Horizontal Travelling Waves on the Lattice

Judith Berendsen

November 5, 2015

Master thesis

Begeleiding: Dr. Hermen Jan Hupkes

Korteweg-de Vries Instituut voor Wiskunde
Faculteit der Natuurwetenschappen, Wiskunde en Informatica
Universiteit van Amsterdam

Abstract

We consider a reaction-diffusion equation on the lattice in two space dimensions and concentrate on the horizontal direction. We prove the stability of horizontal travelling waves under large perturbations and thereby we work out a special case of the stability result obtained in [1]. In order to prove stability we use the comparison principle and construct sub- and supersolution.

Titel: Horizontal Travelling Waves on the Lattice
Auteur: Judith Berendsen,
Begeleiding: Dr. Hermen Jan Hupkes
Tweede beoordelaar: Dr. Ale Jan Homburg
Einddatum: November 5, 2015

Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam
Science Park 904, 1098 XH Amsterdam
`http://www.science.uva.nl/math`

Contents

Dankwoord

Ik ben mijn begeleider Hermen Jan Hupkes enorm dankbaar voor zijn begeleiding bij mijn masterscriptie. Hij wist altijd veel tijd voor mij vrij te maken en mij bij tegenslagen weer met optimisme naar huis te laten gaan. Zijn hulp en enthousiasme werkten enorm motiverend voor mij.
Voor de praktische ondersteuning en hulp in de laatste jaren dank ik het KdVI, in het bijzonder Ale Jan Homburg, Han Peters, Jan Wiegerinck, Iris Hettelingh en Chris Zaal.
Ten slotte dank ik Marion en Stefanie voor al hun hulp en ondersteuning.

1. Introduction

Let the lattice $\mathbb{Z}^2$ be indexed by coordinates $(i,j) \in \mathbb{Z}^2$. In this thesis we consider the following differential equation on the lattice

$$\dot{u}_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} + g(u_{i,j}). \tag{1.1}$$

The function g is assumed to be a nonlinear and multistable function. Throughout this paper we assume g to be bistable and given by

$$g(u) = u(u-a)(1-u) \tag{1.2}$$

for some detuning parameter $0 < a < 1$.
We can think of (1.1) as the discrete analogue of the Nagumo PDE in two real dimensions

$$u_t = \Delta u + g(u). \tag{1.3}$$

We focus on the stability of travelling wave-like solutions for large perturbations. This is an important step towards understanding the effects of obstacles. In this thesis we closely follow [1] to work out stability results for the horizontal direction on the lattice. The horizontal direction is a direction on the lattice causing complications resulting from weaker resonance to disappear. Thereby we are able to make the conditions for the existence, uniqueness and stability of entire asymptotic planar wave solutions found in [1] more explicit. Furthermore, we show that decay of the residual is faster in the horizontal direction.

1.1. Reaction-Diffusion Equations

The PDE (1.3) is a prototype of a reaction-diffusion equation, which is a semi-linear parabolic partial differential equation.

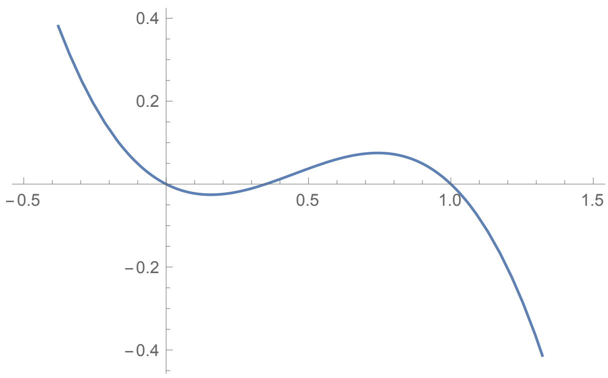

Figure 1.1.: Nonlinearity g with detuning parameter $a = 3.5$

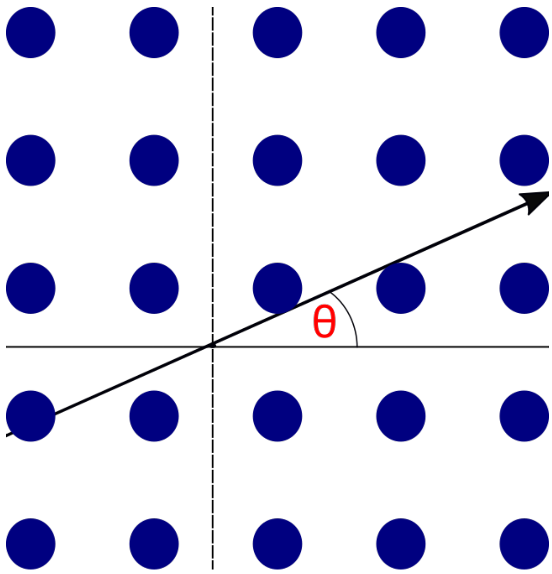

Figure 1.2.: Lattice with direction denoted by θ

The term Δu is the diffusion term and $g(u)$ is the reaction term, they both strive for dominance on the plane. As we can see in Fig. 1.1, the reaction term $g(u)$ causes peaks, because it is bistable. The diffusion operator Δu softens these peaks.
Reaction-diffusion equations are essential for modelling. They are used to describe systems, which are influenced by both a reaction and diffusion. We can think of diffusion as the spreading out in space. Thus diffusion accounts for the second law of thermodynamics, i.e. for the increase in entropy. Note that we get Fick's second law, which is called the heat equation on a multi-dimensional space, for $g \equiv 0$.
If g is not the zero function, the reaction term can account for any influence we can describe with such a function g.
Chemical reactions are the most obvious and common application of reaction-diffusion equations. Yet they can be used as a generic model for any pattern forming process we can describe with (1.3). Therefore another widely used example is population dynamics, in particular predator-prey models, but also population genetics models as seen in [10].
Reaction-diffusion equations on the discrete space with their two-dimensional prototype given in (1.1) have only recently enjoyed great interest. This development is mostly due to the tremendous progress when it comes to computer power in the previous decades. Within this young research field it has already become clear that there is a huge added benefit to studying reaction-diffusion equations on a multidimensional discrete space. The most prominent reason is that assuming continuity is often a mere approximation and we actually lose information about the structure. Furthermore, numerous processes can only be modelled discretely.
On the lattice it matters from which direction we look at the lattice as illustrated in Fig. A.1. In mathematical terms we lose isotropy and gain direction dependence. We also lose translational invariance. These structural differences to the continuous finite-dimensional space $\mathbb{R}^N$ have consequences for the existence and stability of solutions. In the sequel we will make these consequences explicit.

Discrete models based on reaction-diffusion equations are of great applicational interest. Apart from many biological and chemical applications mentioned above, we mention pattern recognition, imaging (think of pixels as lattice points) or material science as highlighted in [9], an overview paper by Mallet-Paret and Shen. For completeness we note that the studied LDE (1.1) is also the standard spatial discretization scheme for models based on the PDE (1.3). As such they are essential to the field of numerical analysis as pointed out in [1].

1.2. Existence and Uniqueness of Solutions

Consider the PDE (1.3) on the space $\mathbb{R}$ for $u = u(x,t) : \mathbb{R}^2 \to \mathbb{R}$

$$u_t = u_{xx} + g(u). \tag{1.4}$$

We can fill in the nonlinearity g

$$g(u) = u(u-a)(1-u) \tag{1.5}$$

and the PDE becomes

$$u_t = u_{xx} + u(u-a)(u-1). \tag{1.6}$$

The solutions $0, 1$ are stable equilibria of g and the solution $u = a$ is unstable as we can see in Fig. 1.1. We immediately see that the constant functions $u = 0, a$ and 1 solve the PDE. We are interested in solutions taking values between 0 and 1. We can use the travelling wave Ansatz already introduced by Kolmogorov, Petrovskii, and, Piskunov in [4] in 1937

$$u(x,t) = \Phi(x+ct)$$

with travelling wave coordinate

$$\xi = \xi(t) = x + ct,$$

which has to solve the wave profile equation

$$c\Phi'(\xi) = \Phi''(\xi) + g(\Phi(\xi), a). \tag{1.7}$$

Note that the travelling wave Ansatz can be seen as a compromise balancing out the reaction and the diffusion term [10]. In particular, we want the travelling wave to connect the solutions 0 and 1. Therefore, the following limits also have to hold

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to +\infty} \Phi(\xi) = 1. \tag{1.8}$$

The ODE can be solved explicitly

$$\Phi(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{2}}{4}\xi\right). \tag{1.9}$$

The graph of the travelling wave $\Phi(\xi)$ is depicted in Fig. 1.3. In Fig. 1.4 and Fig. 1.5 we also see the graphs of the first and second derivative of $\Phi(\xi)$

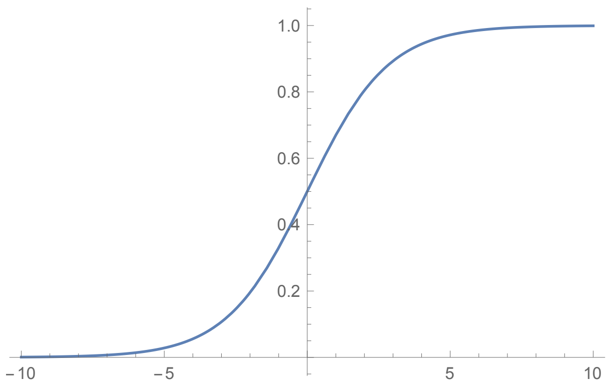

Figure 1.3.: The wave profile Φ

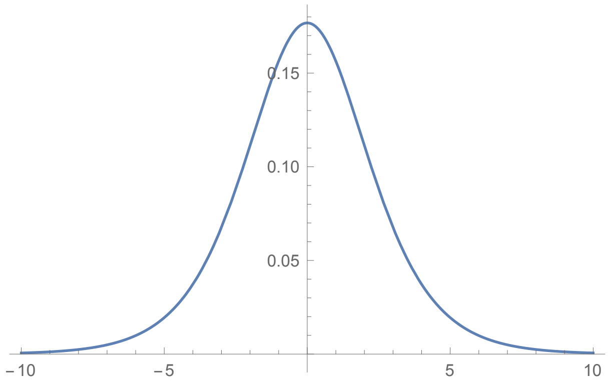

Figure 1.4.: The first derivative Φ′ of the wave profile Φ

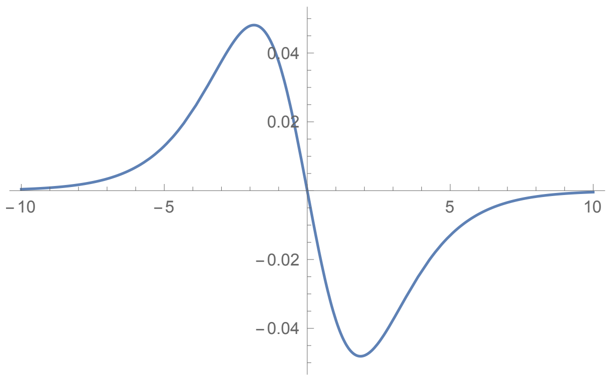

Figure 1.5.: The second derivative Φ″ of the wave profile Φ

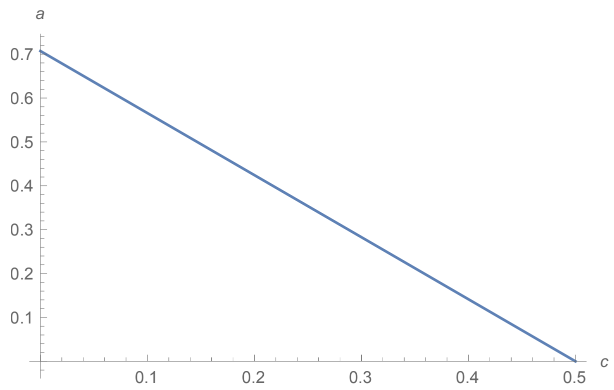

Figure 1.6.: c versus a in the continuous case

$$\begin{aligned}\Phi'(\xi) &= \frac{1}{4\sqrt{2}}\operatorname{sech}^2\left(\frac{1}{2\sqrt{2}}\xi\right) > 0\\ \Phi''(\xi) &= -\frac{1}{8}\tanh\left(\frac{1}{2\sqrt{2}}\xi\right)\operatorname{sech}^2\left(\frac{1}{2\sqrt{2}}\xi\right).\end{aligned} \tag{1.10}$$

We write out an explicit formula for the wave speed c as a function of the detuning parameter a

$$c(a) = \frac{1}{\sqrt{2}}(1-2a)$$

as shown in Fig. 1.6.

Fife and McLeod have shown as early as 1977 that phase plane analysis can be used to show existence of solutions of the Nagumo PDE (1.4) with arbitrary bistable nonlinearities. Furthermore, we can extend the analysis to higher finite dimensions $N \geq 2$ by exploiting radial symmetry. To illustrate this, consider the two-dimensional PDE for $u = u(x, y, t) : \mathbb{R}^3 \to \mathbb{R}$ given by

$$u_t = u_{xx} + u_{yy} + g(u). \tag{1.11}$$

The travelling wave Ansatz in two dimensions for the direction (σ_h, σ_v) with $\sigma_h^2 + \sigma_v^2 = 1$ becomes

$$u(x, y, t) = \Phi(\sigma_h x + \sigma_v y + ct)$$

with travelling wave coordinate

$$\xi = \xi(t) = \sigma_h x + \sigma_v y + ct.$$

But this means for fixed a we get the same wave profile equation as in the one-dimensional case

$$\begin{aligned}\Phi_t(\xi) &= \Phi_{xx}(\xi) + \Phi_{yy}(\xi) + g(\Phi(\xi))\\ c\Phi'(\xi) &= (\sigma_h^2 + \sigma_v^2)\Phi''(\xi) + g(\Phi(\xi))\\ c\Phi'(\xi) &= \Phi''(\xi) + g(\Phi(\xi)).\end{aligned}$$

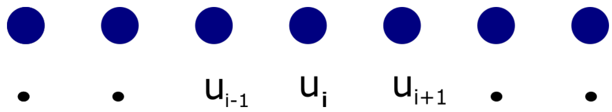

Figure 1.7.: The one-dimensional discrete space with u_i and its direct neighbours

We now want to look at the one-dimensional case on $\mathbb{Z}$ and project the first-dimensonal Nagumo PDE (1.3) from $\mathbb{R}$ to $\mathbb{Z}$. An illustrating sketch is given in Fig. 1.7. Before we consider the discrete analogue of (1.4) we want to have an intuition the discrete analogue of f''. Let $f \in C^2(\mathbb{R}, \mathbb{R})$. The symmetric definition of the first derivative of f is given by

$$f'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}.$$

We manipulate the difference quotient for f''

$$\begin{aligned} f''(x) &= \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \frac{1}{2h}\left(\frac{f(x+2h) - f(x)}{2h} - \frac{f(x) - f(x-2h)}{2h}\right) \\ &= \frac{1}{4h^2}\Big(f(x+2h) - f(x) + f(x-2h) - f(x)\Big) \\ &= [h' = 2h]\frac{1}{h'^2}\Big(f(x+h') + f(x-h') - 2f(x)\Big) \end{aligned}$$

Associate h' with the step size of the lattice points, here assumed to be $h' = 1$. Then we see that can think of the second derivative on $\mathbb{Z}$ as the difference of the difference of two neighbouring points. Thus the LDE on $\mathbb{Z}$ is given by

$$u_i' = u_{i+1} + u_{i-1} - 2u_i + g(u_i).$$

The travelling wave Ansatz here becomes

$$u_i(t) = \Phi(i + ct).$$

Analogously to the continuous case we will use the travelling wave constant

$$\xi = \xi(t) = i + ct$$

such that the travelling wave equation takes the form

$$c\Phi'(\xi) = \Phi(\xi + 1) + \Phi(\xi - 1) - 2\Phi(\xi) + g(\Phi(\xi)).$$

But here the nature of the equation changes with the wave speed. For $c = 0$, we have a difference equation and a differential equation whenever $c \neq 0$. For $c \neq 0$ we in fact get a differential equation of mixed type (MFDE). We can express c as function in a numerically, this has been done for multiple dimensions in [17]. Let a^* be the intersection

of the graph of c versus a with the $a-$axis. If $a^* < \frac{1}{2}$ the wave fails to propagate for a range of values for a, a phenomenon which is called pinning.
Lastly, consider (1.1), the discrete analogue of the Nagumo PDE on the lattice $\mathbb{Z}^2$

$$u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} + g(u_{i,j}).$$

Let (σ_h, σ_v) denote the direction on the lattice from which we consider solutions. For convenience we only consider directions normalized by the condition $\sigma_h^2 + \sigma_v^2 = 1$. We use the travelling wave Ansatz given by

$$u_{i,j}(t) = \Phi(i\sigma_h + j\sigma_v + ct),$$

with travelling wave coordinate

$$\xi = \xi(t) = i\sigma_h + j\sigma_v + ct.$$

Phase plane analysis is not possible in the discrete case, which becomes apparent if we look at the wave profile equation

$$c\Phi'(\xi) = \Phi(\xi + \sigma_h) + \Phi(\xi - \sigma_h) + \Phi(\xi + \sigma_v) + \Phi(\xi - \sigma_v) - 4\Phi(\xi) + g(\Phi(\xi), a).$$

Not only does the equation become a difference equation for $c = 0$ like in the one-dimensional case, but also is there directional dependence. Again c can be written as a function of θ and a and the results can be determined numerically. This has been done in [17].
Whether pinning occurs depends on the direction and on the nonlinearity g. In [14] it has been shown that pinning occurs in all rational directions for g resembling a sawtooth. More specifically, in [13] it has been shown that pinning occurs in the horizontal and vertical direction if g is bistable and satisfies a monotonicity condition.
The existence and uniqueness of solutions of (1.1) with respect to nonzero values of c for all directions (σ_h, σ_v) on the lattice has been shown in [9]. These results allow us to not consider pinning here. Instead we focus on stability of travelling waves in this thesis with detuning parameter a chosen such that they do not fail to propagate.

1.3. Stability of Waves

In the last section, we have seen that extending existence results for the PDE (1.3) to higher dimensions is straightforward. In this section we consider the more difficult problem of stability. Starting with the one-dimensional continuous case, we have to refer to Fife and McLeod's landmark paper from 1977 once more. In [6], Fife and McLeod consider an additively initially perturbed wave-like solution. Then they use squeezing techniques to prove that these solutions uniformly converge to travelling wave solutions in time with adjacent ranges. We can think of these wave solutions as a stacked combination of wave fronts. This way they prove stability for the continuous case in one dimension. Their approach can be used to solve FitzHugh-Nagumo PDE's. Berestycki, Hamel and Matano have established uniqueness of the entire solution based on [6].

However, stability of travelling wave solutions of the PDE in two dimensions is more difficult to show. The problem is that level sets of plane waves are now lines rather than points. There are two major consequences. Firstly, phase shifts are harder to trigger by local perturbations. Secondly, in the direction transversal to the propagation of the waves one has deformations as well, which makes uniform convergence difficult to show on the plane.

Stability for solutions of the PDE in the critical two-dimensional case has only been shown in 1997 by [5]. The author Kapitula uses spectral methods, Green's functions and fixed-points arguments. The result can be extended to $\mathbb{R}^N$ for $N \geq 2$ by exploiting radial symmetry.

In four or more dimensions, stability has been shown in 1992 by Xin in [7]. Xin decomposes the perturbations into normal and transversal components in $L^2(\mathbb{R})$ before using spectral estimates of the heat kernel and iteration techniques to show decay of the perturbations.

