and γ on some interval J. Suppose that Assumption 5.7.6 holds and that*

$$x_1(\xi) = x_2(\xi), \quad \tau + r_{\min} \leq \xi \leq \tau + r_{\max}, \tag{5.135}$$

for some $\tau \in J$ for which $[\tau + r_{\min}, \tau + r_{\max}] \subseteq J$. Then

$$x_1(\xi) = x_2(\xi), \quad \xi \in J^{\#}. \tag{5.136}$$

We remark here that in combination with Lemma 5.7.2 the above result yields the following useful corollary.

Corollary 5.7.8. *Consider the linear differential difference equation (5.3) and suppose that Assumption 5.2.9 holds. Let $x_j : J \to \mathbb{R}$ for $j = 1, 2$ be two solutions to (5.3) with the same parameters c and γ on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. If for all $\xi \in J^{\#}$ we have*

$$x_1(\xi) \geq x_2(\xi), \tag{5.137}$$

with equality $x_1(\xi_0) = x_2(\xi_0)$ for some $\xi_0 \in J$, then we have

$$x_1(\xi) = x_2(\xi), \quad \xi \in J^{\#}. \tag{5.138}$$

Suppose that x_1 and x_2 are both bounded solutions of the nonlinear autonomous differential difference equation (5.133) with the same parameters c and γ, where $\gamma > 0$. We have seen in Section 5.3 that the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear homogeneous equation (5.4) with coefficients given by

$$A_j(\xi) = \int_0^1 \left. \frac{\partial F(u)}{\partial u_j} \right|_{u = t\pi(x_1, \xi) + (1-t)\pi(x_2, \xi)} dt. \tag{5.139}$$

If Assumption 5.7.6 holds for the equation (5.133), it is easy to see that $A_j(\xi) > 0$ for all $\xi \in \mathbb{R}$ and for all $1 \leq j \leq N$. Since the derivatives $\frac{\partial F(u)}{\partial u_j}$ are continuous, we can use the fact that $x_1(\xi)$ and $x_2(\xi)$ are uniformly bounded to establish that the coefficients $A_j(\xi)$ are uniformly bounded for $0 \le j \le N$. The continuity of these coefficients follows from the Lipschitz condition on the partial derivatives of F. This means that our linear equation (5.4) with coefficients (5.139) satisfies all the assumptions of Lemma 5.7.2. Applying this result to the difference $x_1(\xi) - x_2(\xi)$ and invoking Lemma 5.7.7, we obtain the following useful comparison principle.

Lemma 5.7.9. *Let $x_j : J^\# \to \mathbb{R}$ for $j = 1, 2$ be two bounded solutions of the nonlinear differential difference equation (5.133) with the same parameters c and γ on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Suppose also that Assumption 5.7.6 holds and that*

$$x_1(\xi) \ge x_2(\xi), \quad \xi \in J^\#. \tag{5.140}$$

Then if $x_1(\xi_0) = x_2(\xi_0)$ for some ξ_0, we have $x_1(\xi) = x_2(\xi)$ for all $\xi \in J^\#$.

In order to establish uniqueness of solutions to (5.1), we shall need a comparison principle for solutions to (5.133) which have different wavespeeds.

Lemma 5.7.10. *Let $x_j : J^\# \to \mathbb{R}$ for $j = 1, 2$ be two bounded solutions of the nonlinear autonomous differential difference equation (5.133) with parameters $\gamma = \gamma_j$ and $c = c_j$ on some interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Suppose that Assumption 5.7.6 holds and that $\gamma_1 = \gamma_2 > 0$, but that $c_1 > c_2$. Also assume that*

$$x_1(\xi) \ge x_2(\xi), \quad \xi \in J^\# \tag{5.141}$$

and that $x_2(\xi)$ is monotonically increasing. Then if $x_1(\xi_0) = x_2(\xi_0)$ for some ξ_0, we have that $x_1(\xi) = x_2(\xi)$ is constant for all $\xi \ge \xi_0$.

Proof. We start out by noticing that the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear equation

$$y''(\xi) = -\frac{c_1}{\gamma}x_1'(\xi) + \frac{c_2}{\gamma}x_2'(\xi) - \frac{1}{\gamma}\sum_{j=0}^{N} A_j(\xi)y(\xi + r_j), \tag{5.142}$$

where the coefficients A_j are again given by (5.139).

We have already seen that the coefficients $A_j(\xi)$ are uniformly bounded for $0 \le j \le N$ and that $A_j(\xi) > 0$ for all $\xi \in \mathbb{R}$ and for $1 \le j \le N$. We can thus write $A_0(\xi) \ge \alpha_0$, for some $\alpha_0 < 0$. Now using the fact that $x_2'(\xi) \ge 0$, we have

$$\frac{c_2}{\gamma}x_2'(\xi) \le \frac{c_1}{\gamma}x_2'(\xi), \tag{5.143}$$

which allows us to conclude

$$y''(\xi) \le -\frac{c_1}{\gamma}y'(\xi) - \frac{\alpha_0}{\gamma}y(\xi). \tag{5.144}$$

Upon defining $z(\xi) = e^{\frac{c_1}{2\gamma}\xi}y(\xi)$, we obtain

$$z''(\xi) \le (\frac{c_1^2}{4\gamma^2} - \frac{\alpha_0}{\gamma})z(\xi) = Bz(\xi), \tag{5.145}$$

where $B > 0$. We now proceed as in the proof of Lemma 5.7.1 to conclude that $z(\xi) = 0$ for all $\xi \geq \xi_0$, which implies $x_1(\xi) = x_2(\xi)$ for all $\xi \geq \xi_0$. Referring back to (5.133), we see that for $\xi \geq \xi_0 + r_{\min}$ we must have $c_1 x_1'(\xi) = c_2 x_2'(\xi)$. However, as also $x_1'(\xi) = x_2'(\xi)$, we must have $x_1'(\xi) = x_2'(\xi) = 0$. This establishes the claim. □

We are now ready to provide the proof of Theorem 5.2.10 and we note here that the preparations in this section allow us to follow closely the proof of [113, Theorem 4.1].

Proof of Theorem 5.2.10. Denote the limiting constant coefficient operators at $\pm\infty$ by $L_\pm$. Then it follows from Lemma 5.7.4 that the equations (5.126) with $L_\pm$ are both hyperbolic. In fact, the same result holds for the family of constant coefficient operators $\frac{1}{2}\big((1-\rho)L_- + (1+\rho)L^+\big)$ for $-1 \leq \rho \leq 1$, which connects L_- to L_+. Theorem 5.2.4 thus guarantees that $\Lambda_{c,\gamma,L}$ is a Fredholm operator with $\mathrm{ind}(\Lambda_{c,\gamma,L}) = 0$. Corollary 5.7.8 immediately implies that the nontrivial solution p satisfies $p > 0$. Using Proposition 5.2.6 and Lemma 5.7.4, we obtain the asymptotic expressions

$$p(\xi) = \begin{cases} C_-^p e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ C_+^p e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{5.146}$$

for some $\epsilon > 0$, with finite exponents

$$-\infty < \lambda_+^s < 0 < \lambda_-^u < \infty. \tag{5.147}$$

Since p does not decay superexponentially and is strictly positive, Proposition 5.2.6 and Lemma 5.2.7 imply that both $C_\pm^p > 0$. Suppose that there exists some $x \in \mathcal{K}(\Lambda_{c,\gamma,L})$ which is linearly independent of p. By adding some multiple of p and replacing x by $-x$ if necessary, we may assume that x satisfies a similar asymptotic expansion (5.146) with $C_-^x \leq 0$ and $C_+^x = 0$. Because x is not identically zero, Lemma 5.7.7 implies that there exist arbitrarily large ξ for which $x(\xi) \neq 0$. If $x(\xi) \leq 0$ for all large ξ, then the same reasoning as applied above to conclude that $C_+^p > 0$ in the expansion (5.146) leads to a contradiction with $C_+^x = 0$. This means there even are arbitrarily large ξ for which $x(\xi) > 0$. From this it immediately follows that there exists $\mu_0 > 0$ such that

$$p(\xi) - \mu_0 x(\xi) < 0, \tag{5.148}$$

for some $\xi \in \mathbb{R}$. We now consider the family $p - \mu x \in \mathcal{K}_{c,\gamma,L}$ for $0 \leq \mu \leq \mu_0$. The asymptotic expressions for p and x ensure that there exist $\tau,\ K,\ \lambda \in \mathbb{R}$ such that

$$p(\xi) - \mu x(\xi) \geq K e^{-\lambda|\xi|} > 0, \quad |\xi| > \tau, \quad 0 \leq \mu \leq \mu_0. \tag{5.149}$$

Now define

$$\mu_* = \sup\{\mu \in [0, \mu_0] \mid p(\xi) - \mu x(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}\}. \tag{5.150}$$

By definition it follows from (5.148) that $\mu_* < \mu_0$. Obviously, we have the inequality $\mu_* x(\xi) \leq p(\xi)$ for all $\xi \in \mathbb{R}$, but actually it is easy to see that also $\mu_* x(\xi_0) = p(\xi_0)$ for some $\xi_0 \in [-\tau, \tau]$. From Corollary 5.7.8 it now immediately follows that $\mu_* x(\xi) = p(\xi)$, but this contradicts the linear independence of x and p, establishing $\dim \mathcal{K}(\Lambda_{c,\gamma,L}) = 1$. To complete the proof, it is enough to show that there exists a $p^* \in \mathcal{K}(\Lambda^*_{c,\gamma,L})$ which satisfies $p^* \geq 0$, as the strict positivity then follows immediately from Corollary 5.7.8. Thus assume to the contrary that $p^*(\xi_1) > 0 > p^*(\xi_2)$ for some $\xi_1, \xi_2 \in \mathbb{R}$. Lemma 5.7.7 guarantees that we may assume that $|\xi_1 - \xi_2| \leq r_{\max} - r_{\min}$. This means that there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} p^*(\xi)h(\xi)d\xi = 0$, with $\mathrm{supp}(h) \subset [\tau_1, \tau_2]$ for some $\tau_1, \tau_2 \in \mathbb{R}$ satisfying $\tau_2 - \tau_1 < r_{\max} - r_{\min}$. Theorem 5.2.4 now implies that there exists an $x \in W^{2,\infty}$ such that $\Lambda_{c,\gamma,L}x = h$. We now consider the family of such solutions $x + \mu p$ for $\mu \in \mathbb{R}$. Noting that x satisfies the homogeneous equation (5.4) for large $|\xi|$ and using similar arguments as above, one argues that there exists a $\mu^* \in \mathbb{R}$ such that $y = x + \mu^* p$ satisfies $y \geq 0$ and $y(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Since $(-\infty, \tau_1 + r_{\max}] \cup [\tau_2 + r_{\min}, \infty) = \mathbb{R}$, we may use Lemma 5.7.7 to conclude that $y(\xi)$ does not vanish for all large $|\xi|$. By possibly making the substitution $\xi \to -\xi$, we may assume $y(\xi)$ does not vanish for all large ξ. However, Lemma 5.7.2 now implies $y(\xi) = 0$ for all $\xi \geq \xi_0$, which gives the desired contradiction. □

5.8. Implementation Issues

Performing the iteration step defined in (5.88) with $\mu = 0$ amounts to solving a boundary value problem on the real line. This observation in principle allows one to perform the Newton iterations requiring the help of a boundary value problem solver for ordinary differential equations only, if one truncates the problem to some appropriate finite interval $[T_-, T_+]$ with $T_- < 0 < T_+$. In our C++ implementation, the boundary value problem solver COLMOD [31] was used at each iteration step. Since the boundary value problem which has to be solved has degree three, three boundary conditions need to be specified at each step. These conditions were chosen to be $\phi(T_\pm) = \pm 1$ and $\phi(0) = 0$, in order to pick out the unique translate. In addition, when evaluating the delay and advanced terms in (5.88), the iterates were taken to satisfy $\phi(\xi) = -1$ whenever $\xi \leq T_-$ and $\phi(\xi) = 1$ for $\xi \geq T_+$. In the literature, other sets of boundary conditions have been proposed, which require that at the boundary points $\xi = T_\pm$ solutions are matched up with the exponential eigenfunctions of the corresponding linearization. However, since solutions have exponential behaviour at $\pm\infty$, the distinction between these two sets of boundary conditions vanishes numerically if the interval is chosen to be large enough.

In order to perform the iteration (5.88) with $\mu \neq 0$, the COLMOD code was adapted, roughly along the lines of [1, 10], to collocate the delay and advance terms directly. This required the usage of extra memory space to accomodate the larger matrices involved and the use of a different linear system solver to allow for non block-diagonal matrices.

It remains to specify how a suitable starting value (ϕ_0, c_0) can be supplied for the Newton iterations. It turns out that this is very hard in general: very often the algorithm requires a very accurate initial guess to converge. One has to use the technique of continuation to arrive at a suitable starting value. In general, this means that one starts by solving an "easy" problem to a certain degree of accuracy and gradually moves toward the "hard" problem, using the solution of one problem as the starting value for the next problem which lies

”nearby”. As an example, we mention that a continuation scheme for the family (5.112) can involve varying combinations of the detuning parameter ρ, the size of the delay term α and the size of γ . The continuity in parameter space which was established in Proposition 5.3.3 shows that this is indeed a feasible strategy.

Chapter 6

Lin's Method and Homoclinic Bifurcations

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Abstract. We extend Lin's method for use in the setting of parameter-dependent nonlinear functional differential equations of mixed type (MFDEs). We show that the presence of M-homoclinic and M-periodic solutions that bifurcate from a prescribed homoclinic connection, can be detected by studying a finite dimensional bifurcation equation. As an application, we describe the codimension two orbit-flip bifurcation in the setting of MFDEs.

6.1. Introduction

The main purpose of this chapter is to provide a framework that facilitates the detection of solutions to a parameter-dependent nonlinear functional differential equation of mixed type

$$x'(\xi) = G(x_\xi, \mu), \tag{6.1}$$

that bifurcate from a prescribed homoclinic or heteroclinic connection. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \leq 0$ and $r_{\max} \geq 0$, hence the nonlinearity G may depend on advanced and retarded arguments simultaneously. The parameter μ is taken from an open subset of $\mathbb{R}^p$, for some integer $p \geq 1$.

The fact that travelling wave solutions to lattice differential equations are described by functional differential equations of mixed type (MFDEs), forms one of the primary motivations for this chapter. As exhibited in detail in Chapter 1, lattice differential equations have many modelling applications in a wide range of scientific disciplines. As a consequence they are attracting a considerable amount of interest, both from an applied as well as a theoretical perspective. One of the driving forces in these investigations is the desire to apply the powerful tools that are currently available for ODEs to the infinite dimensional setting of (6.1). The constructions in previous chapters concerning finite dimensional center manifolds, which describe the behaviour of solutions to (6.1) in the vicinity of equilibria and periodic solutions, should be seen in this light.

In the present work we continue this approach, by studying solutions to (6.1) that remain orbitally close to a prescribed homoclinic or heteroclinic solution q that solves (6.1) at $\mu = \mu_0$. We will be particularly interested in the construction of M-homoclinic and M-periodic orbits, which loosely speaking wind around the principal orbit q exactly M times, before converging to an equilibrium or repeating their pattern. More precisely, we will fix a Poincaré section that intersects the trajectory of q at q_0 in a transverse fashion and study solutions that pass through this section M times. We will show that for μ sufficiently close to μ_0, one may construct solutions that satisfy these winding properties, up to M possible discontinuities that occur exactly at the Poincaré section. Moreover, our construction will force these jumps to be contained in some finite dimensional subset of this section. This crucial reduction allows us to search for M-homoclinic and M-periodic orbits by studying the roots of M finite dimensional bifurcation equations, that effectively measure the size of the jumps.

This construction is known as Lin's method and was originally developed by Lin [106] in order to study systems that depend upon a single parameter. Sandstede generalized the method in such a way that bifurcations with higher codimensions could also be incorporated [134]. Our approach here should be seen as a subsequent generalization of this latter framework to the infinite dimensional context of (6.1). In addition, we will show that the bifurcation equations that describe the size of the jumps have a similar asymptotic form as those derived for the ODE version of (6.1). This provides a bridge that will allow classical bifurcation results obtained for ODEs to be directly lifted to the mixed type functional differential equation (6.1).

We mention here that very recently Lin's method was used to study homoclinic solutions to a reversible lattice differential equation, in the neighbourhood of a prescribed symmetric homoclinic connection [69]. The approach in [69] however cannot be used to detect bifurcating periodic solutions. In addition, the choice to use $\mathbb{C}^n \times L_2([r_{\min}, r_{\max}], \mathbb{C}^n)$ as a state space for (6.1), causes the nonlinearity to have a domain and therefore requires the use of a proper functional-analytic setup. In contrast to our approach, this prevents the smoothness of the nonlinearity to be carried over to the bifurcation equations.

Historically, the primary motivation for the work by Lin and Sandstede mentioned above, was the classification of the bifurcations that homoclinic solutions to generic ODEs with one or two parameters may undergo. In a sequence of papers, Shilnikov [140, 141, 142, 143, 144, 145] presented an alternative for generic ODE versions of (6.1) with $p = 1$. In particular, the ODE either admits precisely one branch of large-period periodic solution that bifurcates from the homoclinic orbit q for $\mu > \mu_0$ or $\mu < \mu_0$, or else admits symbolic dynamics for all μ sufficiently close to μ_0. The existence of the unique periodic orbit was generalized to semilinear parabolic PDEs and delay equations by Chow and Deng [33] using semigroup techniques. Sandstede lifted the result concerning the presence of symbolic dynamics to parabolic PDEs that have a sectorial linear part [134].

According to Yanagida [168], the generic bifurcations of codimension two that a hyperbolic homoclinic solution to an ODE may undergo, are the inclination-flip and the orbit-flip bifurcations. The former of these has been analyzed by several authors [80, 95] using Lyapunov-Schmidt techniques, that unfortunately break down when studying the orbit-flip bifurcation. However, employing the adaptation of Lin's method discussed above, Sandstede obtained a general description of this bifurcation for ODEs in [134]. In Section 6.2 we will use our bridge to lift this result and characterize the orbit-flip bifurcation for (6.1).

The first obstacle that needs to be overcome in any bifurcation analysis involving MFDEs, is that the linearized problems one encounters are ill-posed and therefore do not generate a semiflow. It is known that exponential dichotomies form a very powerful tool when dealing with ill-posed problems, since they split the state space into separate parts that do admit a semiflow. The existence of such exponential splittings for parameter-independent homogeneous linear MFDEs, was established independently and simultaneously by Verduyn Lunel and Mallet-Paret [115] on the one hand and Härterich and coworkers [75] on the other, using very different methods. A second obstacle is that there is no immediate way to write down a variation-of-constants formula that solves inhomogeneous MFDEs. This is caused by the fact that the inhomogeneity will simply be a $\mathbb{C}^n$-valued function, while the projections associated to the exponential dichotomies act on the state space $C([r_{\min}, r_{\max}], \mathbb{C}^n)$. In view of this fact, a third obstacle arises when one wishes to study systems that depend on a parameter, since robustness theorems for exponential dichotomies are generally established using a variation-of-constants formula.

In previous chapters, the absence of a variation-of-constants formula was circumvented by utilizing variants of the Greens function that was constructed by Mallet-Paret for autonomous MFDEs [112]. Continuing in this spirit, we will use the Fredholm theory developed in [112] for nonautonomous MFDEs, to construct inverses for inhomogeneous MFDEs on half-lines. By carefully combining these inverses with the exponential splittings developed in [115], we are able to construct exponential dichotomies for parameter-dependent MFDEs without using a variation-of-constants formula. In addition, this setup will allow us to obtain precise estimates on the speed at which the projections associated to these dichotomies approach the limiting spectral projections at $\pm\infty$. We will also be able to isolate the portion of the state space that corresponds to a specific eigenvalue of one of these spectral projections. These results can be found in Sections 6.3 to 6.5 and provide the machinery that we require to construct the bridge between ODEs and MFDEs.

In Section 6.2 we state our main results, which describe Lin's method in the setting of MFDEs and give an explicit expression for the leading order terms in the bifurcation equations. In addition, we characterize the orbit-flip bifurcation for MFDEs. In Section 6.6 we construct the candidate M-homoclinic and M-periodic orbits, that satisfy (6.1) up to M jumps. Our approach in that section broadly follows the presentation in [134], but we avoid the smooth coordinates changes that are used there, since these are often problematic in an infinite dimensional setting. Instead, these coordinate changes are only applied after the problem has been reduced to a finite dimensional one. Finally, in Sections 6.7 and 6.8 we obtain estimates on the size of the error that is made, if one only considers the leading order terms when measuring the size of the M jumps.

6.2. Main Results

Consider for some integer $N \geq 0$ the general nonlinear functional differential equation of mixed type

$$x'(\xi) = G(x(\xi + r_0), \ldots, x(\xi + r_N), \mu) = G(x_\xi, \mu), \tag{6.2}$$

in which x should be seen as a mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some $n \geq 1$. The shifts $r_j \in \mathbb{R}$ may have either sign and we will assume that they are ordered as $r_0 < \ldots < r_N$, with $r_0 \leq 0$ and $r_N \geq 0$. Introducing $r_{\min} = r_0$ and $r_{\max} = r_N$, we write $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ for the state space associated to (6.2). The state of a function x at $\xi \in \mathbb{R}$ will be denoted by $x_\xi \in X$ or alternatively $\mathrm{ev}_\xi x \in X$ and is defined by $x_\xi(\theta) = x(\xi + \theta)$ for $r_{\min} \leq \theta \leq r_{\max}$. The parameter μ is taken from an open subset $U \subset \mathbb{R}^p$ for some integer $p \geq 1$. For convenience, we will use both of the representations for G that were introduced in (6.2) interchangeably throughout the sequel, but the details should be clear from the context.

We will need the following assumptions on the nonlinearity G. We remark that the parameter-independence of the equilibria is not a real restriction, as this can always be achieved by means of a change of variables.

(HG) The nonlinearity $G : X \times U \to \mathbb{C}^n$ is C^{k+2} smooth for some integer $k \geq 2$. In addition, it admits D distinct equilibria $q_* \in \mathbb{C}^n$, which we label as $q_*^{(1)}$ through $q_*^{(D)}$. These equilibria do not depend on the parameter μ, i.e., we have $G(q_*^{(i)}, \mu) = 0$ for all $\mu \in U$ and all integers $1 \leq i \leq D$.

It is important to understand the linearizations of (6.2) around these equilibrium solutions. To this end, we define $L^{(i)}(\mu) = D_1 G(q_*^{(i)}, \mu)$ and consider the homogeneous linear MFDE

$$x'(\xi) = L^{(i)}(\mu) x_\xi = \sum_{j=0}^{N} A_j^{(i)}(\mu) x(\xi + r_j). \tag{6.3}$$

Associated to this linear MFDE one has the characteristic matrix

$$\Delta^{(i)}(z, \mu) = zI - L^{(i)}(\mu) e^{z\cdot} = zI - \sum_{j=0}^{N} A_j^{(i)} e^{z r_j}. \tag{6.4}$$

We will need the following assumption on the linearizations, which basically states that all equilibria are hyperbolic.

(HL) For all integers $1 \leq i \leq D$, the linearization $L^{(i)}(\mu)$ does not depend on μ. In addition, the characteristic equation $\det \Delta^{(i)}(z) = 0$ admits no roots with $\operatorname{Re} z = 0$.

Now let us assume that for $\mu = \mu_0$, equation (6.2) has a heteroclinic solution q that connects the equilibria q_*^- and q_*^+. Inserting $x(\xi) = q(\xi) + v(\xi)$ into (6.2), we find the variational MFDE

$$v'(\xi) = D_1 G(q_\xi, \mu_0) v_\xi + R(\xi, v_\xi, \mu), \tag{6.5}$$

which is no longer autonomous. Associated to the linear part of this equation we define the operator $\Lambda : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ that is given by

$$[\Lambda v](\xi) = v'(\xi) - D_1 G(q_\xi, \mu_0) v_\xi = v'(\xi) - \sum_{j=0}^{N} A_j(\xi) v(\xi + r_j), \tag{6.6}$$

with $A_j(\xi) = D_j G\big(q(\xi + r_0), \ldots, q(\xi + r_N), \mu_0\big)$. It is possible to define an operator $\Lambda^* : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$ that can be interpreted as an adjoint for Λ under suitable conditions. This adjoint is given by

$$[\Lambda^* w](\xi) = w'(\xi) + \sum_{j=0}^{N} A_j(\xi - r_j)^* w(\xi - r_j). \tag{6.7}$$

We will write $Y = C([-r_{\max}, -r_{\min}], \mathbb{C}^n)$ for the state space associated to the adjoint (6.7) and ev^*_ξ for the associated evaluation operator, which now maps into Y. The coupling between Λ and Λ^* is provided through the Hale inner product, which is given by

$$\langle \psi, \phi \rangle_\xi = \psi(0)^* \phi(0) - \sum_{j=0}^{N} \int_0^{r_j} \psi(\theta - r_j)^* A_j(\xi + \theta - r_j) \phi(\theta) d\theta, \tag{6.8}$$

for any $\phi \in X$ and $\psi \in Y$. The following condition on the operator Λ ensures that the Hale inner product is nondegenerate, in the sense that if $\langle \psi, \phi \rangle_\xi = 0$ for all $\psi \in Y$ and some $\phi \in X$, then $\phi = 0$. A proof for this fact can be found in [115].

(HB) The matrices $A_0(\xi)$ and $A_N(\xi)$ are nonsingular for every $\xi \in \mathbb{R}$.

Let $\mathcal{I} \subset \mathbb{R}$ be an interval. To state our results, we use the following family of Banach spaces, parametrized by $\eta \in \mathbb{R}$,

$$BC_\eta(\mathcal{I}, \mathbb{C}^n) = \left\{ x \in C(\mathcal{I}, \mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi \in \mathcal{I}} e^{-\eta \xi} |x(\xi)| < \infty \right\}. \tag{6.9}$$

We also need to consider the finite dimensional kernels

$$\begin{array}{rcl} \mathcal{K} & = & \left\{ b \in BC_0(\mathbb{R}, \mathbb{C}^n) \mid \Lambda b = 0 \right\}, \\ \mathcal{K}^* & = & \left\{ d \in BC_0(\mathbb{R}, \mathbb{C}^n) \mid \Lambda^* d = 0 \right\}. \end{array} \tag{6.10}$$

Let us write $X_0 = \{\phi \in X \mid \phi = b_0 \text{ for some } b \in \mathcal{K}\}$ and choose $\widehat{X}$ in such a way that $X = \widehat{X} \oplus X_0$. In addition, we write $Y_0 = \{\psi \in Y \mid \psi = d_0 \text{ for some } d \in \mathcal{K}^*\}$ and define the space

$$\widehat{X}_\perp = \left\{ \phi \in \widehat{X} \mid \langle \psi, \phi \rangle_0 = 0 \text{ for all } \psi \in Y_0 \right\}. \tag{6.11}$$

We note that $\widehat{X}_\perp \subset \widehat{X}$ is closed and of finite codimension, which allows us to fix a finite dimensional complement Γ and write $X = X_0 \oplus \widehat{X}_\perp \oplus \Gamma$.

Proposition 6.2.1. *Consider the nonlinear equation (6.2) and suppose that (HG), (HL) and (HB) are satisfied. There exists a small neighbourhood $U' \subset U$, with $\mu_0 \in U'$, a small constant $\epsilon > 0$ and two C^{k+1}-smooth maps $u^- : U' \to BC_{+\epsilon}\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ and $u^+ : U' \to BC_{-\epsilon}\big([r_{\min}, \infty), \mathbb{C}^n\big)$, such that the following properties are satisfied.*

(i) *For any $\mu \in U'$, the function $x(\xi) = q(\xi)+u^-(\mu)(\xi)$ satisfies the nonlinear equation (6.2) for all $\xi \leq 0$. In addition, the function $x(\xi) = q(\xi) + u^+(\mu)(\xi)$ satisfies (6.2) for all $\xi \geq 0$.*

(ii) *For all $\mu \in U'$, we have the identities*

$$\begin{array}{rcl} \mathrm{ev}_0 u^-(\mu) & \in & \widehat{X}_\perp \oplus \Gamma, \\ \mathrm{ev}_0 u^+(\mu) & \in & \widehat{X}_\perp \oplus \Gamma. \end{array} \tag{6.12}$$

(iii) *For all $\mu \in U'$, we have $\xi^\infty(\mu) := \mathrm{ev}_0 u^-(\mu) - \mathrm{ev}_0 u^+(\mu) \in \Gamma$.*

(iv) *For any $d \in \mathcal{K}^*$, we have the Melnikov identity*

$$D_\mu[\langle \mathrm{ev}_0^* d, \xi^\infty(\mu)\rangle_0]_{|\mu=\mu_0} = \int_{-\infty}^{\infty} d(\xi')^* D_2 G(q_{\xi'}, \mu_0) d\xi'. \tag{6.13}$$

These maps are locally unique, in the sense that there exists $\delta > 0$ such that any pair $(\widetilde{u}^+, \widetilde{u}^-)$ that satisfies (i) through (iii) for some $\mu \in U'$ and also has $\widetilde{u}^+ \in BC_{-\epsilon}\big([r_{\min}, \infty), \mathbb{C}^n\big)$, $\widetilde{u}^- \in BC_{+\epsilon}\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ and $\|\widetilde{u}^\pm\|_0 < \delta$, must satisfy $\widetilde{u}^+ = u^+(\mu)$ and $\widetilde{u}^- = u^-(\mu)$.

We remark that the condition (HB) ensures that the Hale inner product is nondegenerate, which means that the inner product appearing in (6.13) is a valid way of measuring the gap between the local stable and unstable manifolds of (6.2). If one is merely interested in studying heteroclinic orbits that bifurcate from a prescribed heteroclinic connection, then Proposition 6.2.1 already reduces this problem to a finite dimensional one. Indeed, item (iii) implies that one has to search for the roots of a C^{k+1}-smooth function defined on Γ.

For the purpose of this chapter however, let us consider a family of heteroclinic connections $\{q_j\}_{j\in\mathcal{J}}$, in which $\mathcal{J} \subset \mathbb{Z}$ is a possibly infinite set of subsequent integers. We emphasize here that these connections need not be distinct, thus any heteroclinic connection can appear in the family an arbitrary number of times. We write $\mathcal{J}^* \subset \mathbb{Z} + \frac{1}{2}$ for the set of half-integers $\mathcal{J}^* = \{j \pm \frac{1}{2}\}_{j\in\mathcal{J}}$, that will be related to the boundary conditions that tie the connections together. In particular, we will assume that the family $\{q_j\}_{j\in\mathcal{J}}$ connects the equilibria $\{q_\ell^*\}_{\ell\in\mathcal{J}^*}$, i.e.,

$$\lim_{\xi\to\pm\infty} q_j(\xi) = q^*_{j\pm\frac{1}{2}}. \tag{6.14}$$

Our aim is to construct solutions x to (6.2) that subsequently intersect the Poincaré sections $\mathrm{ev}_0 q_j + \widehat{X}_\perp + \Gamma$ close to $\mathrm{ev}_0 q_j$ at prescribed times T_j. To this end, we look for solutions to (6.2) that can be written as

$$\begin{array}{rclr} x(T_j+\xi) & = & q_j(\xi) + u_j^-(\mu)(\xi) + v_j^-(\mu)(\xi), & \omega_j^- + r_{\min} \leq \xi \leq r_{\max}, \\ x(T_j+\xi) & = & q_j(\xi) + u_j^+(\mu)(\xi) + v_j^+(\mu)(\xi), & r_{\min} \leq \xi \leq \omega_j^+ + r_{\max}, \end{array} \tag{6.15}$$

in which we will take $\omega_j^+ = -\omega_{j+1}^- = \omega_{j+\frac{1}{2}}$, for some family $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ that has $T_{j+1} - T_j = 2\omega_{j+\frac{1}{2}}$, wherever this is defined. If $\mathcal{J}$ is finite, i.e., $\mathcal{J} = \{1, \ldots, M\}$, then we can supply boundary conditions by requiring either $\lim_{\xi \to -\infty} x(\xi) = q^*_{\frac{1}{2}}$ and $\lim_{\xi \to \infty} x(\xi) = q^*_{M+\frac{1}{2}}$ if we are looking for a heteroclinic connection or $\mathrm{ev}_{\omega_1^-} x = \mathrm{ev}_{\omega_M^+} x$ if we are interested in periodic orbits.

The main result of this chapter shows that if the prescribed crossing times T_j are sufficiently far apart, the search for solutions x of the form (6.15) is equivalent to the search for roots of a smooth function defined on the collection of finite dimensional spaces $\{\Gamma^{(j)}\}_{j \in \mathcal{J}}$.

Theorem 6.2.2. *Consider the nonlinear equation (6.2) and suppose that (HG), (HL) and (HB) are satisfied. Furthermore, consider a family of heteroclinic connections $\{q_j\}_{j \in \mathcal{J}}$ that satisfies (6.14). There exists an $\Omega > 0$ and an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, such that for any family $\{\omega_\ell\}_{\ell \in \mathcal{J}^*}$ that has $\omega_\ell \geq \Omega$ for all $\ell \in \mathcal{J}^*$, there exist two families of functions $v_j^- : U' \to C\big([\omega_j^- + r_{\min}, r_{\max}], \mathbb{C}^n\big)$ and $v_j^+ : U' \to C\big([r_{\min}, \omega_j^+ + r_{\max}], \mathbb{C}^n\big)$, defined for $j \in \mathcal{J}$, that satisfy the following properties.*

(i) For any $\mu \in U'$ and $j \in \mathcal{J}$, the function $x(\xi) = q_j(\xi) + u_j^-(\mu)(\xi) + v_j^-(\mu)(\xi)$ satisfies the nonlinear equation (6.2) for all $\omega_j^- \leq \xi \leq 0$. In addition, the function $x(\xi) = q_j(\xi) + u_j^+(\mu)(\xi) + v_j^+(\xi)$ satisfies (6.2) for all $0 \leq \xi \leq \omega_j^+$.

(ii) For any $\mu \in U'$ and any $j \in \mathcal{J}$, we have $\mathrm{ev}_0 v_j^-(\mu) \in \widehat{X}_\perp^{(j)} \oplus \Gamma^{(j)}$ and similarly $\mathrm{ev}_0 v_j^+(\mu) \in \widehat{X}_\perp^{(j)} \oplus \Gamma^{(j)}$.

(iii) For any $\mu \in U'$ and $j \in \mathcal{J}$, the following boundary conditions are satisfied,

$$\mathrm{ev}_{\omega_{j+1}^-} v_{j+1}^-(\mu) - \mathrm{ev}_{\omega_j^+} v_j^+(\mu) = \mathrm{ev}_{\omega_j^+}[q_j + u_j^+(\mu)] - \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu)]. \tag{6.16}$$

If the family $\mathcal{J}$ is finite with M elements and $-\infty < \omega_1^- = -\omega_M^+$, then v_{M+1}^- should be read as v_1^-. If however $\omega_1^- = -\infty$ and $\omega_M^+ = \infty$, then (6.16) holds for all $1 \leq j < M$ and one has the additional limits

$$\lim_{\xi \to -\infty} v_1^-(\mu)(\xi) = 0, \qquad \lim_{\xi \to \infty} v_M^+(\mu)(\xi) = 0. \tag{6.17}$$

(iv) For any $\mu \in U'$ and any $j \in \mathcal{J}$, we have $\xi_j(\mu) \in \Gamma^{(j)}$, in which $\xi_j(\mu)$ denotes the gap $\mathrm{ev}_0[v_j^-(\mu) - v_j^+(\mu)]$.

The two families $\{v_j^\pm\}_{j \in \mathcal{J}}$ are locally unique in a sense similar to the one described in Proposition 6.2.1. In addition, these functions $v_j^\pm$ depend C^k-smoothly on μ, while the shifts ξ_j depend C^k-smoothly on the pair $(\mu, \{\omega_\ell\}_{\ell \in \mathcal{J}^*})$. Finally, for any $d \in \mathcal{K}^*$ and $j \in \mathcal{J}$, we can estimate $\xi_j(\mu)$ according to

$$\begin{aligned} \langle \mathrm{ev}_0^* d, \xi_j(\mu) \rangle_0 &= \langle \mathrm{ev}^*_{\omega_j^+} d, \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu) - q^*_{j+\frac{1}{2}}] \rangle_{\omega_j^+} \\ &\quad - \langle \mathrm{ev}^*_{\omega_j^-} d, \mathrm{ev}_{\omega_{j-1}^+}[q_{j-1} + u_{j-1}^+(\mu) - q^*_{j-\frac{1}{2}}] \rangle_{\omega_j^-} + \mathcal{R}_j. \end{aligned} \tag{6.18}$$

The error term $\mathcal{R}_j$ enjoys the following estimate, for some positive constants C_1 and C_2,

$$\mathcal{R}_j \leq \left\| \mathrm{ev}_0^* d \right\| [C_1 |\mu - \mu_0| e^{-2\alpha\omega} + C_2 e^{-3\alpha\omega}]. \tag{6.19}$$

Here we have introduced $\omega = \min_{\ell \in \mathcal{J}^*}\{\omega_\ell\}$, *while* $\alpha > 0$ *is sufficiently small to ensure that the characteristic equations* $\det \Delta^{(i)} z = 0$ *have no roots with* $|\mathrm{Re}\, z| \le \alpha$ *for all* $1 \le i \le D$.

We note here that sharper estimates for the remainder terms $\mathcal{R}_j$ can be found in Sections 6.7 and 6.8, where we also provide estimates on the derivatives of $\mathcal{R}_j$ with respect to μ and the family $\{\omega_\ell\}_{\ell \in \mathcal{J}^*}$. In combination with these estimates, Theorem 6.2.2 allows bifurcation problems for the infinite dimensional system (6.2) to be treated on a similar footing as bifurcation problems for ODEs.

The orbit-flip bifurcation

To illustrate the application range of Theorem 6.2.2, we lift a result obtained by Sandstede [134] that describes the homoclinic orbit-flip bifurcation for ODEs. We proceed by stating the assumptions on the system (6.2) that we will need.

(OF1) The nonlinearity G is C^{k+2}-smooth with $k \ge 2$. The parameter space U is two dimensional and contains the origin, i.e., $0 \in U \subset \mathbb{R}^2$. The nonlinear differential equation (6.2) has an equilibrium at $x = 0$ for all $\mu \in U$. The linearization $D_1 G(0, \mu)$ around this equilibrium does not depend on μ.

(OF2) There exist constants $\eta_-^f < 0$ and $\eta_+^f > 0$, such that the characteristic equation $\det \Delta(z) = 0$ associated to the equilibrium of (6.2) at $x = 0$ has precisely two eigenvalues $z = \lambda_\pm$ in the strip $\eta_-^f \le \mathrm{Re}\, z \le \eta_+^f$. These eigenvalues are simple roots of the characteristic equation and there exist constants $\eta_\pm^s$ such that the following inequalities are satisfied,

$$\eta_-^f < \lambda_- < \eta_-^s < 0 < \eta_+^s < \lambda_+ < \eta_+^f. \tag{6.20}$$

(OF3) There exists a homoclinic solution q to (6.2) at $\mu = 0$ that satisfies $\lim_{\xi \to \pm\infty} q(\xi) = 0$. The kernel $\mathcal{K} = \mathcal{K}(\Lambda) \subset BC_0(\mathbb{R}, \mathbb{C}^n)$ associated to the linearization (6.6) of the nonlinear equation (6.2) around this orbit q, is one dimensional and satisfies

$$\mathcal{K} = \mathrm{span}\{q'\}. \tag{6.21}$$

(OF4) The kernel $\mathcal{K}^* = \mathcal{K}(\Lambda^*)$ associated to the adjoint of the linearization (6.6) is one dimensional, i.e., for some $d \in BC_0(\mathbb{R}, \mathbb{C}^n)$ we have

$$\mathcal{K}^* = \mathrm{span}\{d\}. \tag{6.22}$$

The spectral splitting in (OF2) ensures that we can decompose the state spaces X and Y as

$$X = \mathcal{M}_c \oplus \mathcal{M}_{\lambda_-} \oplus \mathcal{M}_{\lambda_+}, \qquad Y = \mathcal{M}_c^* \oplus \mathcal{M}_{-\lambda_-}^* \oplus \mathcal{M}_{-\lambda_+}^*, \tag{6.23}$$

in which $\mathcal{M}_{\lambda_\pm}$ are the one dimensional eigenspaces associated to the eigenvalues $\lambda_\pm$ and $\mathcal{M}_c$ is a closed complement, while the starred spaces are defined similarly. The spectral projections $\Pi_{\mathcal{M}_{\lambda_\pm}}$ and $\Pi_{\mathcal{M}^*_{-\lambda_\pm}}$ onto these eigenspaces can be written in terms of the Hale inner product [72]. More precisely, there exist $\psi_\pm \in Y$ and $\phi_\pm \in X$ such that

$$\Pi_{\mathcal{M}_{\lambda_\pm}}\phi = \langle \psi_\pm, \phi \rangle_\infty \phi_\pm, \qquad \Pi_{\mathcal{M}^*_{-\lambda_\pm}}\psi = \langle \psi, \phi_\pm \rangle_\infty \psi_\pm. \tag{6.24}$$

Let us now consider the functions $u^\pm(\mu)$ introduced in Proposition 6.2.1, together with the jump $\xi^\infty(\mu)$. We also need to introduce the function $\Phi^\pm : U' \to \mathbb{R}$ given by

$$\Phi_\pm(\mu) \quad = \quad \lim_{\xi \to \pm\infty} e^{-\lambda_\mp \xi} \langle \psi_\mp, \mathrm{ev}_\xi\big(q + u^\pm(\mu)\big) \rangle_{\pm\infty}. \tag{6.25}$$

In a similar fashion we define the scalars

$$\Phi^*_\pm \quad = \quad \lim_{\xi \to \pm\infty} e^{\lambda_\pm \xi} \langle \mathrm{ev}^*_\xi d, \phi_\pm \rangle_{\pm\infty}. \tag{6.26}$$

Using arguments very similar to those given in [112, Section 7], one may show that both $\Phi^\pm$ depend C^k-smoothly on μ.

(OF5) We have the identities $\Phi^+(0) = 0$, $\Phi^-(0) \neq 0$ and $\Phi^*_\pm \neq 0$. In particular, q approaches its limit in forward time at an exponential rate faster than η^f_-, but behaves generically as $\xi \to -\infty$, while d behaves generically at both $\pm\infty$.

(OF6) The Melnikov integral $\int_{-\infty}^{\infty} d(\xi')^* D_2 G(q_{\xi'}, 0) d\xi' \in \mathbb{R}^2$ and the derivative $[D\Phi^+](0) \in \mathbb{R}^2$ are linearly independent.

This condition allows us to redefine the coordinates on the parameter space U to ensure that

$$\begin{array}{rcl} \mu_1 & = & \Phi^+(\mu_1, \mu_2), \\ \mu_2 & = & \langle d, \xi^\infty(\mu_1, \mu_2) \rangle_0. \end{array} \tag{6.27}$$

In the event that $\lambda_+ > -\eta^f_-$ we need to strengthen the condition (OF2) and give a more detailed description of the negative part of the spectrum associated to the limiting equation.

(OF7) There exist constants $\eta^{ff}_- < \eta^f_- < 0$ and $\eta^f_+ > 0$ such that the characteristic equation $\det \Delta(z) = 0$ associated to the equilibrium of (6.2) at $x = 0$ has precisely three eigenvalues $z = \lambda_\pm$ and $z = \lambda^f_-$ in the strip $\eta^{ff}_- \leq \operatorname{Re} z \leq \eta^f_+$. These eigenvalues are simple roots of the characteristic equation and there exist constants $\eta^s_\pm$ such that the following inequalities are satisfied,

$$\eta^{ff}_- < \lambda^f_- < \eta^f_- < \lambda_- < \eta^s_- < 0 < \eta^s_+ < \lambda_+ < \eta^f_+. \tag{6.28}$$

Writing Φ^f_+ and Φ^{*f}_- for the quantities associated to this eigenvalue λ^f_- that are analogous to those defined for $\lambda_\pm$ in (6.25) and (6.26), we have $\Phi^f_+(0) \neq 0$ and $\Phi^{*f}_- \neq 0$.

After all these preparations, we are almost ready to apply Theorem 6.2.2 and describe the orbit-flip bifurcation for functional differential equations of mixed type. It merely remains to define the type of solutions to (6.2) in which we are interested. To this end, consider any pair of positive constants (δ, Ω), where δ should be seen as small and Ω as large. Let us consider a solution x to (6.2) that satisfies the limits $\lim_{\xi \to \pm\infty} x(\xi) = 0$. Suppose that x remains δ-close to q, in the sense that for any $\xi \in \mathbb{R}$ there is a $\xi' \in \mathbb{R}$ such that $\left\| \mathrm{ev}_\xi x - \mathrm{ev}_{\xi'} q \right\| < \delta$. Suppose furthermore that there exist exactly M distinct values $\{\xi_j\}_{j=1}^M$ for which $\mathrm{ev}_{\xi_j} x \in \mathrm{ev}_0 q + X_\perp + \Gamma$, with $\left\| \mathrm{ev}_{\xi_j} x - \mathrm{ev}_0 q \right\| < \delta$. Finally, suppose that for any pair $1 \leq j_1, j_2 \leq M$, we have $\left| \xi_{j_1} - \xi_{j_2} \right| > \Omega$. Then we will refer to x as a (δ, Ω, M)-homoclinic solution. Similarly, let us consider a periodic solution x to (6.2) with minimal period ω. If x also satisfies the conditions above, where the values ξ_j should now be interpreted modulo ω, then we will call x a (δ, Ω, M)-periodic solution.

Theorem 6.2.3. *Consider the nonlinear equation (6.2) and assume that the conditions (OF1) through (OF6) and (HB) are satisfied, with $\lambda_+ \neq -\lambda_-$. In the event that $\lambda_+ \geq -\eta_-^f$, assume furthermore that (OF7) is satisfied and that $\lambda_+ \neq -\lambda_-^f$. Then upon fixing $\delta > 0$ sufficiently small and $\Omega > 0$ sufficiently large, one of the following three alternatives must hold.*

(A) (Homoclinic Continuation) We have $\lambda_+ < -\lambda_-$. For all sufficiently small pairs (μ_1, μ_2), with $\mu_2 > 0$, equation (6.2) admits precisely one $(\delta, \Omega, 1)$-periodic solution. For all sufficiently small $|\mu_1|$, there exists precisely one $(\delta, \Omega, 1)$-homoclinic solution to (6.2) with $\mu_2 = 0$. For all integers $M \geq 2$, there are no (δ, Ω, M)-periodic and (δ, Ω, M)-homoclinic solutions to (6.2).

(B) (Homoclinic Doubling) We have $-\lambda_- < \lambda_+ < -\eta_-^f$. Excluding the line $\mu_2 = 0$, there are two curves that extend from the origin in parameter space on which codimension one bifurcations occur. More precisely, there is a branch of $(\delta, \Omega, 2)$-homoclinic solutions that passes through the origin and a curve emanating from the origin at which a period-doubling bifurcation takes place, turning $(\delta, \Omega, 1)$-periodic solutions into $(\delta, \Omega, 2)$-periodic solutions.

(C) (Homoclinic Cascade) We have $-\lambda_-^f < \lambda_+$. For every $M \geq 1$ there is a branch of (δ, ω, M)-homoclinic solutions to (6.2) that emerges from the origin in parameter space. In addition, branches of codimension-one period-fold and period-doubling bifurcations emerge from the origin and there is an open wedge in parameter space in which (6.2) admits symbolic dynamics.

We refer to [134] for a more graphic description of these three bifurcation scenarios.

6.3. Preliminaries

In this section we recall the basic linear theory that was developed for the linear inhomogeneous system

$$x'(\xi) = L(\xi)x_\xi + f(\xi) = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j) + f(\xi), \tag{6.29}$$

in which we take $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$. We will assume throughout this section that the complex $n \times n$ matrix valued functions A_j are continuous and that the shifts r_j are ordered according to $r_0 < \ldots < r_N$, again with $r_0 \le 0$ and $r_N \ge 0$.

The system (6.29) is said to be asymptotically hyperbolic if the limits $A_j^\pm = \lim_{\xi \to \pm\infty} A_j(\xi)$ exist for all integers $0 \le j \le N$, while the characteristic equations $\det \Delta^\pm(z) = 0$ associated to these limiting equations do not have any roots on the imaginary axis. Here we have defined

$$\Delta^\pm(z) = zI - \sum_{j=0}^{N} A_j^\pm e^{zr_j}. \tag{6.30}$$

We recall the linear operator $\Lambda : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ associated to (6.29) that is given by

$$[\Lambda x](\xi) = x'(\xi) - \sum_{j=0}^{N} A_j(\xi) x(\xi + r_j), \tag{6.31}$$

together with the formal adjoint $\Lambda^* : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$[\Lambda^* y](\xi) = y'(\xi) + \sum_{j=0}^{N} A_j(\xi - r_j)^* y(\xi - r_j). \tag{6.32}$$

The following important result, that describes the relation between the Fredholm operators Λ and Λ^*, is due to Mallet-Paret and can be found in [112].

Theorem 6.3.1. *Assume that (6.29) is asymptotically hyperbolic. Then both Λ and Λ^* are Fredholm operators, with Fredholm indices given by*

$$\mathrm{ind}(\Lambda) = -\mathrm{ind}(\Lambda^*) = \dim \mathcal{K}(\Lambda) - \dim \mathcal{K}(\Lambda^*). \tag{6.33}$$

Every element in the kernels $\mathcal{K}(\Lambda)$ and $\mathcal{K}(\Lambda^*)$ decays exponentially as $\xi \to \pm\infty$, while the relation between Λ and Λ^* is given by the following identities,

$$\begin{aligned} \mathcal{R}(\Lambda) &= \left\{ h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} d(\xi')^* h(\xi') d\xi' = 0 \text{ for every } d \in \mathcal{K}(\Lambda^*) \right\}, \\ \mathcal{R}(\Lambda^*) &= \left\{ h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} b(\xi')^* h(\xi') d\xi' = 0 \text{ for every } b \in \mathcal{K}(\Lambda) \right\}. \end{aligned} \tag{6.34}$$

In the special case that the functions $A_j(\xi)$ do not depend on ξ, the operator Λ is invertible and there exists a Greens function $G : \mathbb{R} \to \mathbb{C}^{n\times n}$ such that

$$[\Lambda^{-1} f](\xi) = \int_{-\infty}^{\infty} G(\xi - \xi') f(\xi') d\xi'. \tag{6.35}$$

The Fourier transform of the function G is given by $\widehat{G}(\eta) = \Delta^{-1}(i\eta)$, which implies that G decays exponentially at both $\pm\infty$.

For our purposes in this chapter, we will need to study the action of Λ on function spaces with exponentially weighted norms. We therefore introduce the notation $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$. In addition, we introduce the family of exponentially weighted spaces

$$\begin{array}{rcl} L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in L^\infty(\mathbb{R}, \mathbb{C}^n)\right\}, \\ W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)\right\}, \end{array} \tag{6.36}$$

with norms given by $\|x\|_{L^\infty_\eta} = \left\|e_{-\eta} x\right\|_{L^\infty}$ and similarly $\|x\|_{W^{1,\infty}_\eta} = \left\|e_{-\eta} x\right\|_{W^{1,\infty}}$.

To study how Λ behaves under the action of e_η, let us define the shifted operator $\Lambda_\eta : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$[\Lambda_\eta x](\xi) = x'(\xi) - \eta x(\xi) - \sum_{j=0}^{N} A_j(\xi) e^{-\eta r_j} x(\xi + r_j). \tag{6.37}$$

In addition, we write $\Delta^\pm_\eta$ for the characteristic equations associated to the shifted operator Λ_η. It is not hard to check that

$$\begin{array}{rcl} \Lambda e_\eta x & = & e_\eta \Lambda_{-\eta} x, \\ \Delta_\eta(z) & = & \Delta(z - \eta). \end{array} \tag{6.38}$$

Using the definition of the adjoint Λ^* in (6.32), one may also easily conclude that we have the identity

$$(\Lambda_\eta)^* = (\Lambda^*)_{-\eta}. \tag{6.39}$$

In this fashion we can define the Fredholm operator $\Lambda_{(\eta)} : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ by means of

$$\Lambda_{(\eta)} = e_\eta \circ \Lambda_{-\eta} \circ e_{-\eta}. \tag{6.40}$$

In a similar fashion we define $\Lambda^*_{(\eta)} : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ by

$$\Lambda^*_{(\eta)} = e_\eta \circ (\Lambda^*)_{-\eta} \circ e_{-\eta}. \tag{6.41}$$

The next proposition provides the appropriate generalization of Theorem 6.3.1.

Proposition 6.3.2. *Assume that (6.29) is asymptotically autonomous and in addition that the characteristic equations* $\det \Delta^\pm(z) = 0$ *have no roots with* $\operatorname{Re} z = \eta$. *Then both* $\Lambda_{(\eta)} : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ *and* $\Lambda^*_{(-\eta)} : W^{1,\infty}_{-\eta}(\mathbb{R}, \mathbb{C}^n) \to L^\infty_{-\eta}(\mathbb{R}, \mathbb{C}^n)$ *are Fredholm operators, with*

$$\operatorname{ind}(\Lambda_{(\eta)}) = -\operatorname{ind}(\Lambda^*_{(-\eta)}) = \dim \mathcal{K}(\Lambda_{(\eta)}) - \dim \mathcal{K}(\Lambda^*_{(-\eta)}). \tag{6.42}$$

For every element b in $\mathcal{K}(\Lambda_{(\eta)})$, the function $e_{-\eta} b$ decays exponentially at both $\pm\infty$, while for any d in $\mathcal{K}(\Lambda^*_{(-\eta)})$ we have that $e_\eta d$ decays exponentially at both $\pm\infty$. The relation between $\Lambda_{(\eta)}$ and $\Lambda^*_{(-\eta)}$ is given by the following identities,

$$\begin{array}{rcl} \mathcal{R}(\Lambda_{(\eta)}) & = & \left\{h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} d(\xi')^* h(\xi') = 0 \textit{ for every } d \in \mathcal{K}(\Lambda^*_{(-\eta)})\right\}, \\ \mathcal{R}(\Lambda^*_{(-\eta)}) & = & \left\{h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} b(\xi')^* h(\xi') = 0 \textit{ for every } b \in \mathcal{K}(\Lambda_{(\eta)})\right\}. \end{array} \tag{6.43}$$

Proof. The result follows using Theorem 6.3.1 and the identities

$$\begin{array}{lcl} \mathcal{K}(\Lambda_{(\eta)}) & = & e_{\eta}\mathcal{K}(\Lambda_{-\eta}), \\ \mathcal{K}(\Lambda^*_{(-\eta)}) & = & e_{-\eta}\mathcal{K}((\Lambda^*)_{\eta}) = e_{-\eta}\mathcal{K}((\Lambda_{-\eta})^*), \\ \mathcal{R}(\Lambda_{(\eta)}) & = & e_{\eta}\mathcal{R}(\Lambda_{-\eta}), \\ \mathcal{R}(\Lambda^*_{(-\eta)}) & = & e_{-\eta}\mathcal{R}((\Lambda^*)_{\eta}) = e_{-\eta}\mathcal{R}((\Lambda_{-\eta})^*), \end{array} \tag{6.44}$$

together with the identity $\Delta_{-\eta}(z) = \Delta(z + \eta)$. □

We now introduce parameter dependence into our main linear equation (6.29). In particular, we study the system

$$x'(\xi) = L(\xi, \mu)x_{\xi} + f(\xi) = \sum_{j=0}^{N} A_j(\xi, \mu)x(\xi + r_j) + f(\xi), \tag{6.45}$$

in which the parameter μ is taken from an open set $U \subset \mathbb{R}^p$ for some $p \geq 1$. We write $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ for the parameter-dependent version of (6.31). Throughout the remainder of this section, we will assume that Λ depends C^k-smoothly on the parameter $\mu \in U$.

We set out here to define a solution operator for (6.45) on half-lines that also depends smoothly on the parameter μ, in the neighbourhood of some fixed parameter $\mu_0 \in U$. To this end, let us introduce the shorthands $\mathcal{K} = \mathcal{K}\big(\Lambda(\mu_0)\big)$ and $\mathcal{R} = \mathcal{R}\big(\Lambda(\mu_0)\big)$. Consider two arbitrary complements $\mathcal{K}^{\perp}$ for $\mathcal{K}$ and $\mathcal{R}^{\perp}$ for $\mathcal{R}$, which allow us to write

$$W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \;=\; \mathcal{K} \oplus \mathcal{K}^{\perp}, \qquad L^{\infty}(\mathbb{R}, \mathbb{C}^n) \;=\; \mathcal{R} \oplus \mathcal{R}^{\perp}. \tag{6.46}$$

The projections associated to this splitting of $L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ will be denoted by $\pi_{\mathcal{R}}$ and $\pi_{\mathcal{R}^{\perp}}$. Note that for μ sufficiently close to μ_0, we have that $\pi_{\mathcal{R}}\Lambda(\mu) : \mathcal{K}^{\perp} \to \mathcal{R}$ is invertible, with a C^k-smooth inverse $\mu \mapsto [\pi_{\mathcal{R}}\Lambda(\mu)]^{-1} \in \mathcal{L}(\mathcal{R}, \mathcal{K}^{\perp})$. Upon choosing a sufficiently small neighbourhood $U' \subset U$, with $\mu_0 \in U'$, we can hence define a C^k-smooth function $h : U' \to \mathcal{L}(\mathcal{K}, \mathcal{K}^{\perp})$ via

$$h(\mu)(b) = -[\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}\Lambda(\mu)b. \tag{6.47}$$

Observe first that we have $h(\mu_0) = 0$ by construction. In addition, this definition ensures that for $\mu \in U'$ the infinite dimensional problem to find $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ that solves $\Lambda(\mu)x = f$, is equivalent to the search for a solution $b \in \mathcal{K}$ of

$$\pi_{\mathcal{R}^{\perp}}[\Lambda(\mu)](b + h(\mu)b) = \pi_{\mathcal{R}^{\perp}}f - \pi_{\mathcal{R}^{\perp}}\Lambda(\mu)[\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}f. \tag{6.48}$$

This can be seen by substituting

$$x = [\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}f + b + h(\mu)b. \tag{6.49}$$

These considerations allow us to define a quasi-inverse for Λ that solves (6.45) in the sense of the following result.

Proposition 6.3.3. *Consider the parameter-dependent inhomogeneous system (6.45) and fix a parameter $\mu_0 \in U$ for which (6.45) is asymptotically hyperbolic. Then there exists an open subset $U' \subset U$, with $\mu_0 \in U'$, together with a C^k-smooth function*

$$\mathcal{C} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), \mathcal{R}^\perp\big) \tag{6.50}$$

and a C^k-smooth quasi-inverse

$$\Lambda^{\mathrm{qinv}} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)\big), \tag{6.51}$$

such that the following properties hold.

(i) For all $\mu \in U'$ we have

$$\dim \mathcal{K}\big(\Lambda(\mu)\big) \le \dim \mathcal{K}\big(\Lambda(\mu_0)\big). \tag{6.52}$$

(ii) For all $\mu \in U'$ and all $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ we have the identity

$$\Lambda(\mu)\Lambda^{\mathrm{qinv}}(\mu) f = f + \mathcal{C}(\mu) f. \tag{6.53}$$

In addition, the restriction of $\mathcal{C}(\mu_0)$ to the set $\mathcal{R}$ vanishes identically.

Proof. Item (i) can be confirmed by noting that

$$\dim \mathcal{K}(\Lambda(\mu)) = \dim \mathcal{K} - \operatorname{rank} \pi_{\mathcal{R}^\perp} \Lambda(\mu)[I + h(\mu)] \le \dim \mathcal{K}. \tag{6.54}$$

To establish item (ii), we choose $\mathcal{C}$ and Λ^{qinv} according to

$$\begin{array}{rcl} \Lambda^{\mathrm{qinv}}(\mu) f & = & [\pi_{\mathcal{R}} \Lambda(\mu)]^{-1} \pi_{\mathcal{R}} f, \\ \mathcal{C}(\mu) f & = & -\pi_{\mathcal{R}^\perp} f + \pi_{\mathcal{R}^\perp} \Lambda(\mu) [\pi_{\mathcal{R}} \Lambda(\mu)]^{-1} \pi_{\mathcal{R}} f. \end{array} \tag{6.55}$$

A simple calculation is now sufficient to conclude the proof. □

In order to define a solution operator for (6.45) on half-lines, we will need to utilize the freedom we still have to choose the complements $\mathcal{K}^\perp$ and $\mathcal{R}^\perp$ in a special fashion. To do this, we will need to assume that condition (HB) holds, i.e., we demand that both $\det A_0(\xi, \mu_0)$ and $\det A_N(\xi, \mu_0)$ are non-zero for all $\xi \in \mathbb{R}$.

Lemma 6.3.4. *Consider the parameter-dependent linear system (6.45) and suppose that condition (HB) holds for this system at $\mu = \mu_0$, for some $\mu_0 \in U$. Write $n_d = \dim \mathcal{K}\big(\Lambda^*(\mu_0)\big)$ and choose a basis $\{d^i\}_{i=1}^{n_d}$ for $\mathcal{K}\big(\Lambda^*(\mu_0)\big)$. For any $\xi \in \mathbb{R}$ there exists a set of functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ such that for any pair of integers $1 \le i, j \le n_d$ we have*

$$\int_{-r_{\max}}^{-r_{\min}} d^i(\xi + \theta)^* \psi^j(\theta) d\theta = \delta_{ij}. \tag{6.56}$$

Proof. First notice that the condition (HB) implies that the set of elements $\{\mathrm{ev}_\xi^* d^i\}_{i=d}^{n_d} \subset Y$ is linearly independent. In particular, this means that the $n_d \times n_d$ matrix Z with entries $Z_{ij} = \big(\mathrm{ev}_\xi^* d^i, \mathrm{ev}_\xi^* d^j\big)$ is invertible, where $(,)$ denotes the integral inner product

$$(\psi, \phi) = \int_{-r_{\max}}^{-r_{\min}} \psi(\theta)^* \phi(\theta) d\theta. \tag{6.57}$$

For any integer $1 \le j \le n_d$ we now choose

$$\psi^j = \sum_{k=1}^{n_d} \mathrm{ev}_\xi^* d^k Z_{kj}^{-1}. \tag{6.58}$$

A simple calculation shows that indeed

$$(\mathrm{ev}_\xi^* d^i, \psi^j) = \sum_{k=1}^{n_d} (\mathrm{ev}_\xi^* d^i, \mathrm{ev}_\xi^* d^k) Z_{kj}^{-1} = \sum_{k=1}^{n_d} Z_{ik} Z_{kj}^{-1} = \delta_{ij}. \tag{6.59}$$

□

We will use Lemma 6.3.4 to explicitly construct a representation for $\pi_{\mathcal{R}}$ and $\pi_{\mathcal{R}^\perp}$. Indeed, let us write $r = r_{\max} - r_{\min}$ and fix an arbitrary $\xi_0 \le -4r$. In addition, for any integer $1 \le i \le n_d$ we let $g^i \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ denote the function that has $\mathrm{ev}_{\xi_0}^* g^i = \psi^i$, while $g^i(\xi') = 0$ for all $\xi' < \xi_0 - r_{\max}$ and $\xi' > \xi_0 - r_{\min}$. Here the functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ arise from an application of Lemma 6.3.4 with $\xi = \xi_0$. Since the set $\{g^i\}_{i=1}^{n_d}$ is linearly independent, we can now explicitly define the projection

$$\pi_{\mathcal{R}^\perp} f \quad = \quad \textstyle\sum_{i=1}^{n_d} [\int_{-\infty}^{\infty} d^i(\xi')^* f(\xi') d\xi'] g^i. \tag{6.60}$$

This enables us to define an inverse for $\Lambda(\mu)$ on the positive half-line. Indeed, consider the operator $\Lambda_+^{-1}(\mu) : L^\infty\big([0, \infty), \mathbb{C}^n\big) \to W^{1,\infty}\big([r_{\min}, \infty), \mathbb{C}^n\big)$ given by

$$\Lambda_+^{-1}(\mu) f = \Lambda^{\mathrm{qinv}}(\mu) E f, \tag{6.61}$$

in which $[Ef](\xi) = 0$ for all $\xi < 0$ and $[Ef](\xi) = f(\xi)$ for all $\xi \ge 0$. Since $g^i(\xi) = 0$ for all $\xi \ge 0$ and all integers $1 \le i \le n_d$, an application of (6.53) immediately implies that for all $\xi \ge 0$ we have

$$[\Lambda(\mu)\Lambda_+^{-1}(\mu) f](\xi) = f(\xi). \tag{6.62}$$

In a similar fashion an inverse $\Lambda_-^{-1}(\mu) : L^\infty\big((-\infty, 0], \mathbb{C}^n\big) \to W^{1,\infty}\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ can be constructed for the negative half-line. Both these inverses depend C^k-smoothly on the parameter $\mu \in U'$.

6.4. Exponential Dichotomies

In this section we study exponential splittings for the homogeneous counterpart of the linear system (6.29), which we will write as

$$x'(\xi) = L(\xi)x_\xi = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j). \tag{6.63}$$

Throughout this entire section we will assume that the functions A_j are continuous. In addition, we will assume that (6.63) is asymptotically hyperbolic and that the condition (HB) holds. We start by stating the main theorem which we set out to prove in this section. We remark that a similar result was previously obtained in a Hilbert space setting [75].

Theorem 6.4.1. *Consider the linear system (6.63). There exist constants $K > 0$, $\alpha_S > 0$ and $\alpha_Q > 0$, such that for every $\xi \geq 0$ there is a splitting*

$$X = Q(\xi) \oplus S(\xi), \tag{6.64}$$

such that each $\phi \in Q(\xi)$ can be extended to a solution $E\phi \in C\big([\xi + r_{\min}, \infty), \mathbb{C}^n\big)$ of the homogeneous equation (6.63) on the interval $[\xi, \infty)$, while each $\psi \in S(\xi)$ can be extended to a function $E\psi \in C\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big)$ that satisfies the homogeneous equation (6.63) on the interval $[0, \xi]$. In addition, we have the exponential estimates

$$\begin{array}{lllll} \big|[E\phi](\xi')\big| & \leq & Ke^{-\alpha_Q|\xi'-\xi|}\,\|\phi\| & \text{for every } \phi \in Q(\xi) & \text{and} \quad \xi' \geq \xi, \\ \big|[E\psi](\xi')\big| & \leq & Ke^{-\alpha_S|\xi'-\xi|}\,\|\psi\| & \text{for every } \psi \in S(\xi) & \text{and} \quad 0 \leq \xi' \leq \xi. \end{array} \tag{6.65}$$

These spaces are invariant, in the sense that for any $0 \leq \xi' \leq \xi$ and any $\psi \in S(\xi)$, we have $\mathrm{ev}_{\xi'}E\psi \in S(\xi')$, together with a similar identity for $\phi \in Q(\xi)$. Finally, the projections $\Pi_{Q(\xi)}$ and $\Pi_{S(\xi)}$ depend continuously on $\xi \geq 0$ and there exists a constant C such that $\big\|\Pi_{Q(\xi)}\big\| \leq C$ and $\big\|\Pi_{S(\xi)}\big\| \leq C$ for all $\xi \geq 0$.