In 2001 Bates and Chen have shown stability of travelling waves for the non-local Allen-Cahn equation on $\mathbb{R}^N$ with $N \geq 4$ with spectral analysis in [11]. In 2014 Hoffman, Hupkes, and Van Vleck in [3] have been able to extend these techniques for the LDE (1.1) on the space $\mathbb{Z}^N$ with $N \geq 2$.

However, we want to deal with large perturbations, because obstacles cause large perturbations. Therefore, the method of choice is the comparison principle. Note that we may apply the comparison principle, because the coefficients of all off-site terms of the LDE are positive. Unlike spectral analysis techniques, the comparison principle requires structure of the equation, but allows for estimates strong enough to deal with large perturbations.

Therefore, Hoffman, Hupkes, and Van Vleck have shown stability for solutions of the LDE on $\mathbb{Z}^2$ in the same year in [1] using a different technique. In this way [1] can be seen as a companion paper to [3] giving an alternative proof of stability in the unobstructed case as a preparation for their obstacle results.

In this thesis we follow the paper [1], stability results for travelling waves on the lattice are obtained by the comparison principle. We use a different and more direct technique by exploiting the alignment of the horizontal direction along the lattice direction. In particular, we use estimates found by Pal'tsev in his paper [12] from 1999 to obtain the stability result we are after. We are constructing a subsolution and thereby a supersolution by symmetry. We prove the following formulation of the stability result obtained in [1], which states that horizontal travelling waves $u_{i,j}(t) = \Phi(i + ct)$ satisfying (4.11) and $c \neq 0$ are stable under large but localized perturbations.

Theorem 1.1. *If* $U : [0, \infty) \to l^\infty(\mathbb{Z}^2, \mathbb{R})$ *is a* C^1-*smooth function satisfying* (1.1) *for all* $t \geq 0$ *and*

$$|U_{i,j}(0) - \Phi(i)| \to 0 \quad \textit{as} \quad |i| + |j| \to \infty, \tag{1.12}$$

whereby $0 \leq U_{i,j}(0) \leq 1$, *then we have the uniform convergence*

$$\sup_{(i,j)\in\mathbb{Z}^2} |U_{i,j}(t) - \Phi(i + ct)| \to 0 \quad \textit{for} \quad t \to \infty.$$

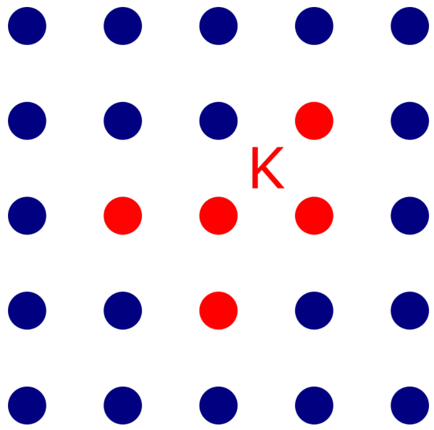

Figure 1.8.: Compact obstacle K on the lattice

1.4. Solutions on the Obstructed Space

Consider the Nagumo PDE (1.3) on the space $\mathbb{R}^N$ obstructed by some subset K such that $\mathbb{R}^N \setminus K$ is compact. Furthermore, assume that the Neumann boundary condition

$$\nu \cdot \nabla u = 0 \quad \text{on} \quad \partial K,$$

where ν denotes the outward normal holds. As recently as 2008 existence and uniqueness of entire wave-like solutions of this problem for a star-shaped or directionally convex obstacle $K \subset \mathbb{R}^N$ has been shown in [2]. The authors Berestycki, Hamel, and Matano also show stability of the entire wave-like solutions under use of the comparison principle. We remove a set of points K from the lattice and denote the lattice obstructed by K as $\Lambda = \mathbb{Z}^2 \setminus K$. In Fig. 1.8 we can see an example of the lattice obstructed by a compact obstacle K. To consider the LDE (1.1) on Λ we have to adjust (1.1) slightly. On the one hand, we have to define the Laplacian in terms of the four direct neighbors of $u_{i,j}$ on Λ. On the other hand, we need boundary conditions on $\partial\Lambda$ such that $u \equiv 0, u \equiv a$ and $u \equiv 1$ are not only zeros of g, but also solutions of the LDE.
The paper by Berestycki, Hamel, and Matano has been generalized to the two-dimensional discrete case by Hoffman, Hupkes, and Van Vleck in 2014. In [1] stability of entire asymptotic planar wave solutions has been shown for the LDE obstructed by a bounded and directionally convex obstacle K such that $\mathbb{Z}^2 \setminus K$ is connected. An entire asymptotic planar wave solution $u_{i,j}$ must be defined for all times $t \in \mathbb{R}$ and satisfy the limit

$$\lim_{|t|\to\infty} \sup_{(i,j)\in\Lambda} |u_{i,j}(t) - \Phi(i\sigma_h + j\sigma_v + ct)| = 0.$$

Organization

In chapter 2 we investigate the stability of the PDE on $\mathbb{R}^2$ and look at the continuous heat kernel. In chapter 3 we introduce the LDE on $\mathbb{Z}^2$. We consider the discrete heat kernel and perform preliminary calculations. In chapter 4 we determine a subsolution by sharp estimates of the residual and we prove the stability of the travelling wave solution. Finally, we will prove stability of the travelling waves.

2. Stability for the Nagumo PDE

In this chapter, we prepare the stability results we are after for the LDE on the lattice by looking at the Nagumo PDE on $\mathbb{R}^2$. We give an outline of the subsolution Ansatz and calculate its residual explicitly. Then we take a look at the heat kernel.

2.1. The Continuous Subsolution and its Residual

The two-dimensional Nagumo PDE takes the form

$$u_t = \Delta u + g(u). \tag{2.1}$$

We have seen in the introduction that the travelling wave Ansatz can be applied to (2.1). This has been performed in [6] in order to show existence and uniqueness with phase plane analysis. Therefore, we can now focus on stability following [2]. In order to show stability the comparison principle is used. The discrete version is formulated in the appendix in theorem A.12.
In [2] the subsolution takes the form

$$u^-(x, y, t) = \Phi(x + ct - \theta(y, t) - Z(t)) - z(t) \tag{2.2}$$

and the symmetric supersolution looks like

$$u^+(x, y, t) = \Phi(x + ct + \theta(y, t) + Z(t)) + z(t),$$

where three external functions $\theta(y, t)$, $z(t)$, and $Z(t)$ are introduced. We begin by studying

$$z \in C^1([0, \infty), \mathbb{R})$$

and its integral

$$Z \in C^1([0, \infty), \mathbb{R}), Z(t) = K_Z \int_0^t z(s)ds$$

with $K_Z > 1$ a constant.
Hereby Berestycki, Hamel, and Matano use results of [6], where initial perturbations are controlled asymptotically with phase shifts. An additive initial perturbation can be described by the decreasing function z. The increasing function Z describes the asymptotic phaseshift. Their relation becomes clear by calculating the residual $\theta(y, t) = 0$ directly. As we have seen in the introduction, we may even restrict ourselves to the one-dimensional subsolution $u^-(x, t) = \theta(x + ct - Z(t)) - z(t)$.

$$\begin{aligned}
J &= u_t^-(x, t) - \Delta u^-(x, t) - g(u^-(x, t)) \\
&= \frac{d}{dt}(\Phi(x + ct - Z(t)) - z(t)) - \partial_{xx}(\Phi(x + ct - Z(t)) - z(t)) - g(\Phi(x + ct - Z(t)) - z(t)) \\
&= (c - Z'(t))\Phi'(x + ct - Z(t)) - z'(t) - \Phi''(x + ct - Z(t)) - g(\Phi(x + ct - Z(t)) - z(t)).
\end{aligned}$$

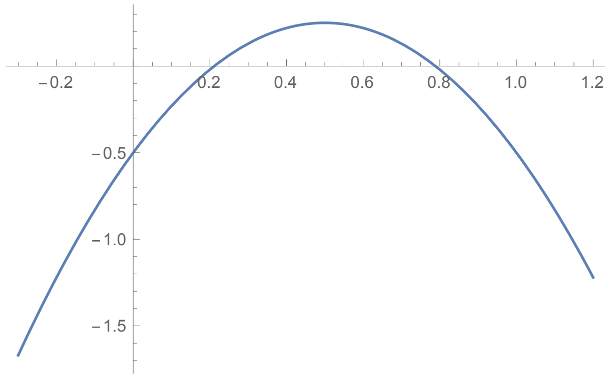

Figure 2.1.: Graph of the derivative g' of the nonlinearity g with $a = \frac{1}{2}$

The wave profile equation is satisfied, because we know that the travelling wave solution $\Phi = \Phi(x + ct - Z(t))$ solves the PDE

$$c\Phi' = \Phi'' + g(\Phi).$$

Using the wave equation in the calculation of the residual gives

$$J = -Z'(t)\Phi' - z'(t) + g(\Phi) - g(\Phi - z(t)).$$

In order for u^- to be a proper subsolution we need the residual to be negative. Looking at the residual, we notice that

$$g(\Phi) - g(\Phi - z(t)) \sim g'(\Phi)z(t)$$

by the mean value theorem. So the residual becomes

$$J = -Z'(t)\Phi' - z'(t) + g'(\Phi)z(t).$$

Remark that we have seen in (1.10) that $\Phi' > 0$. Furthermore, we assumed z to be decreasing and Z to be increasing, so $z' < 0$ and $Z' > 0$. We consider the derivative of g, the parabola

$$g'(u) = -3u^2 + 2(a+1)u - a, \tag{2.3}$$

which is depicted in Fig. 2.1. In the region where g' is positive, the term $Z'\Phi$ must dominate both z and z'. In the region close to 0 and 1 where g' is negative, z must dominate its own derivative z'. Therefore, we choose z to be a slowly decaying exponential function. Let ϵ and η_z be positive constants and define

$$z(t) = \epsilon e^{-\eta_z t}.$$

Then we find

$$Z(t) = K_Z \int_0^t z(s)ds = \epsilon K_Z \int_0^t e^{-\eta_z s} ds = \epsilon K_Z \Big(-\frac{1}{\eta_z} e^{-\eta_z t} + \frac{1}{\eta_z} \Big).$$

Specifically,

$$Z^\infty \sim \int_0^\infty z(t)dt = \frac{\epsilon}{\eta_z} \sim z_0$$

where Z^∞ can be thought of as the asymptotic phase shift and z_0 the size of the initial perturbation. From now on, we take the transversal dependency $\theta(y,t)$ unequal to the zero function.

In order to find a valid subsolution, we must prove negativity of the residual. For the discrete case we will see this in chapter 3 and chapter 4. For the continuous case we restrict ourselves to the calculation of the residual.

Lemma 2.1. *The residual of the subsolution*

$$J = u_t^-(t) - \Delta u^-(t) - g(u^-(t))$$

is given by

$$\begin{aligned} J_{global} &= -z'(t) - Z'(t)\Phi'(\xi(t)) + g(\Phi(\xi)) - g(\Phi(\xi) - z(t)) \\ J_{nl} &= -\Phi''(\xi(t))\theta_y(y,t)^2 \\ J_{heat} &= -\Phi'(\xi(t))(\theta_t(y,t) - \theta_{yy}(y,t)). \end{aligned}$$

such that

$$J = J_{global} + J_{nl} + J_{heat}.$$

Proof. We calculate the residual directly by using the subsolution of the form (2.2) and the travelling wave constant $\xi(t) = x + ct - \theta(y,t) - Z(t)$. Differentiating the subsolution in the residual gives

$$\begin{aligned} J &= (c - \theta_t(y,t) - Z'(t))\Phi'(\xi(t)) - z'(t) \\ &\quad - \frac{\partial^2}{\partial x^2}(\Phi(\xi(t)) - z(t)) - \frac{\partial^2}{\partial y^2}(\Phi(\xi(t)) - z(t)) \\ &\quad - g(\Phi(\xi(t)) - z(t)) \\ &= c\Phi'(\xi(t)) - \theta_t(y,t)\Phi'(\xi(t)) - Z'(t)\Phi'(\xi(t)) - z'(t) \\ &\quad - \frac{\partial}{\partial x}(\Phi'(\xi(t))) - \frac{\partial}{\partial y}(-\theta_y(y,t)\Phi'(\xi(t))) \\ &\quad - g(\Phi(\xi(t)) - z(t)) \\ &= c\Phi'(\xi(t)) - \theta_t(y,t)\Phi'(\xi(t)) - Z'(t)\Phi'(\xi(t)) - z'(t) \\ &\quad - \Phi''(\xi(t)) - (-\theta_{yy}(y,t)\Phi'(\xi(t)) + \theta_y(y,t)^2\Phi'(\xi(t))) \\ &\quad - g(\Phi(\xi(t)) - z(t)) \\ &= c\Phi'(\xi(t)) - \theta_t(y,t)\Phi'(\xi(t)) - Z'(t)\Phi'(\xi(t)) - z'(t) \\ &\quad - \Phi''(\xi(t)) + \theta_{yy}(y,t)\Phi'(\xi(t)) - \theta_y(y,t)^2\Phi'(\xi(t)) \\ &\quad - g(\Phi(\xi(t)) - z(t)). \end{aligned}$$

In order to further simplify the expression, we make use of the wave equation in $\xi(t)$

$$c\Phi'(\xi(t)) = \Phi''(\xi(t)) + g(\Phi(\xi(t))$$

to find

$$\begin{aligned}J &= \Phi''(\xi(t)) + g(\Phi(\xi(t)) - \theta_t(y,t)\Phi'(\xi(t)) - Z'(t)\Phi'(\xi(t)) - z'(t)\\&\quad - \Phi''(\xi(t)) + \theta_{yy}(y,t)\Phi'(\xi(t)) - \theta_y(y,t)^2\Phi'(\xi(t)))\\&\quad - g(\Phi(\xi(t)) - z(t))\\&= g(\Phi(\xi(t)) - \theta_t(y,t)\Phi'(\xi(t)) - Z'(t)\Phi'(\xi(t)) - z'(t)\\&\quad + \theta_{yy}(y,t)\Phi'(\xi) - \theta_y(y,t)^2\Phi'(\xi(t)) - g(\Phi(\xi(t)) - z(t)).\end{aligned}$$

Splitting the terms according to their quality gives the required result. □

The difference in quality of the three terms J_{global}, J_{nl}, and J_{heat} is the reason for splitting the residual. In the proof of the validity of the subsolution they are considered separately. We end this section with a closer look at the three terms of the residual found in lemma 2.1.

The global residual J_{global} depends on the choice of the external functions $z(t)$ and $Z(t)$. By carefully choosing $z(t)$ and $Z(t)$, i.e. the constants ϵ, η_z, and K_Z, we keep J_{global} negative.

The other two residuals depend on the derivatives of Φ. We have seen both in the introduction in (1.10). The nonlinear residual J_{nl} carries a quadratic dependency of the transversal function $\theta(y,t)$. Note in particular that J_{nl} comes with no obvious sign, because the sign of Φ'' is unknown as we have seen in 1.10 and Fig. 1.5.

The heat residual J_{heat} depends on the residual of the continuous heat equation

$$\theta_{yy}(y,t) - \theta_t(y,t) = 0 \tag{2.4}$$

in $\theta(y,t)$. Furthermore, we have seen that Φ' is positive everywhere in Fig. 1.5. Therefore, by determining the sign of (2.4), we can determine the sign of J_{heat}. This property is exploited by dominating J_{nl} by J_{heat}.

In the next section, we take a closer look at the external function $\theta(y,t)$.

2.2. The Continuous Heat Kernel

The choice for the function $\theta(y,t)$ is based on the continuous heat kernel. Consider the heat equation

$$h_t(y,t) = h_{yy}(y,t)$$

with formal initial condition

$$\lim_{t\to 0} h(y,t) = \delta(y).$$

The heat equation is solved by the heat kernel, which is defined as

$$h(y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}, \tag{2.5}$$

because a quick calculation with the product rule shows us that we get

$$\frac{1}{\sqrt{4\pi t}}\Big(-\frac{1}{2t} + \frac{y^2}{4t}\Big)e^{-\frac{y^2}{4t}}$$

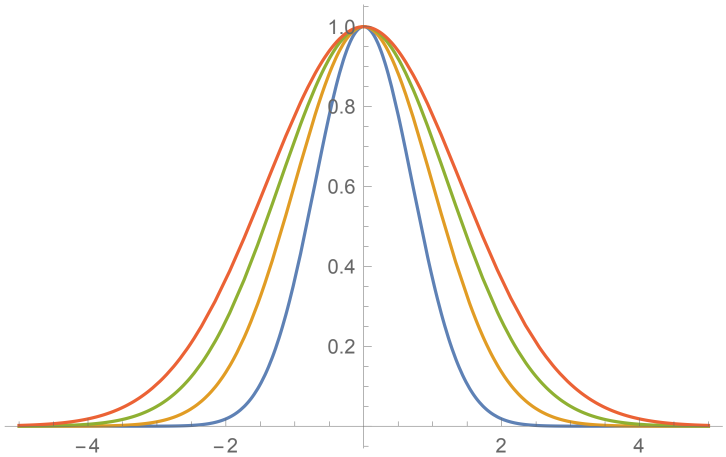

Figure 2.2.: The graph of $e^{\frac{-y^2}{t}}$ for $t = 1$ (blue) to $t = 4$ (red)

on both sides of the heat equation (2.5). Now define $v(y,t) = e^{-\frac{y^2}{4t}}$ such that

$$h(y,t) = \frac{1}{\sqrt{4\pi t}} v(y,t).$$

The asymptotic behaviour of $v(y,t)$ is straightforwardly determined as

$$v(y,t) \sim e^{-\frac{y^2}{4t}}, \quad v_t(y,t) \sim \frac{y^2}{4t^2} e^{-\frac{y^2}{4t}}, \quad v_y(y,t) \sim -\frac{y}{2t} e^{-\frac{y^2}{4t}}, \quad v_{yy}(y,t) \sim \left(\frac{y^2}{4t^2} - \frac{1}{2t}\right) e^{-\frac{y^2}{4t}}.$$

Now the transversal dependence $\theta(y,t)$, which has been introduced in [2] in order to deal with long-term perturbations, is a modification of $v(y,t)$ under use of the positive constants $\beta \gg 1, \gamma \gg 1$, and $0 < \alpha \ll 1$, given by

$$\theta(y,t) = \beta t^{-\alpha} v(y, \gamma t).$$

Hereby β is chosen according to the imposed intial condition. Furthermore, $\alpha \ll 1$ softens the decay and $\gamma \gg 1$ accelerates diffusion. In Fig. 2.2 we see the graph of $e^{\frac{-y^2}{t}}$ as a function in y for different values of t to illustrate the effect of large values of t.
But we want to show the effects of γ and α more clearly by comparing $\theta(y,t)$ with the heat kernel $h(y,t)$.
Let us start with α. We isolate the factor $t^{-\alpha}$ by considering

$$\theta(0,t) = \beta t^{-\alpha} \quad \text{and} \quad h(0,t) = \frac{1}{\sqrt{4\pi t}}$$

and compare

$$\frac{\theta(0,t)}{\theta(0,1)} = \frac{\beta t^{-\alpha}}{\beta} = \frac{1}{t^\alpha} \quad \text{to} \quad \frac{h(0,t)}{h(0,1)} = \frac{h(0,t)}{h(0,1)} = \frac{\sqrt{4\pi}}{\sqrt{4\pi t}} = \frac{1}{\sqrt{t}}.$$

The constant α is chosen to be smaller than $\frac{1}{2}$. Therefore, the fact that $t^{-\alpha}$ decays more slowly than $\frac{1}{\sqrt{t}}$ causes $\theta(y,t)$ to decay more slowly than the heat kernel $h(y,t)$ does.