Throughout this section, we will follow the notation employed in [115]. In particular, we introduce the spaces

$$\begin{array}{lll} \mathcal{P}(\xi) & = & \big\{x \in BC_0\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big) \mid x'(\xi') = L(\xi')v_{\xi'} \text{ for all } \xi' \in (-\infty, \xi]\big\}, \\ \mathcal{Q}(\xi) & = & \big\{x \in BC_0\big([\xi + r_{\min}, \infty), \mathbb{C}^n\big) \mid x'(\xi') = L(\xi')v_{\xi'} \text{ for all } \xi' \in [\xi, \infty)\big\}, \\ \widehat{\mathcal{P}}(\xi) & = & \big\{x \in \mathcal{P}(\xi) \mid \int_{-\infty}^{\min(\xi + r_{\max}, 0)} b(\xi')^* x(\xi')d\xi' = 0 \text{ for all } b \in \mathcal{K}\big\}, \\ \widehat{\mathcal{Q}}(\xi) & = & \big\{x \in \mathcal{Q}(\xi) \mid \int_{+\infty}^{\max(\xi + r_{\min}, 0)} b(\xi')^* x(\xi')d\xi' = 0 \text{ for all } b \in \mathcal{K}\big\}. \end{array} \tag{6.66}$$

We remark here that these definitions of $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ differ slightly from those given in [115], in the sense that the upper bounds of the defining integrals are now constant for $\xi \geq 0$ respectively $\xi \leq 0$. All the results obtained in [115] remain unaffected by this choice, which we make here to ensure that $\widehat{P}(\xi)$ is invariant on the positive half-line and $\widehat{Q}(\xi)$ is invariant on the negative half-line. As in [115], we also introduce the following spaces, that describe the initial conditions associated to the spaces above and the kernels $\mathcal{K}$ and $\mathcal{K}^*$.

$$\begin{array}{lcl} P(\xi) & = & \{\phi \in X \mid \phi = x_\xi \text{ for some } x \in \mathcal{P}(\xi)\}, \\ Q(\xi) & = & \{\phi \in X \mid \phi = x_\xi \text{ for some } x \in \mathcal{Q}(\xi)\}, \\ \widehat{P}(\xi) & = & \{\phi \in X \mid \phi = x_\xi \text{ for some } x \in \widehat{\mathcal{P}}(\xi)\}, \\ \widehat{Q}(\xi) & = & \{\phi \in X \mid \phi = x_\xi \text{ for some } x \in \widehat{\mathcal{Q}}(\xi)\}, \\ B(\xi) & = & \{\phi \in X \mid \phi = b_\xi \text{ for some } b \in \mathcal{K}\}, \\ B^*(\xi) & = & \{\phi \in Y \mid \phi = d_\xi \text{ for some } d \in \mathcal{K}^*\}. \end{array} \tag{6.67}$$

The following result was obtained in [115] and shows that $P(\xi)$ and $Q(\xi)$ together span X up to a finite dimensional complement, that can be described explicitly in terms of the Hale inner product.

Proposition 6.4.2. *Consider the homogeneous linear system (6.63). For any $\xi \in \mathbb{R}$, let $Z(\xi) \subset X$ be the closed subspace of finite codimension that is given by*

$$Z(\xi) = \{\phi \in X \mid \langle \psi, \phi \rangle_\xi = 0 \text{ for every } \psi \in B^*(\xi)\}. \tag{6.68}$$

Then we have the direct sum decomposition

$$Z(\xi) = \widehat{P}(\xi) \oplus \widehat{Q}(\xi) \oplus B(\xi). \tag{6.69}$$

Our main contribution in this section is to provide an explicit complement for $Z(\xi)$ that will allow us to enlarge the space $\widehat{P}(\xi)$ and obtain a set $S(\xi)$ that satisfies the properties in Theorem 6.4.1. To do this, we will employ a very useful property of the Hale inner product. In particular, fix an interval $[\xi_-, \xi_+]$ and consider an arbitrary function $z \in W^{1,1}_{\text{loc}}([\xi_- - r_{\max}, \xi_+ - r_{\min}])$ together with an arbitrary function $x \in W^{1,1}_{\text{loc}}[\xi_- + r_{\min}, \xi_+ + r_{\max}]$. Then for every $\xi \in [\xi_-, \xi_+]$, we can perform the computation

$$\begin{array}{rcl} D_\xi \langle \text{ev}^*_\xi z, \text{ev}_\xi x \rangle_\xi & = & D_\xi\big[z(\xi)^* x(\xi) - \sum_{j=0}^N \int_\xi^{\xi+r_j} z(\theta - r_j)^* A_j(\theta - r_j) x(\theta) d\theta\big] \\ & = & z'(\xi)^* x(\xi) + z(\xi)^* x'(\xi) - \sum_{j=0}^N z(\xi)^* A_j(\xi) x(\xi + r_j) \\ & & + z(\xi - r_j)^* A_j(\xi - r_j) x(\xi) \\ & = & z(\xi)^* [\Lambda x](\xi) + [\Lambda^* z](\xi)^* x(\xi). \end{array} \tag{6.70}$$

Lemma 6.4.3. *Consider the homogeneous linear system (6.63). Let $\{d^i\}_{i=1}^{n_d}$ be a basis for the kernel $\mathcal{K}^*$ and recall the constant $r = r_{\max} - r_{\min}$. Then for every $\xi \geq 0$ and every integer $1 \leq i \leq n_d$, there exists a function $y^i_{(\xi)} \in C\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big)$ that satisfies the following properties.*

(i) For every $\xi \geq 0$ and every integer $1 \leq i \leq n_d$, we have $[\Lambda y^i_{(\xi)}](\xi') = 0$ for all $\xi' \geq -3r$ and all $\xi' \leq -5r$.

(ii) For any pair $0 \leq \xi' \leq \xi$ and any pair of integers $1 \leq i, j \leq n_d$, we have the identity

$$\langle \text{ev}^*_{\xi'} d^i, \text{ev}_{\xi'} y^j_{(\xi)} \rangle_{\xi'} = \delta_{ij}. \tag{6.71}$$

(iii) Fix an integer $1 \leq i \leq n_d$ *and a constant* $0 \leq \xi'$. *Then the function* $\xi \mapsto \mathrm{ev}_{\xi'} y^i_{(\xi)}$ *depends continuously on* ξ, *for* $\xi' \leq \xi$.

(iv) Consider any triple $0 \leq \xi' \leq \xi_1 \leq \xi_2$. *Then for any integer* $1 \leq i \leq n_d$ *we have*

$$\mathrm{ev}_{\xi'}[y^i_{(\xi_1)} - y^i_{(\xi_2)}] \in \widehat{P}(\xi'). \tag{6.72}$$

(v) For every $\xi \geq 0$ *and every integer* $1 \leq i \leq n_d$, *we have the integral condition*

$$\int_{-\infty}^{0} b(\xi')^* y_{(\xi)}(\xi') d\xi' = 0, \tag{6.73}$$

which holds for all $b \in \mathcal{K}(\Lambda)$.

Proof. Fix $\xi_0 = -4r$ and consider the functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ that were constructed in Lemma 6.3.4 for $\xi = \xi_0$. As in Section 6.3, define the functions $g^i \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ that have $\mathrm{ev}^*_{\xi_0} g^i = \psi^i$, while $g^i = 0$ elsewhere. For the remainder of this proof, fix an integer $1 \leq i \leq n_d$. Consider a sequence $\xi_k = k \to \infty$ and define $y_{(k)} = \Lambda^{-1}_{(k)} g^i$, where the inverse $\Lambda^{-1}_{(k)}$ should be interpreted as the analogue of Λ^{-1}_{-} for the half-line $(-\infty, \xi_k]$. Note that by adding an appropriate element in $\mathcal{K}$ to $y_{(k)}$ we can ensure that the integral condition (6.73) is satisfied. For any integer $1 \leq j \leq n_d$ we can use (6.70) together with the exponential decay of d^j at $-\infty$ to compute

$$\langle \mathrm{ev}^*_\xi d^j, \mathrm{ev}_\xi y_{(k)} \rangle_\xi = \int_{-\infty}^{\xi} d^j(\xi')^* [\Lambda y_{(k)}](\xi') d\xi' = (\mathrm{ev}^*_{\xi_0} d^j, \psi^i) = \delta_{ij}. \tag{6.74}$$

Choose a continuous function $\chi : [0, \infty) \to [0, 1]$ such that χ is zero near even integers and one near odd integers. Write

$$y_{(\xi)} = \chi(2\xi) y_{(\lceil \xi \rceil)} + [1 - \chi(2\xi)] y_{(\lceil \xi + \frac{1}{2} \rceil)}, \tag{6.75}$$

in which $\lceil \xi \rceil$ denotes the smallest integer that is larger or equal to ξ. With this definition it is easy to see that the properties (i) through (v) all hold. □

The functions defined in Lemma 6.4.3 are sufficient to construct the space $S(\xi)$ appearing in Theorem 6.4.1. Indeed, we will use the spaces

$$\begin{array}{rcl} \mathcal{S}(\xi) & = & \widehat{\mathcal{P}}(\xi) \oplus \mathrm{span}\{y^i_{(\xi)}\}_{i=1}^{n_d}, \\ S(\xi) & = & \widehat{P}(\xi) \oplus \mathrm{span}\{\mathrm{ev}_\xi y^i_{(\xi)}\}_{i=1}^{n_d}. \end{array} \tag{6.76}$$

The following result should be seen as the appropriate generalization of Theorem 4.2 in [115] and shows that functions in $\mathcal{S}$ automatically decay exponentially.

Proposition 6.4.4. *Consider the homogeneous linear system (6.63). Let the sets* $\mathcal{S}(\xi) \subset X$ *for* $\xi \geq 0$ *be defined as in (6.76). Then there exist constants* $K > 0$ *and* $\alpha_S > 0$ *such that for all* $\xi \geq 0$ *and all* $\xi' \leq \xi$, *we have*

$$\left| x(\xi') \right| \leq K e^{-\alpha_S (\xi - \xi')} \left\| x_\xi \right\|, \tag{6.77}$$

for every $x \in \mathcal{S}(\xi)$.

Proof. As in [115] it suffices to prove the following two statements.

(i) There exists $\sigma > -r_{\max}$ such that for all $\xi \geq 0$ and all $y \in \mathcal{S}(\xi)$, we have

$$\left|y(\xi')\right| \leq \frac{1}{2}\sup_{s<\xi+r_{\max}} |y(s)| \text{ for all } \xi' \leq \xi - \sigma. \tag{6.78}$$

(ii) There exists $K > 0$ such that for all $\xi \geq 0$ and all $y \in \mathcal{S}(\xi)$, we have

$$\left|y(\xi')\right| \leq K \left\|\mathrm{ev}_{\xi} y\right\| \text{ for all } \xi' \leq \xi + r_{\max}. \tag{6.79}$$

Assuming that (i) fails, we have sequences $\sigma^j \to \infty$, $\xi^j \geq 0$ and $y^j \in \mathcal{S}(\xi^j)$ such that

$$\left|y^j(-\sigma^j + \xi^j)\right| \geq \tfrac{1}{2}, \qquad \sup_{s<\xi^j+r_{\max}} \left|y^j(s)\right| = 1. \tag{6.80}$$

Suppose first that $-\sigma^j + \xi^j$ is unbounded, i.e., $-\sigma^j + \xi^j \to \pm\infty$ after passing to a subsequence. Writing $z^j(\xi') = y^j(\xi' - \sigma^j + \xi^j)$, an application of Ascoli's theorem yields a convergent subsequence $z^j \to z$. Notice that $z(0) \geq \frac{1}{2}$, which means that z is a nontrivial bounded solution on $\mathbb{R}$ of one of the limiting equations at $\pm\infty$. This situation is however precluded by the hyperbolicity of this limiting equation.

Now suppose that, possibly after passing to a subsequence, we have $-\sigma^j + \xi^j \to \beta^0$. Using the fact that $[\Lambda y^j](\xi') = 0$ for $\xi \geq r_{\min}$, together with the limit $\xi^j \to \infty$, we may apply Ascoli-Arzela to conclude that $y^j \to y_*$ uniformly on compact subsets of $[r_{\min}, \infty)$. Since we also have $[\Lambda y_*](\xi') = 0$ for all $\xi' \geq 0$, we conclude that $\mathrm{ev}_0 y_* \in Q(0)$. However, this immediately implies that for any $\psi \in B^*(0)$ we have $\langle \psi, \mathrm{ev}_0 y_* \rangle_0 = 0$. In view of the identity

$$\Lambda y^j = \sum_{i=1}^{n_d} g^i \langle \mathrm{ev}_0^* d^i, \mathrm{ev}_0 y^j \rangle_0, \tag{6.81}$$

this however implies that $\Lambda y^j \to 0$ uniformly on every compact subset of $\mathbb{R}$. This allows us to apply Ascoli-Arzela on the entire line, by means of which we obtain the convergence $y^j \to y_*$, which is again uniform on compacta. In addition, we have $\Lambda y_* = 0$, which now means that $y_* \in \mathcal{K}$. However, this is precluded by the integral condition (6.73).

Let us now suppose that (ii) fails, which implies that for some sequence $K^j \to \infty$, $\xi^j \geq 0$ and $y^j \in \mathcal{S}(\xi^j)$, we have

$$\sup_{s<\xi^j+r_{\max}} \left|y^j(s)\right| = K^j \left\|\mathrm{ev}_{\xi^j} y^j\right\| = 1. \tag{6.82}$$

In view of (i), this means that there exists a sequence $\sigma^j \in [-r_{\min}, \sigma]$ such that $\left|y^j(-\sigma^j + \xi^j)\right| = 1$.

Suppose that ξ^j is unbounded. We find $y^j(\xi' + \xi^j) \to z(\xi')$ where $z : (-\infty, r_{\max}] \to \mathbb{C}^n$ is a bounded solution of the limiting equation at $+\infty$. Since the sequence σ^j is bounded, z does not vanish identically. Since $\left\|\mathrm{ev}_{\xi^j} y^j\right\| = 1/K^j \to 0$, we have $\|z_0\| = 0$ and hence z can be extended to a bounded nontrivial solution of the limiting equation at $+\infty$ on the entire line. Again, this is precluded by the hyperbolicity of this limiting equation.

Now assume that, possibly after passing to a subsequence, we have $\xi^j \to \xi^* \geq 0$. Since $\mathrm{ev}_{\xi^j} y^j \to 0$, we can use Ascoli-Arzela to find the convergence $y^j \to y_*$, which is now uniform on the interval $[-r + r_{\mathrm{min}}, \xi^* + r_{\mathrm{max}}]$. In addition, we have $[\Lambda y_*](\xi') = 0$ for all $\xi' \in [-r, \xi^*]$. If $\xi^* \geq \sigma$, this fact is precluded by the non-degeneracy condition (HB), since we also have $\mathrm{ev}_{\xi^*} y_* = 0$. In the case where $\xi^* < \sigma$, we can again use (6.81) to obtain the convergence $y^j \to y_*$, which this time is uniform on compact subsets of $(-\infty, \xi^* + r_{\mathrm{max}}]$. As before, the condition (HB) now leads to a contradiction. □

Notice that we have now obtained a splitting

$$X = S(\xi) \oplus Q(\xi) \tag{6.83}$$

that satisfies nearly all of the properties stated in Theorem 6.4.1. It remains only to consider the statements concerning the projections $\Pi_{S(\xi)}$ and $\Pi_{Q(\xi)}$. We will address these issues in the remainder of this section by establishing the continuity of these projections and studying the limiting behaviour as $\xi \to \infty$. To this end, we recall the splitting

$$X = P(\infty) \oplus Q(\infty) \tag{6.84}$$

associated to the autonomous limit of (6.63) at $+\infty$, which was established in [115].

Lemma 6.4.5. *Consider the linear homogeneous system (6.63). The following limit holds in the spaces $\mathcal{L}(S(\xi), X)$,*

$$[I - \Pi_{P(\infty)}]_{|S(\xi)} \to 0 \quad as \quad \xi \to \infty. \tag{6.85}$$

In addition, in the spaces $\mathcal{L}(Q(\xi), X)$ we have the similar limit

$$[I - \Pi_{Q(\infty)}]_{|Q(\xi)} \to 0 \quad as \quad \xi \to \infty. \tag{6.86}$$

Proof. The second limit was established in [115], so we restrict ourselves to the first limit here. Choose an arbitrarily small $\epsilon > 0$ and fix $C > 0$ sufficiently large to ensure that for all $\xi \in \mathbb{R}$, the inequality

$$\sum_{j=0}^{N} \left| A_j(\xi) e^{\alpha_S r_j} \right| \leq C \tag{6.87}$$

holds. Recalling the constants K and α_S from Proposition 6.4.4, pick $\xi_0 > 0$ sufficiently large to ensure that $4(1 + C)^2 K \exp(-\alpha_S \xi_0) < \frac{\epsilon}{2}$ and also

$$\sum_{j=0}^{N} \left| A_j(\xi') - A_j^+ \right| < \frac{\epsilon}{2K} \tag{6.88}$$

for all $\xi' \geq \xi_0$. Fix any $\xi \geq 2\xi_0 + r_{\mathrm{max}}$. Consider an arbitrary $y \in \mathcal{S}(\xi)$ and write $\phi = \mathrm{ev}_\xi y \in S(\xi)$. Notice first that $\phi_{|[r_{\mathrm{min}}, 0]} \in C^1([r_{\mathrm{min}}, 0], \mathbb{C}^n)$. We can hence approximate ϕ with a sequence of C^1-smooth functions ϕ^k that have $\phi^k(\theta) = \phi(\theta)$ for all $\theta \in [-1, 0]$. Let us extend these functions to C^1-smooth functions y^k on the line, with $\mathrm{ev}_\xi y^k = \phi^k$ but also $y^k(\xi') = y(\xi')$ for all $0 \le \xi' \le \xi$. Notice that we may construct the functions y^k in such a way to ensure that the following estimate holds for all $\xi' \le 0$,

$$\left|Dy^k(\xi')\right| + \left|y^k(\xi')\right| \le 2[\left|Dy^k(0)\right| + \left|y^k(0)\right|]. \tag{6.89}$$

In particular, this means that for all $\xi' \le \xi$ we have the bound

$$\left|Dy^k(\xi')\right| + \left|y^k(\xi')\right| \le 2K(1+C)[e^{-\alpha s\xi} + e^{-\alpha s(\xi-\xi')}] \left\|\phi^k\right\|. \tag{6.90}$$

Now, for any C^1-smooth function y we have the representation

$$\Pi_{Q(\infty)}\mathrm{ev}_\xi y = \mathrm{ev}_\xi \Lambda_\infty^{-1}[I - H_\xi]\Lambda_\infty y, \tag{6.91}$$

in which we have introduced the notation $[\Lambda_\infty x](\xi') = x'(\xi') - \sum_{j=0}^N A_j^+ x(\xi' + r_j)$, together with the Heaviside function H_ξ that satisfies $H_\xi(\xi') = I$ if $\xi' \ge \xi$ and zero otherwise. Observing that

$$[\Lambda_\infty y^k](\xi') = [\Lambda y^k](\xi') + \sum_{j=0}^{N}[A_j(\xi') - A_j^+]\mathrm{ev}_{\xi'} y^k, \tag{6.92}$$

we may compute

$$\begin{array}{rcl} \left\|[I - H_\xi]\Lambda_\infty y^k\right\|_{L^\infty(\mathbb{R},\mathbb{C}^n)} & \le & \sup_{\xi'\le\xi_0} Dy^k(\xi') + C\left\|\mathrm{ev}_{\xi'} y^k\right\| \\ & & \quad + \sup_{\xi_0\le\xi'\le\xi} \frac{\epsilon}{2K}\left\|\mathrm{ev}_{\xi'} y^k\right\| \\ & \le & 4K(1+C)^2 e^{-\alpha s\xi_0}\left\|\phi^k\right\| + \frac{\epsilon}{2}\left\|\phi^k\right\| \\ & \le & \epsilon\left\|\phi^k\right\|. \end{array} \tag{6.93}$$

This however means that for some constant $C' > 0$ we have

$$\left\|\Pi_{Q(\infty)}\phi^k\right\| \le \epsilon C' \left\|\phi^k\right\|, \tag{6.94}$$

which concludes the proof due to the continuity of $\Pi_{Q(\infty)}$. □

Lemma 6.4.6. *Consider the system (6.63) and suppose that (HB) is satisfied. Fix an arbitrary $\xi \ge 0$. Write $\Gamma(\xi) = \mathrm{span}\{\mathrm{ev}_\xi y^i_{(\xi)}\}_{i=1}^{n_d}$ and consider the splitting*

$$X = \widehat{P}(\xi) \oplus \Gamma(\xi) \oplus Q(\xi) \tag{6.95}$$

with the corresponding projection operators $\Pi_{\widehat{P}(\xi)}$, $\Pi_{\Gamma(\xi)}$ and $\Pi_{Q(\xi)}$. Then we have the following limits,

$$\begin{array}{rcll} [I - \Pi_{\widehat{P}(\xi_0)}]_{|\widehat{P}(\xi)} & \to & 0 & \text{as } \xi \to \xi_0, \\ [I - \Pi_{Q(\xi_0)}]_{|Q(\xi)} & \to & 0 & \text{as } \xi \to \xi_0, \\ [I - \Pi_{\Gamma(\xi_0)}]_{|\Gamma(\xi)} & \to & 0 & \text{as } \xi \to \xi_0. \end{array} \tag{6.96}$$

Proof. The statements concerning $\widehat{P}(\xi)$ and $Q(\xi)$ were established in [115]. The limit involving $\Gamma(\xi)$ follows easily using the finite dimensionality of $\Gamma(\xi)$ and item (iii) in Lemma 6.4.3. □

Lemma 6.4.7. *Consider an arbitrary $\xi_0 \geq 0$. The projections $\Pi_{Q(\xi)}$ can be uniformly bounded for all $\xi \geq \xi_0$.*

Proof. Assuming the statement is false, let us consider a sequence ξ^j and $\phi^j \in X$ that has $\xi^j \geq \xi_0$ and $\|\phi^j\| = 1$ for all integers $j \geq 1$, while $\|\Pi_{Q(\xi^j)}\phi^j\| \to \infty$ as $j \to \infty$. Let us first assume that ξ^j is bounded, which after passing to a subsequence implies that $\xi^j \to \xi_*$ for some $\xi_* \geq \xi_0$. Let us write $\sigma^j = \Pi_{\Gamma(\xi^j)}\phi^j$, $p^j = \Pi_{\widehat{P}(\xi^j)}\phi^j$ and $q^j = \Pi_{Q(\xi^j)}\phi^j$. Defining $\kappa_j = \|\sigma^j\| + \|p^j\| + \|q^j\|$, let us also introduce the bounded sequence $\widetilde{\sigma}^j = \kappa_j^{-1}\sigma^j$ and similarly defined sequences $\widetilde{p}^j$ and $\widetilde{q}^j$. In addition, we introduce $\widetilde{\sigma}_*^j = \Pi_{\Gamma(\xi_*)}\widetilde{\sigma}^j$ and similarly $\widetilde{p}_*^j = \Pi_{\widehat{P}(\xi_*)}\widetilde{p}^j$ and $\widetilde{q}_*^j = \Pi_{Q(\xi_*)}\widetilde{q}^j$. Using Lemma 6.4.6 we obtain the following limits as $j \to \infty$,

$$\begin{array}{lcl} \widetilde{\sigma}^j + \widetilde{p}^j + \widetilde{q}^j & \to & 0, \\ \widetilde{\sigma}^j - \widetilde{\sigma}_*^j & \to & 0, \\ \widetilde{p}^j - \widetilde{p}_*^j & \to & 0, \\ \widetilde{q}^j - \widetilde{q}_*^j & \to & 0. \end{array} \tag{6.97}$$

Since $\Gamma(\xi_*)$ is finite dimensional, we can pass to a subsequence and obtain $\widetilde{\sigma}_*^j \to \sigma_*$ This implies the following limit as $j \to \infty$,

$$\sigma_* + \widetilde{p}_*^j + \widetilde{q}_*^j \to 0. \tag{6.98}$$

We now introduce the truncation operators $\pi^+ : X \to C\big([0, r_{\max}], \mathbb{C}^n\big)$ and $\pi^- : X \to C\big([r_{\min}, 0], \mathbb{C}^n\big)$. Using the exponential estimates on $\mathcal{Q}(\xi_*)$ and $\widehat{\mathcal{P}}(\xi_*)$, it is not hard to see that the restriction of π^+ to $Q(\xi_*)$ is compact, as is the restriction of π^- to $\widehat{P}(\xi_*)$. After passing to a subsequence, we thus find that $\pi^+\widetilde{q}_*^j$ and hence also $\pi^+\widetilde{p}_*^j$ converge uniformly on $[0, r_{\max}]$. Invoking a similar argument involving π^- we conclude that as $j \to \infty$, we must have $\widetilde{p}_*^j \to p_*$ and $\widetilde{q}_*^j \to q_*$ for some $p_* \in \widehat{P}(\xi_*)$ and $q_* \in Q(\xi_*)$. In view of (6.98), this leads to a contradiction, since $\|\sigma_*\| + \|p_*\| + \|q_*\| = 1$.

It remains to consider the case that $\xi^j \to \infty$. However, using the splitting $X = S(\xi) \oplus Q(\xi)$ and the limits in Lemma 6.4.5, we can obtain a contradiction in the same fashion as above. □

Corollary 6.4.8. *Consider the linear homogeneous system (6.63) and recall the splittings*

$$X = S(\xi) \oplus Q(\xi), \tag{6.99}$$

that hold for $\xi \geq 0$. The projections $\Pi_{S(\xi)}$ and $\Pi_{Q(\xi)}$ depend continuously on $\xi \in \mathbb{R}$. In addition, we have the limits

$$\lim_{\xi \to \infty} \big\|\Pi_{Q(\xi)} - \Pi_{Q(\infty)}\big\| = 0, \qquad \lim_{\xi \to \infty} \big\|\Pi_{S(\xi)} - \Pi_{P(\infty)}\big\| = 0. \tag{6.100}$$

Proof. The limit for $\Pi_{Q(\xi)}$ as $\xi \to \infty$ can be seen by writing

$$\Pi_{Q(\xi)} - \Pi_{Q(\infty)} = [I - \Pi_{Q(\infty)}]\Pi_{Q(\xi)} - [I - \Pi_{P(\infty)}]\Pi_{S(\xi)} \tag{6.101}$$

and using the limits in Lemma 6.4.5, together with the uniform bounds for $\Pi_{Q(\xi)}$ and $\Pi_{S(\xi)}$ that follow from Lemma 6.4.7. The other statements follow analogously. □

6.5. Parameter-Dependent Exponential Dichotomies

In this section we show how homogeneous linear systems of the form

$$x'(\xi) = L(\xi, \mu)x_\xi = \sum_{j=0}^{N} A_j(\xi, \mu)x(\xi + r_j), \tag{6.102}$$

which depend on a parameter $\mu \in U$, can be incorporated into the framework developed in the previous section. Throughout this section we will assume that the linear operators $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ associated to (6.102) by means of (6.31), depend C^k-smoothly on the parameter μ. In addition, we will assume that (HB) holds for some parameter $\mu_0 \in U$. Our main result shows that the exponential splittings can be constructed in such a way, that the relevant spaces and projections depend smoothly on the parameter μ. The price we have to pay is that we lose the invariance of $S(\xi, \mu)$, but for our purposes this will be irrelevant.

Theorem 6.5.1. *Consider the linear homogeneous system (6.102). There exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, such that for all $\mu \in U'$ and all $\xi \geq 0$ we have the splitting*

$$X = Q(\xi, \mu) \oplus S(\xi, \mu). \tag{6.103}$$

In addition, there exist constants $K > 0$, $\alpha_S > 0$ and $\alpha_Q > 0$, such that each $\phi \in Q(\xi, \mu)$ can be extended to a solution $E\phi$ of the homogeneous equation (6.102) on $[\xi, \infty)$, while each $\psi \in S(\xi, \mu)$ can be extended to a function $E\psi$ that is defined on the interval $[r_{\min}, \xi + r_{\max}]$ and satisfies the homogeneous equation (6.102) on $[0, \xi]$. The maps $\mu \mapsto \Pi_{Q(\xi,\mu)}$ and $\mu \mapsto \Pi_{S(\xi,\mu)}$ are C^k-smooth and all derivatives can be bounded independently of $\xi \geq 0$. Moreover, we have the following exponential estimates for all integers $0 \leq \ell \leq k$,

$$\begin{array}{lcll} \left\| D^\ell \mathrm{ev}_{\xi'} E\Pi_{Q(\xi,\mu)} \right\| & \leq & K e^{-\alpha_Q |\xi' - \xi|} & \text{for every } \xi' \geq \xi, \\ \left\| D^\ell \mathrm{ev}_{\xi'} E\Pi_{S(\xi,\mu)} \right\| & \leq & K e^{-\alpha_S |\xi' - \xi|} & \text{for every } 0 \leq \xi' \leq \xi, \end{array} \tag{6.104}$$

in which the differentiation operator D acts with respect to the parameter μ.

Our approach towards establishing Theorem 6.5.1 will be to construct the parameter-dependent spaces $Q(\xi, \mu)$ and $S(\xi, \mu)$ separately, using the implicit function theorem to represent these spaces as graphs over $Q(\xi, \mu_0)$ and $S(\xi, \mu_0)$. The exponential estimates will follow essentially from those established in the previous section for (6.102) with $\mu = \mu_0$.

Lemma 6.5.2. *Consider the exponential splitting $X = Q(\xi) \oplus S(\xi)$ for $\xi \geq 0$, as defined in Theorem 6.4.1 for the system (6.102) with $\mu = \mu_0$. Then there exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, together with a family of C^k-smooth functions $u^*_{Q(\xi)} : U' \to \mathcal{L}(Q(\xi), X)$, parametrized by $\xi \geq 0$, such that for all $\mu \in U'$ we have $\mathcal{R}(u^*_{Q(\xi)}(\mu)) = Q(\xi, \mu)$, with $\Pi_{Q(\xi)} u^*_{Q(\xi)}(\mu) = I$ and $[I - \Pi_{Q(\xi)}]u^*_{Q(\xi)}(\mu) \to 0$ as $\mu \to \mu_0$, uniformly for $\xi \geq 0$. In addition, there exist constants $K > 0$ and $\alpha_Q > 0$ such that for all $\mu \in U'$, all pairs $\xi' \geq \xi \geq 0$ and all integers $0 \leq \ell \leq k$, we have*

$$\left\| D^\ell \mathrm{ev}_{\xi'} E u^*_{Q(\xi)}(\mu) \right\|_{\mathcal{L}(Q(\xi), X)} \leq K e^{-\alpha_Q |\xi' - \xi|}. \tag{6.105}$$

Proof. We recall the C^k-smooth operator

$$\mathcal{C} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), \mathcal{R}(\Lambda(\mu_0))^\perp\big) \tag{6.106}$$

defined in Proposition 6.3.3 and we choose a basis for $\mathcal{R}(\Lambda(\mu_0))^\perp$ in such a way that the support of each basis function is contained in $[-4r - r_{\max}, -4r - r_{\min}] \subset (-\infty, 0)$. We also recall the constants $K > 0$ and $\alpha_Q > 0$ obtained by an application of Theorem 6.4.1 to the system (6.102) at $\mu = \mu_0$.

For any $\xi \geq 0$, let us consider the map $\mathcal{G} : U \to \mathcal{L}\big(BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$ that is given by

$$\mathcal{G}(\mu)u = \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u - E\Pi_{Q(\xi)}\mathrm{ev}_\xi\Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u. \tag{6.107}$$

Here we have introduced the notation $[L(\mu)u](\xi) = L(\xi, \mu)u_\xi$. We first note that $\mathcal{G}$ is well-defined, since the extension operator E indeed maps $Q(\xi)$ into $BC_{-\alpha_Q}\big([r_{\min} + \xi, \infty), \mathbb{C}^n\big)$ due to the exponential estimates in Theorem 6.4.1. To be more precise, note that the $\mathcal{L}\big(Q(\xi), BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$-norm of this extension is given by

$$\|E\| \leq Ke^{\alpha_Q\xi}. \tag{6.108}$$

Notice also that for some constant $C_1 > 0$ the $\mathcal{L}\big(BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n), X\big)$-norm of the evaluation operator $\mathrm{ev}_{\xi'}$ is bounded by

$$\big\|\mathrm{ev}_{\xi'}\big\| \leq C_1 e^{-\alpha_Q\xi'}. \tag{6.109}$$

The C^k-smoothness of $\mu \mapsto L(\mu)$ now implies that $\mathcal{G}$ is C^k-smooth as a map from U into $\mathcal{L}\big(BC_{-\alpha_Q}([r_{\min}+\xi, \infty), \mathbb{C}^n)\big)$. By taking μ sufficiently close to μ_0 we can achieve the following bounds, simultaneously for all $\xi \geq 0$ and every integer $1 \leq \ell \leq k$,

$$\begin{array}{lcl} \|\mathcal{G}(\mu)\| & \leq & \frac{1}{2}, \\ \big\|D^\ell\mathcal{G}(\mu)\big\| & \leq & C_2, \end{array} \tag{6.110}$$

in which we have introduced a constant $C_2 > 0$. The first estimate in (6.110) implies that for all μ sufficiently close to μ_0 and all $\xi \geq 0$, we can define the linear maps

$$v^*_{Q(\xi)}(\mu) : Q(\xi) \to BC_{-\alpha_Q}\big([r_{\min} + \xi, \infty), \mathbb{C}^n\big), \ \phi \mapsto [1 - \mathcal{G}(\mu)]^{-1}E\phi, \tag{6.111}$$

together with $u^*_{Q(\xi)}(\mu) = \mathrm{ev}_\xi v^*_{Q(\xi)}(\mu)$. The exponential estimates (6.105) follow directly from this representation of $u^*_{Q(\xi)}(\mu)$, together with (6.108), (6.109) and (6.110). In addition, it is immediately clear from our choice of $\mathcal{G}$ that $\Pi_{Q(\xi)}u^*_{Q(\xi)}(\mu) = I$. The remainder term can be bounded using the identity

$$[I - \Pi_{Q(\xi)}]u^*_{Q(\xi)}(\mu) = \mathrm{ev}_\xi\big[[I - \mathcal{G}(\mu)]^{-1} - I\big]E, \tag{6.112}$$

which approaches 0 as $\mu \to \mu_0$. Again, this limit can be obtained simultaneously for all $\xi \geq 0$ by using (6.108) and (6.109).

We now set out to prove that $\mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big) = \mathcal{Q}(\xi,\mu)$. Suppose therefore that $u = v^*_{Q(\xi)}(\mu)\phi$ for some $\phi \in Q(\xi)$. Notice that u necessarily satisfies the following identity for all $\xi' \geq \xi$,

$$\begin{array}{rcl} [\Lambda(\mu)u](\xi') & = & [L(\xi',\mu_0) - L(\xi',\mu)]\mathrm{ev}_{\xi'}E\phi + [L(\xi',\mu) - L(\xi',\mu_0)]\mathrm{ev}_{\xi'}u \\ & & + [L(\xi',\mu_0) - L(\xi',\mu)]\mathrm{ev}_{\xi'}\Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u \\ & & - [L(\xi',\mu_0) - L(\xi',\mu)]\mathrm{ev}_{\xi'}E\Pi_{Q(\xi)}\mathrm{ev}_{\xi} \\ & & \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u \\ & = & [L(\xi',\mu_0) - L(\xi',\mu)]\mathrm{ev}_{\xi'}u + [L(\xi',\mu) - L(\xi',\mu_0)]\mathrm{ev}_{\xi'}u = 0. \end{array} \tag{6.113}$$

This means that $v^*_{Q(\xi)}(\mu)$ indeed maps into $\mathcal{Q}(\xi,\mu)$.

It remains to show that $\mathcal{Q}(\xi,\mu) \subset \mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big)$. Supposing this is not the case, pick $q^1_\mu \in \mathcal{Q}(\xi,\mu)$ with $q^1_\mu \notin \mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big)$ and write $\phi = \Pi_{Q(\xi)}\mathrm{ev}_\xi q^1_\mu$ and $q^2_\mu = v^*_{Q(\xi)}(\mu)\phi$. Writing $q_\mu = q^1_\mu - q^2_\mu$, we have $q_\mu \in \mathcal{Q}(\xi,\mu)$ with $\Pi_{Q(\xi)}\mathrm{ev}_\xi q_\mu = 0$. Noticing that $[L(\mu) - L(\mu_0)]q_\mu = \Lambda(\mu_0)q_\mu$, we find that for some $q_{\mu_0} \in \mathcal{Q}(\xi)$ we must have

$$\begin{array}{rcl} \mathcal{G}(\mu)q_\mu & = & q_\mu + q_{\mu_0} - E\Pi_{Q(\xi)}\mathrm{ev}_\xi[q_\mu + q_{\mu_0}] \\ & = & q_\mu + q_{\mu_0} - q_{\mu_0} = q_\mu \end{array} \tag{6.114}$$

and hence $q_\mu \in \mathcal{K}\big(I - \mathcal{G}(\mu)\big) = \{0\}$, which concludes the proof. □

In the next proposition, a similar approach is used to construct $S(\xi,\mu)$. Notice however that this construction will be treated as a definition, as there is no canonical way to define $S(\xi,\mu)$ as was possible for $Q(\xi,\mu)$.

Lemma 6.5.3. *Consider the exponential splitting $X = Q(\xi) \oplus S(\xi)$ for $\xi \geq 0$ as defined in Theorem 6.4.1 for the system (6.102) with $\mu = \mu_0$. Then there exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, together with a family of C^k-smooth functions $u^*_{S(\xi)} : U \to \mathcal{L}(S(\xi,\mu_0), X)$, parametrized by $\xi \geq 0$, such that for all $\mu \in U'$ we have $\Pi_{S(\xi)}u^*_{S(\xi)}(\mu) = I$ and $[I - \Pi_{S(\xi)}]u^*_{S(\xi)}(\mu) \to 0$ as $\mu \to \mu_0$, uniformly for $\xi \geq 0$. In addition, there exist constants $K > 0$ and $\alpha_S > 0$, such that for all $\mu \in U'$, all pairs $0 \leq \xi' \leq \xi$ and all integers $0 \leq \ell \leq k$, we have*

$$\left\| D^\ell \mathrm{ev}_{\xi'} E u^*_{S(\xi)}(\mu) \right\|_{\mathcal{L}(S(\xi),X)} \leq K e^{-\alpha_S|\xi'-\xi|}. \tag{6.115}$$

Finally, for all $\mu \in U'$ and all $\xi \geq 0$, the range $\mathcal{R}\big(u^*_{S(\xi)}(\mu)\big) \subset X$ is closed.

Proof. We can proceed in the same fashion as in the proof of Lemma 6.5.2, although we here need to use the function space $BC_{\alpha_S}\big([r_{\min}, \xi + r_{\max}], \mathbb{C}^n\big)$. To see that $\mathcal{R}\big(u^*_{S(\xi)}(\mu)\big)$ is closed, consider a sequence $\phi^j \in S(\xi)$, write $\psi^j = u^*_{S(\xi)}(\mu)\phi^j$ and assume that $\psi^j \to \psi_*$. Since $\Pi_{S(\xi)}\psi^j = \phi^j$, we also have $\phi^j \to \Pi_{S(\xi)}\psi_* := \phi_*$. Since $u^*_{S(\xi)}(\mu)$ is bounded, we have $u^*_{S(\xi)}(\mu)[\phi^j - \phi_*] \to 0$ and hence $\psi_* = u^*_{S(\xi)}(\mu)\phi_*$. □

Proof of Theorem 6.5.1. We first establish the splitting $X = Q(\xi, \mu) \oplus S(\xi, \mu)$. To this end, consider the family of maps $U^*_\xi : U' \to \mathcal{L}\big(Q(\xi) \oplus S(\xi)\big)$ defined by

$$U^*_\xi(\mu)(\phi, \psi) = \big(\Pi_{Q(\xi)}\big[u^*_{Q(\xi)}(\mu)\phi + u^*_{S(\xi)}(\mu)\psi\big], \Pi_{S(\xi)}\big[u^*_{Q(\xi)}(\mu)\phi + u^*_{S(\xi)}(\mu)\psi\big]\big). \tag{6.116}$$

Since $\Pi_{Q(\xi)}u^*_{S(\xi)} \to 0$ as $\mu \to \mu_0$ and similarly $\Pi_{S(\xi)}u^*_{Q(\xi)} \to 0$, uniformly for $\xi \geq 0$, we find that by choosing the neighbourhood U' small enough, we can ensure that $U^*_\xi(\mu)$ is invertible for all $\mu \in U'$ and all $\xi \geq 0$, with a bound on the inverse and the first k derivatives of this inverse with respect to μ that is uniform for $\mu \in U$ and $\xi \geq 0$. This allows us to define the projections

$$\begin{array}{rcl} \Pi_{S(\xi,\mu)} & = & u^*_{S(\xi)}(\mu)\Pi_{S(\xi)}[U^*_\xi(\mu)]^{-1}, \\ \Pi_{Q(\xi,\mu)} & = & u^*_{Q(\xi)}(\mu)\Pi_{Q(\xi)}[U^*_\xi(\mu)]^{-1}. \end{array} \tag{6.117}$$

It is easy to see that indeed $\Pi^2_{Q(\xi,\mu)} = \Pi_{Q(\xi,\mu)}$ and similarly $\Pi^2_{S(\xi,\mu)} = \Pi_{S(\xi,\mu)}$. Also $\Pi_{Q(\xi,\mu)} + \Pi_{S(\xi,\mu)} = I$. These functions $\mu \mapsto \Pi_{Q(\xi,\mu)}$ and $\mu \mapsto \Pi_{S(\xi,\mu)}$ are C^k-smooth as functions $U' \to \mathcal{L}(X)$, which follows from the C^k-smoothness of $u^*_{Q(\xi)}$, $u^*_{S(\xi)}$ and U^*_ξ. In addition, since we have estimates on the first k derivatives of these functions with respect to μ, that are uniform for $\mu \in U'$ and $\xi \geq 0$, the same holds for the derivatives of the projections. The exponential estimates (6.104) now follow from (6.105) and (6.115). □

Throughout the remainder of this section we will consider the limiting behaviour of the projections $\Pi_{S(\xi,\mu)}$ and $\Pi_{Q(\xi,\mu)}$ as $\xi \to \infty$. The next result describes the speed at which these projections approach their limiting values $\Pi_{P(\infty)}$ and $\Pi_{Q(\infty)}$.

Theorem 6.5.4. *Consider the linear system (6.102) and suppose that for some $\alpha_- < 0$, the characteristic equation* $\det \Delta^+(z) = 0$ *has no roots in the strip $\alpha_- \leq \operatorname{Re} z \leq 0$, where Δ^+ is the characteristic matrix associated to the limiting system at $+\infty$. Suppose furthermore that for some $\alpha^f_- \leq \alpha_-$, all $\xi \in \mathbb{R}$, all $\mu \in U'$ and some constant $C > 0$ we have the bound*

$$\big\|L(\xi, \mu) - L^+\big\|_{\mathcal{L}(X,\mathbb{C}^n)} \leq C\big[\,|\mu - \mu_0|\, e^{\alpha_-\xi} + e^{\alpha^f_-\xi}\big], \tag{6.118}$$

in which L^+ denotes the linear operator (6.29) associated to the limiting system at $+\infty$. Then there exists a constant $K > 0$, such that the following bound holds for all $\xi \geq 0$,

$$\big\|\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty)}\big\| \leq K\big[\,|\mu - \mu_0|\, e^{\alpha_-\xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha^f_-\xi}\big]. \tag{6.119}$$

In addition, suppose that for some $C > 0$ and all integers $0 \leq \ell \leq k$, we have

$$\Big\|D^\ell[L(\xi, \mu) - L^+]\Big\|_{\mathcal{L}(X,\mathbb{C}^n)} \leq C\big[\,|\mu - \mu_0|\, e^{\alpha_-\xi} + e^{\alpha^f_-\xi}\big]. \tag{6.120}$$

Then there exists a constant $K > 0$, such that for all integers $0 \leq \ell \leq k$ and all $\xi \geq 0$, we have the bound

$$\Big\|D^\ell[\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty)}]\Big\| \leq K\big[\,|\mu - \mu_0|\, e^{\alpha_-\xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha^f_-\xi}\big]. \tag{6.121}$$

Our approach towards proving these bounds, will be to provide sharper versions of the results previously established in Lemma 6.4.5. We will need to study the quantity $\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty)}$ separately from its derivatives $D^{\ell}\Pi_{Q(\xi,\mu)}$ for $1 \le \ell \le k$.

Lemma 6.5.5. *Consider the setting of Theorem 6.5.4 and suppose that (6.118) holds. Then there exists a constant K_1 such that*

$$[I - \Pi_{P(\infty)}]_{|S(\xi,\mu)} \le K_1\big[\,|\mu - \mu_0|\, e^{\alpha_- \xi} + e^{\alpha_-^f \xi} + e^{(\alpha_- - \alpha_S)\xi}\big], \tag{6.122}$$

for all $\mu \in U'$ and $\xi \ge 0$.

Proof. Consider a $\phi \in S(\xi, \mu)$. We recall the sequences ϕ^j and y^j of C^1-smooth functions that were introduced in the proof of Lemma 6.4.5, with $\phi^j \to \phi$ as $j \to \infty$ and $\mathrm{ev}_\xi y^j = \phi^j$. We will give a detailed estimate of the quantity $\mathrm{ev}_\xi z^j$ defined by

$$z^j = \Lambda_\infty^{-1}[I - H_\xi]\Lambda_\infty y^j. \tag{6.123}$$

To this end, we recall the Greens function G from Theorem 6.3.1 that satisfies $G(\xi) \le K_2 e^{\alpha_- \xi}$ for $\xi \ge 0$ and allows us to write

$$\Lambda_\infty^{-1} f = \int_{-\infty}^{\infty} G(\xi - s) f(s) ds. \tag{6.124}$$

Using this representation, we introduce the shorthands $z = z^j$ and $y = y^j$ and calculate

$$\begin{array}{rcl} z(\xi) & = & \int_{-\infty}^{0} G(\xi - s)[\Lambda_\infty y](s) + \int_0^\xi G(\xi - s)[\Lambda_\infty y](s) ds \\ & \le & K_3 e^{-\alpha_S \xi} \, \|\phi\| \int_{-\infty}^{0} e^{\alpha_-(\xi - s)} ds + \int_0^\xi e^{\alpha_-(\xi - s)} \left\| L(\xi, \mu) - L^+ \right\| \mathrm{ev}_s y ds \\ & \le & K_4 e^{-\alpha_S \xi} e^{\alpha_- \xi} \, \|\phi\| \\ & & \quad + K_5 \, \|\phi\| \int_0^\xi e^{\alpha_-(\xi - s)} [|\mu - \mu_0|\, e^{\alpha_- s} + e^{\alpha_-^f s}] e^{-\alpha_S(\xi - s)} ds \\ & \le & K_4 e^{(\alpha_- - \alpha_S)\xi} \, \|\phi\| \\ & & \quad + K_6 \, \|\phi\| \, e^{(\alpha_- - \alpha_S)\xi} \big[\,|\mu - \mu_0|\, [e^{\alpha_S \xi} - 1] + [e^{(\alpha_S + \alpha_-^f - \alpha_-)\xi} - 1]\big]. \end{array} \tag{6.125}$$

Similar estimates for $z(\xi + \theta)$, with $r_{\min} \le \theta \le r_{\max}$, complete the proof. □

Lemma 6.5.6. *Consider the setting of Theorem 6.5.4 and suppose that (6.118) holds. Then there exists a constant $K_1 > 0$ such that*

$$[I - \Pi_{Q(\infty)}]_{|Q(\xi,\mu)} \le K_1\big[\,|\mu - \mu_0|\, e^{\alpha_- \xi} + e^{\alpha_-^f \xi}\big] \tag{6.126}$$

for all $\mu \in U'$ and all $\xi \ge 0$.

Proof. Consider a similar setup as in the proof of Lemma 6.5.5, now with $y^j \in \mathcal{Q}(\xi, \mu)$. This time, we need to estimate the quantity $\mathrm{ev}_\xi z^j$, with

$$z^j = \Lambda_\infty^{-1} H_\xi \Lambda_\infty y^j. \tag{6.127}$$

For all $\xi' \ge \xi$ we compute

$$[\Lambda_\infty y^k](\xi') = [\Lambda(\mu) y^j](\xi') + [L(\xi', \mu) - L^+] y^j_{\xi'} = [L(\xi', \mu) - L^+] y^j_{\xi'}, \tag{6.128}$$

since $y^j \in \mathcal{Q}(\xi, \mu)$. The estimate is now immediate. □

Lemma 6.5.7. *Consider the setting of Theorem 6.5.4 and suppose that (6.120) holds. Then there exists a constant $K_1 > 0$, such that for all integers $1 \leq \ell \leq k$, we have*

$$\left\| D^\ell u^*_{S(\xi)} \right\| \leq K_1 \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi} + e^{\alpha_-^f \xi} + e^{(\alpha_- - \alpha_S)\xi} \big], \tag{6.129}$$

for all $\mu \in U'$ and $\xi \geq 0$.

Proof. We will be interested in the derivatives of $u^*_{S(\xi)}(\mu)$ with respect to μ. To get the estimates we need, we will work in the space $BC_{\alpha_-}\big([r_{\min}, \xi + r_{\max}], \mathbb{C}^n\big)$. By contrast, our construction of $u^*_{S(\xi)}$ involved the function space $BC_{\alpha_S}\big([r_{\min}, \xi + r_{\max}], \mathbb{C}^n\big)$. Indeed, let us recall the family of operators $\mathcal{G}(\mu)$ defined by

$$\mathcal{G}(\mu)u = \Lambda^{\mathrm{qinv}}_{(\alpha_S)}(\mu_0)[L(\mu) - L(\mu_0)]u - E\Pi_{S(\xi)}\mathrm{ev}_\xi \Lambda^{\mathrm{qinv}}_{(\alpha_S)}(\mu_0)[L(\mu) - L(\mu_0)]u. \tag{6.130}$$

For our purposes here we wish to consider $\mathcal{G}(\mu)$ as an operator in $\mathcal{L}\big(BC_{\alpha_-}([r_{\min}, \xi + r_{\max}], \mathbb{C}^n)\big)$. The key point is that there are no roots of the characteristic equation $\det \Delta^+(z) = 0$ in the strip $\alpha_- \leq \operatorname{Re} z \leq \alpha_S$, from which we conclude that $\Lambda^{\mathrm{qinv}}_{(\alpha_S)}$ and $\Lambda^{\mathrm{qinv}}_{(\alpha_-)}$ agree on the space $BC_{\alpha_-}\big([r_{\min}, \infty), \mathbb{C}^n\big)$. The norm of the extension operator E in the space $\mathcal{L}\big(S(\xi), BC_{\alpha_-}([r_{\min}, \xi + r_{\max}], \mathbb{C}^n)\big)$ can now be bounded by

$$\|E\| \leq K_2 e^{-\alpha_- \xi}. \tag{6.131}$$

In addition, the norm of the evaluation operator ev_ξ in the space $\mathcal{L}\big(BC_{\alpha_-}([r_{\min}, \xi + r_{\max}], \mathbb{C}^n), X\big)$ can be bounded by

$$\left\| \mathrm{ev}_{\xi'} \right\| \leq K_3 e^{\alpha_- \xi'}. \tag{6.132}$$

This means that we can again bound the family $\mathcal{G}(\mu)$ uniformly for $\mu \in U'$ and $\xi \geq 0$, with a norm that goes to zero as $\mu \to \mu_0$. Now, recall that $v^*_{S(\xi)}(\mu) = [I - \mathcal{G}(\mu)]^{-1}E$. In particular, for any integer $1 \leq \ell \leq k$, this allows us to compute

$$\begin{array}{rcl} D^\ell v^*_{S(\xi)}(\mu) & = & \sum_{(f_1, \ldots f_q)} c_{(f_1, \ldots, f_q)} [I - \mathcal{G}(\mu)]^{-1} [D^{f_1}\mathcal{G}] [I - \mathcal{G}(\mu)]^{-1} \\ & & \quad \ldots [I - \mathcal{G}(\mu)]^{-1} [D^{f_q}\mathcal{G}] [I - \mathcal{G}(\mu)]^{-1} E \\ & = & \sum_{(f_1, \ldots f_q)} c_{(f_1, \ldots, f_q)} [I - \mathcal{G}(\mu)]^{-1} [D^{f_1}\mathcal{G}] [I - \mathcal{G}(\mu)]^{-1} \\ & & \quad \ldots [I - \mathcal{G}(\mu)]^{-1} [D^{f_q}\mathcal{G}] v^*_{S(\xi)}, \end{array} \tag{6.133}$$

in which the sum is taken over tuples $(f_1, \ldots, f_q)$ with $f_i \geq 1$ and $f_1 + \ldots + f_q = \ell$. We will focus on the last part of this expression, namely $[D^{f_q}\mathcal{G}]v^*_{S(\xi)}$ and consider this as a linear operator from $S(\xi)$ into $BC_{\alpha_-}([r_{\min}, \xi + r_{\max}], \mathbb{C}^n)$. In particular, we compute

$$\begin{array}{rcl} e^{-\alpha_- \xi'} \left| \big[[D^{f_q}\mathcal{G}] v^*_{S(\xi)} \phi \big](\xi') \right| & \leq & K_4 e^{-\alpha_- \xi'} \left| [D^{f_q} L(\xi', \mu)] \mathrm{ev}_{\xi'} v^*_{S(\xi)}(\mu) \phi \right| \\ & \leq & K_5 e^{-\alpha_- \xi'} \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi'} + e^{\alpha_-^f \xi'} \big] \\ & & e^{-\alpha_S(\xi - \xi')} \, \|\phi\| \, , \end{array} \tag{6.134}$$

which, after taking the supremum over $0 \le \xi' \le \xi + r_{\max}$, implies that

$$\left\| [D^{f_q} L(\xi', \mu)] u^*_{S(\xi)} \phi \right\|_{\alpha_-} \le K_6 \big[|\mu - \mu_0| + e^{-\alpha_S \xi} + e^{(\alpha_-^f - \alpha_-)\xi} \big] \, \|\phi\| \, . \tag{6.135}$$

Upon using the uniform bounds of the operators $[I - \mathcal{G}(\mu)]^{-1}$ and $D^{f_i} \mathcal{G}(\mu)$ acting on the space $BC_{\alpha_-}\big([r_{\min}, \xi + r_{\max}], \mathbb{C}^n\big)$, we finally find the bound

$$\left\| D^\ell u^*_{S(\xi)}(\mu) \right\| \le K_1 \big[|\mu - \mu_0| \, e^{\alpha_- \xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha_-^f \xi} \big], \tag{6.136}$$

which holds for all $\mu \in U'$ and all $\xi \ge 0$, as desired. □

Lemma 6.5.8. *Consider the setting of Theorem 6.5.4 and suppose that (6.120) holds. Then there exists a constant $K_1 > 0$, such that for all integers $1 \le \ell \le k$, we have*

$$\left\| D^\ell u^*_{Q(\xi)} \right\| \le K_1 \big[|\mu - \mu_0| \, e^{\alpha_- \xi} + e^{\alpha_-^f \xi} \big], \tag{6.137}$$

for all $\mu \in U'$ and $\xi \ge 0$.

Proof. We can proceed much as in the proof of Lemma 6.5.7, but now we may work directly in the space $BC_{\alpha_-}\big([\xi + r_{\min}, \infty), \mathbb{C}^n\big)$, which simplifies the analysis considerably. □

Theorem 6.5.4 now follows easily from the results above by invoking the identity (6.101). To conclude this section, we show how we can isolate the part of $S(\xi, \mu)$ that decays at the rate of the leading positive eigenvalue of the characteristic matrix Δ^+. To this end, consider any $\nu > 0$ such that the characteristic equation $\det \Delta^+(z) = 0$ has no roots with $\operatorname{Re} z = \nu$. This allows us to perform the spectral decomposition

$$X = P_\nu(\infty) \oplus Q(\infty) \oplus \Gamma_{0,\nu}, \tag{6.138}$$

in which $\Gamma_{0,\nu}$ is the finite dimensional generalized eigenspace associated to the roots of $\det \Delta^+(z) = 0$ that have $0 < \operatorname{Re} z < \nu$. By nature of the spectral projection, we have the identity $\Pi_{\Gamma_{0,\nu}} + \Pi_{P_\nu(\infty)} = \Pi_{P(\infty)}$.

Let us now introduce the operator $U_\xi \in \mathcal{L}\big(P_\nu(\infty) \oplus \Gamma_{0,\nu} \oplus Q(\infty)\big)$, that is defined by

$$(\psi, \gamma, \phi) \quad \mapsto \quad [\Pi_{P_\nu(\infty)} \oplus \Pi_{\Gamma_{0,\nu}} \oplus Q(\infty)][\Pi_{S_\nu(\xi,\mu)} \psi + \Pi_{S(\xi,\mu)} \gamma + \Pi_{Q(\xi,\mu)} \phi]. \tag{6.139}$$

Here we have introduced the space $S_\nu(\xi, \mu)$, that should be seen as the analogue of $S(\xi, \mu)$ after application of an exponential shift $e_{-\nu}$ to the system (6.102). We claim that U_ξ is close to the identity for ξ large enough. To see this, we compute

$$\begin{array}{rcl} \Pi_{P_\nu(\infty)} U_\xi(\psi, \gamma, \phi) & = & \psi + \Pi_{P_\nu(\infty)}[\Pi_{S_\nu(\xi,\mu)} - \Pi_{P_\nu(\infty)}]\psi \\ & & + \, \Pi_{P_\nu(\infty)}[\Pi_{S(\xi,\mu)} - \Pi_{P(\infty)}]\gamma \\ & & + \, \Pi_{P_\nu(\infty)}[\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty)}]\phi. \end{array} \tag{6.140}$$

Similar estimates for the other projections complete the proof of the claim. This allows us to obtain the following splitting, for all sufficiently large ξ,

$$X = S^f(\xi, \mu) \oplus S^s(\xi, \mu) \oplus Q(\xi, \mu), \tag{6.141}$$

in which we have $\Pi_{S^f(\xi,\mu)} + \Pi_{S^s(\xi,\mu)} - \Pi_{P(\infty)} \to 0$ as $\xi \to \infty$. In addition, we have the identities

$$\begin{array}{rcl} S^f(\xi,\mu) & = & \Pi_{S_\nu(\xi,\mu)}\big(P_\nu(\infty)\big), \\ S^s(\xi,\mu) & = & \Pi_{S(\xi,\mu)}\big(\Gamma_{0,\nu}\big). \end{array} \tag{6.142}$$

6.6. Lin's Method for MFDEs

Now that the necessary machinery for linear systems has been developed, we are ready to consider the nonlinear functional differential equation of mixed type,

$$x'(\xi) = G(x_\xi, \mu) \tag{6.143}$$

and study bifurcations from heteroclinic connections. Our approach in this section was strongly inspired by the presentation in [134], but the notation here will differ somewhat. This is primarily due to the fact that we have to adapt the framework developed by Sandstede to an infinite dimensional setting and need to avoid the use of a variation-of-constants formula.

To set the stage, let q be a heteroclinic solution to (6.143) at some parameter $\mu = \mu_0$, that connects the two equilibria $q_\pm \in \mathbb{C}^n$. We set out to find solutions to (6.143) that remain close to q, for parameters that have $\mu \approx \mu_0$. We therefore write $x = q + u$ and find the variational equation

$$\begin{array}{rcl} u'(\xi) & = & G(q_\xi + u_\xi, \mu) - q'(\xi) \\ & = & G(q_\xi + u_\xi, \mu) - G(q_\xi, \mu_0) \\ & = & [G(q_\xi + u_\xi, \mu) - G(q_\xi, \mu_0) - D_1G(q_\xi, \mu_0)u_\xi - D_2G(q_\xi, \mu_0)(\mu - \mu_0)] \\ & & \; + D_1G(q_\xi, \mu_0)u_\xi + D_2G(q_\xi, \mu_0)(\mu - \mu_0) \\ & = & \mathcal{N}(\xi, u_\xi, \mu) + D_1G(q_\xi, \mu_0)u_\xi + D_2G(q_\xi, \mu_0)(\mu - \mu_0), \end{array} \tag{6.144}$$

in which the nonlinearity $\mathcal{N}$ is given explicitly by

$$\mathcal{N}(\xi, \phi, \mu) = G(q_\xi + \phi, \mu) - G(q_\xi, \mu_0) - D_1G(q_\xi, \mu_0)\phi - D_2G(q_\xi, \mu_0)(\mu - \mu_0). \tag{6.145}$$

Throughout this entire section we will assume that the conditions (HG), (HL) and (HB) are satisfied. We therefore obtain the bound $\mathcal{N}(\xi, \phi, \mu) = O\big((|\mu - \mu_0| + \|\phi\|)^2\big)$ as $\mu \to \mu_0$ and $\phi \to 0$. This estimate holds uniformly for all $\xi \in \mathbb{R}$, due to the fact that the heteroclinic connection q can be uniformly bounded.

We write Λ for the operator (6.6) associated to the linear part of (6.144), i.e., for $u \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ we have

$$[\Lambda u](\xi) = u'(\xi) - D_1G(q_\xi, \mu_0)u_\xi. \tag{6.146}$$

Throughout the sequel, we use the following splitting of the state space X, that is associated to the linearization (6.146),

$$X = \widehat{P}(0) \oplus \widehat{Q}(0) \oplus B(0) \oplus \Gamma(0). \tag{6.147}$$

We pick two constants $\alpha_- < 0 < \alpha_+$ in such a way that the characteristic equations $\det \Delta^{\pm}(z) = 0$ associated to (6.146) have no roots in the strip $\alpha_- \le \operatorname{Re} z \le \alpha_+$. To ease the notation throughout this section, we now introduce the shorthands

$$\begin{array}{rclrcl} BC^{+}_{\alpha_-} & = & BC_{\alpha_-}\big([0,\infty),\mathbb{C}^n\big), & BC^{-}_{\alpha_+} & = & BC_{\alpha_+}\big((-\infty,0],\mathbb{C}^n\big), \\ BC^{\oplus}_{\alpha_-} & = & BC_{\alpha_-}\big([r_{\min},\infty),\mathbb{C}^n\big), & BC^{\ominus}_{\alpha_+} & = & BC_{\alpha_+}\big((-\infty,r_{\max}],\mathbb{C}^n\big). \end{array} \tag{6.148}$$

We recall the inverses for Λ on half-lines that were constructed in (6.61). In particular, we will use the appropriately defined inverses $\Lambda_+^{-1} = \Lambda^{\mathrm{qinv}}_{(\alpha_-)}(\mu_0)$ to ensure that for any $f \in BC^{+}_{\alpha_-}$ we can find $x \in BC^{\oplus}_{\alpha_-}$ with $\Lambda x = f$ on $[0,\infty)$, with the analogous properties for $\Lambda_-^{-1} = \Lambda^{\mathrm{qinv}}_{(\alpha_+)}(\mu_0)$.

Lemma 6.6.1. *Consider the linearization (6.146). For every pair of functions (g^-, g^+) that has $g^- \in BC^{-}_{\alpha_+}$ and $g^+ \in BC^{+}_{\alpha_-}$, there exists a unique pair $(u^-, u^+) = L_1(g^-, g^+)$, with $u^- \in BC^{\ominus}_{\alpha_+}$ and $u^+ \in BC^{\oplus}_{\alpha_-}$, such that the following properties hold.*

(i) We have the identities

$$\begin{array}{rclll} [\Lambda u^-](\xi') & = & g^-(\xi') & \text{for all} & \xi' \le 0, \\ [\Lambda u^+](\xi') & = & g^+(\xi') & \text{for all} & \xi' \ge 0. \end{array} \tag{6.149}$$

(ii) We have $\mathrm{ev}_0 u^- \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$ *and similarly* $\mathrm{ev}_0 u^+ \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$.

(iii) We have $\mathrm{ev}_0[u^- - u^+] \in \Gamma(0)$*, with*

$$\langle \mathrm{ev}_0^* d, \mathrm{ev}_0[u^- - u^+]\rangle_0 = \int_{-\infty}^{0} d(\xi')^* g^-(\xi') d\xi' + \int_0^{\infty} d(\xi')^* g^+(\xi') d\xi', \tag{6.150}$$

for any $d \in \mathcal{K}(\Lambda^*)$.

The linear map L_1 is bounded as a map from $BC^{-}_{\alpha_+} \times BC^{+}_{\alpha_-}$ into $BC^{\ominus}_{\alpha_+} \times BC^{\oplus}_{\alpha_-}$.

Proof. One may easily check that the choice

$$\begin{array}{rcl} u^- & = & \Lambda_-^{-1} g^- - E\Pi_{B(0)} \mathrm{ev}_0 \Lambda_-^{-1} g^- + E\Pi_{\widehat{P}(0)}[\Lambda_+^{-1} g^+ - \Lambda_-^{-1} g^-], \\ u^+ & = & \Lambda_+^{-1} g^+ - E\Pi_{B(0)} \mathrm{ev}_0 \Lambda_+^{-1} g^+ + E\Pi_{\widehat{Q}(0)}[\Lambda_-^{-1} g^- - \Lambda_+^{-1} g^+], \end{array} \tag{6.151}$$

ensures that all the required properties hold, using the identity (6.70) to verify (iii). □

Proof of Proposition 6.2.1. In order to find the functions $u^-(\mu)$ and $u^+(\mu)$ that satisfy the properties stated in Proposition 6.2.1, it suffices to solve the nonlinear fixed point problem

$$(u^-, u^+) = L_1\big(\mathcal{N}(u^-, \mu) + D_2 G(q, \mu_0)(\mu - \mu_0), \mathcal{N}(u^+, \mu) + D_2 G(q, \mu_0)(\mu - \mu_0)\big). \tag{6.152}$$

Here the maps $\mathcal{N}$ and $D_2 G$ should be viewed as substitution operators, i.e., for any $\xi' \ge 0$ we have $\mathcal{N}(u^+, \mu)(\xi') = \mathcal{N}(\xi', \mathrm{ev}_{\xi'} u^+, \mu)$, together with similar identities for $D_2 G(q, \mu_0)$ and $\mathcal{N}(u^-, \mu)$. By construction we have that $(0, 0)$ is a solution to this problem at $\mu = \mu_0$.

The definition of $\mathcal{N}$ in (6.145) ensures that, by taking μ sufficiently close to μ_0 and by restricting u^+ to a small ball in $BC^{\oplus}_{\alpha_-}$, we may achieve

$$\big\| [D_2\mathcal{N}](\xi, \mathrm{ev}_\xi u^+, \mu) \big\| \le C[\big\| \mathrm{ev}_\xi u^+ \big\| + |\mu - \mu_0|] \qquad (6.153)$$

for all $\xi \ge 0$. Now consider the ball $B_\delta(0) \subset BC^{\ominus}_{\alpha_+} \times BC^{\oplus}_{\alpha_-}$ around the pair $(0, 0)$ that has radius $\delta > 0$. Choosing δ sufficiently small, (6.153) implies that the right hand side of (6.152) is a contraction on $B_\delta(0)$. In addition, choosing a sufficiently small neighbourhood $U' \subset U$ ensures that the right hand side of (6.152) maps $B_\delta(0)$ into itself. Together with the implicit function theorem, these observations show that for each $\mu \in U'$, equation (6.152) has a unique solution in $B_\delta(0)$, that depends C^{k+1}- smoothly on μ. We lose one order of smoothness here, due to the fact that the substitution operator $\mathcal{N}$ is only C^{k+1}-smooth [134]. □

We now proceed towards establishing Theorem 6.2.2. In order to meet the boundary conditions in item (iii), we will need to insert $x^\pm = q(\xi) + u^\pm(\mu)(\xi) + v^\pm(\xi)$ into the nonlinear equation (6.143). We find that $v^\pm$ must solve the equations

$$\begin{array}{rclr} [D_\xi v^-](\xi) &=& \mathcal{M}^-(\xi, \mathrm{ev}_\xi v^-, \mu) + D_1 G(q_\xi + \mathrm{ev}_\xi u^-(\mu), \mu)\mathrm{ev}_\xi v^-, & \xi \le 0, \\ {[D_\xi v^+](\xi)} &=& \mathcal{M}^+(\xi, \mathrm{ev}_\xi v^+, \mu) + D_1 G(q_\xi + \mathrm{ev}_\xi u^+(\mu), \mu)\mathrm{ev}_\xi v^+, & \xi \ge 0, \end{array} \qquad (6.154)$$

in which the nonlinearities $\mathcal{M}^\pm$ are given by

$$\begin{array}{rcl} \mathcal{M}^\pm(\xi, \phi, \mu) &=& G\big(q_\xi + \mathrm{ev}_\xi u^\pm(\mu) + \phi, \mu\big) - D_1 G\big(q_\xi + \mathrm{ev}_\xi u^\pm(\mu), \mu\big)\phi \\ && - G(q_\xi + \mathrm{ev}_\xi u^\pm, \mu). \end{array} \qquad (6.155)$$

Let us write $\Lambda(\mu)$ for the operator (6.6) associated to the inhomogeneous linearization

$$v'(\xi) = D_1 G\big(q_\xi + \widetilde{\mathrm{ev}}_\xi u(\mu), \mu\big) v_\xi + f(\xi), \qquad (6.156)$$

in which we have $\widetilde{\mathrm{ev}}_\xi u(\mu) = \mathrm{ev}_\xi u^+(\mu)$ for $\xi \ge 0$ and $\widetilde{\mathrm{ev}}_\xi u(\mu) = \mathrm{ev}_\xi u^-(\mu)$ for $\xi \le 0$. We remark here that the matrix-valued functions $A_j(\xi, \mu)$ associated to (6.156) that were introduced in (6.45) are no longer continuous at $\xi = 0$ for $\mu \ne \mu_0$, but this will not matter for our purposes here. For convenience, we introduce the following shorthands for $\omega_+ > 0$ and $\omega_- < 0$,

$$\begin{array}{rclrcl} C^+_{(\omega^+)} &=& C\big([0, \omega^+], \mathbb{C}^n\big), & C^-_{(\omega^-)} &=& C\big([\omega^-, 0], \mathbb{C}^n\big), \\ C^{\oplus}_{(\omega^+)} &=& C\big([r_{\min}, \omega^+ + r_{\max}], \mathbb{C}^n\big), & C^{\ominus}_{(\omega^-)} &=& C\big([\omega^- + r_{\min}, r_{\max}], \mathbb{C}^n\big). \end{array} \qquad (6.157)$$

We also recall the splitting $X = Q(\xi, \mu) \oplus S(\xi, \mu)$ that holds for all $\xi \ge 0$. Similarly, for all $\xi \le 0$ we will use the splitting $X = P(\xi, \mu) \oplus R(\xi, \mu)$. Here we have introduced the spaces $R(\xi, \mu)$, that should be seen as the natural counterparts of $S(\xi, \mu)$ on the negative half-line.