The effect of γ becomes apparent when we compare for which y the heat kernel $h(y,t)$ reaches half of its initial value $h(0,t)$ to the equivalent for its modification $\theta(y,t)$. We do not need much more than the definitions of $h(y,t)$ and $v(y,t)$ to calculate

$$\begin{aligned} h(y,t) &= \frac{1}{2}h(0,t) \\ \frac{1}{\sqrt{4\pi t}}v(y,t) &= \frac{1}{2}\frac{1}{\sqrt{4\pi t}}v(0,t) \\ e^{-\frac{y^2}{4t}} &= \frac{1}{2} \\ -\frac{y^2}{4t} &= -ln2 \\ y &= \pm 2\sqrt{ln2}\sqrt{t} \end{aligned}$$

and analogously

$$\begin{aligned} \theta(y,t) &= \frac{1}{2}\theta(0,t) \\ \beta t^{-\alpha}v(y,\gamma t) &= \frac{1}{2}\beta t^{-\alpha}v(0,t) \\ e^{-\frac{y^2}{4\gamma t}} &= \frac{1}{2} \\ -\frac{y^2}{4\gamma t} &= -ln2 \\ y &= \pm 2\sqrt{ln2}\sqrt{\gamma t}. \end{aligned}$$

Thus $\theta(y,t)$ reaches half of its initial value for y dilated by a factor $\sqrt{\gamma}$ compared to $h(y,t)$. At each point in time t the graph of the heat kernel $h(y,t)$ is narrower with a factor $\frac{1}{\sqrt{\gamma}}$ than the graph of its modification $\theta(y,t)$. In other words the factor γ lets $\theta(y,t)$ spread faster.

3. The Discrete Heat Kernel and its Shape Profile

We turn to the LDE on the lattice. In this chapter, we want to prepare the construction of the subsolution for the stability result by looking at the discrete heat kernel, which is crucial for describing the transversal dependence as seen in section 2.2 for the continuous case.

3.1. The Discrete Heat Kernel

Consider the discrete heat equation

$$\frac{d}{dt}h_j(t) = h_{j+1}(t) + h_{j-1}(t) - 2h_j(t)$$

with initial conditions

$$h_j(0) = 0 \quad \text{for} \quad j \neq 0 \quad \text{and} \quad h_0(0) = 1.$$

In the following lemma, we calculate the fundamental solution of the discrete heat equation by using the continuous heat kernel from section 2.2. The fundamental solution of the discrete heat equation is called the discrete heat kernel.

Lemma 3.1. *The discrete heat equation is solved by the discrete heat kernel*

$$h_j(t) = e^{-2t}\mathcal{I}_j(2t),$$

where $\mathcal{I}_j(t)$ is the modified Bessel function of the first kind as defined in A.10.

Proof. We make use of discrete Fourier transformation from definition A.11

$$(h_j(t))_j \in l^2(\mathbb{R}) \iff \hat{h}_\omega(t) \in L^2_{per}[-\pi, \pi]$$

to solve the discrete heat equation

$$h_j'(t) = h_{j+1}(t) + h_{j-1}(t) - 2h_j(t).$$

We set initial conditions, which can be shown to be consistent with our definition of $h_j(t)$ under use of the properties of $\mathcal{I}_j(t)$ found in A.10. We find

$$h_0(0) = e^0\mathcal{I}_0(0) = 1$$

and it follows that

$$h_k(0) = 0 \text{ for } k \neq 0.$$

We apply continuous Fourier transformation

$$\frac{d}{dt}(\hat{h}_\omega(t)) = (e^{i\omega} + e^{-i\omega} - 2)\hat{h}_\omega(t) = (2\cos\omega - 2)\hat{h}_\omega(t).$$

This is a simple ODE we can solve instantly

$$\hat{h}_\omega(t) = e^{2(\cos\omega - 1)t}\hat{h}_\omega(0) = \frac{1}{2\pi}e^{2(\cos\omega-1)t}.$$

Now we retransform with the help of discrete Fourier transformation from A.11. We make use of the fact that the sine function is an odd function. Furthermore, we use that the cosine function is an even function, which implies that $e^{2t\cos(\omega)}$ is also an even function

$$\begin{aligned} h_j(t) &= \int_{-\pi}^{\pi} e^{i\omega j}\hat{h}_\omega(t)d\omega \\ &= \int_{-\pi}^{\pi} e^{i\omega j}\frac{1}{2\pi}e^{2(\cos\omega-1)t}d\omega \\ &= \frac{e^{-2t}}{2\pi}\int_{-\pi}^{\pi} e^{i\omega j + 2t\cos(\omega)}d\omega \\ &= \frac{e^{-2t}}{2\pi}\int_{-\pi}^{\pi} (cos(\omega j) + i\sin(\omega j))e^{2t\cos(\omega)}d\omega \\ &= \frac{e^{-2t}}{\pi}\int_{0}^{\pi} cos(\omega j)e^{2t\cos(\omega)}d\omega. \end{aligned}$$

In order to simplify h_j, we use the modified Bessel function of the first order for whole j

$$\mathcal{I}_j(t) = \frac{1}{\pi}\int_0^\pi e^{t\cos\omega}cos(j\omega)d\omega.$$

Finally, we may write

$$h_j(t) = e^{-2t}\mathcal{I}_j(2t).$$

□

Of course, $h_j(t)$ solves the discrete heat equation by construction, i.e.

$$h'_j(t) = h_{j+1}(t) + h_{j-1}(t) - 2h_j(t).$$

We calculate the time derivative of $h_j(t)$ by the product and chain rule to be

$$h'_j(t) = -2e^{-2t}\mathcal{I}_j(2t) + 2e^{-2t}\mathcal{I}'_j(2t). \tag{3.1}$$

Note that we have implicitly used a property of the modified Bessel function of the first kind, which we get back by inserting (3.1) in the discrete heat equation

$$\begin{aligned} -2e^{-2t}\mathcal{I}_j(2t) + 2e^{-2t}\mathcal{I}'_j(2t) &= e^{-2t}\mathcal{I}_{j+1}(2t) + e^{-2t}\mathcal{I}_{j-1}(2t) - 2e^{-2t}\mathcal{I}_j(2t) \qquad (3.2) \\ 2\mathcal{I}'_j(2t) &= \mathcal{I}_{j+1}(2t) + \mathcal{I}_{j-1}(2t). \end{aligned}$$

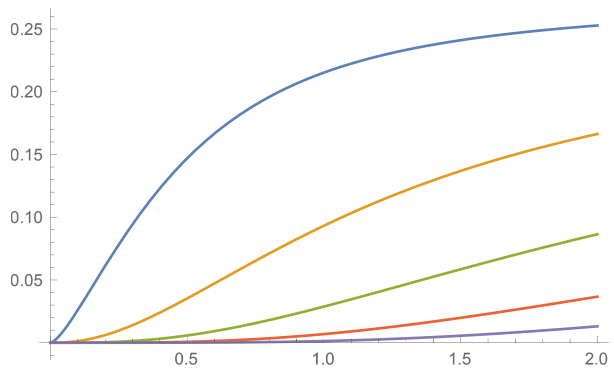

Figure 3.1.: Graph of $v_j(t)$ for $j = 1$ (blue) to $j = 5$ (purple)

3.2. Properties of the Shape Profile

Let

$$v_j(t) = \sqrt{t}h(t), \tag{3.3}$$

i.e. by lemma 3.1

$$v_j(t) = \sqrt{t}e^{-2t}\mathcal{I}_j(2t).$$

We call v_j the shape profile of the discrete heat kernel. We define $v_j(t)$ in anology to the continuous heat kernel considered in (2.5). Its graph is shown in Fig. 3.1. In this section we want to explore the behaviour and properties of $v_j(t)$, because ultimately we use it to describe the transversal dependency $\theta_j(t)$ in our modified subsolution. We begin with direct estimates of $v_j(t)$, respectively $\frac{j}{t}v_j(t)$.

3.2.1. Upper and Lower Bounds of the Shape Profile

In definition A.10 we are given that the modified Bessel function is positive for whole j. Thus it is obvious from the definition that $v_j(t) = \sqrt{t}e^{-2t}\mathcal{I}_j(2t)$ is positive, because each one of its multiple factors is positive. We use the estimate from theorem A.9 to prove the following lemma.

Lemma 3.2. *For $j = 0, t = 1$ or $1 \leq j \leq t$ there is a constant $0 < C < 1$ such that*

$$v_j(t) \leq Ce^{-2t(\frac{j^2}{4t^2})^{\frac{1}{8}}}.$$

Proof. Theorem A.9 gives us the estimate from above

$$\mathcal{I}_j(2t) \leq \frac{1}{\sqrt{2\pi}}(4t^2+j^2)^{-\frac{1}{4}}e^{\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}e^{\frac{1}{2\sqrt{4t^2+j^2}}}$$

and thus

$$v_j(t) \leq \frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}}e^{-2t+\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}e^{\frac{1}{2\sqrt{4t^2+j^2}}}.$$

We can estimate the factor $\frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}}$ directly

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}} &= \frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2)^{-\frac{1}{4}}\left(1+\frac{j^2}{4t^2}\right)^{-\frac{1}{4}} \\ &= \frac{1}{2\sqrt{\pi}}\left(1+\frac{j^2}{4t^2}\right)^{-\frac{1}{4}} \leq \frac{1}{2\sqrt{\pi}}. \end{aligned} \tag{3.4}$$

Furthermore, we estimate

$$e^{\frac{1}{2\sqrt{4t^2+j^2}}} \leq e^{\frac{1}{4t}} \leq e^{\frac{1}{4}}, \tag{3.5}$$

because the exponent is maximal for $j=0$ and $t=1$.
For convenience we define

$$C = \frac{1}{2\sqrt{\pi}}e^{\frac{1}{4}} \approx 0.22 \tag{3.6}$$

and remark that $0 < C < 1$ as required.
In order to estimate the remaining factor

$$e^{-2t+\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}, \tag{3.7}$$

we consider its exponent

$$-2t+\sqrt{4t^2+j^2}+j\left(ln\left(\frac{2t}{j+\sqrt{4t^2+j^2}}\right)\right) \tag{3.8}$$

$$= -2t+2t\sqrt{1+\frac{j^2}{4t^2}} - j\left(ln\left(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\right)\right) \tag{3.9}$$

$$= 2t\left(-1+\sqrt{1+\frac{j^2}{4t^2}} - \frac{j}{2t}ln\left(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\right)\right). \tag{3.10}$$

We use lemma A.2 for $x=\frac{j^2}{4t^2}$ to simplify

$$-1+\sqrt{1+\frac{j^2}{4t^2}} \leq -1+1+\frac{1}{2}\frac{j^2}{4t^2} = \frac{1}{2}\frac{j^2}{4t^2}.$$

We use that the logarithm is an increasing function and lemma A.5 for $x=\frac{j}{2t}\leq 1$ to estimate

$$-\frac{j}{2t}ln\left(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\right) \leq -\frac{j}{2t}ln\left(\frac{j}{2t}+1+\frac{3}{8}\frac{j^2}{4t^2}\right).$$

Now we want to get rid off the logarithm by using lemma A.6 with $x=\frac{j}{2t}+\frac{3}{8}\frac{j^2}{4t^2}$

$$\begin{aligned}&-\frac{j}{2t}ln\left(\frac{j}{2t}+1+\frac{3}{8}\frac{j^2}{4t^2}\right) \\ &\leq -\frac{j}{2t}\left(\frac{j}{2t}+\frac{3}{8}\frac{j^2}{4t^2}-\frac{1}{2}\left(\frac{j}{2t}+\frac{3}{8}\frac{j^2}{4t^2}\right)^2\right).\end{aligned}$$

Inserting the estimates above into (3.10) gives

$$\begin{aligned}
&2t\Bigg(-1+\sqrt{1+\frac{j^2}{4t^2}}-\frac{j}{2t}ln\Bigg(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\Bigg)\Bigg)\\
&\leq 2t\Bigg(\frac{1}{2}\frac{j^2}{4t^2}-\frac{j}{2t}\Bigg(\frac{j}{2t}+\frac{3}{8}\frac{j^2}{4t^2}-\frac{1}{2}\Bigg(\frac{j}{2t}+\frac{3}{8}\frac{j^2}{4t^2}\Bigg)^2\Bigg)\Bigg)\\
&=2t\Bigg(\frac{j^2}{4t^2}\Bigg(-\frac{1}{2}-\frac{3}{8}\frac{j}{2t}+\frac{1}{2}\frac{j}{2t}\Bigg(1+\frac{9}{64}\frac{j^2}{4t^2}+\frac{3}{4}\frac{j}{2t}\Bigg)\Bigg)\Bigg).
\end{aligned}$$

The estimate of the worst case scenario of the term

$$-\frac{1}{2}-\frac{3}{8}\frac{j}{2t}+\frac{1}{2}\frac{j}{2t}\Bigg(1+\frac{9}{64}\frac{j^2}{4t^2}+\frac{3}{4}\frac{j}{2t}\Bigg)$$

proves to be sufficient here. The negative term is maximal for $\frac{j}{t}=0$, i.e.

$$-\frac{1}{2}-\frac{3}{8}\frac{j}{2t}\leq -\frac{1}{2}.$$

While the positive term is maximal for $\frac{j}{t}=1$, i.e.

$$\frac{1}{4}\Bigg(1+\frac{9}{256}+\frac{3}{8}\Bigg)=\frac{361}{1024}.$$

Therefore, we estimate (3.10) from above by

$$2t\Bigg(\frac{j^2}{4t^2}\Bigg(-\frac{1}{2}+\frac{361}{1024}\Bigg)\Bigg)=-2t\Bigg(\frac{j^2}{4t^2}\Bigg)\frac{151}{1024}\leq -2t\Bigg(\frac{j^2}{4t^2}\Bigg)\frac{1}{8}.$$

Now we have established an estimate of (3.7). Together with (3.5) and (3.4) we find

$$\begin{aligned}
v_j(t) &\leq \frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}}e^{-2t+\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}e^{\frac{1}{2\sqrt{4t^2+j^2}}}\\
&\leq Ce^{-2t(\frac{j^2}{4t^2})\frac{1}{8}},
\end{aligned}$$

which proves the claim. □

There are two direct consequences of the lemma we need for the estimate of the residual in section 4.3.

Corollary 3.3. *For* $0\leq j\leq t$ *we find*

$$0<v_j(t)\leq 1.$$

Proof. We have seen in lemma 3.2 that

$$v_j(t)\leq Ce^{-2t(\frac{j^2}{4t^2})\frac{1}{8}}$$

with C positive and smaller than 1. The exponential factor has a negative exponent since $t\geq 1$ and $(\frac{j^2}{4t^2})\frac{1}{8}>0$. But then

$$e^{-2t(\frac{j^2}{4t^2})\frac{1}{8}}\leq e^0=1$$

holds and the result follows. □

Corollary 3.4. *For $1 \le t^{\frac{3}{4}} \le j \le t$ we find*

$$0 < v_j(t) \le Ce^{-\frac{\sqrt{t}}{16}}.$$

Proof. In lemma 3.2 we have seen that

$$v_j(t) \le Ce^{-2t(\frac{j^2}{4t^2})\frac{1}{8}}.$$

Now for $t^{\frac{3}{4}} \le j \le t$ we can use that the exponential function is increasing, to find

$$2t\left(\frac{t^{\frac{3}{2}}}{4t^2}\right) = \frac{\sqrt{t}}{2} \le 2t\left(\frac{j^2}{4t^2}\right) \le \frac{t}{2}$$
$$e^{-\frac{t}{2}} \le e^{-2t(\frac{j^2}{4t^2})} \le e^{-\frac{\sqrt{t}}{2}}.$$

Summarizing with lemma 3.2 we have

$$v_j(t) \le Ce^{-2t(\frac{j^2}{4t^2})\frac{1}{8}} \le Ce^{-\frac{\sqrt{t}}{2}\frac{1}{8}} = Ce^{-\frac{\sqrt{t}}{16}}$$

as required. □

We have already seen that $v_j(t)$ is positive, now we need to improve the lower bound of $v_j(t)$.

Lemma 3.5. *There is a constant $0 < C' < 1$ such that for $j = 0, t = 1$ or $1 \le j \le \sqrt{t}$*

$$v_j(t) \ge C'e^{-(\frac{1}{4}+\frac{1}{8\sqrt{t}}+\frac{1}{64t})}.$$

Proof. We use the estimate from below for the Bessel function found in theorem A.9

$$\mathcal{I}_j(2t) \ge \frac{1}{\sqrt{2\pi}}(4t^2+j^2)^{-\frac{1}{4}}e^{\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}e^{-\frac{1}{2\sqrt{4t^2+j^2}}}.$$

and thus

$$v_j(t) \ge \frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}}e^{-2t+\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}e^{-\frac{1}{2\sqrt{4t^2+j^2}}}.$$

We can estimate the factor $\frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}}$ directly by using the maximum $j = \sqrt{t}$ and the minimum $t = 1$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2+j^2)^{-\frac{1}{4}} &= \frac{1}{\sqrt{2\pi}}\sqrt{t}(4t^2)^{-\frac{1}{4}}\left(1+\frac{j^2}{4t^2}\right)^{-\frac{1}{4}} \\
&= \frac{1}{2\sqrt{\pi}}\left(1+\frac{j^2}{4t^2}\right)^{-\frac{1}{4}} \\
&\ge \frac{1}{2\sqrt{\pi}}\left(1+\frac{\sqrt{t}^2}{4t^2}\right)^{-\frac{1}{4}} \\
&= \frac{1}{2\sqrt{\pi}}\left(1+\frac{1}{4t}\right)^{-\frac{1}{4}} \\
&\ge \frac{1}{2\sqrt{\pi}}\left(\frac{4}{5}\right)^{\frac{1}{4}}.
\end{aligned}$$

Furthermore we estimate

$$e^{-\frac{1}{2\sqrt{4t^2+j^2}}} \geq e^{-\frac{1}{4}}, \tag{3.11}$$

because the minus sign enables us to just reverse the estimate (3.5) as seen in lemma 3.2. For convenience we define

$$C' = \frac{1}{2\sqrt{\pi}} \left(\frac{4}{5}\right)^{\frac{1}{4}} e^{-\frac{1}{4}} \approx 0.21 \tag{3.12}$$

and remark that $0 < C' < 1$ as required.
The remaining factor

$$e^{-2t+\sqrt{4t^2+j^2}+j(ln(\frac{2t}{j+\sqrt{4t^2+j^2}}))}$$

can be estimated from above by considering its exponent

$$\begin{aligned} &-2t + \sqrt{4t^2+j^2} + j\Bigg(ln\bigg(\frac{2t}{j+\sqrt{4t^2+j^2}}\bigg)\Bigg) \\ &= 2t\Bigg(-1+\sqrt{1+\frac{j^2}{4t^2}} - \frac{j}{2t}\Bigg(ln\bigg(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\bigg)\Bigg)\Bigg). \end{aligned} \tag{3.13}$$

We use lemma A.1 for $x = \frac{j}{2t}$ to simplify the first term

$$-1 + \sqrt{1+\frac{j^2}{4t^2}} \geq \frac{1}{2}\frac{j^2}{4t^2} - \frac{1}{8}\frac{j^4}{16t^4}.$$

We use the fact that the logarithm is an increasing function and lemma A.2 for $x = \frac{j}{2t}$ to estimate the second term

$$-\frac{j}{2t}ln\bigg(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\bigg) \geq -\frac{j}{2t}ln\bigg(1+\frac{j}{2t}+\frac{1}{2}\frac{j^2}{4t^2}\bigg).$$

In a last step, we apply lemma A.4 for $x = \frac{j}{2t}$ to find

$$-\frac{j}{2t}ln\bigg(1+\frac{j}{2t}+\frac{1}{2}\frac{j^2}{4t^2}\bigg) \geq -\frac{j}{2t}\bigg(\frac{j}{2t}+\frac{1}{2}\frac{j^2}{4t^2}\bigg).$$

Summarizing the estimates we find for the exponent (3.13)

$$
\begin{aligned}
&2t\Big(1+\sqrt{1+\frac{j^2}{4t^2}}-\frac{j}{2t}\Big(ln\Big(\frac{j}{2t}+\sqrt{1+\frac{j^2}{4t^2}}\Big)\Big)\\
&\geq 2t\Big(\frac{1}{2}\frac{j^2}{4t^2}-\frac{1}{8}\frac{j^4}{16t^4}-\frac{j}{2t}\Big(\frac{j}{2t}+\frac{1}{2}\frac{j^2}{4t^2}\Big)\Big)\\
&=2t\frac{j^2}{4t^2}\Big(-\frac{1}{2}-\frac{1}{2}\frac{j}{2}t-\frac{1}{8}\frac{j^2}{4t^2}\Big)\\
&=-t\frac{j^2}{4t^2}\Big(1+\frac{j}{2t}+\frac{1}{4}\frac{j^2}{4t^2}\Big)\\
&\geq -t\frac{\sqrt{t}^2}{4t^2}\Big(1+\frac{\sqrt{t}}{2t}+\frac{1}{4}\frac{\sqrt{t}^2}{4t^2}\Big)\\
&=-\frac{1}{4}\Big(1+\frac{1}{2\sqrt{t}}+\frac{1}{16t}\Big)\\
&\geq -\Big(\frac{1}{4}+\frac{1}{8\sqrt{t}}+\frac{1}{64t}\Big),
\end{aligned}
$$

where we have used that $j \leq \sqrt{t}$. Together with (3.6), the required lower bound becomes

$$v_j(t) \geq C'e^{-(\frac{1}{4}+\frac{1}{8\sqrt{t}}+\frac{1}{64t})}.$$

□

Lemma 3.6. *For $2 \leq t \leq j$ we find the following upper bound of $\frac{j}{t}v_j(t)$*

$$\frac{j}{t}v_j(t) \leq \frac{1}{\sqrt{2}}e^{\frac{1}{4}}e^{-\frac{t}{4}} \leq \frac{1}{\sqrt{2}}e^{\frac{1}{4}} \approx 0.55.$$

Proof. For convenience we rename $1 \leq \frac{j}{t} = x$. We may apply theorem A.9 since we set $t \geq 2$ and find

$$xv_j(t) \leq x\sqrt{t}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2t}(1+x^2)^{\frac{1}{4}}}e^{4t\sqrt{1+x^2}}e^{-2t(1+\sqrt{1+x^2}-xln(x+\sqrt{1+x^2})}.$$

Again we begin with the non-exponential factor:

$$
\begin{aligned}
\sqrt{t}x(4t^2+j^2)^{-\frac{1}{4}} &= \sqrt{t}x(4t^2)^{-\frac{1}{4}}(1+\frac{j^2}{4t^2})^{-\frac{1}{4}}\\
&=\frac{x}{\sqrt{2}(1+x^2)^{\frac{1}{4}}} \leq \frac{x}{\sqrt{2}}.
\end{aligned}
$$

We also estimate $e^{\frac{1}{2\sqrt{4t^2+j^2}}}$ right away, which is bounded by $e^{\frac{1}{4}}$ as shown in (3.5). We consider the exponent of the remaining factor and denote

$$g(x) = \sqrt{1+x^2}-1-xln(x+\sqrt{1+x^2}).$$

We want to find a $\kappa > 0$ such that

$$g(x) \leq -\kappa x ln(x). \tag{3.14}$$

Under use of lemma A.3 we can write

$$g(x) \leq x + \frac{1}{2x} - 1 - x ln(x + \sqrt{1 + x^2}).$$

We can use that $\sqrt{x^2 + 1} \geq x$ to write

$$\begin{aligned} g(x) &\leq x + \frac{1}{2x} - 1 - x ln(x + \sqrt{1 + x^2}) \\ &\leq x + \frac{1}{2x} - 1 - x ln(2x) \\ &\leq x + \frac{1}{2x} - 1 - x ln2 - x ln(x). \end{aligned}$$

We want to find a $B = -\kappa + 1 < 1$ such that for all $x \geq 1$

$$x + \frac{1}{2x} - 1 - x ln2 \leq B x ln(x),$$

because then we can determine the κ in (3.14).
First we fill in $x = 1$ and we obtain a trivially true statement for every choice of B

$$1 + \frac{1}{2} - 1 - ln2 \leq 0.$$

For $B = 1$ the functions on both sides of the inequality are depicted in Fig. 3.2. Now we take the derivative on both sides of the inequality

$$1 - \frac{1}{2x^2} - ln2 \leq B(1 + ln(x)).$$

We can estimate the left hand side from above by

$$1 - \frac{1}{2x^2} - ln2 \leq 1 - ln2$$

and we can estimate the right hand side from below by

$$B(1 + ln(x)) \geq B.$$

It follows that we can choose

$$B = 1 - ln2 \approx 0.31.$$

Therefore

$$\kappa = 1 - B = ln2 \approx 0.69.$$

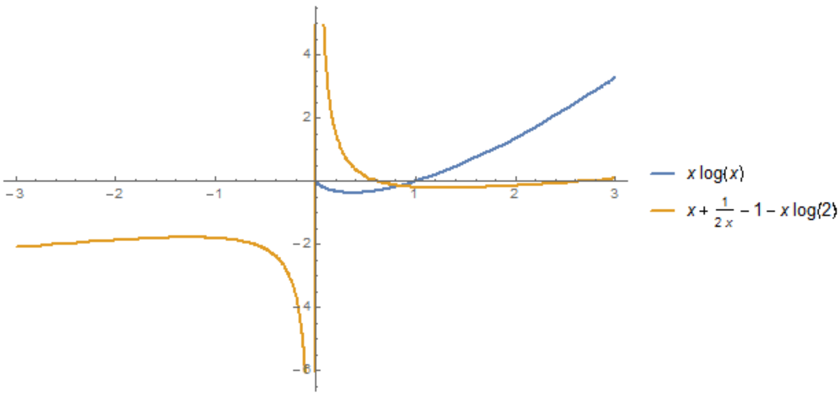

Figure 3.2.: The graphs of $xlnx$ and $x + \frac{1}{2x} - 1 - xln2$

Combining the results we find

$$\begin{aligned} xv_j(t) &\leq \frac{x}{\sqrt{2}(1+x)^{\frac{1}{4}}} e^{\frac{1}{2t\sqrt{4+x^2}}} e^{2tg(x)} \\ &\leq e^{\frac{1}{4}} \frac{x}{\sqrt{2}} e^{-2t\kappa xln(x)}. \end{aligned}$$

It remains to prove that

$$-2t\kappa xln(x) \leq -\frac{1}{4}t$$

i.e.

$$\kappa xlnx \geq \frac{1}{8}.$$

But for $x \geq 1$, $xln(x)$ is a nonneqative function and clearly, $\frac{1}{8} < \kappa \approx 0.69$. It follows that

$$e^{-2t\kappa xln(x)} \leq e^{-\frac{1}{4}t} \leq 1$$

and the lemma is proven. □

3.2.2. Relative Bounds of the Shape Profile

In this subsection we want to obtain bounds of $v_j'(t)$ and the differences $|v_{j\pm1}(t) - v_j(t)|$ in $v_j(t)$, where we use the expression $v_{j\pm1}$ to denote that we can fill in either v_{j+1} or v_{j-1}. In the discrete case we can exploit properties of the Bessel functions to get estimates for the asymptotic behaviour of $v_j'(t)$. A direct calculation of the derivative of $v_j(t)$ gives

$$\begin{aligned} v_j'(t) &= \frac{1}{2\sqrt{t}} e^{-2t}\mathcal{I}_j(2t) - 2\sqrt{t}\mathcal{I}_j(2t) + 2\sqrt{t}e^{-2t}\mathcal{I}'_j(2t) \\ &= \frac{1}{2t}v_j(t) - 2v_j(t) + \frac{2a_j(2t)}{2t}v_j(t) \\ &= \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t}\right)v_j(t), \end{aligned} \tag{3.15}$$

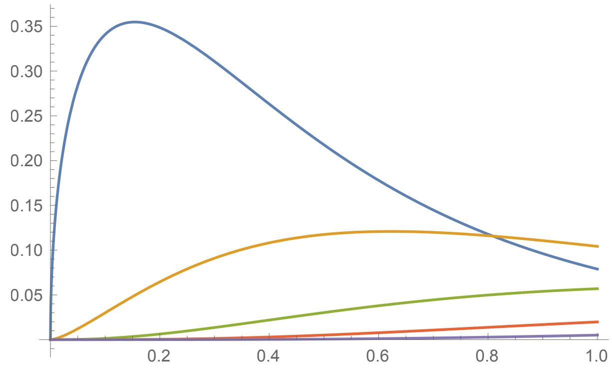

Figure 3.3.: Graph of $v_j'(t)$ for $j = 1$ (blue) to $j = 5$ (purple) on $[0, 1]$

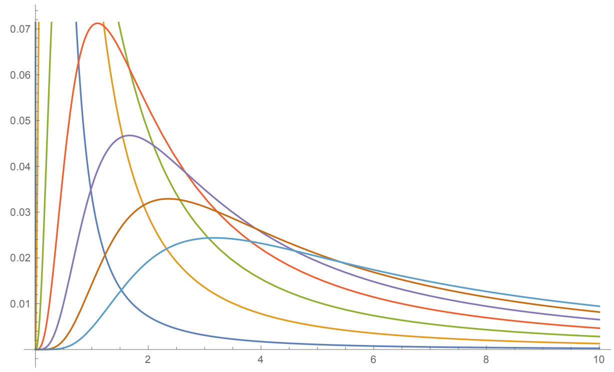

Figure 3.4.: Graph of $v_j'(t)$ for $j = 1$ (blue) to $j = 5$ (purple) on $[0, 10]$

where we have used the expression

$$a_j(2t) = \frac{2t\mathcal{I}'_j(2t)}{\mathcal{I}_j(2t)}$$

as defined in [12]. In Fig. 3.3 and Fig. 3.4 we can see the graph of $v_j'(t)$ in small zoom and in large zoom.

Remark 3.7. With theorem A.8 we can find an accurate estimate of $v_j'(t)$. In order to get a better intuition for the asymptotic behaviour of $v_j'(t)$ first, we take a rough estimate $a_j(2t) \sim \sqrt{4t^2 + j^2}$ taken from theorem A.8 in our calculation of $v_j'(t)$.

$$\begin{aligned} v_j'(t) &\sim v_j(t)\left(\frac{1}{2t} - 2 + \frac{\sqrt{4t^2 + j^2}}{t}\right) \\ &= v_j(t)\left(\frac{1}{2t} - 2 + 2\sqrt{1 + \frac{j^2}{4t^2}}\right) \end{aligned}$$

For small $|\frac{j}{t}| \leq 1$ we use the first order Taylor approximation on the square root term to see that

$$v_j'(t) \sim v_j(t)\left(\frac{1}{2t} - 2 + 2\left(1 + \frac{1}{2}\frac{j^2}{4t^2}\right)\right) \sim \frac{j^2}{4t^2}v_j(t).$$

For large $|\frac{j}{t}| \geq 1$ we apply lemma A.2 for $x = \frac{j^2}{4t^2}$. The lemma provides an accurate estimate for large x. Furthermore, we use that the terms $\frac{2t}{j} \leq 1$ and $\frac{1}{2t}$ are neglible for large $t \leq j$ to see

$$v_j'(t) \sim v_j(t)\left(\frac{1}{2t} - 2 + 2\left(\frac{j}{2t} + \frac{t}{4j}\right)\right) \sim \frac{j}{t}v_j(t).$$

We want to take a closer look at $v_j(t)$ now. Starting out from the property of the Bessel function given in A.10

$$2\mathcal{I}'_j(2t) = \mathcal{I}_{j+1}(2t) + \mathcal{I}_{j-1}(2t)$$

and the recursion relation found in A.10 the Bessel functions satisfy

$$\mathcal{I}'_j(2t) = \mathcal{I}_{j+1}(2t) + \frac{j}{2t}\mathcal{I}_j(2t). \tag{3.16}$$

We can write

$$\mathcal{I}'_j(2t) = \mathcal{I}_{j-1}(2t) - \frac{j}{2t}\mathcal{I}_j(2t). \tag{3.17}$$

Adding up (3.16) and (3.17) and multiplying with $\sqrt{t}e^{-2t}$ gives

$$v_{j+1}(t) + v_{j-1}(t) = 2e^{-2t}\sqrt{t}\mathcal{I}'_j(2t).$$

Under use of the expression $a_j(2t) = \frac{2t\mathcal{I}'_j(2t)}{\mathcal{I}_j(2t)}$, we find

$$v_{j+1}(t) + v_{j-1}(t) = 2e^{-2t}\sqrt{t}\frac{a_j(2t)\mathcal{I}_j(2t)}{2t} = \frac{a_j(2t)}{t}v_j(t).$$

Finally, using the expression for $v_j'(t)$ we found in (3.15) we can rewrite as

$$v_{j+1}(t) + v_{j-1}(t) = v_j'(t) - \frac{1}{2t}v_j(t) + 2v_j(t),$$

which reflects that $v_j(t)$ does not solve the heat equation exactly. With the Bessel function recursion relation found in A.10 we can rewrite $a_j(t)$ as

$$a_j(t) = j + t\frac{\mathcal{I}_{j+1}(t)}{\mathcal{I}_j(t)}$$

and use this to write the difference $v_{j+1} - v_j$ as a multiple of v_j

$$\begin{aligned} v_{j+1}(t) - v_j(t) &= \sqrt{t}e^{-2t}(\mathcal{I}_{j+1}(2t) - \mathcal{I}_j(2t)) \\ &= \sqrt{t}e^{-2t}\left(\frac{a_j(2t)}{2t} - \frac{j}{2t} - 1\right)\mathcal{I}_j(2t) \\ &= \left(\frac{a_j(2t)}{2t} - \frac{j}{2t} - 1\right)v_j(t). \end{aligned}$$

Analogously we find

$$\begin{aligned}
v_{j-1}(t) - v_j(t) &= \sqrt{t}e^{-2t}(\mathcal{I}_{j-1}(2t) - \mathcal{I}_j(2t)) \\
&= \sqrt{t}e^{-2t}\left(\frac{a_j(2t)}{2t} + \frac{j}{2t} - 1\right)\mathcal{I}_j(2t) \\
&= \left(\frac{a_j(2t)}{2t} + \frac{j}{2t} - 1\right)v_j(t).
\end{aligned}$$

We formulate a corollary of theorem A.8 for a better estimate of the expressions in v_j.

Corollary 3.8. *For $j \geq 0$ and $t \geq 2$ we find*

$$a_j(t) = \sqrt{t^2 + j^2} - \frac{t^2}{2(t^2 + j^2)} + b_j(t)$$

where $b_j(t)$ denotes the error satisfying

$$|b_j(t)| \leq \frac{t^2}{2(t^2 + j^2)^{\frac{3}{2}}}. \tag{3.18}$$

Proof. We apply theorem A.8. In order for the bound (3.18) to hold, we need $j \geq 0$ that $t > 0$ and

$$\sqrt{j^2 + t^2} \geq \frac{\sqrt{7} + 2}{3} \approx 1.55.$$

Therefore, the expression for $a_j(t)$ and its error $b_j(t)$ is valid for $j \geq 0$ and $t \geq 2$. □

Thus we can write

$$\begin{aligned}
v_j'(t) &= \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t}\right)v_j(t) \\
&= \left(\frac{1}{2t} - 2 + \frac{\sqrt{4t^2 + j^2} - \frac{4t^2}{2(4t^2+j^2)} + b_j(2t)}{t}\right)v_j(t) \\
&= \left(\frac{1}{2t} - 2 + 2\sqrt{1 + \frac{j^2}{4t^2}} - \frac{1}{2t(1 + \frac{j^2}{4t^2})}\right)v_j(t) + \frac{b_j(2t)}{t}v_j(t)
\end{aligned}$$

as well as

$$\begin{aligned}
v_{j-1}(t) - v_j(t) &= \left(\frac{a_j(2t)}{2t} + \frac{j}{2t} - 1\right)v_j(t) \\
&= \left(\frac{\sqrt{4t^2 + j^2}}{2t} - \frac{4t^2}{4t(4t^2 + j^2)} + \frac{b_j(2t)}{2t} + \frac{j}{2t} - 1\right)v_j(t) \\
&= \left(\sqrt{1 + \frac{j^2}{4t^2}} - \frac{1}{4t(1 + \frac{j^2}{4t^2})} + \frac{j}{2t} - 1\right)v_j(t) + \frac{b_j(2t)}{2t}v_j(t),
\end{aligned} \tag{3.19}$$

and

$$v_{j+1}(t) - v_j(t) = \left(\frac{a_j(2t)}{2t} - \frac{j}{2t} - 1 \right) v_j(t) \tag{3.20}$$
$$= \left(\frac{\sqrt{4t^2 + j^2}}{2t} - \frac{4t^2}{4t(4t^2 + j^2)} + \frac{b_j(2t)}{2t} - \frac{j}{2t} - 1 \right) v_j(t)$$
$$= \left(\sqrt{1 + \frac{j^2}{4t^2}} - \frac{1}{4t(1 + \frac{j^2}{4t^2})} - \frac{j}{2t} - 1 \right) v_j(t) + \frac{b_j(2t)}{2t} v_j(t).$$

We have seperated the exact terms from the errors. These expressions become useful when we estimate the heat residual and the nonlinear residual. First, we want to estimate the time derivative v_j' of v_j in terms of v_j. We see that

$$v_j'(t) = \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t} \right) v_j(t) \tag{3.21}$$

is positive if the term

$$\frac{1}{2t} - 2 + \frac{a_j(2t)}{t} \tag{3.22}$$

is positive, which we want to estimate from below. We have to distinguish four cases, $j = 0$ or $j = 1$, $j \leq t$ and finally $j \geq t$. We treat the cases $j = 0$ and $j = 1$ separately, because for such small j we have to estimate with a negative coefficient of $v_j(t)$.