Lemma 6.6.2. *Consider the parameter-dependent inhomogeneous linear system (6.156). Then there exists a neighbourhood $U' \subset U$, with $\mu_0 \in U'$ and a constant $\Omega > 0$, such that for every $\mu \in U'$, every pair $\omega^- < -\Omega < \Omega < \omega^+$, every pair $(g^-, g^+) \in C^-_{(\omega^-)} \times C^+_{(\omega^+)}$ and every pair $(\phi^-, \phi^+) \in Q(-\infty) \times P(\infty)$, there exists a unique pair $(v^-, v^+) \in C^{\ominus}_{(\omega^-)} \times C^{\oplus}_{(\omega^+)}$ that satisfies the following properties.*

(i) The functions $v^{\pm}$ satisfy the linear system

$$\begin{array}{llll} [\Lambda(\mu)v^-](\xi') = g^-(\xi') & \text{for all} & \omega^- \le \xi' \le 0, \\ [\Lambda(\mu)v^+](\xi') = g^+(\xi') & \text{for all} & 0 \le \xi' \le \omega^+. \end{array} \tag{6.158}$$

(ii) We have $\mathrm{ev}_0 v^-(\mu) \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$ *and similarly* $\mathrm{ev}_0 v^+(\mu) \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$.

(iii) The gap between v^- and v^+ at zero satisfies $\mathrm{ev}_0[v^-(\mu) - v^+(\mu)] \in \Gamma(0)$.

(iv) The functions $v^{\pm}$ satisfy the boundary conditions

$$\begin{array}{lll} \Pi_{Q(-\infty)}\mathrm{ev}_{\omega^-}v^- & = & \phi^-, \\ \Pi_{P(\infty)}\mathrm{ev}_{\omega^+}v^+ & = & \phi^+. \end{array} \tag{6.159}$$

This pair (v^-, v^+) will be denoted by

$$(v^-, v^+) = L_3\big(g^-, g^+, \phi^-, \phi^+, \mu, \omega^-, \omega^+\big), \tag{6.160}$$

in which L_3 is a linear operator with respect to the first four variables that depends C^{k+1}-smoothly on μ, with a norm that can be bounded independently of $\omega^{\pm}$.

In addition, consider any $d \in \mathcal{K}^*$ and write $d^+ = Eu^*_{Q^*(0)}\mathrm{ev}_0^* d$ and $d^- = Eu^*_{P^*(0)}\mathrm{ev}_0^* d$. Then the following identity holds for the gap at zero,

$$\begin{array}{lll} \langle \mathrm{ev}_0^* d^-, \mathrm{ev}_0 v^- \rangle_{0,\mu} & = & \langle \mathrm{ev}_0^* d^+, \mathrm{ev}_0 v^+ \rangle_{0,\mu} \\ & & + \langle \mathrm{ev}^*_{\omega^-} d^-, \mathrm{ev}_{\omega^-} v^- \rangle_{\omega^-,\mu} - \langle \mathrm{ev}^*_{\omega^+} d^+, \mathrm{ev}_{\omega^+} v^+ \rangle_{\omega^+,\mu} \\ & & + \int_{\omega^-}^0 d^-(\xi')^* g^-(\xi') d\xi' + \int_0^{\omega^+} d^+(\xi')^* g^+(\xi') d\xi'. \end{array} \tag{6.161}$$

Proof. We first define the functions $w^+ = \Lambda_+^{-1}(\mu)g^+$ and $w^- = \Lambda_-^{-1}(\mu)g^-$. In order to satisfy the conditions (ii) through (iv), we now set out to find $\psi^{B+} \in B(0)$, $\psi^{B-} \in B(0)$, $\psi^{\widehat{Q}} \in \widehat{Q}(0)$, $\psi^{\widehat{P}} \in \widehat{P}(0)$, $\psi^S \in P(\infty)$ and $\psi^R \in Q(-\infty)$ that satisfy the linear system

$$\begin{array}{lll} -\Pi_{B(0)} w_0^+ & = & \psi^{B+} + \Pi_{B(0)}\mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)}\psi^S, \\ -\Pi_{B(0)} w_0^- & = & \psi^{B-} + \Pi_{B(0)}\mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)}\psi^R, \\ -\Pi_{\widehat{Q}(0)}[w_0^- - w_0^+] & = & -\psi^{\widehat{Q}} + \Pi_{\widehat{Q}(0)} u^*_{P(0)}(\mu)[\psi^{\widehat{P}} + \psi^{B-}] \\ & & + \Pi_{\widehat{Q}(0)}\big[\mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)}\psi^R - \mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)}\psi^S\big], \\ -\Pi_{\widehat{P}(0)}[w_0^- - w_0^+] & = & \psi^{\widehat{P}} - \Pi_{\widehat{P}(0)} u^*_{Q(0)}(\mu)[\psi^{\widehat{Q}} + \psi^{B+}] \\ & & + \Pi_{\widehat{P}(0)}\big[\mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)}\psi^R - \mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)}\psi^S\big], \\ \Pi_{P(\infty)}[\phi^+ - w^+_{\omega^+}] & = & \psi^S + \Pi_{P(\infty)}\mathrm{ev}_{\omega^+} E u^*_{Q(0)}(\mu)[\psi^{\widehat{Q}} + \psi^{B+}] \\ & & + \Pi_{P(\infty)}[\Pi_{S(\omega^+,\mu)} - \Pi_{P(\infty)}]\psi^S, \\ \Pi_{Q(-\infty)}[\phi^- - w^-_{\omega^-}] & = & \psi^R + \Pi_{Q(-\infty)}\mathrm{ev}_{\omega^-} E u^*_{P(0)}(\mu)[\psi^{\widehat{P}} + \psi^{B-}] \\ & & + \Pi_{Q(-\infty)}[\Pi_{R(\omega^-,\mu)} - \Pi_{Q(-\infty)}]\psi^R. \end{array} \tag{6.162}$$

Then upon writing $\psi^Q = \psi^{\widehat{Q}} + \psi^{B+}$ and $\psi^P = \psi^{\widehat{P}} + \psi^{B-}$ and defining

$$\begin{array}{rcl} v^+ & = & w^+ + u^*_{Q(0)}(\mu)\psi^Q + E\Pi_{S(\omega^+,\mu)}\psi^S, \\ v^- & = & w^- + u^*_{P(0)}(\mu)\psi^P + E\Pi_{R(\omega^-,\mu)}\psi^R, \end{array} \tag{6.163}$$

we see that the properties (i) through (iv) are satisfied. The exponential estimates in Theorem 6.5.1, together with the results established in Lemma 6.5.2 and Theorem 6.5.4, ensure that by choosing a sufficiently small neighbourhood $U' \subset U$, with $\mu_0 \in U'$ and a sufficiently large constant $\Omega > 0$, the system (6.162) can always be solved. Moreover, the inverse of the linear operator associated to (6.162) depends C^{k+1}-smoothly on μ.

To verify the identity (6.161), it suffices to observe that for any continuous function d that satisfies $\Lambda(\mu)d = 0$ on the interval $[0, \xi]$, we have

$$\langle \mathrm{ev}_0^* d, \mathrm{ev}_0 x\rangle_{0,\mu} \quad = \quad \langle \mathrm{ev}_\xi^* d, \mathrm{ev}_\xi x\rangle_{\xi,\mu} + \int_\xi^0 d(\xi')^*[\Lambda(\mu)x](\xi')d\xi'. \tag{6.164}$$

To see the uniqueness of the pair (v^-, v^+) that has now been constructed, consider any continuous function $y \in C^{\oplus}_{(\omega^+)}$ that has $\Lambda(\mu)y = 0$ on $[0, \omega^+]$. Writing $z = E\Pi_{S(\omega^+,\mu)}\mathrm{ev}_{\omega^+}y$, we find that $\Lambda(\mu)(y - z) = 0$ on $[0, \omega^+]$, while $\mathrm{ev}_{\omega^+}[y - z] \in Q(\omega^+, \mu)$. This implies that $\mathrm{ev}_0[y - z] \in Q(0, \mu)$, which in turn means $y \in \mathcal{Q}(0, \mu) + \mathcal{S}(\omega^+, \mu)$, with some abuse of notation. It is thus sufficient to show that

$$\begin{array}{rcl} \mathcal{S}(\omega^+, \mu) & = & \Pi_{S(\omega^+,\mu)}\big(P(\infty)\big), \\ \mathcal{R}(\omega^-, \mu) & = & \Pi_{R(\omega^-,\mu)}\big(Q(-\infty)\big), \end{array} \tag{6.165}$$

but these identities follow directly from the discussion at the end of Section 6.5. □

We are now ready to consider a family of heteroclinic connections $\{q_j\}_{j\in\mathcal{J}}$ that connect the equilibria $\{q^*_\ell\}_{\ell\in\mathcal{J}^*}$, i.e.,

$$\lim_{\xi\to\pm\infty} q_j(\xi) = q^*_{j\pm\frac{1}{2}}. \tag{6.166}$$

For any $j \in \mathcal{J}$, let us write $\Lambda^{(j)}(\mu)$ for the linear operator (6.156) that is associated to the heteroclinic connection q_j.

Lemma 6.6.3. *Consider the nonlinear equation (6.143) and a family of heteroclinic connections $\{q_j\}_{j\in\mathcal{J}}$ that satisfy (6.166). Then there exists a neighbourhood $U' \subset U$, with $\mu_0 \in U'$ and a constant $\Omega > 0$, such that for every $\mu \in U'$, every family $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ that has $\omega_\ell > \Omega$ for all $\ell \in \mathcal{J}^*$, every family $\{g_j^-, g_j^+\}_{j\in\mathcal{J}}$ with $(g_j^-, g_j^+) \in C^-_{(\omega_j^-)} \times C^+_{(\omega_j^+)}$, and every family $\{\Phi_\ell\}_{\ell\in\mathcal{J}^*} \in X^{\mathcal{J}^*}$, there is a unique family $\{v_j^-, v_j^+\}_{j\in\mathcal{J}}$ with $(v_j^-, v_j^+) \in C^{\ominus}_{(\omega_j^-)} \times C^{\oplus}_{(\omega_j^+)}$, that satisfies the following properties.*

(i) For every $j \in \mathcal{J}$, the pair (v_j^-, v_j^+) solves the linear system

$$\begin{array}{lll} [\Lambda(\mu)v_j^-](\xi') = g_j^-(\xi') & \textit{for all} & \omega_j^- \le \xi' \le 0, \\ [\Lambda(\mu)v_j^+](\xi') = g_j^+(\xi') & \textit{for all} & 0 \le \xi' \le \omega_j^+. \end{array} \tag{6.167}$$

(ii) For every $j \in \mathcal{J}$, we have $\mathrm{ev}_0 v_j^- \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$ *and similarly* $\mathrm{ev}_0 v_j^+ \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$.

(iii) For every $j \in \mathcal{J}$, the gap between $v_j^\pm$ at zero satisfies $\mathrm{ev}_0[v_j^- - v_j^+] \in \Gamma(0)$.

(iv) For every $\ell \in \mathcal{J}^*$, we have the boundary condition

$$\mathrm{ev}_{\omega^+_{\ell-\frac{1}{2}}} v^+_{\ell-\frac{1}{2}} - \mathrm{ev}_{\omega^-_{\ell+\frac{1}{2}}} v^-_{\ell+\frac{1}{2}} = \Phi_\ell, \tag{6.168}$$

which should be interpreted in the sense of item (iii) in Theorem 6.2.2.

This family $\{v_j^-, v_j^+\}$ will be denoted by

$$\{v_j^-, v_j^+\} = L_4\big(\{g_j^-, g_j^+\}, \{\Phi_\ell\}, \mu, \{\omega_\ell\}\big), \tag{6.169}$$

in which L_4 is a linear operator with respect to the first two variables that depends C^{k+1}-smoothly on μ, with a norm that can be bounded independently of the family $\{\omega_\ell\}$.

Proof. It suffices to choose a family $\{\phi_j^-, \phi_j^+\}_{j\in\mathcal{J}}$, with $\phi_j^- \in Q^{(j)}(-\infty)$ and $\phi_j^+ \in P^{(j)}(\infty)$, such that the family of solutions defined by $(v_j^-, v_j^+) = L^3\big(g_j^-, g_j^+, \phi_j^-, \phi_j^+, \mu, \omega_j^-, \omega_j^+\big)$ satisfies the following boundary condition for every $\ell \in \mathcal{J}^*$,

$$\begin{array}{rcl} \Pi^{(\ell-\frac{1}{2})}_{P(\infty)}\big[\Phi_\ell + \mathrm{ev}_{\omega^-_{\ell+\frac{1}{2}}} L_3^-(g^-_{\ell+\frac{1}{2}}, g^+_{\ell+\frac{1}{2}}, 0, 0)\big] & = & \phi^+_{\ell-\frac{1}{2}} + K^+_{\ell-\frac{1}{2}}\big(\phi^-_{\ell+\frac{1}{2}}, \phi^+_{\ell+\frac{1}{2}}\big), \\ \Pi^{(\ell+\frac{1}{2})}_{Q(-\infty)}\big[\mathrm{ev}_{\omega^+_{\ell-\frac{1}{2}}} L_3^+(g^-_{\ell-\frac{1}{2}}, g^+_{\ell-\frac{1}{2}}, 0, 0) - \Phi_\ell\big] & = & \phi^-_{\ell+\frac{1}{2}} + K^-_{\ell+\frac{1}{2}}\big(\phi^-_{\ell-\frac{1}{2}}, \phi^+_{\ell-\frac{1}{2}}\big). \end{array} \tag{6.170}$$

Here we have introduced the obvious shorthand $L_3 = (L_3^-, L_3^+)$ and dropped the dependence of L_3 on μ and $\omega^\pm$. For any $j \in \mathcal{J}$ we can inspect (6.162) and obtain the bounds

$$\begin{array}{rcl} \big\| K_j^+ \big\| & \le & K_1 e^{\alpha_P \omega^-_{j+1}} + K_2 \left\| \Pi^{(j+1)}_{R(\omega^-_{j+1},\mu)} - \Pi^{(j+1)}_{Q(-\infty)} \right\|, \\ \big\| K_j^- \big\| & \le & K_1 e^{-\alpha_Q \omega^+_{j-1}} + K_2 \left\| \Pi^{(j-1)}_{S(\omega^+_{j-1},\mu)} - \Pi^{(j-1)}_{P(\infty)} \right\|, \end{array} \tag{6.171}$$

which ensures that the right hand side of (6.170) is close to the identity, for sufficiently large $\Omega > 0$ and a sufficiently small neighbourhood $U' \subset U$. □

With this result we are ready to establish the existence of the family $\{v_j^-, v_j^+\}_{j\in\mathcal{J}}$ that appears in Theorem 6.2.2. We will defer the proof of the estimates (6.19) to the next section.

Proof of Theorem 6.2.2. In order to find the family $\{v_j^-, v_j^+\}$, we will first fix the family $\{\omega_\ell\}$ and solve the fixed point problem

$$\{v_j^-, v_j^+\} = L_4\big(\{\mathcal{M}^-(v_j^-, \mu), \mathcal{M}^+(v_j^+, \mu)\}, \{\Phi_\ell\}, \mu, \{\omega_\ell\}\big). \tag{6.172}$$

First note that for some $C > 0$ we can make the estimate

$$\left\| D_2\mathcal{M}^+(\xi, \phi, \mu) \right\| \leq C \, \|\phi\| \, , \tag{6.173}$$

uniformly for $\xi \geq 0$ and $\mu \in U'$. This allows us to proceed as in the proof of Proposition 6.2.1, to obtain families $v_j^{\pm}\big(\{\Phi_\ell\}, \mu, \{\omega_\ell\}\big)$ that solve (6.172), for small values of $\{\Phi_\ell\}$ and μ sufficiently close to μ_0. Moreover, these families depend C^k-smoothly on $\{\Phi_\ell\}$ and μ, where we have again lost an order of smoothness due to our use of the substitution operators $\mathcal{M}^{\pm}$. Upon choosing a sufficiently large constant $\Omega > 0$ and subsequently using (6.16) to pick the appropriate (small) values for $\Phi_\ell = \Phi_\ell(\mu)$, the fixed point of (6.172) will satisfy the properties (i) through (iv) in Theorem 6.2.2. Since $\Phi_\ell(\mu)$ depends C^k-smoothly on μ, the fixed point of (6.172) will share this property.

It remains to consider the smoothness of the jumps with respect to the family $\{\omega_\ell\}$. Let us therefore fix a sufficiently large $\overline{\Omega} > \Omega$ and reconsider the setting of Lemma 6.6.2. Instead of looking for a pair $(v^-, v^+) \in C^{\ominus}_{(\omega^-)} \times C^{\oplus}_{(\omega^+)}$ that satisfies the properties (i) through (iv), we will look for a pair $(v^-, v^+) \in C^{\ominus}_{(-\overline{\Omega})} \times C^{\oplus}_{(\overline{\Omega})}$ that satisfies these properties, still with the original quantities $\omega^{\pm}$ that have $\left|\omega^{\pm}\right| < \overline{\Omega}$. In order to solve this modified problem, let us adapt the action of the extension operator E on the space $S(\omega^+, \mu)$, to ensure that $E\psi \in C^{\oplus}_{\overline{\Omega}}$ for $\psi \in S(\omega^+, \mu)$, with a similar modification for the space $R(\omega^-, \mu)$. The exact details are irrelevant, as long as we still have $\Lambda(\mu)E\psi = 0$ on the interval $[0, \omega^+]$. After this modification, it again suffices to solve the linear system (6.162). To see that $\Pi_{S(\omega^+,\mu)}$ depends smoothly on ω^+, we note that for any ω_* we can redefine the space $\mathcal{S}(\omega_*, \mu)$ so that it contains solutions to (6.102) on the slightly larger half-line $[-1, \omega_*]$. We can then obtain solutions to (6.102) on the interval $[0, \omega^+]$ with $\omega^+ = \omega_* + \Delta\omega$, by solving (6.102) with $A_j(\xi, \mu, \Delta\omega) = A_j(\xi + \Delta\omega, \mu)$ and shifting the resulting function to the right by $\Delta\omega$. This observation allows us to treat the parameter ω^+ on the same footing as μ. We emphasize that these modifications do not affect the pair (v^-, v^+) when viewed as functions in $C^{\oplus}_{(\omega^-)} \times C^{\ominus}_{(\omega^+)}$, due to the uniqueness result in Lemma 6.6.2. Applying similar modifications to Lemma 6.6.3 and the construction above now completes the proof, using the estimates for $\mathcal{R}_j$ that are obtained in the next section. □

6.7. The remainder term

Our goal in this section is to obtain estimates on the size of the remainder term $\mathcal{R}_j$ that features in (6.18). To set the stage, assume that for some $j \in \mathcal{J}$ we have $d \in \mathcal{K}\big((\Lambda^{(j)})^*\big)$ with $\left\|\mathrm{ev}_0^* d\right\| = 1$. We also recall the functions $d^+(\mu) \in \mathcal{Q}^*(0, \mu)$ and $d^-(\mu) \in \mathcal{P}^*(0, \mu)$ that are defined by

$$\begin{aligned} d^+(\mu) &= Eu^*_{Q^*(0)}(\mu)\mathrm{ev}_0^* d, \\ d^-(\mu) &= Eu^*_{P^*(0)}(\mu)\mathrm{ev}_0^* d. \end{aligned} \tag{6.174}$$

In this section, we will study the slightly modified remainder term $\widetilde{\mathcal{R}}_j$, that is given by

$$\begin{array}{rcl} \widetilde{\mathcal{R}}_j & = & \langle \mathrm{ev}_0^* d^-, \mathrm{ev}_0 v^- \rangle_{0,\mu} - \langle \mathrm{ev}_0^* d^+, \mathrm{ev}_0 v^+ \rangle_{0,\mu} \\ & & - \langle \mathrm{ev}^*_{\omega_j^+} d^+, \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu) - q^*_{j+\frac{1}{2}}] \rangle_{\omega_j^+,\mu} \\ & & + \langle \mathrm{ev}^*_{\omega_j^-} d^-, \mathrm{ev}_{\omega_{j-1}^+}[q_{j-1} + u_{j-1}^+(\mu) - q^*_{j-\frac{1}{2}}] \rangle_{\omega_j^-,\mu}. \end{array} \tag{6.175}$$

In the terminology of Theorem 6.2.2, we see that the difference satisfies $\left|\widetilde{\mathcal{R}_j} - \mathcal{R}_j\right| = O(|\mu - \mu_0| e^{-2\alpha\omega})$. To simplify our arguments, we introduce the following quantities that are associated to the boundary conditions in (6.16).

$$\begin{array}{rcl} \Theta_j^{\pm} & = & \mathrm{ev}_{\omega_j^{\pm}}[q_j + u_j^{\pm}(\mu) - q^*_{j\pm\frac{1}{2}}], \\ \Phi_j^+ & = & \Pi^{(j)}_{P(\infty)}\big(\mathrm{ev}_{\omega_j^+}[q_j + u_j^+(\mu)] - \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu)]\big), \\ \Phi_j^- & = & \Pi^{(j)}_{Q(-\infty)}\big(\mathrm{ev}_{\omega_j^-}[q_j + u_j^-(\mu)] - \mathrm{ev}_{\omega_{j-1}^+}[q_{j-1} + u_{j-1}^+(\mu)]\big). \end{array} \tag{6.176}$$

We also introduce the supremum norms $\|\Theta\| = \sup_{j\in\mathcal{J}} \left\|\Theta_j^{\pm}\right\|$ and similarly $\|\Phi\| = \sup_{j\in\mathcal{J}} \left\|\Phi_j^{\pm}\right\|$. In addition, we introduce the terms

$$\begin{array}{rcl} r_j^+ & = & \left\|\Pi_{S(\omega_j^+,\mu)} - \Pi_{P(\infty)}\right\| + \left\|D[\Pi_{S(\omega_j^+,\mu)} - \Pi_{P(\infty)}]\right\|, \\ r_j^- & = & \left\|\Pi_{R(\omega_j^-,\mu)} - \Pi_{Q(-\infty)}\right\| + \left\|D[\Pi_{R(\omega_j^-,\mu)} - \Pi_{Q(-\infty)}]\right\|, \end{array} \tag{6.177}$$

in which D denotes differentiation with respect to the pairs $(\omega^{\pm}, \mu)$. Our main focus will be to study the rate at which the error terms $\widetilde{R}_j$ decay as the quantities $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ tend towards infinity. In order to eliminate the need to keep track of constants, we introduce the notation

$$a\big(\mu, \{\omega_\ell\}\big) \leq_* b\big(\mu, \{\omega_\ell\}\big) \tag{6.178}$$

to indicate that there exists a constant $C > 0$ such that for all $\mu \in U'$ and families $\{\omega_\ell\}$ that have $\omega_\ell > \Omega$ for all $\ell \in \mathcal{J}^*$, we have the inequality

$$a\big(\mu, \{\omega_\ell\}\big) \leq C b\big(\mu, \{\omega_\ell\}\big). \tag{6.179}$$

As a final preparation, we will assume that for every $j \in \mathcal{J}$ we have obtained the splittings

$$\begin{array}{rcl} X & = & S^s(\omega_j^+, \mu) \oplus S^f(\omega_j^+, \mu) \oplus Q(\omega_j^+, \mu), \\ X & = & R^s(\omega_j^-, \mu) \oplus R^f(\omega_j^-, \mu) \oplus P(\omega_j^-, \mu), \end{array} \tag{6.180}$$

as introduced at the end of Section 6.5. We write α_S^f and α_R^f for the exponential rates associated to the fast spaces $S^f(\omega^+, \mu)$ and $R^f(\omega^-, \mu)$. In view of this more detailed splitting, we modify the definition of $v^{\pm}$ in (6.163) to make it read

$$\begin{array}{rcl} v^+ & = & w^+ + u^*_{Q(0)}(\mu)\phi^Q + E\Pi_{S^f(\omega^+,\mu)}\psi^S + E\Pi_{S^s(\omega^+,\mu)}\psi^S, \\ v^- & = & w^- + u^*_{P(0)}(\mu)\phi^P + E\Pi_{R^f(\omega^-,\mu)}\psi^S + E\Pi_{R^s(\omega^-,\mu)}\psi^R. \end{array} \tag{6.181}$$

Our first goal will be to fix a $j \in \mathcal{J}$, consider small boundary values ϕ^+ and ϕ^- and get estimates on the solution of the nonlinear fixed point problem

$$(v^-, v^+) = L_3\big(\mathcal{M}^-(v^-), \mathcal{M}^+(v^+), \phi^-, \phi^+, \mu, \omega^-, \omega^+\big), \tag{6.182}$$

in terms of ϕ^+ and ϕ^-. We proceed by introducing the notation $\mathbf{w}^- = (w^-, \psi^R, \psi^P)$, $\mathbf{w}^+ = (w^+, \psi^Q, \psi^S)$ and $\mathbf{w} = (w^-, w^+) \in \mathbf{W}$, where $\mathbf{W}$ denotes the space

$$\mathbf{W} = C^{\ominus}_{(\omega^-)} \times C^{\oplus}_{(\omega^+)} \times Q(-\infty) \times P(0) \times Q(0) \times P(\infty). \tag{6.183}$$

The problem (6.182) can now be written as

$$\mathbf{w} = [I-K]^{-1}\Big([J_0 B_0 + J_1 B_1 + J_2 B_2]\big(\mathcal{M}^-(\mathbf{w}^-), \mathcal{M}^+(\mathbf{w}^+)\big) + J_3(\phi^-, \phi^+)\Big), \tag{6.184}$$

in which the operators B_0, B_1 and B_2 act as

$$\begin{array}{rcl} B_0(g^-, g^+) & = & (\Lambda_-^{-1}(\mu) g^-, \Lambda_+^{-1}(\mu) g^+), \\ B_1(g^-, g^+) & = & \mathrm{ev}_0 B_0(g^-, g^+), \\ B_2(g^-, g^+) & = & \big(\mathrm{ev}_{\omega^-}, \mathrm{ev}_{\omega^+}\big) B_0(g^-, g^+), \end{array} \tag{6.185}$$

while the precise form of the operators $K \in \mathcal{L}(\mathbf{W})$ and J_i can be found by inspection of (6.162). Note that for any $\mathbf{b} \in \mathbf{W}$ we have the bound

$$\Big\| [I-K]^{-1}\mathbf{b} - [I+K]\mathbf{b} \Big\| \leq [I - \|K\|]^{-1} \|K\| \, \|K\mathbf{b}\| \, . \tag{6.186}$$

Now consider the first order estimate $\mathbf{w}_0 = [I-K]^{-1} J_3(\phi^-, \phi^+)$. Upon introducing the quantities

$$\begin{array}{rcl} T_0^+ & = & e^{-\alpha_S \omega^+} \left\| \Pi_{S^s(\omega^+, \mu)} \phi^+ \right\| + e^{-\alpha_S^f \omega^+} \left\| \phi^+ \right\|, \\ T_0^- & = & e^{\alpha_R \omega^-} \left\| \Pi_{R^s(\omega^-, \mu)} \phi^- \right\| + e^{\alpha_R^f \omega^-} \left\| \phi^- \right\|, \\ T_1^+ & = & r^+ e^{-\alpha_S \omega^+} \left\| \phi^+ \right\|, \\ T_1^- & = & r^- e^{\alpha_R \omega^-} \left\| \phi^- \right\|, \end{array} \tag{6.187}$$

together with $T_0 = T_0^- + T_0^+$ and $T_1 = T_1^- + T_1^+$, we find $\mathbf{w}_0 = (0, 0, \psi^R, \psi^P, \psi^Q, \psi^S)$, with

$$\begin{array}{lcl} \left\| \psi^R - \phi^- \right\| & \leq_* & r^- \left\| \phi^- \right\| + e^{\alpha_P \omega^-}[T_0 + T_1^+], \\ \left\| \psi^P \right\| & \leq_* & T_0 + T_1, \\ \left\| \psi^Q \right\| & \leq_* & T_0 + T_1, \\ \left\| \psi^S - \phi^+ \right\| & \leq_* & r^+ \left\| \phi^+ \right\| + e^{-\alpha_Q \omega^+}[T_0 + T_1^-]. \end{array} \tag{6.188}$$

In order to see that these are in fact all the terms, we note that we can use a separate norm on $\mathbf{W}$ for each of the components. In particular, the operator K remains bounded, independently of $\omega^+ \geq \Omega$ and $\omega^- \leq -\Omega$, after the scalings $\widetilde{\psi}^S \sim e^{-\alpha_S \omega^+} \psi^S$ and $\widetilde{\psi}^R \sim e^{\alpha_R \omega^-} \psi^R$, which allows us to get the estimates on ψ^P and ψ^Q. To obtain the estimate on ψ^S, one can use the scalings $\widetilde{\psi}^Q \sim e^{-\alpha_Q \omega^+} \psi^Q$, $\widetilde{\psi}^P \sim e^{-\alpha_Q \omega^+} \psi^P$ and $\widetilde{\psi}^R = e^{\alpha_R \omega^-} e^{-\alpha_Q \omega^+} \psi^R$.

We now include the higher order terms using the expansion

$$\begin{array}{rcl}\mathbf{w} & = & \mathbf{w}_0 + [I-K]^{-1}[J_0B_0 + J_1B_1 + J_2B_2]\big(\mathcal{M}^-(\mathbf{w}_0^-), \mathcal{M}^+(\mathbf{w}_0^+)\big) \\ & & + (V^-, V^+),\end{array} \tag{6.189}$$

in which $\|V^{\pm}\|_0 \leq_* \|\phi\|^3$. We thus find that the fixed point $\mathbf{w} = (\omega^-, \omega^+, \psi^R, \psi^P, \psi^Q, \psi^S)$ of (6.184) can be bounded by

$$\begin{array}{lcl}\|\widehat{\omega}^-\|_0 & \leq_* & e^{-2\alpha_S\omega^+}\|\phi\|^2, \\ \|\widetilde{\omega}^-\|_{-\alpha_R} & \leq_* & e^{\alpha_R\omega^-}\|\phi\|^2, \\ \|\widehat{\omega}^+\|_0 & \leq_* & e^{2\alpha_R\omega^-}\|\phi\|^2, \\ \|\widetilde{\omega}^+\|_{\alpha_S} & \leq_* & e^{-\alpha_S\omega^+}\|\phi\|^2, \\ \|\psi^R - \phi^-\| & \leq_* & r^-\|\phi^-\| + e^{\alpha_P\omega^-}[T_0 + T_1^+] + \|\phi\|^2, \\ \|\psi^P\| & \leq_* & T_0 + T_1 + e^{-\alpha_S\omega^+}\|\phi\|^2 + e^{\alpha_R\omega^-}\|\phi\|^2, \\ \|\psi^Q\| & \leq_* & T_0 + T_1 + e^{-\alpha_S\omega^+}\|\phi\|^2 + e^{\alpha_R\omega^-}\|\phi\|^2, \\ \|\psi^S - \phi^+\| & \leq_* & r^+\|\phi^+\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-] + \|\phi\|^2,\end{array} \tag{6.190}$$

in which we have split $\omega^{\pm} = \widehat{\omega}^{\pm} + \widetilde{\omega}^{\pm}$. Adding higher order terms does not change these estimates.