Lemma 3.9. *For $j = 0$ and $t \geq 2$ we have*

$$v_0'(t) \geq -\frac{1}{4t^2} v_0(t). \tag{3.23}$$

Proof. We easily calculate (3.22) for $j = 0$

$$\frac{1}{2t} - 2 + \frac{a_0(2t)}{t} = \frac{1}{2t} - 2 + \frac{\sqrt{4t^2}}{t} - \frac{4t^2}{2t(4t^2)} + \frac{b_0(2t)}{t} = \frac{b_0(2t)}{t}.$$

In particular, we note that $v_0'(t)$ is only positive for a positive error. By the error estimate (3.18), we know that

$$\left| \frac{b_0(2t)}{t} \right| \leq \frac{1}{4t^2}.$$

□

Lemma 3.10. *For $j = 1$ and $t \geq 2$ we find*

$$v_1'(t) \geq -\frac{1}{64} \frac{1}{t^4} v_1(t). \tag{3.24}$$

Proof. We calculate the term (3.22)

$$
\begin{aligned}
\frac{1}{2t} - 2 + \frac{\sqrt{4t^2+1}}{t} - \frac{2t}{4t^2+1} + \frac{b_1(2t)}{t} &\geq \frac{1}{2t} - 2 + 2\sqrt{1+\frac{1}{4t^2}} - \frac{1}{2t(1+\frac{1}{4t^2})} - \frac{2t}{(4t^2+1)^{\frac{3}{2}}}, \\
&\geq \left(\frac{1}{4}\frac{1}{t^2} - \frac{1}{64}\frac{1}{t^4}\right) + \left(\frac{1}{2t} - \frac{1}{2t(1+\frac{1}{4t^2})}\right) - \frac{1}{4t^2(1+\frac{1}{4t^2})^{\frac{3}{2}}} \\
&\geq \frac{1}{4t^2} - \frac{1}{64t^4} + \frac{1}{2t} - \frac{1}{2t} - \frac{1}{4t^2} \\
&= -\frac{1}{64t^4},
\end{aligned}
$$

where we have used lemma A.1 for $x = \frac{1}{4t^2}$ on the dominant term $-2 + 2\sqrt{1+\frac{1}{4t^2}}$. □

Lemma 3.11. *For $2 \leq j \leq t$ we have*

$$
v_j'(t) \geq \frac{11}{64}\frac{j^2}{t^2} v_j(t). \tag{3.25}
$$

Proof. We calculate

$$
\begin{aligned}
v_j'(t) &= \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t}\right) v_j(t) \\
&\geq \left(\frac{1}{2t} - 2 + \frac{\sqrt{4t^2+j^2}}{t} - \frac{2t}{4t^2+j^2} - \frac{2t}{(4t^2+j^2)^{\frac{3}{2}}}\right) v_j(t) \\
&\geq \left(\frac{1}{2t} + \frac{1}{4}\frac{j^2}{t^2} - \frac{1}{64}\frac{j^4}{t^4} - \frac{2t}{4t^2+j^2} - \frac{2t}{(4t^2+j^2)^{\frac{3}{2}}}\right) v_j(t) \\
&= \left(\left(\frac{1}{4}\frac{j^2}{t^2} - \frac{1}{64}\frac{j^4}{t^4}\right) + \left(\frac{1}{2t} - \frac{1}{2t(1+\frac{j^2}{4t^2})}\right) - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}}\right) v_j(t).
\end{aligned}
$$

In the first line we have inserted the lower bound of the error. In the second line we have estimated the dominant term $-2 + \frac{\sqrt{4t^2+j^2}}{t} = -2 + 2\sqrt{1+\frac{j^2}{4t^2}}$ with lemma A.1 for $x = \frac{1}{4t^2}$. The fact that $0 \leq \frac{j^2}{t^2} \leq 1$ allows for the estimate

$$
\left(\frac{1}{4}\frac{j^2}{t^2} - \frac{1}{64}\frac{j^4}{t^4}\right) \geq \left(\frac{1}{4}\frac{j^2}{t^2} - \frac{1}{64}\frac{j^2}{t^2}\right) = \frac{15}{16}\frac{j^2}{4t^2}. \tag{3.26}
$$

In order to prove the lemma we only need to show that the remaining terms are larger than $-\frac{4}{16}\frac{j^2}{4t^2}$ i.e.

$$
\frac{1}{2t} - \frac{1}{2t(1+\frac{j^2}{4t^2})} - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}} \geq -\frac{4}{16}\frac{j^2}{4t^2}.
$$

We directly calculate

$$\frac{1}{2t} - \frac{1}{2t(1+\frac{j^2}{4t^2})} - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}} \tag{3.27}$$

$$= \frac{\frac{j^2}{4t^2}}{2t(1+\frac{j^2}{4t^2})} - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}}$$

$$= \frac{\frac{j^2}{4t^2}2t\sqrt{1+\frac{j^2}{4t^2}} - 1}{4t^2\left(1+\frac{j^2}{4t^2}\right)^{\frac{3}{2}}}$$

$$= \frac{j^2}{4t^2}\left(\frac{\frac{1}{2t}\sqrt{1+\frac{j^2}{4t^2}} - \frac{1}{j^2}}{(1+\frac{j^2}{4t^2})^{\frac{3}{2}}}\right).$$

We estimate the numerator from below

$$\frac{1}{2t}\sqrt{1+\frac{j^2}{4t^2}} - \frac{1}{j^2} \geq \frac{1}{2t} - \frac{1}{j^2} \geq -\frac{1}{j^2} \geq -\frac{1}{4},$$

where we have used that $\frac{1}{j^2}$ is maximal for $j = 2$ since we only consider $j \geq 2$. We estimate the denominator from above

$$\frac{1}{(1+\frac{j^2}{4t^2})^{\frac{3}{2}}} \leq 1.$$

In conclusion, we can estimate the line (3.27) from below by

$$\frac{j^2}{4t^2}\left(\frac{\frac{1}{2t}\sqrt{1+\frac{j^2}{4t^2}} - \frac{1}{j^2}}{(1+\frac{j^2}{4t^2})^{\frac{3}{2}}}\right) \geq -\frac{j^2}{16t^2}. \tag{3.28}$$

Inserting (3.26) and (3.28) into our estimate of the derivative we get

$$v_j'(t) = \left(\frac{15}{16}\frac{j^2}{4t^2} - \frac{1}{4}\frac{j^2}{4t^2}\right)v_j(t)$$

$$= \frac{11}{16}\frac{j^2}{4t^2}v_j(t).$$

□

Lemma 3.12. *Now we let $3 \leq t \leq j$ and find*

$$v_j'(t) \geq \frac{1}{5}\frac{j}{t}v_j(t). \tag{3.29}$$

Proof. We calculate (3.15) to be

$$
\begin{aligned}
v_j'(t) &= \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t}\right)v_j(t)\\
&\geq \left(\frac{1}{2t} - 2 + \frac{\sqrt{4t^2+l^2}}{t} - \frac{2t}{4t^2+l^2} - \frac{2t}{(4t^2+l^2)^{\frac{3}{2}}}\right)v_j(t)\\
&= \left(\left(-2+2\sqrt{1+\frac{j^2}{4t^2}}\right) + \left(\frac{1}{2t} - \frac{2t}{4t^2+j^2}\right) - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}}\right)v_j(t).
\end{aligned}
$$

In the first step we have again inserted the lower bound of the error. In the second step we have ordered the terms according to dominance. The most dominant term $-2+2\sqrt{1+\frac{j^2}{4t^2}}$ can be estimated by A.1 for $x = \frac{j^2}{4t^2}$. Furthermore, we use that $j \geq t$

$$\frac{2t}{4t^2+j^2} \leq \frac{2t}{5t^2} = \frac{2}{5t}$$

and for the error

$$\frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}} \leq \frac{1}{4t^2}.$$

Summarizing we obtain the expression

$$
\begin{aligned}
v_j'(t) &= \left(\frac{1}{2t} - 2 + \frac{a_j(2t)}{t}\right)v_j(t)\\
&\geq \left(\left(-2+\sqrt{4+\frac{j^2}{t^2}}\right) + \left(\frac{1}{2t} - \frac{2t}{4t^2+j^2}\right) - \frac{1}{4t^2(1+\frac{j^2}{t^2})^{\frac{3}{2}}}\right)v_j(t)\\
&\geq \left(\frac{1}{5}\frac{j}{t} + \frac{1}{2t} - \frac{2}{5t} - \frac{1}{4t^2}\right)v_j(t)\\
&= \left(\frac{1}{5}\frac{j}{t} + \frac{4t-10}{40t^2}\right)v_j(t)\\
&\geq \frac{1}{5}\frac{j}{t}v_j(t),
\end{aligned}
$$

where we have used that for $t \geq 3$

$$\frac{4t-10}{40t^2} \geq 0.$$

□

Now we want to estimate the following expression, using (3.19) and (3.20)

$$v_{j\pm1}(t) - v_j(t) = \left(\frac{a_j(2t)}{2t} \mp \frac{j}{2t} - 1\right)v_j(t). \tag{3.30}$$

We again distinguish between the cases $j = 0, j = 1, j \leq t$ and $j \geq t$.

Lemma 3.13. *For $j = 0$ and $t \geq 2$ we have*

$$|v_{\pm 1}(t) - v_0(t)| \leq \left(\frac{1}{4t} + \frac{1}{8t^2}\right) v_0(t). \tag{3.31}$$

Proof. We calculate the difference directly. In the first step we use the error estimate (3.18) and in the second step we use the triangle inequality

$$\begin{aligned}
|v_{\pm 1}(t) - v_0(t)| &= \left| \left(\frac{a_0(2t)}{2t} \mp \frac{0}{2t} - 1 \right) v_0(t) \right| \\
&\leq \left| \frac{\sqrt{4t^2}}{2t} - \frac{4t^2}{4t(4t^2)} + \frac{b_0(2t)}{2t} - 1 \right| v_0(t) \\
&\leq \left| -\frac{1}{4t} + \frac{b_0(2t)}{2t} \right| v_0(t) \\
&\leq \left(\left| -\frac{1}{4t} \right| + \left| \frac{b_0(2t)}{2t} \right| \right) v_0(t) \\
&\leq \left(\frac{1}{4t} + \frac{1}{8t^2} \right) v_0(t).
\end{aligned}$$

□

Note that the largest bound of $|v_{\pm 1}(t) - v_0(t)|$ is attained in $t = 1$ with absolute value $\frac{3}{8}$ and the larger the time the sharper the bound.

Lemma 3.14. *For $j = 1, t = 2$ or $2 \leq j \leq t$ we have*

$$|v_{j\pm 1}(t) - v_j(t)| \leq \frac{j}{t} v_j(t). \tag{3.32}$$

Proof. As in the previous proof we calculate (3.30) directly and use the error estimate (3.18) first and then the triangle inequality. In a next step we use lemma A.1 on the most dominant term $\sqrt{1 + \frac{j^2}{4t^2}} - 1$ for $x = \frac{j^2}{4t^2}$. Finally, we use that $\frac{j^2}{t^2} \leq \frac{j}{t} \leq 1$ and $j^2 \geq j \geq 2$

to simplify

$$
\begin{aligned}
|v_{l\pm1}(t) - v_j(t)| &= \left| \left(\frac{a_j(2t)}{2t} \mp \frac{j}{2t} - 1 \right) v_j(t) \right| \\
&\le \left| \left(\frac{\sqrt{4t^2+j^2}}{2t} - \frac{4t^2}{4t(4t^2+j^2)} + \frac{1}{2t}\left(\frac{4t^2}{2(4t^2+j^2)^{\frac{3}{2}}} \right) \mp \frac{j}{2t} - 1 \right) \right| v_j(t) \\
&\le \left(\left| \sqrt{1+\frac{j^2}{4t^2}} - 1 \right| + \left| -\frac{t}{4t^2+j^2} \right| + \left| \mp \frac{j}{2t} \right| + \left| \frac{t}{(4t^2+j^2)^{\frac{3}{2}}} \right| \right) v_j(t) \\
&\le \left(\frac{1}{2}\frac{j^2}{4t^2} + \frac{1}{4t(1+\frac{j^2}{t^2})} + \frac{j}{2t} + \frac{1}{8t^2(1+\frac{j^2}{4t^2})^{\frac{3}{2}}} \right) v_j(t) \\
&\le \left(\frac{1}{8}\frac{j^2}{t^2} + \frac{1}{4t} + \frac{j}{2t} + \frac{1}{8t^2} \right) v_j(t) \\
&\le \left(\frac{1}{8}\frac{j^2}{t^2} + \frac{j}{4t} + \frac{j}{2t} + \frac{1}{8}\frac{j^2}{t^2} \right) v_j(t) \\
&\le \left(\frac{3}{4}\frac{j}{t} + \frac{1}{4}\frac{j^2}{t^2} \right) v_j(t) \\
&\le \frac{j}{t} v_j(t).
\end{aligned}
$$

□

Lemma 3.15. *For $2 \le t \le j$ we have*

$$
|v_{j\pm1}(t) - v_j(t)| \le \frac{13}{8}\frac{j}{t} v_j(t). \tag{3.33}
$$

Proof. We estimate (3.30) directly. As in the previous two proofs we start by using the error and the triangle inequality respectively. However, for the dominant term $\sqrt{1+\frac{j^2}{4t^2}} -$

1 we use lemma A.7 for $x = \frac{j}{2t}$ to obtain

$$\begin{aligned}
\Big|v_{j\pm1}(t) - v_j(t)\Big| &= \left|\left(\frac{a_j(2t)}{2t} \mp \frac{j}{2t} - 1\right)v_j(t)\right| \\
&\leq \left|\frac{\sqrt{4t^2+j^2}}{2t} - \frac{4t^2}{4t(4t^2+j^2)} + \frac{1}{2t}\left(\frac{4t^2}{2(4t^2+j^2)^{\frac{3}{2}}}\right) \mp \frac{j}{2t} - 1\right|v_j(t) \\
&= \left|\left(\sqrt{1+\frac{j^2}{4t^2}} - 1\right) + \left(-\frac{2t}{4t^2+j^2} \mp \frac{j}{2t}\right) + \frac{t}{(4t^2+j^2)^{\frac{3}{2}}}\right|v_j(t) \\
&\leq \left(\frac{j}{2t} + \left|-\frac{2t}{4t^2+j^2}\right| + \left|\mp\frac{j}{2t}\right| + \frac{1}{8t^2(1+\frac{j^2}{4t^2})^{\frac{3}{2}}}\right)v_j(t) \\
&\leq \left(\frac{j}{t} + \frac{1}{2t(1+\frac{j^2}{4t^2})} + \frac{1}{8t^2}\right)v_j(t) \\
&\leq \left(\frac{j}{t} + \frac{1}{2t} + \frac{1}{8t}\right)v_j(t) \\
&\leq \left(\frac{j}{t} + \frac{j}{2t} + \frac{j}{8t}\right)v_j(t) \\
&\leq \frac{13}{8}\frac{j}{t}v_j(t).
\end{aligned}$$

□

4. Stability of the Nagumo LDE

We want to apply the comparison principle for the discrete case and look at the subsolution with a travelling wave Ansatz

$$u^-_{i,j}(t) = \Phi(i + ct - \theta_j(t) - Z(t)) - z(t) = \Phi(\xi_{i,j}(t)) - z(t),$$

where we define for convenience

$$\xi_{i,j}(t) = i + ct - \theta_j(t) - Z(t). \tag{4.1}$$

Furthermore, we use the external functions z and Z as in the continuous case. We are able to make these external functions explicit in section 4.4.

4.1. Horizontal Travelling Waves on the Lattice

In the discrete case we have to focus on the solution in one direction, because of the direction dependency on the lattice. We call the direction

$$(cos\theta, sin\theta) = (\sigma_h, \sigma_v)$$

rational, if $\tan\theta \in \mathbb{Q}$. We can rewrite the travelling wave Ansatz as

$$u_{i,j}(t) = \Phi((\cos\theta, \sin\theta) * (i, j) + ct) = \Phi(\xi)$$

such that $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$. The travelling wave equation becomes

$$c\Phi'(\xi) = \Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi) + g(\Phi(\xi)).$$

We perform a coordinate transformation

$$\begin{aligned} n &= +i\sigma_h + j\sigma_v \quad \text{parallel to lattice} \\ l &= -i\sigma_v + j\sigma_h \quad \text{transversal to lattice} \end{aligned}$$

such that we can rewrite the Laplace operator in terms of the four direct neighbors of u_{nl}. The neighbor set is defined as

$$\mathcal{N}(n, l) = \{(n + \sigma_h, l + \sigma_v), (n + \sigma_v, l - \sigma_h), (n - \sigma_h, l - \sigma_v), (n - \sigma_v, l + \sigma_h)\}.$$

The Laplace operator becomes

$$[\Delta u]_{nl} = \Sigma_{(n',l')\in\mathcal{N}(n,l)}[u_{n'l'} - u_{nl}].$$

In this thesis we focus on the horizontal direction

$$(\sigma_h, \sigma_v) = (1, 0).$$

Then $\theta = 0$ and the coordinate transformation becomes

$$(n, l) = (i, j).$$

In the horizontal case we do not have to perform a coordinate transformation and we can use (i, j). In the horizontal case we can abbreviate the discrete Laplace operator as

$$\Delta^+ u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \tag{4.2}$$

where the plus sign refers to the four neighbours around $u_{i,j}$. In this thesis we also narrow down the problem by fixing a detuning parameter a such that there is no pinning involved. the The wave speed $c > 0$ is also fixed from now on, because $c = c(a, \theta)$ as we have seen in the introduction.

4.2. Transversal Dependency

In order to describe the transversal dependency we take

$$\theta \in C^1([0, \infty), l^2(\mathbb{Z}, \mathbb{R}))$$

in analogy to the continuous case. We remark that $(\theta_j(t))_{j\in\mathbb{Z}} \in l^2(\mathbb{R})$ depends on j, i.e. the transversal direction on the lattice, for fixed t.
We are ready to specify $\theta_j(t)$ under use of constants named in the same way as in the continuous case discussed in section 2.1. We define $\theta_j(t)$ under the use of $v_j(t)$ from (3.3) as

$$\theta_j(t) = \beta t^{-\alpha} \sqrt{t\gamma} e^{-2\gamma t} \mathcal{I}_j(2\gamma t) = \beta t^{-\alpha} v_j(\gamma t),$$

where $\beta, \gamma \gg 1$ and $0 < \alpha \ll 1$ are constants. Ultimately, we want to express $0 < \alpha \ll 1$ as a function of γ and γ as a function of β. The constant β in turn, depends on the initial condition as we will see in section 4.4. For completeness, we note that the relative expressions for $v_j(t)$ we have found in section 3.2 become

$$\frac{d}{dt} v_j(\gamma t) = \gamma v_j'(\gamma t) = \left(\frac{1}{2t} - 2\gamma + \frac{a_j(2\gamma t)}{t}\right) v_j(\gamma t)$$

and

$$v_{j+1}(\gamma t) + v_{j-1}(\gamma t) = \frac{a_j(2\gamma t)}{\gamma t} v_j(\gamma t).$$

We know the asymptotic behaviour of $\theta_j(t)$ from our preliminary calculations in section 3.1.

Lemma 4.1. *We have the following limit for θ_j*

$$\lim_{t\to\infty} \theta_j(t) = 0. \tag{4.3}$$

Proof. By definition of the modified θ_j we get

$$\lim_{t\to\infty} \theta_j(t) = \lim_{t\to\infty} \beta t^{-\alpha} v_j(\gamma t) = \beta \lim_{t\to\infty} t^{-\alpha} v_j(\gamma t). \tag{4.4}$$

We choose $\alpha \ll 1$, so $\lim_{t\to\infty} t^{-\alpha} = 0$ and it suffices to show that $v_j(t)$ is bounded as $t \to \infty$. We want to use the estimates we have found in section 3.2.
If $\frac{j}{t} \leq 1$ we can use corollary 3.3 stating that $|v_j(t)|$ is bounded from above by 1 and the limit follows.
If $\frac{j}{t} \geq 1$ we can use lemma 3.6 stating that $|v_j(t)| = \frac{t}{j}\frac{j}{t}|v_j(t)|$ is bounded from above by $\frac{t}{j}\frac{1}{\sqrt{2}}e^{-\frac{1}{4}} \leq \frac{t}{j} \leq 1$ and the limit follows as well. □

4.3. Estimate of the Residual

We have gained enough understanding to consider the residual of the LDE in two space dimensions. We have to prove one preliminary lemma, before we can calculate the discrete residual analogously to the continuous residual calculated in lemma 2.1.