We are now ready to move on to the full system. We will use (6.170) to find the family $\{\phi_j^-, \phi_j^+\}$ in terms of the boundary conditions $\{\Phi_j^-\}$ and $\{\Phi_j^+\}$. To this end, we reformulate (6.170) as follows,

$$\begin{array}{rcl}\phi_j^+ & = & \Phi_j^+ + \Pi_{P(\infty)}\big[\mathrm{ev}_{\omega_{j+1}^-}\omega_{j+1}^- + \mathrm{ev}_{\omega_{j+1}^-}u_{P(0)}^*\psi_{j+1}^P\big] \\ & & + \Pi_{P(\infty)}[\Pi_{R^s(\omega_{j+1}^-,\mu)} + \Pi_{R^f(\omega_{j+1}^-,\mu)} - \Pi_{Q(-\infty)}]\psi_{j+1}^R, \\ \phi_j^- & = & \Phi_j^- + \Pi_{Q(-\infty)}\big[\mathrm{ev}_{\omega_{j-1}^+}\omega_{j-1}^+ + \mathrm{ev}_{\omega_{j-1}^+}u_{Q(0)}^*\psi_{j-1}^Q\big] \\ & & + \Pi_{Q(-\infty)}[\Pi_{S^s(\omega_{j-1}^+,\mu)} + \Pi_{S^f(\omega_{j-1}^+,\mu)} - \Pi_{P(\infty)}]\psi_{j-1}^S.\end{array} \tag{6.191}$$

We first set out to find the lowest order terms, i.e., we compute

$$\{\phi_j^{(1)-}, \phi_j^{(1)+}\} = [D\{\phi_j^-, \phi_j^+\}](0)\big(\{\Phi_j^-\}, \{\Phi_j^+\}\big) = [I - \mathbf{K}]^{-1}(\{\Phi_j^-\}, \{\Phi_j^+\}), \tag{6.192}$$

for some linear operator $\mathbf{K}$. We can use the estimate (6.190) to bound the components of $\mathbf{K}$

by

$$
\begin{array}{rcl}
\left|K_j^{+}(\{c^-\},\{c^+\})\right| & \leq_* & r_{j+1}^{-}\left\|c_{j+1}^{-}\right\| + r_{j+1}^{-}e^{\alpha_P\omega_{j+1}^{-}}e^{-\alpha_S\omega_{j+1}^{+}}\left\|c_{j+1}^{+}\right\| \\
& & + e^{\alpha_P\omega_{j+1}^{-}}e^{\alpha_R\omega_{j+1}^{-}}\left\|\Pi_{R^s(\omega_{j+1}^{-},\mu)}c_{j+1}^{-}\right\| \\
& & + e^{\alpha_P\omega_{j+1}^{-}}e^{-\alpha_S\omega_{j+1}^{+}}\left\|\Pi_{S^s(\omega_{j+1}^{+},\mu)}c_{j+1}^{+}\right\| \\
& & + e^{\alpha_P\omega_{j+1}^{-}}[e^{\alpha_R^f\omega_{j+1}^{-}}\left\|c_{j+1}^{-}\right\| + e^{-\alpha_S^f\omega_{j+1}^{+}}\left\|c_{j+1}^{+}\right\|] \\
& & + e^{\alpha_P\omega_{j+1}^{-}}e^{\alpha_R\omega_{j+1}^{-}}r_{j+1}^{-}\left\|c_{j+1}^{-}\right\| \\
& & + e^{\alpha_P\omega_{j+1}^{-}}e^{-\alpha_S\omega_{j+1}^{+}}r_{j+1}^{+}\left\|c_{j+1}^{+}\right\|, \\
\left|K_j^{-}(\{c^-\},\{c^+\})\right| & \leq_* & r_{j-1}^{+}\left\|c_{j-1}^{+}\right\| + r_{j-1}^{+}e^{-\alpha_Q\omega_{j-1}^{+}}e^{\alpha_R\omega_{j-1}^{-}}\left\|c_{j-1}^{-}\right\| \\
& & + e^{-\alpha_Q\omega_{j-1}^{+}}e^{\alpha_R\omega_{j-1}^{-}}\left\|\Pi_{R^s(\omega_{j-1}^{-},\mu)}c_{j-1}^{-}\right\| \\
& & + e^{-\alpha_Q\omega_{j-1}^{+}}e^{-\alpha_S\omega_{j-1}^{+}}\left\|\Pi_{S^s(\omega_{j+1}^{+},\mu)}c_{j-1}^{+}\right\|] \\
& & + e^{-\alpha_Q\omega_{j-1}^{+}}[e^{\alpha_R^f\omega_{j-1}^{-}}\left\|c_{j-1}^{-}\right\| + e^{-\alpha_S^f\omega_{j-1}^{+}}\left\|c_{j-1}^{+}\right\|] \\
& & + e^{-\alpha_Q\omega_{j-1}^{+}}e^{\alpha_R\omega_{j-1}^{-}}r_{j-1}^{-}\left\|c_{j-1}^{-}\right\| \\
& & + e^{-\alpha_Q\omega_{j-1}^{+}}e^{-\alpha_S\omega_{j-1}^{+}}r_{j-1}^{+}\left\|c_{j-1}^{+}\right\|.
\end{array}
\tag{6.193}
$$

Let us now introduce the scaling factors

$$
\begin{array}{rcl}
\widetilde{\phi}_j^{+} & \sim & e^{-\alpha_P\omega_{j+1}^{-}}\phi_j^{+}, \\
\widetilde{\phi}_j^{-} & \sim & e^{\alpha_Q\omega_{j-1}^{+}}\phi_j^{-}.
\end{array}
\tag{6.194}
$$

In terms of these scaled variables, the operator K can still be bounded independently of the family $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$, as long as $\omega_\ell > \Omega$ for all $\ell \in \mathcal{J}^*$. We wish to invoke the general estimate (6.186) using these scaled variables. Let us therefore split up $\mathbf{K}(\{\Phi^-\},\{\Phi^+\}) = \{a^-,a^+\} + \{b^-,b^+\}$, in which

$$
\begin{array}{rcl}
\left\|b_j^{-}\right\| & \leq_* & r_{j-1}^{+}\left\|\Phi_{j-1}^{+}\right\|, \\
\left\|b_j^{+}\right\| & \leq_* & r_{j+1}^{-}\left\|\Phi_{j+1}^{-}\right\|,
\end{array}
\tag{6.195}
$$

while the family $\{a^-,a^+\}$ can be estimated using the scaled norm according to

$$
\left\|\{a^-,a^+\}\right\|_{\mathrm{sc}} \leq_* S_0, \tag{6.196}
$$

where we have introduced the quantity

$$
\begin{array}{rcl}
S_0 & = & \sup_{j\in\mathcal{J}}\Big\{e^{\alpha_R\omega_j^{-}}\left\|\Pi_{R^s(\omega_j^{-},\mu)}\Phi_j^{-}\right\| + e^{-\alpha_S\omega_j^{+}}\left\|\Pi_{S^s(\omega_j^{+},\mu)}\Phi_j^{+}\right\| \\
& & + e^{\alpha_R^f\omega_j^{-}}\left\|\Phi_j^{-}\right\| + e^{-\alpha_S^f\omega_j^{+}}\left\|\Phi_j^{+}\right\| \\
& & + e^{\alpha_R\omega_j^{-}}[r_j^{-} + r_j^{+}]\left\|\Phi_j^{-}\right\| + e^{-\alpha_S\omega_j^{+}}[r_j^{+} + r_j^{-}]\left\|\Phi_j^{+}\right\| \\
& & + e^{\alpha_R\omega_j^{-}}r_{j-1}^{+}\left\|\Phi_{j-1}^{+}\right\| + e^{-\alpha_S\omega_j^{+}}r_{j+1}^{-}\left\|\Phi_{j+1}^{-}\right\|\Big\}.
\end{array}
\tag{6.197}
$$

We now compute $\mathbf{K}(\{b^-, b^+\}) = \{e^-, e^+\} + \{f^-, f^+\}$ and obtain the bounds

$$\begin{array}{lcl} \left\| e_j^- \right\| & \le_* & r_{j-1}^+ r_j^- \left\| \Phi_j^- \right\|, \\ \left\| e_j^+ \right\| & \le_* & r_{j+1}^- r_j^+ \left\| \Phi_j^+ \right\|, \\ \left\| \{f^-, f^+\} \right\|_{\mathrm{sc}} & \le_* & S_0. \end{array} \quad (6.198)$$

Since the family $\{e^-, e^+\}$ is now bounded componentwise by the family $\{\Phi^-, \Phi^+\}$, we may write

$$\begin{array}{lcl} \left\| \phi_j^{(1)-} - \Phi_j^- \right\| & \le_* & r_{j-1}^+ \left\| \Phi_{j-1}^+ \right\| + e^{-\alpha_Q \omega_{j-1}^+} S_0, \\ \left\| \phi_j^{(1)+} - \Phi_j^+ \right\| & \le_* & r_{j+1}^- \left\| \Phi_{j+1}^- \right\| + e^{\alpha_P \omega_{j+1}^-} S_0. \end{array} \quad (6.199)$$

Adding the second order terms, we arrive at

$$\begin{array}{lcl} \left\| \phi_j^- - \Phi_j^- \right\| & \le_* & r_{j-1}^+ [\left\| \Phi_{j-1}^+ \right\| + \|\Phi\|^2] + e^{-\alpha_Q \omega_{j-1}^+} [S_0 + \|\Phi\|^2], \\ \left\| \phi_j^+ - \Phi_j^+ \right\| & \le_* & r_{j+1}^- [\left\| \Phi_{j+1}^- \right\| + \|\Phi\|^2] + e^{\alpha_P \omega_{j+1}^-} [S_0 + \|\Phi\|^2]. \end{array} \quad (6.200)$$

We are now finally in a position to estimate the error term. To this end, we write $\widetilde{\mathcal{R}}_j = \widetilde{\mathcal{R}}_j^+ + \widetilde{\mathcal{R}}_j^-$ and represent the two parts in the following manner,

$$\begin{array}{rcl} \widetilde{\mathcal{R}}_j^- & = & \int_{\omega_j^-}^0 d^-(\xi') \mathcal{M}^-(\xi', \mathrm{ev}_{\xi'} v_j^-, \mu) d\xi' + \langle \mathrm{ev}_{\omega_j^-}^* d^-, \phi_j^- - \Phi_j^- \rangle_{\omega_j^-, \mu} \\ & & + \langle \mathrm{ev}_{\omega_j^-}^* d^-, [\Pi_{R(\omega_j^-, \mu)} - \Pi_{Q(-\infty)}] \mathrm{ev}_{\omega_j^-} v_j^- \rangle_{\omega_j^-, \mu} \\ & & - \langle \mathrm{ev}_{\omega_j^-}^* d^-, [\Pi_{R(\omega_j^-, \mu)} - \Pi_{Q(-\infty)}][\Theta_j^- - \Theta_{j-1}^+] \rangle_{\omega_j^-, \mu} \\ & & + \langle \mathrm{ev}_{\omega_j^-}^* d^-, \Pi_{R(\omega_j^-, \mu)} \Theta_j^- \rangle_{\omega_j^-, \mu}, \\ \widetilde{\mathcal{R}}_j^+ & = & \int_0^{\omega_j^+} d^+(\xi') \mathcal{M}^+(\xi', \mathrm{ev}_{\xi'} v_j^+, \mu) d\xi' - \langle \mathrm{ev}_{\omega_j^+}^* d^+, \phi_j^+ - \Phi_j^+ \rangle_{\omega_j^+, \mu} \\ & & - \langle \mathrm{ev}_{\omega_j^+}^* d^+, [\Pi_{S(\omega_j^+, \mu)} - \Pi_{P(\infty)}] \mathrm{ev}_{\omega_j^+} v_j^+ \rangle_{\omega_j^+, \mu} \\ & & + \langle \mathrm{ev}_{\omega_j^+}^* d^+, [\Pi_{S(\omega_j^+, \mu)} - \Pi_{P(\infty)}][\Theta_j^+ - \Theta_{j+1}^-] \rangle_{\omega_j^+, \mu} \\ & & - \langle \mathrm{ev}_{\omega_j^+}^* d^+, \Pi_{S(\omega_j^+, \mu)} \Theta_j^+ \rangle_{\omega_j^+, \mu}. \end{array} \quad (6.201)$$

In order to complete our estimate, observe that $\Pi_{S(\omega_j^+, \mu)} \Theta_j^+ \le_* \|\Theta\|^2$, because the function $q_j + u_j^+(\mu)$ is contained in the stable manifold of $q^*_{j+\frac{1}{2}}$. Notice also that for some small constant $\epsilon > 0$, we may write $d^+(\xi) = O(e^{-(\alpha_S + \epsilon)\xi})$ as $\xi \to \infty$, since the characteristic equation $\det \Delta^+(z) = 0$ associated to the equilibrium $q^*_{j+\frac{1}{2}}$ has no roots in the strip $0 \le \mathrm{Re}\, z \le \alpha_S$. Putting everything together we obtain the following result, which completes the proof of Theorem 6.2.2.

Lemma 6.7.1. *Consider the setting of Theorem 6.2.2. For every $j \in \mathcal{J}$, we have the following estimate for the error term $\widetilde{\mathcal{R}}_j$ that is defined in (6.175),*

$$\begin{array}{rcl} \widetilde{\mathcal{R}}_j & \le_* & e^{-\alpha_S \omega_+^j} [\|\Theta\|^2 + (r_{j+1}^- + r_j^+) \|\Theta\| + e^{\alpha_P \omega_{j+1}^-} S_0] \\ & & + e^{\alpha_R \omega_j^-} [\|\Theta\|^2 + (r_{j-1}^+ + r_j^-) \|\Theta\| + e^{-\alpha_Q \omega_{j-1}^+} S_0]. \end{array} \quad (6.202)$$

6.8. Derivative of the remainder term

The main goal of this section is to provide an estimate for the quantities $D_{\omega_\ell}\widetilde{\mathcal{R}}_j$, for $j \in \mathcal{J}$ and $\ell \in \mathcal{J}^*$. Recalling the fixed point problem (6.184), together with the solution $\mathbf{w} = \mathbf{w}(\phi^-, \phi^+, \omega^-, \omega^+) \in \mathbf{W}$, we set out to compute the derivatives $D_{\phi^\pm}\mathbf{w}$ and $D_{\omega^\pm}\mathbf{w}$. We start with the observation

$$D_{\omega^\pm}\mathbf{w} = [I - K]^{-1}[D_{\omega^\pm}K]\mathbf{w} + [I - K]^{-1} J_2 D_{\omega^\pm}\big[B_2\big(\mathcal{M}^-(\mathbf{w}^-), \mathcal{M}^+(\mathbf{w}^+)\big)\big]. \tag{6.203}$$

Inspection of (6.162) yields the identities

$$\begin{array}{lll} [D_{\omega^-}K_{\psi^R}]\mathbf{w} & \leq_* & r^- \left\|\psi^R\right\| + e^{\alpha_P\omega^-}\left\|\psi^P\right\|, \\ [D_{\omega^-}K_{\psi^P}]\mathbf{w} & \leq_* & e^{\alpha_R\omega^-}\left\|\Pi_{R^s(\omega^-,\mu)}\psi^R\right\| + e^{\alpha_R^f\omega^-}\left\|\psi^R\right\|, \\ [D_{\omega^+}K_{\psi^P}]\mathbf{w} & \leq_* & e^{-\alpha_S\omega^+}\left\|\Pi_{S^s(\omega^+,\mu)}\psi^S\right\| + e^{-\alpha_S^f\omega^+}\left\|\psi^S\right\|, \\ [D_{\omega^-}K_{\phi^Q}]\mathbf{w} & \leq_* & e^{\alpha_R\omega^-}\left\|\Pi_{R^s(\omega^-,\mu)}\psi^R\right\| + e^{\alpha_R^f\omega^-}\left\|\psi^R\right\|, \\ [D_{\omega^+}K_{\phi^Q}]\mathbf{w} & \leq_* & e^{-\alpha_S\omega^+}\left\|\Pi_{S^s(\omega^+,\mu)}\psi^S\right\| + e^{-\alpha_S^f\omega^+}\left\|\psi^S\right\|, \\ [D_{\omega^+}K_{\psi^S}]\mathbf{w} & \leq_* & r_+\left\|\psi^S\right\| + e^{-\alpha_Q\omega^+}\left\|\psi^Q\right\|. \end{array} \tag{6.204}$$

Let us write $\mathbf{Dw_0^\pm} = [I - K]^{-1}[D_{\omega^\pm}K]\mathbf{w}$. Utilizing the bounds (6.190) and performing a calculation in the spirit of the previous section now yields the estimates

$$\begin{array}{lll} (\mathbf{Dw_0^+})_{\psi^R} & \leq_* & e^{\alpha_P\omega^-}\big[T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\|\phi\|^2\big], \\ (\mathbf{Dw_0^+})_{\psi^P} & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\|\phi\|^2, \\ (\mathbf{Dw_0^+})_{\psi^Q} & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\|\phi\|^2, \\ (\mathbf{Dw_0^+})_{\psi^S} & \leq_* & r^+\left\|\phi^+\right\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-] + \|\phi\|^2. \end{array} \tag{6.205}$$

Inserting this back into (6.203), we find the following estimate for the derivative $D_{\omega^+}\mathbf{w}$, where $w = (w^-, w^+, \psi^R, \psi^P, \psi^Q, \psi^S)$ is the solution of the fixed point problem (6.184),

$$\begin{array}{lll} \left\|D_{\omega^+}\widehat{w}^-\right\|_0 & \leq_* & e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\|\phi\|^2, \\ \left\|D_{\omega^+}\widetilde{w}^-\right\|_{-\alpha_R} & \leq_* & e^{\alpha_R\omega^-}e^{\alpha_P\omega^-}e^{-\alpha_S\omega^+}\|\phi\|^2, \\ \left\|D_{\omega^+}\widehat{w}^+\right\|_0 & \leq_* & e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\|\phi\|^2, \\ \left\|D_{\omega^+}\widetilde{w}^+\right\|_{\alpha_S} & \leq_* & e^{-\alpha_S\omega^+}r^+\|\phi\|^2, \\ \left\|D_{\omega^+}\psi^R\right\| & \leq_* & e^{\alpha_P\omega^-}\big[T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-]\big] \\ & & \quad + e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-} + e^{\alpha_P\omega^-}]\|\phi\|^2, \\ \left\|D_{\omega^+}\psi^P\right\| & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\|\phi\|^2, \\ \left\|D_{\omega^+}\psi^Q\right\| & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\|\phi\|^2, \\ \left\|D_{\omega^+}\psi^S\right\| & \leq_* & r^+\left\|\phi^+\right\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-] + \|\phi\|^2. \end{array} \tag{6.206}$$

Using a similar calculation, we also obtain

$$\begin{array}{lll}
\left\|[D_\phi \widehat{w}^-](\Delta\phi^+, \Delta\phi^-)\right\|_0 & \le_* & e^{-2\alpha_S\omega^+} \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \widetilde{w}^-](\Delta\phi^+, \Delta\phi^-)\right\|_{-\alpha_R} & \le_* & e^{\alpha_R\omega^-} \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \widehat{w}^+](\Delta\phi^+, \Delta\phi^-)\right\|_0 & \le_* & e^{2\alpha_R\omega^-} \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \widetilde{w}^+](\Delta\phi^+, \Delta\phi^-)\right\|_{\alpha_S} & \le_* & e^{-\alpha_S\omega^+} \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \psi^R](\Delta\phi^+, \Delta\phi^-)\right\| & \le_* & \left\|\Delta\phi^-\right\| + e^{-\alpha_S\omega^+} e^{\alpha_P\omega^-} \left\|\Delta\phi^+\right\| \\
 & & \quad + e^{-\alpha_S\omega^+} \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \psi^P](\Delta\phi^+, \Delta\phi^-)\right\| & \le_* & \Delta T_0 + \Delta T_1 + [e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}] \, \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \psi^Q](\Delta\phi^+, \Delta\phi^-)\right\| & \le_* & \Delta T_0 + \Delta T_1 + [e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}] \, \|\phi\| \, \|\Delta\phi\| , \\
\left\|[D_\phi \psi^S](\Delta\phi^+, \Delta\phi^-)\right\| & \le_* & \left\|\Delta\phi^+\right\| + e^{\alpha_R\omega^-} e^{-\alpha_Q\omega^+} \left\|\Delta\phi^-\right\| \\
 & & \quad + e^{\alpha_R\omega^-} \|\phi\| \, \|\Delta\phi\| .
\end{array} \tag{6.207}$$

These expressions can be used to determine the derivatives $D_{\omega_\ell}\phi_j^\pm$ for $j \in \mathcal{J}$ and $\ell \in \mathcal{J}^*$ using the boundary conditions in (6.191). Let us therefore fix some $j^* \in \mathcal{J}$ and determine the family $\{b_j^-, b_j^+\}_{j\in\mathcal{J}}$ that describes the derivatives of the family $\{\phi^-, \phi^+\}$ with respect to $\omega_{j^*}^+$ up to first order in $\|\Phi\|$, i.e., $\left\| D_{\omega_{j^*}^+}\phi_j^\pm - b_j^\pm \right\| \le_* \|\Phi\|^2$. Careful inspection of (6.191) shows that we must solve the coupled system

$$\begin{array}{lll}
b_j^+ & = & \mathbf{B}_j^+ + L_j^+\big(b_{j+1}^-, b_{j+1}^+\big), \\
b_j^- & = & \mathbf{B}_j^- + L_j^-\big(b_{j-1}^-, b_{j-1}^+\big),
\end{array} \tag{6.208}$$

in which the norms of $L_j^\pm$ share the estimates for $K_j^\pm$ given in (6.193), while the initial value $\mathbf{B}$ can be bounded as

$$\begin{array}{lll}
\left\|\mathbf{B}_j^+ - \delta_{jj^*} D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| & \le_* & \delta_{j,j^*-1}\big[e^{\alpha_P\omega_{j^*}^-}\left\|D_{\omega_{j^*}^+}\psi_{j^*}^P\right\| + r_{j^*}^-\left\|D_{\omega_{j^*}^+}\psi_{j^*}^R\right\|\big] \\
\left\|\mathbf{B}_j^-\right\| & \le_* & \delta_{j,j^*+1} e^{-\alpha_Q\omega_{j^*}^+}[\left\|\psi_{j^*}^Q\right\| + \left\|D_{\omega_j^+}\psi_{j^*}^Q\right\|] \\
 & & \quad + \delta_{j,j^*+1} r_{j^*}^+[\left\|\psi_{j^*}^S\right\| + \left\|D_{\omega_{j^*}^+}\psi_j^S\right\|].
\end{array} \tag{6.209}$$

As in the previous section, a small number of applications of the operator family $\{L_j^-, L_j^+\}$, together with the scaling (6.194), enables us to obtain an estimate on the solution to the coupled system (6.208). We obtain

$$\begin{array}{lll}
\left\|b_j^+ - \delta_{jj^*} D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| & \le_* & \delta_{jj^*} r_{j^*+1}^- r_{j^*}^+ \left\|\phi_{j^*}^+\right\| + e^{\alpha_P\omega_{j+1}^-}\mathbf{E}, \\
\left\|b_j^-\right\| & \le_* & \delta_{j,j^*+1}[r_{j^*}^+\left\|D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| + r_{j^*}^+\left\|\phi_{j^*}^+\right\|] + e^{-\alpha_Q\omega_{j-1}^+}\mathbf{E},
\end{array} \tag{6.210}$$

in which we have defined the quantity

$$\begin{aligned}\mathbf{E} = \mathbf{E}^{+}_{(j^*)} \quad = \quad & T_0^{j^*} + T_1^{j^*} + r^{+}_{j^*} e^{\alpha_R \omega^{-}_{j^*+1}} \left\| \phi^{+}_{j^*} \right\| + r^{-}_{j^*} e^{-\alpha_S \omega^{+}_{j^*}} \left\| \phi^{+}_{j^*} \right\| \\ & + e^{-\alpha_S \omega^{+}_{j^*}} \left\| \Pi_{S^s(\omega^{+}_{j^*}, \mu)} D_{\omega^{+}_{j^*}} \Phi^{+}_{j^*} \right\| + e^{-\alpha^f_S \omega^{+}_{j^*}} \left\| D_{\omega^{+}_{j^*}} \Phi^{+}_{j^*} \right\| \\ & + [e^{-\alpha_S \omega^{+}_{j^*}} + e^{\alpha_R \omega^{-}_{j^*+1}}] r^{+}_{j^*} \left\| D_{\omega^{+}_{j^*}} \Phi^{+}_{j^*} \right\| \\ & + r^{-}_{j^*} e^{-\alpha_S \omega^{+}_{j^*}} \left\| D_{\omega^{+}_{j^*}} \Phi^{+}_{j^*} \right\| .\end{aligned} \tag{6.211}$$

Of course, similar estimates can be obtained for the derivatives with respect to $\omega^{-}_{j^*+1}$. In order to combine these estimates, we now fix $\ell^* \in \mathcal{J}^*$ and introduce the following quantities for any $\ell \in \mathcal{J}^*$,

$$\begin{aligned}|\Phi_\ell|_1 &= \left|\Phi^{+}_{\ell-\frac{1}{2}}\right| + \left|\Phi^{-}_{\ell+\frac{1}{2}}\right| + \left|D_{\omega_{\ell^*}} \Phi^{+}_{\ell-\frac{1}{2}}\right| + \left|D_{\omega_{\ell^*}} \Phi^{-}_{\ell+\frac{1}{2}}\right| , \\ \left|\Phi^S_\ell\right|_{1,s} &= \left|\Pi_{S^s(\omega_\ell, \mu)} \Phi^{+}_{\ell-\frac{1}{2}}\right| + \left|\Pi_{S^s(\omega_\ell, \mu)} D_{\omega_{\ell^*}} \Phi^{+}_{\ell-\frac{1}{2}}\right| , \\ \left|\Phi^R_\ell\right|_{1,s} &= \left|\Pi_{R^s(-\omega_\ell, \mu)} \Phi^{-}_{\ell+\frac{1}{2}}\right| + \left|\Pi_{R^s(-\omega_\ell, \mu)} D_{\omega_{\ell^*}} \Phi^{-}_{\ell+\frac{1}{2}}\right| , \\ r_\ell &= r^{+}_{\ell-\frac{1}{2}} + r^{-}_{\ell+\frac{1}{2}} .\end{aligned} \tag{6.212}$$

We also introduce the quantity S_1, which should be seen as the sum of the quantities $\mathbf{E}^{+}_{(\ell^*-\frac{1}{2})} + \mathbf{E}^{-}_{(\ell^*+\frac{1}{2})}$, after insertion of the inequalities (6.200),

$$\begin{aligned}S_1 \quad = \quad & e^{-\alpha_S \omega_{\ell^*}} \left|\Phi^S_{\ell^*}\right|_{1,s} + e^{-\alpha_R \omega_{\ell^*}} \left|\Phi^R_{\ell^*}\right|_{1,s} \\ & + [e^{-\alpha^f_S \omega_{\ell^*}} + e^{-\alpha^f_R \omega_{\ell^*}}] \left|\Phi_{\ell^*}\right|_1 \\ & + [e^{-\alpha_S \omega_{\ell^*}} + e^{-\alpha_R \omega_{\ell^*}}] r_\ell \left|\Phi_{\ell^*}\right|_1 \\ & + e^{-\alpha_R \omega_{\ell^*-1}} \left|\Phi^R_{\ell^*-1}\right|_{1,s} + e^{-\alpha^f_R \omega_{\ell^*-1}} \left|\Phi_{\ell^*-1}\right|_1 + r_{\ell^*-1} e^{-\alpha_R \omega_{\ell^*-1}} \left|\Phi_{\ell^*-1}\right|_1 \\ & + e^{-\alpha_S \omega_{\ell^*+1}} \left|\Phi^S_{\ell^*+1}\right|_{1,s} + e^{-\alpha^f_S \omega_{\ell^*+1}} \left|\Phi_{\ell^*+1}\right|_1 + r_{\ell^*+1} e^{-\alpha_S \omega_{\ell^*+1}} \left|\Phi_{\ell^*+1}\right|_1 \\ & + e^{-\alpha_S \omega_{\ell^*}} e^{-\alpha_P \omega_{\ell^*}} S_0 + e^{-\alpha_R \omega_{\ell^*-1}} e^{-\alpha_Q \omega_{\ell^*-1}} S_0 \\ & + e^{-\alpha_R \omega_{\ell^*}} e^{-\alpha_Q \omega_{\ell^*}} S_0 + e^{-\alpha_S \omega_{\ell^*+1}} e^{-\alpha_P \omega_{\ell^*+1}} S_0 \\ & + \|\Phi\|^2 .\end{aligned} \tag{6.213}$$

We are now ready to put everything together. Using (6.210) together with the definitions above and inserting the second order terms in the appropriate places, we obtain the estimates

$$
\begin{array}{lll}
\left\| D_{\omega_{\ell^*}} [\phi^{+}_{\ell^*-\frac{1}{2}} - \Phi^{+}_{\ell^*-\frac{1}{2}}] \right\| & \leq_* & e^{-\alpha_P \omega_{\ell^*}} S_1 + r^{-}_{\ell^*+\frac{1}{2}} |\Phi_{\ell^*}|_1 \\
 & & + r^{-}_{\ell^*+\frac{1}{2}} [e^{-\alpha_Q \omega_{\ell^*}} S_0 + r^{+}_{\ell^*-\frac{1}{2}} e^{-\alpha_P \omega_{\ell^*}} S_0] + \|\Phi\|^2 , \\
\left\| D_{\omega_{\ell^*}} [\phi^{-}_{\ell^*+\frac{1}{2}} - \Phi^{-}_{\ell^*+\frac{1}{2}}] \right\| & \leq_* & e^{-\alpha_Q \omega_{\ell^*}} S_1 + r^{+}_{\ell^*-\frac{1}{2}} |\Phi_{\ell^*}|_1 \\
 & & + r^{+}_{\ell^*-\frac{1}{2}} [e^{-\alpha_P \omega_{\ell^*}} S_0 + r^{-}_{\ell^*+\frac{1}{2}} e^{-\alpha_Q \omega_{\ell^*}} S_0] + \|\Phi\|^2 , \\
\left\| D_{\omega_{\ell^*}} \phi^{+}_{j} \right\| & \leq_* & e^{\alpha_P \omega^{-}_{j+1}} S_1, \quad \text{for all } j \neq \ell^* - \frac{1}{2}, \\
\left\| D_{\omega_{\ell^*}} \phi^{-}_{j} \right\| & \leq_* & e^{-\alpha_Q \omega^{+}_{j-1}} S_1, \quad \text{for all } j \neq \ell^* + \frac{1}{2}.
\end{array}
\tag{6.214}
$$

With these estimates in hand, we can move on and analyze (6.201) in order to obtain estimates for the quantities $D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_j$. Care has to be taken to distinguish the terms in (6.201) that depend directly on ω_{ℓ^*}, from those that only depend on this quantity through the family of boundary terms $\{\phi^-, \phi^+\}$. Using methods similar to those employed here to estimate the derivatives $\left|D_\mu \widetilde{\mathcal{R}}_j\right|$ and $\left|D_\mu D_{\omega^*_\ell} \widetilde{\mathcal{R}}_j\right|$, we obtain the following result.

Lemma 6.8.1. *Consider the setting of Theorem 6.2.2 and recall the error terms (6.175). Fix an $\ell^* \in \mathcal{J}^*$ and let $j \in \mathcal{J}$ be such that $j \neq \ell^* \pm \frac{1}{2}$. Then the following estimates hold for the error terms $\{\widetilde{\mathcal{R}}\}$,*

$$
\begin{array}{lll}
\left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_{\ell^*-\frac{1}{2}} \right| & \leq_* & \left| \mathcal{R}_{\ell^*-\frac{1}{2}} \right| + e^{-\alpha_S \omega_{\ell^*}} [\|\Theta\| + r_{\ell^*}] \, |\Phi_\ell|_1 \\
 & & + e^{-\alpha_S \omega_{\ell^*}} e^{-\alpha_P \omega_{\ell^*}} S_1 \\
 & & + e^{-\alpha_R \omega_{\ell^*-1}} e^{-\alpha_Q \omega_{\ell^*-1}} S_1, \\
\left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_{\ell^*+\frac{1}{2}} \right| & \leq_* & \left| \mathcal{R}_{\ell^*+\frac{1}{2}} \right| + e^{-\alpha_R \omega_{\ell^*}} [\|\Theta\| + r_{\ell^*}] \, |\Phi_\ell|_1 \\
 & & + e^{-\alpha_R \omega_{\ell^*}} e^{-\alpha_Q \omega_{\ell^*}} S_1 \\
 & & + e^{-\alpha_S \omega_{\ell^*+1}} e^{-\alpha_P \omega_{\ell^*+1}} S_1, \\
\left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_j \right| & \leq_* & e^{-\alpha_S \omega^{+}_{j}} e^{\alpha_P \omega^{-}_{j+1}} S_1 + e^{\alpha_R \omega^{-}_{j}} e^{-\alpha_Q \omega^{+}_{j-1}} S_1.
\end{array}
\tag{6.215}
$$

In addition, for all $j \in \mathcal{J}$ we have the estimates

$$
\begin{array}{lll}
\left| D_\mu \widetilde{\mathcal{R}}_j \right| & \leq_* & e^{-3\alpha\omega}, \\
\left| D_\mu D_{\omega^*_\ell} \widetilde{\mathcal{R}}_j \right| & \leq_* & e^{-3\alpha\omega},
\end{array}
\tag{6.216}
$$

in which α and ω are defined as in Theorem 6.2.2.

We are now ready to consider the orbit-flip bifurcation for (6.2). An application of Theorem 6.2.2 to the setting of Theorem 6.2.3 yields a finite dimensional bifurcation equation, that is very similar to the one obtained in Chapter 4 of [134]. The calculations contained in that chapter carry over to our setting and can hence be used to establish Theorem 6.2.3.

Appendix

A. Embedded Contractions

In this appendix we outline a version of the embedded contraction theorem which we used to prove that center manifolds are C^k-smooth. The presentation given here contains slight adaptations of results given in [159].

Let Y_0, Y, Y_1 and Λ be Banach spaces with norms denoted respectively by

$$\|\cdot\|_0, \quad \|\cdot\|, \quad \|\cdot\|_1 \quad \text{and} \quad |\cdot|, \tag{A.1}$$

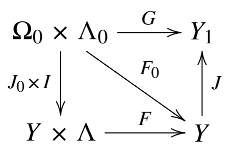
and suppose that we have continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$. Let $\Omega_0 \subset Y_0$ and $\Lambda_0 \subset \Lambda$ be two convex open subsets of Y_0 respectively Λ. We consider the fixed-point equation

$$y = F(y, \lambda) \tag{A.2}$$

for some $F : Y \times \Lambda \to Y$. Associated to F we define a function $F_0 : \Omega_0 \times \Lambda_0 \to Y$ via

$$F_0(y_0, \lambda_0) = F(J_0 y_0, \lambda_0) \tag{A.3}$$

and also a function $G : \Omega_0 \times \Lambda_0 \to Y_1$ by $G = J \circ F_0$. The situation is illustrated by the following commutative diagram.

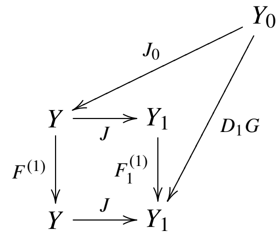


(A.4)

We shall need the following assumptions on F and G.

(HC1) The function G is of class C^1. Fix any $\omega_0 \in \Omega_0$ and $\lambda_0 \in \Lambda_0$ and consider the partial derivative $D_1 G(\omega_0, \lambda_0) \in \mathcal{L}(Y_0, Y_1)$. Then there exist $F^{(1)}(\omega_0, \lambda_0) \in \mathcal{L}(Y)$

and $F_1^{(1)}(\omega_0, \lambda_0) \in \mathcal{L}(Y_1)$ such that the following diagram is commutative,



(A.5)

i.e., for any $v_0 \in Y_0$ we have

$$\begin{aligned} D_1 G(\omega_0, \lambda_0) v_0 &= J F^{(1)}(\omega_0, \lambda_0) J_0 v_0, \\ J F^{(1)}(\omega_0, \lambda_0) y &= F_1^{(1)}(\omega_0, \lambda_0) J y. \end{aligned} \tag{A.6}$$

(HC2) There exists some $\kappa_1 \in [0, 1)$ such that for all $\omega_0 \in \Omega_0$ and $\lambda_0 \in \Lambda_0$ we have

$$\left\| F^{(1)}(\omega_0, \lambda_0) \right\|_{\mathcal{L}(Y)} \le \kappa_1 \quad \text{and} \quad \left\| F_1^{(1)}(\omega_0, \lambda_0) \right\|_{\mathcal{L}(Y_1)} \le \kappa_1. \tag{A.7}$$

(HC3) The mapping $(\omega_0, \lambda_0) \mapsto J \circ F^{(1)}(\omega_0, \lambda_0)$ is continuous as a map from $\Omega_0 \times \Lambda_0$ into $\mathcal{L}(Y, Y_1)$.

(HC4) The function F_0 has a continuous partial derivative

$$D_2 F_0 : \Omega_0 \times \Lambda_0 \to \mathcal{L}(\Lambda, Y). \tag{A.8}$$

(HC5) There exists some $\kappa_2 \in [0, 1)$ such that for all $y, \overline{y} \in Y$ and all $\lambda_0 \in \Lambda_0$ we have

$$\| F(y, \lambda_0) - F(\overline{y}, \lambda_0) \| \le \kappa_2 \| y - \overline{y} \| . \tag{A.9}$$

It follows from (HC5) that (A.2) has for each $\lambda_0 \in \Lambda_0$ a unique solution $\Psi = \Psi(\lambda_0)$. We assume that

(HC6) For some continuous $\Phi : \Lambda_0 \to \Omega_0$ we have $\Psi = J_0 \circ \Phi$.

We define $\kappa = \max(\kappa_1, \kappa_2)$.

Lemma A.1. *Assume that assumptions (HC1) through (HC6) hold, except possibly (HC3). Then Ψ is locally Lipschitz continuous.*

Proof. We calculate

$$\begin{aligned} \| \Psi(\lambda_0) - \Psi(\mu_0) \| &= \| F(\Psi(\lambda_0), \lambda_0) - F(\Psi(\mu_0), \mu_0) \| \\ &\le \| F(\Psi(\lambda_0), \lambda_0) - F(\Psi(\mu_0), \lambda_0) \| \\ &\quad + \| F_0(\Phi(\mu_0), \lambda_0) - F_0(\Phi(\mu_0), \mu_0) \| \\ &\le |\lambda_0 - \mu_0| \sup_{s \in [0,1]} \| D_2 F_0(\Phi(\mu_0), s\lambda_0 + (1-s)\mu_0) \| \\ &\quad + \kappa \| \Psi(\lambda_0) - \Psi(\mu_0) \| . \end{aligned} \tag{A.10}$$

Now fix some $\lambda_0 \in \Lambda_0$ and let $C(\lambda_0) > \|D_2F_0(\Phi(\lambda_0), \lambda_0)\|$. Since both D_2F_0 and Φ are continuous, there exists some $\delta > 0$ such that for all $\mu_0 \in \Lambda$ with $|\mu_0 - \lambda_0| < \delta$ we have $\mu_0 \in \Lambda_0$ and

$$\sup_{s\in[0,1]} \|D_2F_0(\Phi(\mu_0), s\lambda_0 + (1-s)\mu_0)\| \leq C(\lambda_0). \tag{A.11}$$

Using (A.10) we immediately conclude that for such μ_0 we have

$$\|\Psi(\lambda_0) - \Psi(\mu_0)\| \leq C(\lambda_0)(1-\kappa)^{-1} |\lambda_0 - \mu_0|, \tag{A.12}$$

which concludes the proof. □

Assuming that (HC1) through (HC6) hold, we can consider the following equation for $A \in \mathcal{L}(\Lambda, Y)$,

$$A = F^{(1)}(\Phi(\lambda_0), \lambda_0)A + D_2F_0(\Phi(\lambda_0), \lambda_0). \tag{A.13}$$

Since $\left\|F^{(1)}\right\|_{\mathcal{L}(Y)} \leq \kappa < 1$ by (HC2), we see that $I - F^{(1)}(\Phi(\lambda_0), \lambda_0)$ is invertible in $\mathcal{L}(Y)$ and hence for each $\lambda_0 \in \Lambda_0$ (A.13) has a unique solution $A = \mathcal{A}(\lambda_0) \in \mathcal{L}(\Lambda, Y)$.

Lemma A.2. *Assume that (HC1) through (HC6) hold. Then the mapping $J \circ \Psi$ is of class C^1 and $D(J \circ \Psi)(\lambda_0) = J \circ \mathcal{A}(\lambda_0)$ for all $\lambda_0 \in \Lambda_0$.*

Proof. Fix $\lambda_0 \in \Lambda_0$. For any $\mu_0 \in \Lambda_0$ write $S(\mu_0) = J\Psi(\mu_0) - J\Psi(\lambda_0) - J\mathcal{A}(\lambda_0)(\mu_0 - \lambda_0)$ and calculate

$$\begin{aligned}
S(\mu_0) &= JF(\Psi(\mu_0), \mu_0) - JF(\Psi(\lambda_0), \lambda_0) - JF^{(1)}(\Phi(\lambda_0), \lambda_0)\mathcal{A}(\lambda_0)(\mu_0 - \lambda_0) \\
&\quad - JD_2F_0(\Phi(\lambda_0), \lambda_0)(\mu_0 - \lambda_0) \\
&= G(\Phi(\mu_0), \mu_0) - G(\Phi(\lambda_0), \lambda_0) - JF^{(1)}(\Phi(\lambda_0), \lambda_0)\mathcal{A}(\lambda_0)(\mu_0 - \lambda_0) \\
&\quad - D_2G(\Phi(\lambda_0), \lambda_0)(\mu_0 - \lambda_0) \\
&= G(\Phi(\mu_0), \mu_0) - G(\Phi(\lambda_0), \mu_0) - JF^{(1)}(\Phi(\lambda_0), \lambda_0)\mathcal{A}(\lambda_0)(\mu_0 - \lambda_0) \\
&\quad + G(\Phi(\lambda_0), \mu_0) - G(\Phi(\lambda_0), \lambda_0) - D_2G(\Phi(\lambda_0), \lambda_0)(\mu_0 - \lambda_0) \\
&= JF^{(1)}(\Phi(\lambda_0), \lambda_0)[\Psi(\mu_0) - \Psi(\lambda_0) - \mathcal{A}(\lambda_0)(\mu_0 - \lambda_0)] + R(\lambda_0, \mu_0) \\
&= F_1^{(1)}(\Phi(\lambda_0), \lambda_0)[J\Psi(\mu_0) - J\Psi(\lambda_0) - J\mathcal{A}(\lambda_0)(\mu_0 - \lambda_0)] + R(\lambda_0, \mu_0),
\end{aligned} \tag{A.14}$$

where

$$\begin{aligned}
R(\lambda_0, \mu_0) &= \int_0^1 [JF^{(1)}\big(s\Phi(\mu_0) + (1-s)\Phi(\lambda_0), \mu_0\big) - JF^{(1)}(\Phi(\lambda_0), \lambda_0)] \\
&\quad [\Psi(\mu_0) - \Psi(\lambda_0)]ds \\
&\quad + \int_0^1 [D_2G(\Phi(\lambda_0), s\mu_0 + (1-s)\lambda_0) - D_2G(\Phi(\lambda_0), \lambda_0)] \\
&\quad [\mu_0 - \lambda_0]ds.
\end{aligned} \tag{A.15}$$

Using (HC3) and the continuity of D_2G and Φ, for each $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\begin{aligned}
\sup_{s\in[0,1]} \left\|JF^{(1)}(s\Phi(\mu_0) + (1-s)\Phi(\lambda_0), \mu_0) - JF^{(1)}(\Phi(\lambda_0), \lambda_0)\right\| &< \epsilon, \\
\sup_{s\in[0,1]} \|D_2G(\Phi(\lambda_0), s\mu_0 + (1-s)\lambda_0) - D_2G(\Phi(\lambda_0), \lambda_0)\| &< \epsilon,
\end{aligned} \tag{A.16}$$

whenever $|\mu_0 - \lambda_0| < \delta$. Letting $C(\lambda_0)$ be a Lipschitz constant for Ψ in a neighbourhood of λ_0, we obtain

$$\|R(\lambda_0, \mu_0)\| \leq \epsilon(C(\lambda_0) + 1)\,|\mu_0 - \lambda_0| \tag{A.17}$$

for $|\mu_0 - \lambda_0| < \delta$. From (A.14) and (HC2) it now follows that for such values of μ_0 we have

$$\|S(\mu_0)\|_1 \leq \epsilon \frac{C(\lambda_0)+1}{1-\kappa} |\mu_0 - \lambda_0|, \tag{A.18}$$

which shows that $J \circ \Psi$ is differentiable at λ_0 with $D(J \circ \Psi)(\lambda_0) = J \circ \mathcal{A}(\lambda_0)$. It remains to show that $\lambda_0 \mapsto J \circ \mathcal{A}(\lambda_0)$ is continuous. Since

$$\begin{aligned} J\mathcal{A}(\lambda_0) - J\mathcal{A}(\mu_0) &= JF^{(1)}(\Phi(\lambda_0), \lambda_0)\mathcal{A}(\lambda_0) + D_2G(\Phi(\lambda_0), \lambda_0) \\ &\quad - JF^{(1)}(\Phi(\mu_0), \mu_0)\mathcal{A}(\mu_0) - D_2G(\Phi(\mu_0), \mu_0) \\ &= F_1^{(1)}(J_0\Phi(\lambda_0), \lambda_0)(J\mathcal{A}(\lambda_0) - J\mathcal{A}(\mu_0)) \\ &\quad + \big(JF^{(1)}(\Phi(\lambda_0), \lambda_0) - JF^{(1)}(\Phi(\lambda_0), \mu_0)\big)\mathcal{A}(\mu_0) \\ &\quad + \big(JF^{(1)}(\Phi(\lambda_0), \mu_0) - JF^{(1)}(\Phi(\mu_0), \mu_0)\big)\mathcal{A}(\mu_0) \\ &\quad + D_2G(\Phi(\lambda_0), \lambda_0) - D_2G(\Phi(\mu_0), \mu_0), \end{aligned} \tag{A.19}$$

it follows that

$$\begin{aligned} (1-\kappa)\,\|J\mathcal{A}(\lambda_0) - J\mathcal{A}(\mu_0)\| &\leq \left\|JF^{(1)}(\Phi(\lambda_0), \lambda_0) - JF^{(1)}(\Phi(\lambda_0), \mu_0)\right\| \\ &\qquad \|\mathcal{A}(\mu_0)\| \\ &\quad + \left\|JF^{(1)}(\Phi(\lambda_0), \mu_0) - JF^{(1)}(\Phi(\mu_0), \mu_0)\right\| \\ &\qquad \|\mathcal{A}(\mu_0)\| \\ &\quad + \|D_2G(\Phi(\lambda_0), \lambda_0) - D_2G(\Phi(\mu_0), \mu_0)\|\,. \end{aligned} \tag{A.20}$$

Using the continuity of Φ, D_2G and $JF^{(1)}$, the continuity of $\lambda_0 \mapsto J \circ \mathcal{A}(\lambda_0)$ now easily follows. □

B. Fourier and Laplace Transform

We recall here the definitions of the Fourier transform $\widehat{f}(k)$ of an $L^2(\mathbb{R}, \mathbb{C}^n)$ function f and the inverse Fourier transform $\check{g}(\xi)$ for any $g \in L^2(\mathbb{R}, \mathbb{C}^n)$, given by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi) d\xi, \qquad \check{g}(\xi) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\xi} g(k) dk. \tag{B.1}$$

We remark here that the integrals above are well defined only if $f, g \in L^1(\mathbb{R}, \mathbb{C}^n)$. If this is not the case, the integrals have to be understood as integrals in the Fourier sense, i.e., the functions

$$h_n(k) = \int_{-n}^{n} e^{-ik\xi} f(\xi) d\xi \tag{B.2}$$

satisfy $h_n \to \widehat{f}$ in $L^2(\mathbb{R}, \mathbb{C}^n)$ and in addition there is a subsequence $\{n'\}$ such that $h_{n'}(k) \to \widehat{f}(k)$ almost everywhere. We recall that the Fourier transform takes convolutions into products, i.e., $(\widehat{f * g})(k) = \widehat{f}(k)\widehat{g}(k)$ for almost every k. As another useful tool, we state the Riemann Lebesgue lemma [78, Thm. 21.39].

Lemma B.1. *For any $f \in L^1(\mathbb{R}_+, \mathbb{C}^n)$, we have*

$$\lim_{\omega \to \pm\infty} \left| \int_0^\infty e^{i\omega\xi} f(\xi) d\xi \right| = 0. \tag{B.3}$$

Suppose $f : \mathbb{R} \to \mathbb{C}^n$ satisfies $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$. Then for any z with $\operatorname{Re} z > -a$, define the Laplace transform

$$\widetilde{f}_+(z) = \int_0^\infty e^{-z\xi} f(\xi) d\xi. \tag{B.4}$$

Similarly, if $f(\xi) = O(e^{b\xi})$ as $\xi \to -\infty$, then for any z with $\operatorname{Re} z < b$, define

$$\widetilde{f}_-(z) = \int_0^\infty e^{z\xi} f(-\xi) d\xi. \tag{B.5}$$

The inverse transformation is described in the next result, which can be found in the standard literature on Laplace transforms [165, 7.3-5].

Lemma B.2. *Let $f : \mathbb{R} \to \mathbb{C}^n$ satisfy a growth condition $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ and suppose that f is of bounded variation on bounded intervals. Then for any $\gamma > -a$ and $\xi > 0$ we have the inversion formula*

$$\frac{f(\xi+) + f(\xi-)}{2} = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz, \tag{B.6}$$

whereas for $\xi = 0$ we have

$$\frac{f(0+)}{2} = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{z\xi} \widetilde{f}_+(z) dz. \tag{B.7}$$

C. Hopf Bifurcation Theorem

In this appendix we state the Hopf bifurcation theorem for the finite dimensional system of ODE's

$$x' = g(x, \mu), \tag{C.1}$$

for $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^n$, where g satisfies the following assumptions.

(HH1) For some integer $k \geq 2$ we have $g \in C^k(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, with $g(0, \mu) = 0$ for all $\mu \in \mathbb{R}$.

(HH2) For some $\mu_0 \in \mathbb{R}$ the matrix $A = D_1 g(0, \mu_0)$ has simple (i.e. of algebraic multiplicity one) eigenvalues at $\pm i\omega_0$, where $\omega_0 > 0$. In addition, no other eigenvalue of A belongs to $i\omega_0\mathbb{Z}$.

(HH3) Writing $\sigma(\mu)$ for the branch of eigenvalues of $D_1 g(0, \mu)$ through $i\omega_0$ at $\mu = \mu_0$, we have $\operatorname{Re} D\sigma(\mu_0) \neq 0$.

Finally, we define the non-zero vector $v \in \mathbb{R}^n$ to be an arbitrary eigenvector of the matrix A at the eigenvalue $i\omega_0$ and we let $w \in \mathbb{R}^n$ be an arbitrary eigenvector of A^T at $i\omega_0$ normalized such that $w^T v = 1$, i.e., the spectral projection $P_{i\omega_0}$ corresponding to the eigenvalue $i\omega_0$ is given by $P_{i\omega_0} x = v w^T x$. The following results are stated as in [45] and we refer to a paper by Crandall and Rabinowitz [38] for proofs and additional information.

Theorem C.1. *Consider (C.1) and suppose that (HH1)-(HH3) are satisfied. Then there exist C^{k-1}-smooth functions $\tau \to \mu^*(\tau) \in \mathbb{R}$, $\tau \to \omega^*(\tau) \in \mathbb{R}$ and $\tau \to x^*(\tau) \in C(\mathbb{R}, \mathbb{R}^n)$, all defined for τ sufficiently small, such that at $\mu = \mu^*(\tau)$, $x^*(\tau)$ is a $\frac{2\pi}{\omega^*(\tau)}$ periodic solution of (C.1). Moreover, μ^* and ω^* are even, $\mu(0) = \mu_0$, $\omega(0) = \omega_0$, $x^*(-\tau)(\xi) = x^*(\tau)(\xi + \frac{\pi}{\omega^*(\tau)})$ and $x^*(\tau)(\xi) = \tau \mathrm{Re}\,(e^{i\omega_0\xi}v) + o(\tau)$, as $\tau \to 0$, uniformly on compact subsets of $\mathbb{R}$. In addition, if x is a small periodic solution of this equation with μ close to μ_0 and minimal period close to $\frac{2\pi}{\omega_0}$, then $x(\xi) = x^*(\tau)(\xi + \xi_0)$ and $\mu = \mu^*(\tau)$ for some τ and $\xi_0 \in [0, 2\pi/\omega^*(\tau))$, with τ unique modulo its sign.*

We conclude this appendix with a result on the direction of bifurcation.

Theorem C.2. *Consider (C.1) and suppose that (HH1)-(HH3) are satisfied, but with $k \geq 3$. Let μ^* be as defined in Theorem C.1. Then we have $\mu^*(\tau) = \mu_0 + \mu_2\tau^2 + o(\tau^2)$, with*

$$\mu_2 = -\frac{\mathrm{Re}\, c}{\mathrm{Re}\, D\sigma(\mu_0)}. \tag{C.2}$$

The constant c is uniquely determined by the following identity

$$\begin{aligned} cv \;=\; & \tfrac{1}{2} P_{i\omega_0} D_1^3 g(0,\mu_0)(v, v, \overline{v}) \\ & + P_{i\omega_0} D_1^2 g(0,\mu_0)(v, -D_1 g(0,\mu_0)^{-1} D_1^2 g(0,\mu_0)(v,\overline{v})) \\ & + \tfrac{1}{2} P_{i\omega_0} D_1^2 g(0,\mu_0)(\overline{v}, (2i\omega_0 - D_1 g(0,\mu_0))^{-1} D_1^2 g(0,\mu_0)(v,v)). \end{aligned} \tag{C.3}$$

D. Nested Differentiation

We conclude the appendix with the following result on differentiation in nested spaces.

Lemma D.1. *Consider an integer $\ell > 1$ and a sequence of Banach spaces $Y_0 \subset Y_1 \subset \ldots \subset Y_\ell$, in which each inclusion $\mathcal{J}_{ji} : Y_i \to Y_j$ with $j > i$ is continuous. Let Z_0 and Z_1 be Banach spaces and $[a, b] \subset \mathbb{R}$ be an interval. Consider functions $M : [a, b] \to \mathcal{L}(Z_0, Y_0)$ and $L : [a, b] \to \mathcal{L}(Y_\ell, Z_1)$ with the following properties.*

(i) For each $0 \leq j \leq \ell$, we have that the map $[a, b] \to \mathcal{L}(Z_0, Y_j)$ given by $\xi \mapsto \mathcal{J}_{j0}M(\xi)$ is C^j-smooth.

(ii) For every $0 \leq q \leq j$, we have that

$$D^q \mathcal{J}_{j0} M(\xi) = \mathcal{J}_{jq} D^q \mathcal{J}_{q0} M(\xi). \tag{D.1}$$

(iii) For each $0 \leq j \leq \ell$, we have that the restriction map $[a, b] \to \mathcal{L}(Y_j, Z_1)$ given by $\xi \mapsto (L(\xi))_{|Y_j}$ is $C^{\ell - j}$-smooth.

Then the map $[a, b] \to \mathcal{L}(Z_0, Z_1)$ given by $\xi \mapsto L(\xi)\mathcal{J}_{\ell 0}M(\xi)$ is C^ℓ-smooth.

Proof. For convenience, define the maps $L_j = L_{|Y_j}$ and $M_j = \mathcal{J}_{j0}M$. Notice first that for any $p \geq 0$ and $q \geq 0$ with $p + q \leq \ell$, we have that the function $W^{p,q} : [a, b] \to \mathcal{L}(Z_0, Z_1)$ defined by

$$W^{p,q}(\xi) = D^q L_p(\xi) D^p M_p(\xi) \tag{D.2}$$

is well-defined and continuous. Associated to a given C^j-smooth operator $S : [a,b] \to \Omega$, we define the usual remainder functions $R_S^{(j)} : [a,b] \times [a,b] \to \Omega$ by

$$R_S^{(j)}(\xi,\xi') = S(\xi') - \sum_{k=0}^{j} D^k S(\xi) \frac{(\xi' - \xi)^k}{k!} \tag{D.3}$$

and observe that $\left\| R_S^{(j)}(\xi,\xi') \right\| = o(\left|\xi - \xi'\right|^j)$.

Now notice that

$$\begin{aligned} L(\xi')M_\ell(\xi') &= L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \textstyle\sum_{k=0}^{\ell} L(\xi')D^k M_\ell(\xi) \frac{(\xi'-\xi)^k}{k!} \\ &= L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \textstyle\sum_{k=0}^{\ell} L(\xi')\mathcal{J}_{\ell k} D^k M_k(\xi) \frac{(\xi'-\xi)^k}{k!} \\ &= L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \textstyle\sum_{k=0}^{\ell} L_k(\xi')D^k M_k(\xi) \frac{(\xi'-\xi)^k}{k!}. \end{aligned} \tag{D.4}$$

Recalling that

$$L_k(\xi') = R_{L_k}^{(\ell-k)}(\xi,\xi') + \textstyle\sum_{m=0}^{\ell-k} D^m L_k(\xi) \frac{(\xi'-\xi)^m}{m!}, \tag{D.5}$$

one can write

$$\begin{aligned} L(\xi')M_\ell(\xi') - L(\xi)M_\ell(\xi) \;=\; & \textstyle\sum_{(p\geq 0, q\geq 0)|1\leq p+q\leq \ell} c_{p,q} W^{p,q}(\xi' - \xi)^{p+q} \\ & + \textstyle\sum_{k=0}^{\ell} R_{L_k}^{(\ell-k)}(\xi,\xi')D^k M_k(\xi)(\xi' - \xi)^k \frac{1}{k!} \\ & + L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi'), \end{aligned} \tag{D.6}$$

for appropriate constants $c_{p,q}$, which shows that indeed $D^\ell[L\mathcal{J}_{\ell 0}M]$ can be properly defined in a continuous fashion. □

Bibliography

[1] K. A. Abell, C. E. Elmer, A. R. Humphries, and E. S. Van Vleck (2005), Computation of Mixed Type Functional Differential Boundary Value Problems. *SIAM J. Appl. Dyn. Sys.* **4**, 755–781.

[2] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier (1971), *Theory of Bifurcations of Dynamic Systems on a Plane*. Israel Program of Scientific Translations, Jerusalem.

[3] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier (1973), *Theory of Dynamic Systems on a Plane*. Israel Program of Scientific Translations, Jerusalem.

[4] A. A. Andronov and L. Pontryagin (1937), Systèmes Grossiers. *Dokl. Akad. Nauk. SSSR* **14**, 247–251.

[5] A. A. Andronov, E. A. Vitt, and S. E. Khaiken (1966), *Theory of Oscillators*. Pergamon Press, Oxford.

[6] V. I. Arnold (1973), *Ordinary Differential Equations*. M.I.T. Press, Cambrige, MA (Russian original, Moscow, 1971).

[7] V. I. Arnold (1978), *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, Heidelberg, Berlin (Russian original, Moscow, 1974).

[8] V. I. Arnold (1982), *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer-Verlag, New York, Heidelberg, Berlin (Russian original, Moscow, 1977).

[9] P. K. Asea and P. J. Zak (1999), Time-to-Build and Cycles. *J. Econ. Dyn. and Contr.* **23**, 1155–1175.

[10] G. Bader (1985), Solving Boundary Value Problems for Functional Differential Equations by Collocation. In: U. M. Ascher and R. D. Russel (eds.): *Proceedings in Scientific Computing, Numerical Boundary Value ODEs*, Vol. 5. pp. 227–243.

[11] H. Bart, I. Gohberg, and M. A. Kaashoek (1986), Wiener-Hopf Factorization, Inverse Fourier Transforms and Exponentially Dichotomous Operators. *J. Funct. Anal.* **68**, 1–42.

[12] P. W. Bates, X. Chen, and A. Chmaj (2003), Traveling Waves of Bistable Dynamics on a Lattice. *SIAM J. Math. Anal.* **35**, 520–546.

[13] P. W. Bates and A. Chmaj (1999), A discrete Convolution Model for Phase Transitions. *Arch. Rational Mech. Anal.* **150**, 281–305.

[14] P. W. Bates and C. K. Jones (1989), Invariant Manifolds for Semilinar Partial Differential Equations. *Dynamics Reported* **2**, 1–38.

[15] J. Bell (1981), Some Threshold Results for Models of Myelinated Nerves. *Math. Biosciences* **54**, 181–190.

[16] R. Bellman and K. Cooke (1963), *Differential Difference Equations*. Academic Press, New York.

[17] J. Benhabib and K. Nishimura (1979), The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth. *J. Econ. Theory* **21**, 421–444.

[18] L. M. Benveniste and J. A. Scheinkman (1982), Duality Theory for Dynamic Optimization Models of Economics: The Continuous Time Case. *J. Econ. Theory* **27**, 1–19.

[19] S. Benzoni-Gavage (1998), Semi-Discrete Shock Profiles for Hyperbolic Systems of Conservation Laws. *Physica D* **115**, 109–124.

[20] S. Benzoni-Gavage and P. Huot (2002), Existence of Semi-Discrete Shocks. *Discrete Contin. Dyn. Syst.* **8**, 163–190.

[21] S. Benzoni-Gavage, P. Huot, and F. Rousset (2003), Nonlinear Stability of Semidiscrete Shock Waves. *SIAM J. Math. Anal.* **35**, 639–707.

[22] G. D. Birkhoff (1966), *Dynamical Systems*, With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX. American Mathematical Society, Providence, R.I.

[23] K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Society for Industrial and Applied Mathematics, Philadelphia.

[24] G. D. Byrne and P. R. Ponzi (1988), Differential-Algebraic Systems, Their Applications and Solutions. *Comput. Chem. Engineering* **12**, 377–382.

[25] J. W. Cahn (1960), Theory of Crystal Growth and Interface Motion in Crystalline Materials. *Acta Met.* **8**, 554–562.

[26] J. W. Cahn, J. Mallet-Paret, and E. S. Van Vleck (1999), Traveling Wave Solutions for Systems of ODE's on a Two-Dimensional Spatial Lattice. *SIAM J. Appl. Math.* **59**, 455–493.

[27] S. L. Campbell (1980), *Singular Systems of Differential Equations*, Vol. 40 of *Research Notes in Mathematics*. Pitman Pub. Ltd., London.

[28] S. L. Campbell (1994), Numerical Methods for Unstructured Higher Index DAEs. *Ann. Numer. Math.* **1**, 265–277.

[29] S. L. Campbell and W. Marszalek (1997), DAEs Arising from Traveling Wave Solutions of PDEs. *J. Comp. Applied Math.* **82**, 41–58.

[30] J. Carr (1981), *Applications of Centre Manifold Theory*, Vol. 35 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[31] J. R. Cash, G. Moore, and R. W. Wright (1995), An Automatic Continuation Strategy for the Solution of Singularly Perturbed Linear Two-Point Boundary Value Problems. *J. Comp. Phys.* **122**, 266–279.

[32] H. Chi, J. Bell, and B. Hassard (1986), Numerical Solution of a Nonlinear Advance-Delay-Differential Equation from Nerve Conduction Theory. *J. Math. Bio.* **24**, 583–601.

[33] S. N. Chow and B. Deng (1989), Bifurcation of a Unique Periodic Orbit from a Homoclinic Orbit in Infinite-Dimensional Systems. *Trans. Amer. Math. Soc.* **312**, 539–587.

[34] S. N. Chow and J. K. Hale (1982), *Methods of Bifurcation Theory*. Springer-Verlag, New York.

[35] S. N. Chow, J. Mallet-Paret, and W. Shen (1998), Traveling Waves in Lattice Dynamical Systems. *J. Diff. Eq.* **149**, 248–291.

[36] L. O. Chua and T. Roska (1993), The CNN paradigm. *IEEE Trans. Circ. Syst.* **40**, 147–156.

[37] W. A. Coppel (1978), *Dichotomies in Stability Theory*, Vol. 629 of *Lecture Notes in Mathematics*. Springer Verlag, New York.

[38] M. C. Crandall and P. H. Rabinowitz (1978), The Hopf Bifurcation Theorem in Infinite Dimensions. *Arch. Rat. Mech. Anal.* **67**, 53–72.

[39] H. d'Albis and E. Augeraud-Véron, Endogenous Retirement and Monetary Cycles. *Preprint*.

[40] H. d'Albis and E. Augeraud-Véron (2004), Competitive Growth in a Life-Cycle Model: Existence and Dynamics. *Preprint*.

[41] H. d'Albis and E. Augeraud-Véron (2007), Balanced Cycles in an OLG Model with a Continuum of Finitely-lived Individuals. *J. Econ. Theory* **30**, 181–186.

[42] T. Dauxois, M. Peyrard, and A. R. Bishop (1993), Dynamics and Thermodynamics of a Nonlinear Model for DNA Denaturation. *Phys. Rev. E* **47**, 684–695.

[43] P. J. Davis and P. Rabinowitz (1975), *Methods of Numerical Integration*. Academic Press, New York.

[44] O. Diekmann and S. A. van Gils (1984), Invariant Manifolds for Volterra Integral Equations. *J. Diff. Eq.* **54**, 139–180.

[45] O. Diekmann, S. A. van Gils, S. M. Verduyn-Lunel, and H. O. Walther (1995), *Delay Equations*. Springer-Verlag, New York.

[46] J. Dieudonné (1960), *Foundations of Modern Analysis*. Academic Press, New York.

[47] P. H. Douglas (1948), Are there Laws of Production? *American Economic Review* **38**, 1–41.

[48] G. Duffing (1918), *Erzwungene Schwingungen bei Veränderlicher Eigenfrequenz*. Braunschweig, Vieweg.

[49] C. M. Elliott, A. R. Gardiner, I. Kostin, and B. Lu (1994), Mathematical and Numerical Analaysis of a Mean-Field Equation for the Ising Model with Glauber Dynamics. In: *Chaotic Numerics*, AMS, Vol. 172. pp. 217–242.

[50] C. E. Elmer and E. S. Van Vleck (1996), Computation of Traveling Waves for Spatially Discrete Bistable Reaction-Diffusion Equations. *Appl. Numer. Math.* **20**, 157–169.

[51] C. E. Elmer and E. S. Van Vleck (1999), Analysis and Computation of Traveling Wave Solutions of Bistable Differential-Difference equations. *Nonlinearity* **12**, 771–798.

[52] C. E. Elmer and E. S. Van Vleck (2001), Traveling Wave Solutions for Bistable Differential Difference Equations with Periodic Diffusion. *SIAM J. Appl. Math.* **61**, 1648–1679.

[53] C. E. Elmer and E. S. Van Vleck (2002), A Variant of Newton's Method for the Computation of Traveling Waves of Bistable Differential-Difference Equations. *J. Dyn. Diff. Eq.* **14**, 493–517.

[54] C. E. Elmer and E. S. Van Vleck (2005), Dynamics of Monotone Travelling Fronts for Discretizations of Nagumo PDEs. *Nonlinearity* **18**, 1605–1628.

[55] K. J. Engel and R. Nagel (2000), *One-Parameter Semigroups for Linear Evolution Equations*, Vol. 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[56] K. Engelborghs, T. Luzyanina, and D. Roose (2002), Numerical Bifurcation Analysis of Delay Differential Equations using DDE-BIFTOOL. *ACM Transactions on Mathematical Software* **28**, 1–21.

[57] T. Erneux and G. Nicolis (1993), Propagating Waves in Discrete Bistable Reaction-Diffusion Systems. *Physica D* **67**, 237–244.

[58] N. Fenichel (1979), Geometric Singular Perturbation Theory for Ordinary Differential Eequations. *J. Diff. Eq.* **31**, 53–98.

[59] E. Fermi, J. Pasta, and S. Ulam (1955), Studies of Nonlinear Problems. Technical Report LA-1940, Los Alamos National Laboratory.

[60] B. Fiedler (1994), Global Attractors of One-Dimensional Parabolic Equations: Sixteen Examples. *Tatra Mountains Math. Publ.* **4**, 67–92.

[61] B. Fiedler and C. Rocha (1996), Heteroclinic Orbits of Semilinear Parabolic Equations. *J. Diff. Eq.* **125**, 239–281.

[62] P. Fife and J. McLeod (1977), The Approach of Solutions of Nonlinear Diffusion Equations to Traveling Front Solutions. *Arch. Rat. Mech. Anal.* **65**, 333–361.

[63] N. Flytzanis, S. Pnevmatikos, and M. Remoissenet (1985), Kink, Breather and Asymmetric Envelope or Dark Solitons in Nonlinear Chains. I Monoatomic Chain. *J. Phys. C* **18**, 4603–4629.

[64] C. Foias and R. Temam (1994), Approximation of Attractos by Algebraic or Analytic Sets. *SIAM J. Math. Anal.* **25**, 1269–1302.

[65] M. V. S. Frasson (2005), Large Time Behaviour of Neutral Delay Systems. *PhD Thesis*.

[66] M. V. S. Frasson and S. M. Verduyn-Lunel (2003), Large Time Behaviour of Linear Functional Differential Equations. *Integral Eq. and Operator Theory* **47**, 91–121.

[67] G. Friesecke and J. A. Wattis (1994), Existence Theorem for Solitary Waves on Lattices. *Comm. Math. Phys.* **161**, 391–418.