Lemma 4.2. *For every β there is a γ_* such that for all $\gamma \geq \gamma_*$ we find the following estimate for all $j \in \mathbb{Z}$ and $t \geq 2$*

$$|\theta_{j\pm1}(t) - \theta_j(t)| \leq 1.$$

Proof. First let us write out the difference

$$\begin{aligned}|\theta_{j\pm1}(t) - \theta_j(t)| &= \beta t^{-\alpha}|v_{j\pm1}(\gamma t) - v_j(\gamma t)| \\ &\leq \beta|v_{j\pm1}(\gamma t) - v_j(\gamma t)|,\end{aligned}$$

where we have used that the factor $t^{-\alpha}$ is of no concern, because we choose $0 < \alpha \ll 1$ so $t^{-\alpha} < 1$. Let $j = 0$. Then we estimate the difference from above with lemma 3.13 to be

$$\begin{aligned}|\theta_{\pm1}(t) - \theta_0(t)| &\leq \beta|v_{\pm1}(\gamma t) - v_0(\gamma t)| \\ &\leq \beta\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2}\right)v_0(\gamma t).\end{aligned}$$

Clearly, the sum $\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2} \leq \frac{1}{4\gamma_* t} + \frac{1}{8\gamma_*^2 t^2} \leq 1$ for $\gamma \geq \gamma_*$. Using this and choosing $\gamma_* = \beta$ we can prove the lemma for $j = 0$ right away. The difference becomes

$$\begin{aligned}|\theta_{\pm1}(t) - \theta_0(t)| &\leq \beta\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma t^2}\right)v_0(\gamma t) \\ &\leq \beta\left(\frac{1}{4\gamma_* t} + \frac{1}{8\gamma_* t^2}\right)v_0(\gamma t) \\ &= \left(\frac{1}{4t} + \frac{1}{8\gamma_* t^2}\right)v_0(\gamma t) \\ &\leq v_0(\gamma t) \leq 1,\end{aligned}$$

where we have used corollary 3.3 in the last step.
Now let $j \geq 1$. In lemma 3.14 and lemma 3.15 we have found estimates for the differences

$$|v_{j\pm1}(t) - v_j(t)| \leq K\frac{j}{t}v_j(t)$$

with $K = 1$ for $j = 1, t = 2$ or $2 \leq j \leq t$ and $K = \frac{13}{8}$ for $2 \leq t \leq j$. However, the constant $\beta \gg 1$ keeps us from directly using these estimates. Instead we have to choose $\gamma_* = \gamma_*(\beta)$ carefully and large enough to give an estimate of the difference in $\theta_j(t)$. For convenience we write $\gamma t = \tilde{t}$. We want to prove the following claim.
For every $\epsilon > 0$ there is a time $T = T(\epsilon)$ such that

$$\frac{j}{\tilde{t}}v_j(\tilde{t}) \leq \epsilon \quad \forall \tilde{t} \geq T. \tag{4.5}$$

We have to discern between three cases according to the relation between j and $\tilde{t}$.
First we consider $1 \leq j \leq \tilde{t}^{\frac{3}{4}}$. We estimate $v_j(\tilde{t})$ by 1 as we may by corollary 3.3 and find

$$\frac{j}{\tilde{t}}v_j(\tilde{t}) \leq \frac{j}{\tilde{t}} \leq \frac{\tilde{t}^{\frac{3}{4}}}{\tilde{t}} = \frac{1}{\tilde{t}^{\frac{1}{4}}}.$$

So if we choose $\epsilon = \frac{1}{T^{\frac{1}{4}}}$, we have shown (4.5) for $0 \leq j \leq \tilde{t}^{\frac{3}{4}}$.
Secondly, we consider $\tilde{t}^{\frac{3}{4}} \leq j \leq \tilde{t}$. We apply corollary 3.4 and find

$$0 < \frac{j}{\tilde{t}}v_j(\tilde{t}) \leq \frac{j}{\tilde{t}}Ce^{-\frac{\sqrt{\tilde{t}}}{16}}.$$

We have seen in (3.6) that C is a positive constant smaller than 1. Furthermore, we use that $\frac{j}{\tilde{t}} \leq 1$ in this case. Thus

$$\frac{j}{\tilde{t}}v_j(\tilde{t}) \leq Ce^{-\frac{\sqrt{\tilde{t}}}{16}}$$

and we choose $\epsilon = Ce^{-\frac{\sqrt{T}}{16}}$ in order to prove (4.5).
Thirdly, we consider $\gamma \leq \tilde{t} \leq j$. We apply lemma 3.6 for $\tilde{t}$ to get

$$\frac{j}{\tilde{t}}v_j(\tilde{t}) \leq \frac{1}{\sqrt{2}}e^{\frac{1}{4}}e^{-\frac{\tilde{t}}{4}}.$$

So (4.5) follows for $\epsilon = \frac{1}{\sqrt{2}}e^{\frac{1}{4}}e^{-\frac{1}{4}T}$.
Thus for every $\epsilon = \frac{1}{\beta K} > 0$ there is a γ_* such that for all $\tilde{t} \geq T$

$$\begin{aligned}
|\theta_{j\pm1}(t) - \theta_j(t)| &= \beta t^{-\alpha}|v_{j\pm1}(\tilde{t}) - v_j(\tilde{t})| \\
&\leq \beta|v_{j\pm1}(\tilde{t}) - v_j(\tilde{t})| \\
&\leq \beta K\frac{j}{\tilde{t}}v_j(\tilde{t}) \\
&\leq \beta K\epsilon = 1.
\end{aligned}$$

In fact take $T(\epsilon) = \gamma_*$. Then for $1 \le j \le \tilde{t}^{\frac{3}{4}}$, $K = 1$ and the equation

$$\frac{1}{\beta K} = \frac{1}{T^{\frac{1}{4}}}$$

has solution $\gamma_* = (\beta)^{-\frac{1}{4}}$.
For $\tilde{t}^{\frac{3}{4}} \le j \le \tilde{t}$, $K = 1$ as well and solving

$$\frac{1}{\beta K} = Ce^{-\frac{\sqrt{T}}{16}}$$

gives $\gamma_* = (16ln(\beta C))^2$.
Finally, for $\gamma \le \tilde{t} \le j$ $K = \frac{13}{8}$ we solve

$$\frac{1}{\beta K} = \frac{1}{\sqrt{2}} e^{\frac{1}{4}} e^{-\frac{1}{4}T}$$

has solution $\gamma_* = 4ln(\frac{8\beta}{13\sqrt{2}}) + 1$.
Now the statement holds for all $\gamma \ge \gamma_*(\beta)$ and we have proven the lemma. □

Proposition 4.3. *The residual of the subsolution for the horizontal direction*

$$J = u_{i,j}^{-\prime}(t) - \Delta^+ u_{i,j}^-(t) - g(u_{i,j}^-(t))$$

is given by

$$J_{global} = -z'(t) - Z'(t)\Phi'(\xi_{i,j}(t)) + g(\Phi(\xi_{i,j}(t))) - g(\Phi(\xi_{i,j}(t)) - z(t)) \tag{4.6}$$

$$J_{nl} = -\frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^+(t))(\theta_j(t) - \theta_{j+1}(t))^2 - \frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^-(t))(\theta_j(t) - \theta_{j-1}(t))^2 \tag{4.7}$$

$$J_{heat} = \Phi'(\xi_{i,j}(t))(\theta_{j+1}(t) + \theta_{j-1}(t) - 2\theta_j(t) - \theta_j'(t)). \tag{4.8}$$

such that

$$J = J_{global} + J_{nl} + J_{heat}.$$

For every β we can find a γ_* such that for $\gamma \ge \gamma_*, t \ge 2$ and $(i,j) \in \mathbb{Z}^2$ we can find a $\nu_{i,j}^+(t)$ and $\nu_{i,j}^-(t)$ such that

$$|\nu_{i,j}^+(t)|, |\nu_{i,j}^-(t)| \le 1.$$

Proof. We make use of the travelling wave constant given in (4.1). We begin by calculating the first derivative of the subsolution

$$u_{i,j}^{-\prime}(t) = (c - \theta_j'(t) - Z'(t))\Phi'(\xi_{i,j}(t)) - z'(t). \tag{4.9}$$

Inserting (4.9) into the residual gives

$$
\begin{aligned}
J &= u_{i,j}^{-}{}'(t) - [\Delta^{+}u^{-}(t)]_{ij} - g(u_{i,j}^{-}(t)) \\
&= (c - \theta_j'(t) - Z'(t))\Phi'(i + ct - \theta_j(t) - Z(t)) - z'(t) \\
&\quad - u_{i+1,j}(t) - u_{i-1,j}(t) - u_{i,j+1}(t) - u_{i,j-1}(t) \\
&\quad + 4u_{i,j}(t) - g(u_{i,j}(t)) \\
&= (c - \theta_j'(t) - Z'(t))\Phi'(i + ct - \theta_j(t) - Z(t)) - z'(t) \\
&\quad - \Phi(\xi_{i+1,j}(t)) - \Phi(\xi_{i-1,j}(t)) + 2z(t) - \Phi(\xi_{i,j+1}(t)) - \Phi(\xi_{i,j-1}(t)) + 2z(t) \\
&\quad + 4\Phi(\xi_{i,j}(t)) - 4z(t) - g(\Phi(\xi_{i,j}(t) - z(t)) \\
&= (c - \theta_j'(t) - Z'(t))\Phi'(\xi_{i,j}(t)) - z'(t) \\
&\quad - \Phi(\xi_{i+1,j}(t)) - \Phi(\xi_{i-1,j}(t)) - \Phi(\xi_{i,j+1}(t)) - \Phi(\xi_{i,j-1}(t)) + 4\Phi(\xi_{i,j}(t)) \\
&\quad - g(\Phi(\xi_{i,j}(t) - z(t))
\end{aligned}
$$

In order to simplify further, we look at the wave profile equation in the two-dimensional discrete case

$$c\Phi'(\xi) = \Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi) + g(\Phi(\xi)).$$

For the horizontal direction we have seen in section 4.1 that $\theta = 0$. Thus the wave profile equation becomes

$$c\Phi'(\xi) = \Phi(\xi+1)+\Phi(\xi-1)+2\Phi(\xi)-4\Phi(\xi)+g(\Phi(\xi)) = -2\Phi(\xi)+\Phi(\xi+1)+\Phi(\xi-1)+g(\Phi(\xi)).$$

The local form is

$$c\Phi'(\xi_{i,j}) = -2\Phi(\xi_{i,j}) + \Phi(\xi_{i,j} + 1) + \Phi(\xi_{i,j} - 1) + g(\Phi(\xi_{i,j})).$$

By looking at the definition of the travelling wave constant we see that

$$c\Phi'(\xi_{i,j}) = -2\Phi(\xi_{i,j}) + \Phi(\xi_{i+1,j}) + \Phi(\xi_{i-1,j}) + g(\Phi(\xi_{i,j})).$$

Inserting this into the expression above for the local residual gives

$$
\begin{aligned}
J &= -2\Phi(\xi_{i,j}) + \Phi(\xi_{i+1,j}(t)) + \Phi(\xi_{i-1,j}(t)) + g(\Phi(\xi_{i,j})) - (\theta_j'(t) + Z'(t))\Phi'(\xi_{i,j}(t)) - z'(t) \\
&\quad - \Phi(\xi_{i+1,j}(t)) - \Phi(\xi_{i-1,j}(t)) - \Phi(\xi_{i,j+1}(t)) - \Phi(\xi_{i,j-1}(t))) + 4\Phi(\xi_{i,j}(t)) - g(\Phi(\xi_{i,j}(t) - z(t)) \\
&= 2\Phi(\xi_{i,j}) - \Phi(\xi_{i,j+1}(t)) - \Phi(\xi_{i,j-1}(t)) \\
&\quad - (\theta_j'(t) + Z'(t))\Phi'(\xi_{i,j}(t)) - z'(t) \\
&\quad + g(\Phi(\xi)) - g(\Phi(\xi_{i,j}(t) - z(t)).
\end{aligned}
$$

In order to get an even better rest term we focus on the term

$$2\Phi(\xi_{i,j}(t))-\Phi(\xi_{i,j+1}(t))-\Phi(\xi_{i,j-1}(t)) = (\Phi(\xi_{i,j}(t))-\Phi(\xi_{i,j+1}(t)))+(\Phi(\xi_{i,j}(t))-\Phi(\xi_{i-1,j}(t))).$$

We apply the main value theorem on both terms. The first time we apply the theorem we use the auxiliary constants $\nu_1, \nu_1' \in (0,1)$

$$\begin{aligned}2\Phi(\xi_{i,j}(t)) - \Phi(\xi_{i\pm 1j}(t)) &= (\Phi(i+ct-\theta_j(t)-Z(t)) - 2\Phi(i+ct-\theta_{j+1}(t)-Z(t)))\\&\quad + (\Phi(i+ct-\theta_j(t)-Z(t)) - \Phi(i+ct-\theta_{j-1}(t)-Z(t)))\\&= -\Phi'(\xi_{i,j}+\nu_1(\theta_j-\theta_{j+1}))(\theta_j-\theta_{j+1})\\&\quad - \Phi'(\xi_{i,j}+\nu_1'(\theta_j-\theta_{j-1}))(\theta_j-\theta_{j-1}).\end{aligned}$$

Now we want to use the main value theorem again to simplify further. We can do so by adding a remainder term and by using the auxiliary constants $\nu_2, \nu_2' \in (0,1)$

$$\begin{aligned}&-\Phi'(\xi_{i,j}+\nu_1(\theta_j-\theta_{j+1}))(\theta_j-\theta_{j+1})\\&+\Phi(\xi_{i,j})(\theta_j-\theta_{j+1}) - \Phi(\xi_{i,j})(\theta_j-\theta_{j+1})\\&-\Phi'(\xi_{i,j}+\nu_1'(\theta_j-\theta_{j-1}))(\theta_j-\theta_{j-1})\\&+\Phi(\xi_{i,j})(\theta_j-\theta_{j-1}) - \Phi(\xi_{i,j})(\theta_j-\theta_{j-1})\\=&-\frac{1}{2}\Phi''(\xi_{i,j}+\nu_2(\theta_j-\theta_{j+1}))(\theta_j-\theta_{j+1})^2\\&-\frac{1}{2}\Phi''(\xi_{i,j}+\nu_2'(\theta_j-\theta_{j-1}))(\theta_j-\theta_{j-1})^2\\&+(\theta_{j+1}-\theta_j)\Phi'(\xi_{i,j}(t)) + (\theta_{j-1}-\theta_j)\Phi'(\xi_{i,j}(t)).\end{aligned}$$

We put $\nu_2(\theta_j(t)-\theta_{j+1}(t)) = \nu_{i,j}^+(t)$ and $\nu_2'(\theta_j(t)-\theta_{j-1}(t)) = \nu_{i,j}^-(t)$. For the absolute estimate by 1 observe that $\nu_2 \in (0,1)$ and use lemma 4.2. The lemma gives that for every β in the definition of θ_j there must be a $\gamma_* = \gamma_*(\beta)$ such that

$$|\nu_{i,j}^+(t)| = |\nu_2(\theta_j(t)-\theta_{j+1}(t))| = \nu_2 \leq 1$$

for $\gamma \geq \gamma_*$. The analogue holds for $\nu_{i,j}^-(t)$. Finally, we order the residual just as in the continuous case (2.1) according to their quality to obtain the required result. □

We want to estimate the three parts of the residual seperately. Our aim is to make sure that the residual is negative for our modified subsolution. The supersolution case proves to be completely analogous afterwards and we will apply the comparison principle. The global residual

$$J_{global} = -z'(t) - Z'(t)\Phi'(\xi(t)) + g(\Phi(\xi)) - g(\Phi(\xi)-z(t)) \tag{4.10}$$

can be kept negative by choosing $z(t)$ and $Z(t)$ carefully as we will do in section 4.4. The other two parts of the residual are more difficult to estimate. We will have to exploit what we already know about the behaviour of the wave profile Φ. The wave profile Ansatz assumes that there is a wave $\Phi \in C^1$ for every wave speed $c > 0$ such that Φ is bounded, the wave profile equation is satisfied, and the temporal limits

$$\lim_{\xi(t)\to-\infty}\Phi(\xi(t)) = 0 \quad \text{and} \quad \lim_{\xi(t)\to+\infty}\Phi(\xi(t)) = 1 \tag{4.11}$$

hold. In the proof of proposition 4.3 we have seen that the wave profile equation in $\xi_{i,j}$ for our discrete and horizontal case becomes

$$c\Phi'(\xi_{i,j}) = -2\Phi(\xi_{i,j}) + \Phi(\xi_{i+1,j}) + \Phi(\xi_{i-1,j}) + g(\Phi(\xi_{i,j})). \tag{4.12}$$

The wave profile Φ is unique up to shifts and takes values between 0 and 1. We have assumed that Φ is differentiable, so let us differentiate (4.12)

$$c\Phi''(\xi_{i,j}) = -2\Phi'(\xi_{i,j}) + \Phi'(\xi_{i+1,j}) + \Phi'(\xi_{i-1,j}) + \Phi'(\xi_{i,j})g'(\Phi(\xi_{i,j})). \tag{4.13}$$

Now by the smoothness of g we may conclude that Φ is twice differentiable. But the property of Φ we are really interested in for the estimate of the residual is the convergence rate of Φ for $t \to \pm\infty$.
Consider 4.13 for $\xi = \xi_{ij}$ near $-\infty$. By the limit assumptions on the wave we know that $\Phi(-\infty) = 0$ so we get

$$c\Phi''(\xi) = -2\Phi'(\xi) + \Phi'(\xi+1) + \Phi'(\xi-1) + \Phi'(\xi)g'(0).$$

We want to solve the ODE by using the Ansatz $\Phi'(\xi) = e^{z\xi}$, the ODE becomes

$$\begin{aligned} cze^{z\xi} &= -2e^{z\xi} + e^{z(\xi+1)} + e^{z(\xi-1)} + g'(0)e^{z\xi} \\ cz &= -2 + e^{z} + e^{-z} + g'(0) \\ cz &= -2 + 2cosh(z) + g'(0) \end{aligned}$$

We can follow the same steps for 4.13 with ξ near $+\infty$, where $\Phi(+\infty) = 1$. Then we have found the limiting spatial characteristic functions denoted by

$$\Delta^{+}(z) = cz - 2cosh(z) + 2 - g'(0) \tag{4.14}$$
$$\Delta^{-}(z) = cz - 2cosh(z) + 2 - g'(1). \tag{4.15}$$

We have seen the derivative of g in (2.3), we find $g'(0) = -a$ and $g'(1) = -1 + a$. In the rest of this section we work out lemma 3.3 to corollary 3.6 from [1] for the horizontal direction. The following proposition shows that the roots of $\Delta_{\pm}(z) = 0$ are the spatial exponents of the asymptotic rates of convergence of Φ.

Proposition 4.4. *There are positive constants η_{+} and η_{-} such that*

$$\begin{aligned} c\eta_{+} &= 2cosh(\eta_{+}) - 2 + g'(0) \\ c\eta_{-} &= 2cosh(\eta_{-}) - 2 + g'(1), \end{aligned}$$

which implies that $\Delta^{+}(-\eta^{+}) = 0$ and $\Delta^{-}(\eta^{-}) = 0$.
In addition, we have uniqueness in the following sense. Whenever $\Delta^{+}(\eta) = 0$ or $\Delta^{-}(\eta) = 0$ for some $\eta \geq 0$, we have $\eta = \eta^{+}$ or $\eta = \eta^{-}$.

Proof. The first two derivatives of $\Delta^{\pm}$ are straightforwardly determined to be

$$\begin{aligned} \Delta^{\pm\prime}(z) &= c - 2sinh(z) \\ \Delta^{\pm\prime\prime}(z) &= -2cosh(z). \end{aligned}$$

Therefore, $\Delta^{\pm\prime\prime}(z) < 0$ for every real z and Δ^- and Δ^+ are strictly concave functions. Furthermore, we use that the hyperbolic cosine diverges faster than linearly to ∞ to calculate the limit

$$\lim_{z\to\infty} \Delta^+(z) = \lim_{z\to\infty} (cz - 2cosh(z) + 2 - g'(0)) = -\infty.$$

Since Δ^+ and Δ^- only differ by a constant, the same limit holds for Δ^-. By the symmetry of $cosh(z)$ we also have $\Delta^\pm \to -\infty$ for $z \to -\infty$.
In conclusion Δ^+ and Δ^- are both concave functions diverging to $-\infty$ in both directions of the $z-$axis. Both functions attain positive values in 0

$$\Delta^+(0) = 2 - g'(0) = 2 - a > 0 \quad \text{and} \quad \Delta^-(0) = 2 - g'(1) = 2 - (-1 + a) = 3 - a > 0.$$

It follows that they intersect the $z-$axis only twice. Therefore, Δ^+ and Δ^- must have positive roots η^+ respectively η^-. □

Proposition 4.5. *There are constants $C \geq 1, \kappa > 0$, and $\alpha_\pm > 0$ such that for every $\xi \leq 0$*

$$|\Phi(\xi) - \alpha_- e^{-\eta_-|\xi|}| \leq Ce^{(-\eta_- + \kappa)|\xi|} \tag{4.16}$$
$$|\Phi'(\xi) - \eta_- \alpha_- e^{-\eta_-|\xi|}| \leq Ce^{(-\eta_- + \kappa)|\xi|} \tag{4.17}$$
$$|\Phi''(\xi) - (\eta_-)^2 \alpha_- e^{-\eta_-|\xi|}| \leq Ce^{(-\eta_- + \kappa)|\xi|} \tag{4.18}$$

and for every $\xi \geq 0$

$$|(1 - \Phi(\xi)) - \alpha_+ e^{-\eta_+|\xi|}| \leq Ce^{(-\eta_+ + \kappa)|\xi|} \tag{4.19}$$
$$|\Phi'(\xi) - \eta_+ \alpha_+ e^{-\eta_+|\xi|}| \leq Ce^{(-\eta_+ + \kappa)|\xi|} \tag{4.20}$$
$$|\Phi''(\xi) - (\eta_+)^2 \alpha_+ e^{-\eta_+|\xi|}| \leq Ce^{(-\eta_+ + \kappa)|\xi|}. \tag{4.21}$$

Proof. The proof is a consequence of theorem 2.2 in [18]. □

Corollary 4.6. *There exists a constant $C \geq 1$ such that for all $\xi \in \mathbb{R}$ and $|M| \leq 1$*

$$|\Phi''(\xi + M)| \leq C\Phi'(\xi).$$

Proof. The result follows from proposition 4.5 together with the fact that $\Phi' > 0$. The latter is also a consequence of proposition 4.5. □

We deduce from proposition 4.5 that the graphs of Φ, Φ' and Φ'' for the LDE are similar to the corresponding wave profiles of the PDE we have seen in Fig. 1.3, Fig. 1.4 and Fig. 1.5. But then, we also have the same problem as in the continuous case explained in the end of section 2.1, we do not know the sign of Φ''. This implies that we do not know the sign of the nonlinear residual J_{nl} seen in (4.7) and we have to dominate it absolutely somehow. Luckily, we are able to use J_{heat} for this task in the discrete case as well, because we do know that $\Phi' > 0$. Thus we begin by focussing on estimating the heat residual of $\theta_j(t)$ in

$$J_{heat} = -\Phi'(\xi_{i,j}(t))(\theta_j'(t) - \theta_{j+1}(t) - \theta_{j-1}(t) + 2\theta_j(t)).$$

4.4. Explicit Subsolution and Supersolution

In this section we determine γ in terms of β such that the sum of the heat residual and the nonlinear residual is negative. Hereby we want J_{heat} found in (4.8) to dominate J_{nl} given in (4.7). Finally, we also prove negativity of the global residual and complete our search for an explicit subsolution by determining all three external functions $z(t), Z(t)$, and $\theta_j(t)$. The supersolution only differs in sign and is therefore completely analogous. In the last part of section 4.3 we already observed that the sign of J_{heat} depends on the residual of the discrete heat equation in $\theta_j(t)$ since $\Phi' > 0$. In order to find an estimate of the residual of the discrete heat kernel we use the expression linking $v_j'(\gamma t)$ and $v_{j\pm1}(\gamma t)$

$$v_{j\pm1}(\gamma t) = v_j'(\gamma t) - \frac{1}{2\gamma t} v_j(\gamma t) + 2v_j(\gamma t)$$

to find

$$\begin{aligned}
\theta_j'(t) - \theta_{j\pm1}(t) + 2\theta_j(t) &= \frac{d}{dt}(\beta t^{-\alpha} v_j(\gamma t)) - \beta t^{-\alpha} v_{j\pm1}(\gamma t) + 2\beta t^{-\alpha} v_j(\gamma t) \\
&= \beta t^{-\alpha}\Big(-\frac{\alpha}{t} v_j(\gamma t) + \gamma v_j'(\gamma t) - v_{j\pm1}(\gamma t) + 2v_j(\gamma t)\Big) \\
&= \beta t^{-\alpha}\Big(-\frac{\alpha}{t} v_j(\gamma t) + \gamma v_j'(\gamma t) - v_j'(\gamma t) + v_j'(\gamma t) - v_{j\pm1}(\gamma t) + 2v_j(\gamma t)\Big) \\
&= \beta t^{-\alpha}\Big(-\frac{\alpha}{t} v_j(\gamma t) + (\gamma - 1) v_j'(\gamma t) + v_j'(\gamma t) \\
&\quad - v_j'(\gamma t) + \frac{1}{2\gamma t} v_j(\gamma t) - 2v_j(\gamma t) + 2v_j(\gamma t)\Big) \\
&= \beta t^{-\alpha}\Big(-\frac{\alpha}{t} v_j(\gamma t) + (\gamma - 1) v_j'(\gamma t) + \frac{1}{2\gamma t} v_j(\gamma t)\Big) \\
&= \beta t^{-\alpha} v_j(\gamma t)\Big(\frac{1 - 2\alpha\gamma}{2\gamma t}\Big) + \beta t^{-\alpha}(\gamma - 1) v_j'(\gamma t). \qquad (4.22)
\end{aligned}$$

We insert expression (4.22) into the heat residual

$$\begin{aligned}
J_{heat} &= -\Phi'(\xi_{i,j})(\theta_j'(t) - \theta_{j\pm1}(t) + 2\theta_j(t)) \qquad (4.23) \\
&= -\Phi'(\xi_{i,j})\beta t^{-\alpha}\Big(v_j(\gamma t)\Big(\frac{1 - 2\alpha\gamma}{2\gamma t}\Big) + (\gamma - 1) v_j'(\gamma t)\Big).
\end{aligned}$$

Now for the heat residual to be negative we need

$$\frac{1 - 2\alpha\gamma}{2\gamma t} v_j(\gamma t) + (\gamma - 1) v_j'(\gamma t) > 0.$$

The first term is positive if $1 > 2\alpha\gamma$ such that $\alpha = \frac{\delta}{\gamma}$ with $0 < \delta < \frac{1}{2}$. From now on we choose

$$\alpha = \frac{1}{4\gamma}.$$

The heat residual (4.22) becomes

$$J_{heat} = -\Phi'(\xi_{i,j}(t))\beta t^{-\frac{1}{4\gamma}}(v_j(\gamma t)\left(\frac{1}{4\gamma t}\right) + (\gamma-1)v_j'(\gamma t)). \tag{4.24}$$

The second term

$$(\gamma-1)v_j'(\gamma t)$$

is more intricate. Luckily, we can use the estimates of $v_j'(t)$ found in section 3.2.
In order to estimate the nonlinear residual 4.7 we need estimates of $\theta_{j\pm1}(t)-\theta_j(t)$. Therefore, we want to use the direct estimates of $v_{j\pm1}-v_j$ in terms of v_j we found in section 3.2, which we can quadrate. We show negativity of the sum of the heat residual and the nonlinear residual in each of the four cases $j=0, j=1, j\le t$, and $j\ge t$ under use of the prepatory work done in 3.2.

Proposition 4.7. *For $j=0$ and $t\ge 3$ we find $J_{heat}+J_{nl}$ is negative for*

$$\gamma \ge \frac{9C\beta}{16}.$$

Proof. We apply lemma 3.9 for modified γt

$$\begin{aligned} v_0'(\gamma t) &= \left(\frac{1}{2t} - 2\gamma + \frac{a_0(2\gamma t)}{t}\right)v_0(\gamma t) \\ &= \frac{1}{4\gamma^2 t^2} v_0(\gamma t) \end{aligned}$$

As $\gamma \gg 1$ we may assume $|\gamma-1|\le 2\gamma$, so

$$|(\gamma-1)v_0'(\gamma t)| \le |2\gamma v_0'(\gamma t)| \le \frac{1}{2\gamma t^2}v_0(\gamma t). \tag{4.25}$$

We use (4.25) to estimate the residual of the discrete heat equation for $j=0$ as expressed in (4.22)

$$\begin{aligned} \theta_0'(t) - \theta_{0\pm1} + 2\theta_0(t) &= \beta t^{-\alpha} v_0(\gamma t)\left(\frac{1-2\alpha\gamma}{2\gamma t}\right) + \beta t^{-\alpha}(\gamma-1)v_0'(\gamma t) \\ &\le \beta t^{-\alpha} v_0(\gamma t)\left(\frac{1}{4\gamma t} - \frac{1}{2\gamma t^2}\right). \end{aligned}$$

The heat residual (4.24) becomes

$$\begin{aligned} J_{heat} &= -\Phi'(\xi_{i,j})\beta t^{-\alpha}(v_0(\gamma t)\left(\frac{1}{4\gamma t}\right) + (\gamma-1)v_0'(\gamma t)) \\ &\le -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_0(\gamma t)\left(\frac{1}{4\gamma t} - \frac{1}{2\gamma t^2}\right). \end{aligned} \tag{4.26}$$

The last term needs to be positive for (4.8) to be negative. But

$$\frac{1}{4\gamma t} - \frac{1}{2\gamma t^2} > 0$$
$$\frac{1}{2\gamma t}\left(\frac{1}{2} - \frac{1}{t}\right) > 0$$

is true for $t \geq 3$. The nonlinear residual (4.7) from proposition 4.3 becomes under use of lemma 3.13, and the corollary 4.6 and corollary 3.3

$$\begin{aligned} J_{nl} &= -\frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^{+}(t))(\theta_0 - \theta_{+1})^2 \\ &- \frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^{+}(t))(\theta_0 - \theta_{-1})^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-2\alpha}(v_{\pm 1}(\gamma t) - v_0(\gamma t))^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha}(v_{\pm 1}(\gamma t) - v_0)^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha} v_0^2(\gamma t)\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2}\right)^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha} v_0(\gamma t)\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2}\right)^2. \end{aligned} \tag{4.27}$$

Now we add (4.26) and (4.27)

$$J_{heat} + J_{nl} \leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_0(\gamma t)\left(\frac{1}{4\gamma t} - \frac{1}{2\gamma t^2} - C\beta\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2}\right)^2\right).$$

In order for the sum to be negative we need

$$\frac{1}{4\gamma t} - \frac{1}{2\gamma t^2} - C\beta\left(\frac{1}{4\gamma t} + \frac{1}{8\gamma^2 t^2}\right)^2 > 0$$

We multiply both sides of the inequality by $16\gamma^2 t^2$ to get

$$4\gamma t - 8\gamma - C\beta\left(1 + \frac{1}{2\gamma t}\right)^2 > 0.$$

We estimate the left hand side of the inequality from below using that $t \geq 3$

$$4\gamma t - 8\gamma - C\beta\left(1 + \frac{1}{2\gamma t}\right)^2 \geq 12\gamma - 8\gamma - C\beta\left(1 + \frac{1}{2\gamma}\right)^2.$$

Now for

$$12\gamma - 8\gamma - C\beta\left(1 + \frac{1}{2\gamma}\right)^2 > 0$$

to hold we must have $\gamma > \frac{C\beta}{4}\left(1 + \frac{1}{2\gamma}\right)^2$. Noting that $\frac{1}{2\gamma} < \frac{1}{2}$ we choose $\gamma \geq \frac{9C\beta}{16}$ to guarantee negativity. $\square$

Proposition 4.8. *For $j = 1$ and $t \geq 2$ the sum $J_{heat} + J_{nl}$ is negative for*

$$\gamma \geq 4\beta C + \frac{1}{8}.$$

Proof. We use the analogue of (4.25) for $j = 1$ and lemma **??** to find

$$|(\gamma - 1)v_1'(\gamma t)| \leq \frac{2\gamma}{64\gamma^4 t^4} v_j(\gamma t) = \frac{1}{32\gamma^3 t^4} v_j(\gamma t). \tag{4.28}$$

Under use of (4.28) the heat residual becomes

$$\begin{aligned} J_{heat} &= -\Phi'(\xi_{i,j})\beta t^{-\alpha}(v_1(\gamma t)\left(\frac{1}{4\gamma t}\right) + (\gamma - 1)v_1'(\gamma t)) \\ &\leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_1(\gamma t)\left(\frac{1}{4\gamma t} - \frac{1}{32\gamma^3 t^4}\right). \end{aligned} \tag{4.29}$$

The nonlinear residual becomes under use of corollary 4.6, lemma 3.14 and corollary 3.3

$$\begin{aligned} J_{nl} &= -\frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^+(t))(\theta_1 - \theta_2)^2 \\ &\quad - \frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu_{i,j}^+(t))(\theta_1 - \theta_0)^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-2\alpha}(v_{1\pm1}(\gamma t) - v_1(\gamma t))^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha}(v_{1\pm1}(\gamma t) - v_1)^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha} v_1^2(\gamma t)\left(\frac{1}{\gamma t}\right)^2 \\ &\leq C\Phi'(\xi_{i,j})\beta^2 t^{-\alpha} v_1(\gamma t)\frac{1}{\gamma^2 t^2}. \end{aligned} \tag{4.30}$$

The sum of (4.29) and (4.30) is

$$J_{heat} + J_{nl} \leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_1(\gamma t)\left(\frac{1}{4\gamma t} - \frac{1}{32\gamma^3 t^4} - C\beta\frac{1}{\gamma^2 t^2}\right),$$

which is negative if

$$\frac{1}{4\gamma t} - \frac{1}{32\gamma^3 t^4} - C\beta\frac{1}{\gamma^2 t^2} > 0.$$

We multiply both sides of the inequality by $4\gamma^2 t^2$

$$\gamma t - \frac{1}{8t^2\gamma} - 4C\beta > 0.$$

Again we estimate the lefthand side from below by using $t \geq 2$ as in the previous proposition

$$\gamma t - \frac{1}{8t^2\gamma} - 4C\beta \geq \gamma - \frac{1}{8\gamma} - 4C\beta.$$

The inequality $\gamma - \frac{1}{8\gamma} - 4C\beta > 0$ in turn motivates our choice of $\gamma \geq \frac{1}{8} + 4C\beta$ to guarantee negativity, because $\gamma \gg 1$. □

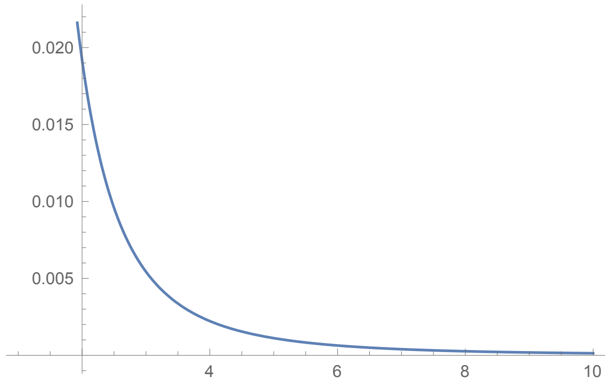

Figure 4.1.: Graph of the factor $\left(\frac{1}{2t} - 2 + 2\sqrt{1+\frac{1}{4t^2}} - \frac{1}{2t(1+\frac{1}{4t^2})} - \frac{2t}{(4t^2+1)^{\frac{3}{2}}}\right)$ linking $v_1(t)$ and $v_1'(t)$ for maximal error

Remark 4.9. For $t = 1$ the derivative $v_1'(t)$ is positive

$$\frac{1}{2} - 2 + \sqrt{5} - \frac{2}{5} + b_1(2) \geq -\frac{19}{15} + \sqrt{5} - \frac{2}{5\sqrt{5}} \approx 0.16$$

and for $t = 2$ as well

$$\frac{1}{4} - 2 + \frac{\sqrt{17}}{2} - \frac{4}{17} + b_1(4) \geq \frac{1}{4} - 2 + \frac{\sqrt{17}}{2} - \frac{4}{17} - \frac{4}{17\sqrt{17}} \approx 0.02.$$

In Fig.4.1 we can see the graph of $\frac{1}{2t} - 2 + 2\sqrt{1+\frac{1}{4t^2}} - \frac{1}{2t(1+\frac{1}{4t^2})} - \frac{2t}{(4t^2+1)^{\frac{3}{2}}}$. In fact we observe that the condition for the derivative $v_1'(t)$ does not actually get negative for maximal error but very close to 0 the larger t becomes. Note therefore, that the bound for $v_1'(t)$ we use in the previous proposition is not sharp, but good enough to ensure negativity of $J_{heat} + J_{nl}$ for $j = 1$.