[68] N. K. Gavrilov (1987), Bifurcations of an Equilibrium State with One Zero Root and a Pair of Purely Imaginary Roots and Additional Degeneration. In: E. A. Leontovich-Andronova (ed.): *Methods of the Qualitative Theory of Differential Equations*. Gor′kov. Gos. Univ., Gorki, pp. 43–51.

[69] M. Georgi (2008), Bifurcations from Homoclinic Orbits to Non-Hyperbolic Equilibria in Reversible Lattice Differential Equations. *Nonlinearity* **21**, 735–763.

[70] J. Giannoulis and A. Mielke (2004), The Nonlinear Schrödinger Equation as a Macroscopic limit for an oscillator chain with Cubic Nonlinearities. *Nonlinearity* **17**, 551–565.

[71] J. Guckenheimer and P. Holmes (1983), *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Vol. 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[72] J. K. Hale and S. M. Verduyn-Lunel (1993), *Introduction to Functional Differential Equations*. Springer–Verlag, New York.

[73] D. Hankerson and B. Zinner (1993), Wavefronts for a Cooperative Tridiagonal System of Differential Equations. *J. Dyn. Diff. Eq.* **5**, 359–373.

[74] J. Harlim and W. F. Langford (2007), The Cusp-Hopf Bifurcation. *Int. J. of Bifurcation and Chaos* **17**, 2547–2570.

[75] J. Härterich, B. Sandstede, and A. Scheel (2002), Exponential Dichotomies for Linear Non-Autonomous Functional Differential Equations of Mixed Type. *Indiana Univ. Math. J.* **51**, 1081–1109.

[76] H. Hemami and B. F. Wyman (1979), Modeling and Control of Constrained Dynamic Systems with Application to Biped Locomotion in the Frontal Plane. *IEEE Trans. Automat. Contrl.* **24**, 526–535.

[77] D. Henry (1981), *Geometric Theory of Semilinear Parabolic Equations*, Vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

[78] E. Hewitt and K. Stromberg (1965), *Real and Abstract Analysis*. Springer-Verlag, Berlin.

[79] E. Hille (1948), *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, New York.

[80] A. J. Homburg, H. Kokubu, and M. Krupa (1994), The Cusp Horseshoe and its Bifurcations from Inclination-Flip Homoclinic Orbits. *Ergod. Theory Dyn. Syst.* **14**, 667–693.

[81] D. K. Hughes (1968), Variational and Optimal Control Problems with Delayed Argument. *J. Optimization Theory Appl.* **2**, 1–14.

[82] H. J. Hupkes and S. M. Verduyn-Lunel (2003), Analysis of Newton's Method to Compute Travelling Wave Solutions to Lattice Differential Equations. Technical Report 2003–09, Mathematical Institute Leiden.

[83] A. F. Huxley and R. Stampfli (1949), Evidence for Saltatory Conduction in Peripheral Meylinated Nerve Fibres. *J. Physiology* **108**, 315–339.

[84] Y. Ioannides and B. Taub (1992), On Dynamics with Time-to-Build Investment Technology and Non-Time-Separable Leisure. *J. Econ. Dyn. and Contr.* **16**, 225–241.

[85] G. Iooss (2000), Traveling Waves in the Fermi-Pasta-Ulam Lattice. *Nonlinearity* **13**, 849–866.

[86] G. Iooss and G. James (2005), Localized Waves in Nonlinear Oscillator Chains. *Chaos* **15**, 015113.

[87] G. Iooss and K. Kirchgässner (2000), Traveling Waves in a Chain of Coupled Nonlinear Oscillators. *Comm. Math. Phys.* **211**, 439–464.

[88] G. James and Y. Sire (2005), Traveling Breathers with Exponentially Small Tails in a Chain of Nonlinear Oscillators. *Comm. Math. Phys.* **257**, 51–85.

[89] M. A. Kaashoek and S. M. Verduyn-Lunel (1992), Characteristic Matrices and Spectral Properties of Evolutionary Systems. *Trans. Amer. Math. Soc.* **334**, 479–517.

[90] M. A. Kaashoek and S. M. Verduyn-Lunel (1994), An Integrability Condition on the Resolvent for Hyperbolicity of the Semigroup. *J. Diff. Eq.* **112**, 374–406.

[91] A. Katok and B. Hasselblatt (1995), *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, NY.

[92] J. Keener and J. Sneed (1998), *Mathematical Physiology*. Springer–Verlag, New York.

[93] J. P. Keener (1987), Propagation and its Failure in Coupled Systems of Discrete Excitable Cells. *SIAM J. Appl. Math.* **47**, 556–572.

[94] K. Kirchgässner (1982), Wave Solutions of Reversible Systems and Applications. *J. Diff. Eqns.* **45**, 113–127.

[95] M. Kisaka, H. Kokubu, and H. Oka (1993), Bifurcation to N-homoclinic Orbits and N-periodic orbits in Vector Fields. *J. Dyn. Diff. Eq.* **5**, 305–358.

[96] C. Kittel (1996), *Introduction to Solid State Physics*. Wiley, New York.

[97] T. A. Kontorova and Y. J. Frenkel (1938), On the Theory of Plastic Deformation and Twinning. *Zh. Eksp. Teor. Fiz.* **8**, 1340–1348.

[98] S. G. Krantz (1999), *Handbook of Complex Variables*. Birkhäuser, Boston, MA.

[99] A. Kumar and P. Daoutidis (1997), High Index DAE Systems in Modeling and Control of Chemical Processes. In: *Preprints of IFAC Conf. on Control of Industrial Systems*, Vol. 1. pp. 518–523.

[100] Y. Kuznetsov (2004), *Elements of Applied Bifurcation Theory*, Vol. 112 of *Applied Mathematical Sciences*. Springer–Verlag, New York.

[101] F. E. Kydland and E. C. Prescott (1982), Time to Build and Aggregate Fluctuations. *Econometrica* **50**, 1345–1370.

[102] W. F. Langford (1983), A Review of Interactions of Hopf and Steady-State Bifurcations. In: G. I. Barenblatt, G. Iooss, and D. D. Joseph (eds.): *Nonlinear Dynamics and Turbulence*. Pitman, Boston, MA, pp. 215–237.

[103] W. F. Langford (1984), Numerical Studies of Torus Bifurcations. In: T. Küpper, H. D. Mittelman, and H. Weber (eds.): *Numerical Methods for Bifurcation Problems*, Vol. ISNM 70. pp. 285–295.

[104] J. P. Laplante and T. Erneux (1992), Propagation Failure in Arrays of Coupled Bistable Chemical Reactors. *J. Phys. Chem.* **96**, 4931–4934.

[105] X. Liao and J. Yu (1998), Qualitative Analysis of Bi-directional Associative Memory with Time Delay. *Int. J. Circ. Theory and Appl.* **26**, 219–229.

[106] X. B. Lin (1990), Using Melnikov's Method to Solve Shilnikov's Problems. *Proc. Roy. Soc. Edinburgh* **116**, 295–325.

[107] E. N. Lorenz (1963), Deterministic Non-Periodic Flow. *J. Atmos. Sci.* **20**, 130–141.

[108] T. Luzyanina and K. Engelborghs (2002), Computing Floquet Multipliers for Functional Differential Equations. *Int. J. of Bifurcation and Chaos* **12**, 2977–2989.

[109] A. M. Lyapunov (1949), *Problème Général de la Stabilité du Mouvement*, Vol. 17 of *Annals of Mathematical Studies*. Princeton University Press, Princeton.

[110] J. Mallet-Paret (1976), Negatively Invariant Sets of Compact Maps and an Extension of a Theorem of Cartwright. *J. Diff. Eq.* **22**, 331–348.

[111] J. Mallet-Paret (1996), Spatial Patterns, Spatial Chaos and Traveling Waves in Lattice Differential Equations. In: *Stochastic and Spatial Structures of Dynamical Systems*, Royal Netherlands Academy of Sciences. Proceedings, Physics Section. Series 1, Vol. 45. Amsterdam, pp. 105–129.

[112] J. Mallet-Paret (1999), The Fredholm Alternative for Functional Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **11**, 1–48.

[113] J. Mallet-Paret (1999), The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems. *J. Dyn. Diff. Eq.* **11**, 49–128.

[114] J. Mallet-Paret (2001), Crystallographic Pinning: Direction Dependent Pinning in Lattice Differential Equations. *Preprint*.

[115] J. Mallet-Paret and S. M. Verduyn-Lunel, Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differential Equations. *J. Diff. Eq., to appear*.

[116] R. Mañé (1981), On the Dimension of the Compact Invariant Sets of Certain Nonlinear Maps. *Springer Lect. Notes Math.* **898**, 230–242.

[117] N. H. McClamroch (1986), Singular Systems of Differential Equations as Dynamic Models for Constrained Robot Systems. In: *Proc. of the IEEE International Conf. on Robotics and Automation*. pp. 21–28.

[118] A. Mielke (1986), A Reduction Principle for Nonautonomous Systems in Infinite-Dimensional Spaces. *J. Diff. Eq.* **65**, 68–88.

[119] A. Mielke (1994), Floquet Theory for, and Bifurcations from Spatially Periodic Patterns. *Tatra Mountains Math. Publ.* **4**, 153–158.

[120] P. Morrel and W. T. Norton (1980), Myeline. *Scientific American* **242**, 88–118.

[121] P. C. Müller (1997), Linear Mechanical Descriptor Systems: Identification, Analysis and Design. In: *Preprints of IFAC Conf. on Control of Industrial Systems*, Vol. 1. pp. 501–506.

[122] R. W. Newcomb (1981), The Semistatic Description of Nonlinear Time-Variable Circuits. *IEEE Trans. Circ. Syst.* **28**, 62–71.

[123] A. Pankov and K. Pflüger (2000), Travelling Waves in Lattice Dynamical Systems. *Math. Meth. Appl. Sci* **23**, 1223–1235.

[124] O. Perron (1928), Über Stabilität und Asymptotisches Verhalten der Integrale von Differentialgleichungssystemen. *Math Zeit* **29**, 129–160.

[125] V. Pliss and G. R. Sell (1999), Robustness of Exponential Dichotomies in Infinite-Dimensional Dynamical Systems. *J. Dyn. Diff. Eq.* **11**, 471–513.

[126] J. F. Pommaret (1983), *Differential Galois Theory*. Gordon & Breach Science Publishers, New York.

[127] M. Remoissenet (1986), Low-Amplitude Breather and Envelope Solitons in Quasi-One-Dimensional Physical Models. *Phys. Rev. B* **33**, 2386–2392.

[128] W. Rudin (1974), *Real and Complex Analysis*. McGraw–Hill, New York.

[129] W. Rudin (1991), *Functional Analysis*. McGraw–Hill, New York.

[130] A. Rustichini (1989), Functional Differential Equations of Mixed Type: the Linear Autonomous Case. *J. Dyn. Diff. Eq.* **11**, 121–143.

[131] A. Rustichini (1989), Hopf Bifurcation for Functional-Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **11**, 145–177.

[132] J. S. W. L. S. Guo and B. W. Rink (2007), Branching Patterns of Wave Trains in the FPU Lattice. *Preprint*.

[133] R. J. Sacker and G. R. Sell (1994), Dichotomies for Linear Evolutionary Equations in Banach Spaces. *J. Diff. Eq.* **113**, 17–67.

[134] B. Sandstede (1993), Verzweigungstheorie Homokliner Verdopplungen. *PhD Thesis*.

[135] B. Sandstede and A. Scheel (2001), On the Structure of Spectra of Modulated Travelling Waves. *Math. Nachr.* **232**, 39–93.

[136] B. Scarpellini (1991), Center Manifolds of Infinite Dimensions I: Main Results and Applications. *Z. Angew. Math. Phys.* **42**, 1–32.

[137] H. Schwetlick and J. Zimmer (2007), Solitary Waves for Nonconvex FPU Lattices. *J. Nonlinear Science* **17**, 1–12.

[138] G. R. Sell and Y. You (2002), *Dynamics of Evolutionary Equations*, Vol. 143 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[139] M. Selva-Soto and C. Tischendorf (2005), Numerical Analysis of DAEs from Coupled Circuit and Semiconductor Simulation. *Appl. Numer. Math* **53**, 471–488.

[140] L. P. Shilnikov (1962), Some Cases of Generation of Periodic Motions in an n-dimensional Space. *Soviet Math. Dokl.* **3**, 394–397.

[141] L. P. Shilnikov (1965), A Case of the Existence of a Countable Number of Periodic Motions. *Soviet Math. Dokl.* **6**, 163–166.

[142] L. P. Shilnikov (1966), On the Generation of a Periodic Motion from a Trajectory which Leaves and Re-enters a Saddle-Saddle State of Equilibrium. *Soviet Math. Dokl.* **7**, 1155–1158.

[143] L. P. Shilnikov (1968), On the Generation of a Periodic Motion from Trajectories Doubly Asymptotic to an Equilibrium State of Saddle Type. *Math. USSR Sbornik* **6**, 427–437.

[144] L. P. Shilnikov (1969), On a New Type of Bifurcation of Multidimensional Dynamical Systems. *Soviet Math. Dokl.* **10**, 1368–1371.

[145] L. P. Shilnikov (1970), A Contribution to a Problem of the Structure of an Extended Neighborhood of a Rough Equilibrium State of Saddle-Focus Type. *Math. USSR Sbornik* **10**, 91–102.

[146] A. J. Sievers and S. Takeno (1988), Intrinsic Localized Modes in Anharmonic Crystals. *Phys. Rev. Lett.* **61**, 970–973.

[147] Y. Sire (2005), Travelling Breathers in Klein-Gordon Lattices as Homoclinic Orbits to p-Tori. *J. Dyn. Diff. Eq.* **17**, 779–823.

[148] Y. Sire and G. James (2004), Traveling Breathers in Klein-Gordon Chains. *CR Acad. Sci. Ser. I: Math.* **338**, 661–666.

[149] A. L. Skubachevskii and H. O. Walther (2006), On the Floquet Multipliers of Periodic Solutions to Non-linear Functional Differential Equations. *J. Dyn. Diff. Eq.* **18**, 257–355.

[150] D. Smets and M. Willem (1997), Solitary Waves with Prescribed Speed on Infinite Lattices. *J. Funct. Anal.* **149**, 266–275.

[151] T. Strunz and F. J. Elmer (1998), Driven Frenkel-Kontorova model I. Uniform Sliding States and Dynamical Domains of Different Particle Densities. *Phys. Rev. E* **58**, 1601–1611.

[152] T. Strunz and F. J. Elmer (1998), Driven Frenkel-Kontorova model II. Chaotic Sliding and Nonequilibrium Melting and Freezing. *Phys. Rev. E* **58**, 1612–1620.

[153] R. Szalai, G. Stépán, and S. J. Hogan (2006), Continuation of Bifurcations in Periodic Delay-Differential Equations using Characteristic Matrices. *SIAM J. Sci. Comput.* **28**, 1301–1317.

[154] R. Temam (1988), *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Vol. 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[155] C. Tischendorf (1999), Topological Index Calculation of Differential-Algebraic Equations in Circuit Simulation. *Surveys Math. Indust.* **8**, 187–199.

[156] A. Tsurui (1972), Wave Modulations in Anharmonic Lattices. *Prog. Theor. Phys.* **48**, 1196–1203.

[157] B. van der Pol (1927), Forced Oscillations in a Circuit with Nonlinear Resistance (Receptance with Reactive Triode). *London, Edinburgh and Dublin Phil. Mag.* **3**, 65–80.

[158] A. Vanderbauwhede and G. Iooss (1992), Center Manifold Theory in Infinite Dimensions. *Dyn. Reported: Expositions in Dyn. Sys.* **1**, 125–163.

[159] A. Vanderbauwhede and S. A. van Gils (1987), Center Manifolds and Contractions on a Scale of Banach Spaces. *J. Funct. Anal.* **72**, 209–224.

[160] S. M. Verduyn-Lunel (1991), The Closure of the Generalized Eigenspace of a Class of Infinitesimal Generators. *Proc. Roy. Soc. Edinburgh Sect. A* **117A**, 171–192.

[161] V. Volterra (1928), Sur la Theorie Mathematique des Phenomenes Hereditares. *Journal de Mathématiques Pures et Appliqueés* **7**, 149–192.

[162] E. von Böhm-Bawerk (1891), *Positive Theory of Capital, translated by W. Smart*. Macmillan, London.

[163] T. Vyhlidal and P. Zitek, Mapping the Spectrum of a Retarded Time Delay System Utilizing Root Distribution Features. *Preprint*.

[164] H. O. Walther (1983), Bifurcation from Periodic Solutions in Functional Differential Equations. *Mathematische Zeitschrift* **182**, 269–289.

[165] D. V. Widder (1946), *The Laplace Transform*. Princeton Univ. Press, Princeton, NJ.

[166] C. C. Wu (1991), New Theory of MHD Shock Waves. In: M. Shearer (ed.): *Viscous Profiles and Numerical Methods for Shock Waves*. pp. 209–236.

[167] J. Wu and X. Zou (1997), Asymptotic and Periodic Boundary Value Problems of mixed FDE's and Wave Solutions of Lattice Differential Equations. *J. Diff. Eq.* **135**, 315–357.

[168] E. Yanagida (1987), Branching of Double Pulse Solutions from Single Solutions in Nerve Axon Equations. *J. Diff. Eq.* **66**, 243–262.

[169] K. Yosida (1948), On the Differentiability and the Representation of One-Parameter Semi-Group of Linear Operators. *J. Math. Soc. Japan* **1**, 15–21.

[170] B. Zinner (1992), Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation. *J. Diff. Eq.* **96**, 1–27.

[171] B. Zinner (1992), Stability of Traveling Wavefronts for the Discrete Nagumo Equation. *J. Diff. Eq.* **96**, 1–27.

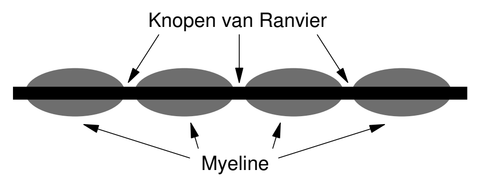
[172] B. Zinner, G. Harris, and W. Hudson (1993), Traveling Wavefronts for the Discrete Fisher's Equations. *J. Diff. Eq.* **105**, 46–62.

Samenvatting

Deze samenvatting is gebaseerd op het artikel "Modelleren in Discrete Ruimtes", dat in april 2008 verschenen is in het Eureka! Magazine.

Handelaren op de beurs kijken vast niet raar op van de bewering dat zenuwen en fluctuaties op de kapitaalmarkt een belangrijke overeenkomst hebben. Ze zullen dit alleen niet zo snel toeschrijven aan het feit dat beiden beschreven kunnen worden door een exotische klasse van differentiaalvergelijkingen, waar wiskundigen momenteel een harde dobber aan hebben.

Vele fysische, chemische en biologische processen spelen zich af in ruimtes die gekenmerkt worden door een discrete achterliggende structuur. Denk bijvoorbeeld aan geluidsgolven die zich voortbewegen door regelmatige kristalroosters. Of kijk naar de dichtheid van mieren in een groot gebied, die in de buurt van elke mierenhoop flink zal toenemen.



Figuur A: Schematische representatie van een axon.

Ter illustratie behandelen we hier een intrigerende toepassing uit de biologische hoek. De verbindingsdraden van ons zenuwstelsel heten axonen en dragen zorg voor de overdracht van elektrische signalen tussen onze hersenen en ledematen. Deze axonen hebben een dikte van ongeveer een micrometer, maar kunnen makkelijk meer dan een meter lang zijn. Op de meeste plaatsen in ons zenuwstelsel worden deze axonen omringd door een vettige insulerende substantie die myeline heet. Dit laagje wordt op regelmatige afstand onderbroken door de zogenaamde knopen van Ranvier, waar de elektrische signalen in de axonen versterkt worden. Zie figuur A voor details.

Het myeline omhulsel heeft een dubbele functie. Het voorkomt dat signalen overspringen naar naburige andere zenuwen, maar zorgt samen met de knopen van Ranvier ook voor een flinke versnelling van het signaal. In axonen zonder omhulsel plant een signaal zich als een golf voort en kan daarbij een snelheid halen van zo'n 2 meter per seconde. De structuur van het omhulsel zorgt ervoor dat dit toeneemt naar ongeveer 50 meter per seconde. Deze versnelling gaat gepaard met het feit dat een signaal zich niet meer regelmatig door het axon voortbeweegt, maar meer lijkt te springen tussen naburige knopen van Ranvier [83].

Rooster Differentiaalvergelijkingen

Om dit sprong-proces wiskundig te modelleren, gaan we kijken we naar het gedrag van de elektrische spanning V_i ter plaatse van de i-de knoop van Ranvier. Hierbij laten we de index i alle positieve en negatieve gehele getallen doorlopen, zodat we doen alsof het axon zich oneindig ver naar links en rechts uitstrekt. Op grond van elektrochemische argumenten kun je nu de volgende differentiaalvergelijking opstellen [32],

$$\frac{dV_i}{dt} = h^{-2}[V_{i+1} + V_{i-1} - 2V_i] - \tfrac{1}{4}(V_i + 1)(V_i - 1)(V_i - \rho), \qquad i \in \mathbb{Z}. \tag{1}$$

Hierin is h de afstand tussen twee opeenvolgende knopen. De parameter ρ ligt in het interval $(-1, 1)$ en is gebaseerd op diverse elektrochemische eigenschappen van het axon en het insulerende myeline.

Als we goed kijken naar (1) zien we dat het volledige systeem in rust is als de potentiaal overal gelijk is aan ± 1 of ρ. Verder valt op dat elke knoop direct beinvloed wordt door zijn twee buren. Om onze gedachten wat verder te bepalen, gaan we nu kijken wat er gebeurt als de onderlinge knoop afstand h steeds kleiner wordt. We voeren daartoe een functie U in die van zowel de plaats x als de tijd t afhangt en voldoet aan $U(ih, t) = V_i(t)$ voor all $i \in \mathbb{Z}$. De lezer die dat leuk vindt, mag als oefening nagaan dat in de limiet $h \to 0$ onze nieuwe functie U voldoet aan de partiële differentiaalvergelijking (pdv)

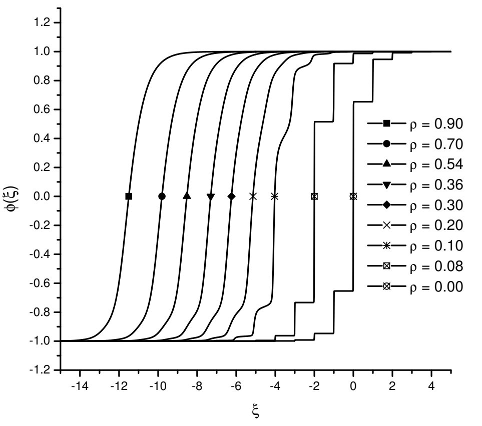
$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - \tfrac{1}{4}(U + 1)(U - 1)(U - \rho). \tag{2}$$

Vergelijking (1) is een één-dimensionaal voorbeeld van wat men inmiddels rooster differentiaalvergelijkingen (rdv's) noemt. Vergeleken met de pdv (2) die inmiddels al uitgebreid is bestudeerd, zijn differentiaalvergelijkingen op roosters wiskundig gezien nog vrijwel onontgonnen gebied. Voordat een wiskundige zal besluiten om zich hier helemaal op te storten, wil hij of zij natuurlijk wel weten of dat rendabel is. Iets concreter gesteld: voegt het echt iets toe als we bij (1) wegblijven van het $h \approx 0$ regime?

Om een beginnetje te maken met het beantwoorden van deze vraag, gaan we nu op zoek naar oplossingen van (1) die een speciale structuur hebben. We gaan kijken naar signalen die zich met een constante vorm ϕ en constante snelheid c door het axon voortbewegen en de evenwichten ± 1 met elkaar verbinden. Deze golven kunnen geschreven worden als $V_i(t) = \phi(i - ct)$.

In figuur B hebben we de profielen ϕ van een aantal van deze numeriek berekende oplossingen getoond, voor verschillende waarden van de parameter ρ. Het is vooral interessant om te zien dat bij $\rho \approx 0.08$ de golven hun gladheid verliezen. Tegelijkertijd blijkt voor de bijbehorende golfsnelheid opeens te gelden dat $c = 0$. De signalen kunnen zich dus bij deze waarden van ρ niet meer door het axon voortbewegen. Vanuit ons model bezien lijkt dit fenomeen wel aannemelijk te zijn, omdat de gaten tussen de knopen zorgen voor een energie barrière die signalen niet altijd zullen kunnen overbruggen. Deze barrière verdwijnt in de limiet $h \to 0$ en inderdaad is deze stagnatie afwezig bij de pdv (2).

Het blijkt dat dit slechts één van de vele verschillen tussen pdv's en rdv's is die de afgelopen jaren aan het licht zijn gekomen. Omdat het ondoenlijk is om vergelijkingen zoals (1) met de hand aan te pakken, is het pas sinds de komst van de computer langzaam duidelijk geworden dat rdv's ongekend rijke dynamica kunnen bezitten. Het is dus niet verwonderlijk dat de belangstelling voor rdv's nu sterk aan het groeien is. Ze zijn inmiddels opgedoken in vele wetenschappelijke disciplines, waaronder beeldverwerking, vaste stof fysica, fysiologie, populatie dynamica en chemische reactie theorie.



Figuur B: Golf profielen voor oplossingen van (1) bij verschillende waarden van ρ, met $h = \sqrt{10}$.

De uitdaging

Wiskundigen zien zich dus nu geconfronteerd met de uitdaging om een rigoreus bouwwerk te ontwikkelen waar rdv's mee kunnen worden aangepakt. Uiteindelijk willen we voor willekeurige rdv's precies kunnen aangeven wat voor verschillende soorten oplossingen er kunnen zijn. Ook willen we kunnen voorspellen hoe dit beeld zal veranderen als gevolg van verstoringen in het systeem. Uiteraard zijn we hier nog lang niet toe in staat, maar er is al een goede start gemaakt.

Vaak blijkt bij het bestuderen van pdv's dat het van cruciaal belang is om eerst de lopende golf oplossingen goed te begrijpen. Voldoende reden dus om dit recept ook op rdv's toe te passen. Laten we daarom $V_i(t) = \phi(i - ct)$ invullen in (1). We vinden dan de volgende vergelijking, waarin de variable ξ staat voor de combinatie $\xi = i - ct$,

$$-c\phi'(\xi) = h^{-2}\big[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big] - \frac{1}{4}(\phi(\xi)+1)(\phi(\xi)-1)(\phi(\xi)-\rho). \quad (3)$$

Vooral het stuk tussen de vierkante haken is hier van belang. Merk op dat we de waarde van ϕ zowel in het 'verleden' $\xi - 1$ als in de 'toekomst' $\xi + 1$ nodig hebben om de afgeleide van ϕ op 'tijdstip' ξ te bepalen. Om deze reden noemen we (3) een differentiaalvergelijking van de gemengde soort. Het is absoluut niet zo dat we hiermee een paragnostisch effect aan ons model hebben toegevoegd, want ξ heeft slechts indirect met een echte tijd te maken.

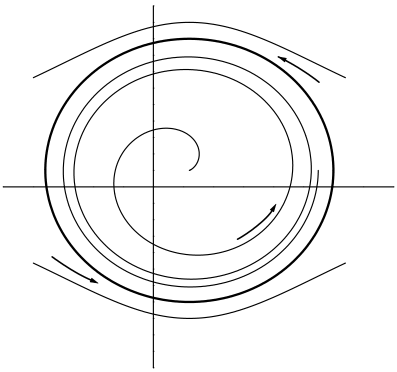
Als we zoeken naar lopende golf oplossingen voor de pdv (2), vinden we de volgende tweede orde gewone differentiaalvergelijking,

$$-c\phi'(\xi) = \phi''(\xi) - \frac{1}{4}(\phi(\xi)+1)(\phi(\xi)-1)(\phi(\xi)-\rho). \quad (4)$$

Wat maakt (3) nu zo anders dan (4)? Dit heeft te maken met de informatie die nodig is om de *toestand* van een systeem vast te leggen. Kijken we bijvoorbeeld naar (4), dan zien we dat zodra we $\phi(0)$ en $\phi'(0)$ kennen, we daarmee $\phi''(0)$ kunnen vastleggen. Dit blijkt voldoende informatie te geven om ook $\phi(\xi)$ uit te rekenen voor alle $\xi \geq 0$. Op grond hiervan zeggen we dat de toestandsruimte van (4) twee-dimensionaal is. Anders wordt het als we kijken naar (3). Als je er even over nadenkt, zie je al snel dat om iets soortgelijks te doen, je uiteraard de waardes van $\phi(0)$, $\phi(-1)$ en $\phi(+1)$ moet kennen, maar ook alle tussenliggende waardes van ϕ. De toestandsruimte wordt dan ook gegeven door de verzameling van continue functies op het interval $[-1, 1]$. In tegenstelling tot de twee-dimensionale toestandsruimte van (4) hebben we hier dus te maken met een oneindig-dimensionale toestandsruimte!

Eindig-Dimensionale Systemen

Onze kennis over differentiaalvergelijkingen in eindig veel dimensies heeft de afgelopen eeuw een stormachtige ontwikkeling doorgemaakt. Tegenwoordig spelen meetkundige methodes daarin een cruciale rol. Elke oplossing ϕ van (4) kunnen we bijvoorbeeld associeren met de kromme in het vlak die bestaat uit alle punten $(\phi(\xi), \phi'(\xi))$. Als ξ toeneemt, loop je dan als het ware langs de kromme. Laten we nu even aannemen dat ϕ een periodieke oplossing voor (4) is. De bijbehorende kromme is dan gesloten, zie figuur C.



Figuur C: De vette gesloten kromme correspondeert met een periodieke oplossing van (4). Andere oplossingen zullen ofwel geheel buiten, ofwel geheel binnen deze kromme blijven.

Nu is er een stelling die je garandeert dat oplossingen van (4) elkaar nooit kunnen snijden. Dit betekent dat als je eenmaal in de gesloten kromme zit, je nooit naar buiten kan komen en vica versa. Hiermee is de totale toestandsruimte al gereduceerd naar twee afzonderlijke componenten, die we vervolgens apart weer verder kunnen bestuderen.

Oneindig-Dimensionale Systemen

Uiteraard werkt deze constructie niet meer in drie of meer dimensies en zeker niet in oneindig veel dimensies. In het laatste geval zijn veel mooie meetkundige eigenschappen van eindig-dimensionale ruimtes helaas niet meer geldig. Gedurende de laatste vijftig jaar is een hele nieuwe tak van de wiskunde ontstaan die zich specifiek bezighoudt met de analyse van dynamische systemen in oneindig-dimensionale ruimtes.

Gelukkig blijkt het zo te zijn dat relevant gedrag in oneindig-dimensionale systemen vaak - zonder gegevensverlies - beschreven kan worden in termen van een eindig-dimensionaal deelsysteem. Het herkennen van deze scenario's bij differentiaalvergelijkingen van de gemengde soort en het ontwikkelen van technieken om deze deelsystemen ook daadwerkelijk te beschrijven, vormen samen het hart van dit proefschrift. We hebben bijvoorbeeld een krachtige stelling die oorspronkelijk is ontwikkeld voor twee dimensies, geschikt gemaakt voor gebruik in onze oneindig-dimensionale context. Hiermee kunnen we voorspellen bij welke waarden van de parameter ρ vergelijking (3) kleine oscillaties om één van de evenwichten zal toestaan.

Maximale Welvaart

Als afsluiting is het interessant om op te merken dat differentiaalvergelijkingen van de gemengde soort op onverwachte wijze ook bij optimalisatie problemen om de hoek komen kijken. Economen gebruiken vaak het volgende welvaarts-optimalisatie model om ontwikkelingen op de kapitaalmarkt te bestuderen [17],

$$\text{maximaliseer} \int_0^\infty W(c(t))dt. \tag{5}$$

Hierbij staat $c(t)$ voor de totale consumptie op een bepaald tijdstip t en W voor de welvaart die dit oplevert. Uiteraard hangt de mogelijkheid om te consumeren af van de investeringen in de productiecapaciteit die *in het verleden* zijn verricht. Fabrieken worden immers niet in één dag gebouwd. Al sinds 1968 is bekend dat deze tijdsvertraging betekent dat de Euler-Lagrange vergelijkingen die de oplossing van (5) typeren, niets anders zijn dan differentiaalvergelijkingen van de gemengde soort [81]. Als wiskundigen zijn we nu dus weer net iets beter in staat om ook de economen de technieken aan te reiken die ze nodig hebben.

Curriculum Vitae

De auteur van dit proefschrift werd geboren op 25 januari 1981 te Gouda. Hij bezocht vanaf 1992 het Rijnlands Lyceum te Oegstgeest, waar hij in 1998 het VWO diploma behaalde. In datzelfde jaar begon hij met de studies wiskunde en natuurkunde aan de Universiteit Leiden. In 2003 schreef hij onder begeleiding van prof. dr. S. M. Verduyn Lunel de scriptie *Analysis of Newton's Method to Compute Travelling Wave Solutions to Lattice Differential Equations,* waarmee hij cum laude afstudeerde in de wiskunde. In de zomer van 2003 werkte hij drie maanden bij CERN in Genève, waar hij bijdroeg aan het ijken van de ATLAS detector. De studie natuurkunde rondde hij in 2004 cum laude af met een onderzoek naar de veiligheid van quantum cryptografische systemen.

In januari 2004 begon hij met zijn promotie onderzoek bij prof. dr. S. M. Verduyn Lunel. Naast het schrijven van de artikelen die in dit proefschrift zijn opgenomen, was hij intensief betrokken bij het onderwijs van de wiskunde opleiding in Leiden. Hij begeleidde diverse werkgroepen en verzorgde ook een introductie college wiskunde voor eerstejaars studenten van de studie Life, Science and Technology. Verder gaf hij voordrachten op diverse nationale en internationale congressen en colloquia.

Eind 2007 werd aan hem een Rubicon subsidie toegewezen door NWO. Hiermee heeft hij de mogelijkheid gekregen om in de periode 2008-2010 verder te werken aan de uitbreiding en toepassing van de in dit proefschrift ontwikkelde theorie. Het is de bedoeling dat dit onderzoek onder begeleiding van prof. dr. B. Sandstede zal worden uitgevoerd aan Brown University in de Verenigde Staten.