Proposition 4.10. *For* $2 \leq j \leq t$ *the sum* $J_{heat} + J_{nl}$ *is negative for*

$$\gamma \geq 10\beta C + 1.$$

Proof. We fill in the estimate of $v_j(\gamma t)$ we found in lemma 3.11 for J_{heat}

$$\begin{aligned} J_{heat} &= -\Phi'(\xi_{i,j})\beta t^{-\alpha}\left(v_j(\gamma t)\left(\frac{1}{4\gamma t}\right) + v_j'(\gamma t)(\gamma - 1)\right) \\ &\leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{\gamma - 1}{10}\frac{j^2}{\gamma^2 t^2}\right). \end{aligned}$$

Likewise we insert the estimate of $v_{j\pm1}(\gamma t) - v_j(\gamma t)$ from lemma 3.14 and use that $v_j(\gamma t) \leq$

1 from corollary 3.3 in J_{nl}

$$\begin{aligned}
J_{nl} &= -\frac{1}{2}\Phi''(\xi_{i,j} + \nu^+_{i,j}(t))\beta^2 t^{-2\alpha}(v_{j+1}(\gamma t) - v_j(\gamma t))^2 \\
&- \frac{1}{2}\Phi''(\xi_{i,j} + \nu^-_{i,j}(t))\beta^2 t^{-2\alpha}(v_{j-1}(\gamma t) - v_j(\gamma t))^2 \\
&\leq -\frac{1}{2}\Phi''(\xi_{i,j} + \nu^+_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t)^2 \frac{j^2}{\gamma^2 t^2} \\
&- \frac{1}{2}\Phi''(\xi_{i,j} + \nu^-_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t)^2 \frac{j^2}{\gamma^2 t^2} \\
&\leq -\frac{1}{2}\Phi''(\xi_{i,j} + \nu^+_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t) \frac{j^2}{\gamma^2 t^2} \\
&- \frac{1}{2}\Phi''(\xi_{i,j} + \nu^-_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t) \frac{j^2}{\gamma^2 t^2}. \\
&\leq C\Phi'(\xi_{i,j})\beta^2 t^{-2\alpha} v_j(\gamma t) \frac{j^2}{\gamma^2 t^2}.
\end{aligned}$$

Again wee have used the wave profile estimate from corollary 4.6 in the last line. Now we can add the two parts of the residual

$$\begin{aligned}
J_{heat} + J_{nl} &\leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{\gamma - 1}{10}\frac{j^2}{\gamma^2 t^2}\right) \\
&- \frac{1}{2}\Phi''(\xi_{i,j} + \nu^+_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t)\frac{j^2}{\gamma^2 t^2} \\
&- \frac{1}{2}\Phi''(\xi_{i,j} + \nu^-_{i,j}(t))\beta^2 t^{-2\alpha} v_j(\gamma t)\frac{j^2}{\gamma^2 t^2} \\
&\leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{\gamma - 1}{10}\frac{j^2}{\gamma^2 t^2} - C\beta t^{-\alpha}\frac{j^2}{\gamma^2 t^2}\right) \\
&\leq -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{j^2}{\gamma^2 t^2}\left(\frac{\gamma - 1}{10} - \beta C\right)\right),
\end{aligned}$$

where we have also used that $t^{-\alpha} < 1$ since $\alpha \ll 1$ in the last line. It remains to show that

$$\frac{j^2}{\gamma^2 t^2}\left(\frac{\gamma - 1}{10} - \beta C\right) \geq 0.$$

Therefore, we have to find a γ such that

$$\frac{\gamma - 1}{10} - \beta C \geq 0,$$

motivating the choice $\gamma \geq 10\beta C + 1$. □

Proposition 4.11. *For $3 \leq t \leq j$ we find $J_{heat} + J_{nl}$ is negative for*

$$\gamma \geq 10\beta C + 1.$$

Proof. We follow the same steps as in the previous proposition. We fill in the estimate of $v_j'(\gamma t)$ we found in lemma 3.12 in J_{heat}

$$\begin{aligned} J_{heat} &= -\Phi'(\xi_{i,j})\beta t^{-\alpha}\left(v_j(\gamma t)\left(\frac{1}{4\gamma t}\right) + v_j'(\gamma t)(\gamma - 1)\right) \\ &\le -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{\gamma-1}{5}\frac{j}{\gamma t}\right). \end{aligned}$$

Likewise we insert the estimate of $|v_{j\pm 1}(\gamma t) - v_j(\gamma t)|$ from lemma 3.15 and use lemma 3.6 in J_{nl}. Again we use that $t^{-\alpha} < 1$ and C from corollary 4.6 to

$$\begin{aligned} J_{nl} &= -\frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{+}(t))\beta^2 t^{-2\alpha}(v_{j+1}(\gamma t) - v_j(\gamma t))^2 \\ &\quad - \frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{-}(t))\beta^2 t^{-2\alpha}(v_{j-1}(\gamma t) - v_j(\gamma t))^2 \\ &\le -\frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{+}(t))\beta^2 t^{-2\alpha}\left(\frac{13}{8}\frac{j}{\gamma t} v_j(\gamma t)\right)^2 \\ &\quad - \frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{-}(t))\beta^2 t^{-2\alpha}\left(\frac{13}{8}\frac{j}{\gamma t} v_j(\gamma t)\right)^2. \\ &\le C\Phi'(\xi_{i,j})\beta t^{-\alpha}\left(\beta t^{-\alpha}\frac{169}{64}\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}\frac{j}{\gamma t} v_j(\gamma t)\right). \end{aligned}$$

Now we can add the two parts of the residual

$$\begin{aligned} J_{heat} + J_{nl} &\le -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \frac{\gamma-1}{5}\frac{j}{\gamma t}\right) \\ &\quad - \frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{+}(t))\beta t^{-\alpha} v_j(\gamma t)\frac{j}{\gamma t}\left(\beta\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}\frac{169}{64}\right) \\ &\quad - \frac{1}{2}\Phi''(\xi_{i,j} + \nu_{i,j}^{-}(t))\beta t^{-\alpha} v_j(\gamma t)\frac{j}{\gamma t}\left(\beta\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}\frac{169}{64}\right) \\ &\le -\Phi'(\xi_{i,j})\beta t^{-\alpha} v_j(\gamma t)\left(\frac{1}{4\gamma t} + \left(\frac{j}{\gamma t}\left(\frac{\gamma-1}{5} - \beta\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}C\frac{169}{64}\right)\right)\right). \end{aligned}$$

Therefore the residual is negative if

$$\frac{\gamma-1}{5} + \beta\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}C\frac{169}{64} \ge 0,$$

which gives

$$\gamma \ge 5\beta C\frac{1}{\sqrt{2}}e^{-\frac{1}{4}}\frac{169}{64} + 1.$$

We choose $\gamma \ge 10\beta C$ as we may, since $\frac{1}{\sqrt{2}}e^{-\frac{1}{4}} \approx 0.55$ and $\frac{169}{64} \approx 3.58$. □

Each of the four propositions give us a choice for γ such that the respective residual is negative. By taking the maximum of the four results for γ

$$\max_{\gamma}\{\frac{9}{16}\beta C, 4\beta C + \frac{1}{8}, 10\beta C + 1, 10\beta C + 1\} = 10\beta C + 1,$$

we can establish global negativity of the residual in the subsolution case. Note that we have implicitly used lemma 4.2 for our definition of the nonlinear residual from proposition 4.3. Therefore, we also have to ensure that $\gamma \geq \gamma_*$ with γ_* the respective threshold value found in lemma 4.2. The explicit subsolution becomes

$$u^-_{i,j}(t) = \Phi(i + ct - \theta_j(t) - Z(t)) - z(t)$$

with transversal dependence

$$\theta_j(t) = \beta t^{-\frac{1}{4\gamma}} \sqrt{\gamma t} e^{-2\gamma t} \mathcal{I}_j(2\gamma t).$$

For completeness we formulate the analogous supersolution

$$u^+_{i,j}(t) = \Phi(i + ct + \theta_j(t) + Z(t)) + z(t).$$

We have completely determined the constants now as we will see in the next section how β depends on the initial condition and C depends on the travelling wave Φ associated with the wave speed $c > 0$.
It remains to precisely determine the external functions z and Z and ensure negativity of J_{global}, because then the residual $J = J_{nl} + J_{heat} + J_{global}$ is ensured to be negative and the subsolution is a valid subsolution.
Reconsider J_{global} from 4.6. In section 2.1 we have seen the calculation of the residual in the continuous case for $\theta_j \equiv 0$. We have seen that z is exponentially decaying and Z is its integral multiplied by a constant. We make the same choices for the external functions up to a shift by 3

$$z(t) = \epsilon z_{hom}(t) = \epsilon e^{-\eta_z(t-3)}, z'(t) = -\epsilon \eta_z e^{-\eta_z(t-3)}$$

such that $Z(t)$ becomes

$$Z(t) = K_Z \int_0^{t-3} z(s) ds = K_Z \epsilon \Big(-\frac{1}{\eta_z} e^{-\eta_z(t-3)} + \frac{1}{\eta_z} \Big), Z'(t) = K_Z \epsilon e^{-\eta_z(t-3)} = K_Z z(t).$$

The shift by 3 is related to lemma 3.15, where we had to consider $3 \leq t \leq j$ for the estimate. For our stability result given in theorem 1.1, we have to shift the starting time of the subsolution to $t = 3$ due to lemma 3.15. In the proof we want to use that $z(0) = \epsilon$, therefore we need to shift z as well. Shifting the starting point is not a problem here, because we are interested in large perturbations. The global residual J_{global} deserves its name, for it lacks dependency on local coordinates. In particular, J_{global} is independent of $\theta_j(t)$, which implies that J_{global} is independent of the initial condition β determining γ and α. Therefore, we drop the local coordinates in the travelling wave constant and write $\xi_{ij} = \xi$ in the following lemma.

Lemma 4.12. *For $\eta_z > 0$ and $\epsilon > 0$ both sufficiently small and K_Z sufficiently large, the global residual expression*

$$J_{global} = -z'(t) - Z'(t)\Phi'(\xi) + g(\Phi(\xi)) - g(\Phi(\xi) - z(t))$$

is negative for our choices of z and Z for all $\xi \in \mathbb{R}$ and $t \geq 3$.

Proof. The function $z(t) = \epsilon e^{-\eta_z(t-3)}$ is positive everywhere and we use that $Z'(t) = K_Z z(t)$. The mean value theorem applied to the smooth function g on the interval $[\Phi(\xi) - z(t), \Phi(\xi)]$ gives that there must be a $\mu \in (0,1)$ such that

$$\begin{aligned} g(\Phi(\xi) - g(\Phi(\xi) - z(t)) &= g'(\Phi(\xi) + \mu z(t)))(\Phi(\xi) - (\Phi(\xi) - z(t)) \\ &= g'(\Phi(\xi) + \mu z(t)))z(t). \end{aligned}$$

The global residual becomes

$$J_{global} = -z'(t) - K_Z z(t)\Phi'(\xi) + z(t)g'(\Phi(\xi) + \mu z(t))).$$

We follow the proof of lemma 5.29 from [1]. In order to keep the residual J_{global} negative, it suffices to show that we can choose $K_Z > 1$ and $0 < \eta_z < 1$ such that for all $\xi \in \mathbb{R}$

$$-z'(t) - K_Z z(t)\Phi'(\xi) + g'(\Phi(\xi) + \mu z(t))z(t) \leq 0.$$

Using the derivative $z'(t) = -\eta_z z(t)$ we can rewrite the last inequality as

$$\begin{aligned} \eta_z z(t) - K_Z z(t)\Phi'(\xi) + g'(\Phi(\xi + \mu z(t))z(t) &\leq 0 \\ \eta_z - K_Z\Phi'(\xi) + g'(\Phi(\xi) + \mu z(t)) &\leq 0 \\ -K_Z\Phi'(\xi) + g'(\Phi(\xi) + \mu z(t)) &\leq -\eta_z. \end{aligned} \tag{4.31}$$

In section 2.2 we have seen the bounded derivative of g

$$g'(u) = -3u^2 + 2(a+1)u - a,$$

which is depicted in Fig. 2.1. It is negative for u close to 0 and 1 and positive in the middle part. In order to formalize the regions seperated by the sign of g', choose $\alpha > 0$ and $\eta_z > 0$ such that $y \geq 1 - \alpha$ implies $g'(y) \leq -\eta_z$ and $y \leq \alpha$ also implies $g'(y) \leq -\eta_z$ Furthermore, we use that we have seen that $\Phi' > 0$ everywhere in the proof of proposition 4.5 so $-K_Z\Phi'(\xi) < 0$. We remark that $|z| \leq \epsilon$ follows directly from the definition of $z(t)$. We pick L large enough and ϵ small enough such that

$$\begin{aligned} &\text{For } \xi \geq +L \text{ and } |z| \leq \epsilon: \quad \Phi(\xi + z) \geq 1 - \alpha \\ &\text{For } \xi \leq -L \text{ and } |z| \leq \epsilon: \quad \Phi(\xi + z) \leq \alpha. \end{aligned}$$

In the first case, $|\xi| \geq L$ and $|z(t)| \leq \epsilon$ for all $t \in \mathbb{R}$. Hence we may conclude

$$g'(\Phi(\xi) + \mu z(t)) \leq -\eta_z \leq -\eta_z + K_Z\Phi'(\xi)$$

and (4.31) follows.
We turn to the second case. The function Φ' is positive, so there is a $\kappa > 0$ such that

$$\Phi'(\xi + z) > \kappa$$

for $|\xi| \leq L$ and $|z(t)| \leq \epsilon$ for all $t \in \mathbb{R}$. Now choose $K_Z > 0$ large enough to ensure

$$\kappa K_Z \geq \eta_z + \|g\|_\infty.$$

We may conclude (4.31)

$$-K_Z\Phi'(\xi) + g'(\Phi(\xi) + \mu z) \leq -\eta_z - \|g\|_\infty + g'(\Phi(\xi) + \mu z) \leq -\eta_z.$$

Thus (4.31) follows for all $\xi \in \mathbb{R}$ and $t \geq 3$ and the lemma is proven. □

In particular, we may conclude that the residual of the subsolution is negative and $u^-_{i,j}(t)$ is a valid subsolution for $t \geq 3$.

4.5. Stability of the Horizontal Travelling Wave

We want to prove stability of the solutions of 4.2 and prove theorem 1.1 under use of the preparation from chapter 3 and the previous sections of chapter 4. In lemma 3.15 we took $t \geq 3$ for the estimate, that is why we have to partially shift the subsolution and define

$$w_{i,j}^{-}(t) = \Phi(i + ct - \theta_j(t+3) - Z(t+3)) - z(t+3)$$

to be able to start at $t = 0$. Note that $w_{i,j}^{-}$ is a valid subsolution, because $u_{i,j}^{-}$ is and the computations of the previous subsections also hold for the wave profile Φ shifted by $-3c$.

Proof of Theorem 1.1. Let $\delta > 0$. We want to show that

$$\liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (U_{i,j}(t) - \Phi(i + ct)) \geq -\delta. \tag{4.32}$$

We take $\epsilon > 0$ from $z(0) = \epsilon$ such that

$$\epsilon K_Z \frac{1}{\eta_z} \|\Phi'\|_\infty \leq \delta.$$

We know that Φ' is bounded from proposition 4.5. From the initial condition 1.12, we know that there exists a finite set S depending on ϵ such that on $\mathbb{Z}^2 \setminus S$

$$|U_{i,j}(0) - \Phi(i)| \leq \epsilon.$$

We want to prove the following claim

$$w_{i,j}^{-}(0) \leq U_{i,j}(0) \tag{4.33}$$

for all choices $(i, j) \in \mathbb{Z}^2$.
First, we prove (4.33) on $\mathbb{Z}^2 \setminus S$. We note that with our choice of Z and z

$$w_{i,j}^{-}(0) = \Phi(i - \theta_j(3) - Z(3)) - z(3) = \Phi(i - \theta_j(3)) - \epsilon.$$

We can use that θ_j is a positive function and Φ is an increasing function as seen in proposition 4.5 to conclude

$$w_{i,j}^{-}(0) = \Phi(i - \theta_j(3)) - \epsilon \leq \Phi(i) - \epsilon \leq U_{i,j}.$$

Secondly, we prove (4.33) on S. Let L denote the maximal diameter of the set S. Then $|i|, |j| \leq L$. Choose Ω_{phase} in such a way that

$$\Phi(i - \Omega_{phase}) - \epsilon \leq U_{i,j}(0).$$

In addition, choose γ_{min} such that

$$L \leq \sqrt{3\gamma_{min}}.$$

By lemma 3.5 there is a $0 < C' < 1$ such that

$$v_j(t) \geq C$$

for all $0 \leq j \leq \sqrt{t}$. But then for all $|j| \leq L$ and $\gamma \geq \gamma_{min}$ we have

$$\theta_j(3) = \beta 3^{-\frac{1}{4\gamma}} v_j(3\gamma) \geq \frac{\beta}{3} C'.$$

Now we pick β so that $\frac{\beta}{3} C = \Omega_{phase}$. This is the link between the imposed initial condition and the modifying constant β in $\theta_j(t)$. Furthermore, ensure that $\gamma \geq 10\beta C + 1$ as seen in the previous section and in the light of lemma 4.2 ensure that $\gamma \geq \{\gamma_*(\beta), \gamma_{min}\}$. It follows that

$$w^-_{i,j}(0) = \Phi(i - \theta_j(3)) - \epsilon \leq \Phi(i - \Omega_{phase}) - \epsilon \leq U_{i,j}(0).$$

In conclusion, we have $w^-_{i,j}(0) \leq U_{i,j}(0)$ on the whole lattice. By the comparison principle given in theorem A.12 with $w^-_{i,j}$ the subsolution and $U_{i,j}$ the supersolution, we know that

$$w^-_{i,j}(t) \leq U_{i,j}(t)$$

holds on the whole lattice. We calculate the limit

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i + ct - Z(t)) \\
&= \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (\Phi(i + c(t+3) - \theta_j(t+3) - Z(t+3)) - z(t+3) - \Phi(i + ct - Z(t))
\end{aligned}$$

We apply the mean value theorem on Φ. There must be a $\mu \in (0,1)$ such that

$$\begin{aligned}
&= \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (Z(t+3) - Z(t) + \theta_j(t+3))\Phi'(i + ct + \mu(Z(t+3) - Z(t) + \theta_j(t+3))) \\
&= \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (Z(t+3) - Z(t) + \theta_j(t+3))\|\Phi'\|_\infty - z(t+3)) \\
&= \|\Phi'\|_\infty \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} \theta_j(t+3) + \|\Phi'\|_\infty \lim_{t\to\infty}(Z(t+3) - Z(t)) - \epsilon \lim_{t\to\infty} e^{-\eta_z t}) \\
&= 0,
\end{aligned}$$

where we have used the definition of $Z, \eta_z > 0$, the boundedness of $\|\Phi'\|_\infty$ shown in proposition 4.5 and the limit of $\theta_j(t)$ as shown in lemma 4.1. Furthermore, we calculate

$$\begin{aligned}
&|\Phi(i + ct) - \Phi(i + ct - Z(t))| \\
&\leq \|\Phi'\|_\infty |Z(t)| \\
&\leq \|\Phi'\|_\infty \left| K_Z \epsilon \left(\frac{1}{\eta_z} e^{-\eta_z t + \frac{1}{\eta_z}} \right) \right| \\
&\leq \|\Phi'\|_\infty \frac{K_Z}{\eta_z} \epsilon \leq \delta.
\end{aligned}$$

Now we use these limits to show (4.32)

$$\begin{aligned}
&\liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (U_{i,j}(t) - \Phi(i+ct)) \\
&\geq \liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i+ct)) \\
&= \liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i+ct-Z(t)) + \Phi(i+ct-Z(t)) - \Phi(i+ct)) \\
&= \liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i+ct-Z(t))) + \liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (\Phi(i+ct-Z(t)) - \Phi(i+ct)) \\
&= \lim_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i+ct-Z(t))) + \liminf_{t\to\infty} \inf_{(i,j)\in\mathbb{Z}^2} (\Phi(i+ct-Z(t)) - \Phi(i+ct)) \\
&\geq \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (w^-_{i,j}(t) - \Phi(i+ct-Z(t))) - \delta = -\delta.
\end{aligned}$$

Now

$$\limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (U_{i,j}(t) - \Phi(i+ct)) \leq \delta$$

follows by symmetry under use of the shifted supersolution $w^+_{i,j}(t) = u^+_{i,j}(t+3)$. We can use the same set S to find Ω_{phase} and Ω to let

$$w^+_{i,j}(0) \geq U_{i,j}(0)$$

hold on the whole lattice. The function $U_{i,j}$ is a subsolution w.r.t. $w^+_{i,j}(t)$, so we can apply the comparison principle again to get

$$w^+_{i,j}(t) \geq U_{i,j}(t)$$

for $t \geq 0$ on the whole lattice. The calculations get the same result, because the sign changes do not affect the calculation. Therefore we can calculate the limit as

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (U_{i,j}(t) - \Phi(i+ct)) \\
&\leq \limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (w^+_{i,j}(t) - \Phi(i+ct)) \\
&= \limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (w^+_{i,j}(t) - \Phi(i+ct+Z(t)) + \Phi(i+ct+Z(t)) - \Phi(i+ct)) \\
&= \lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (w^+_{i,j}(t) - \Phi(i+ct+Z(t))) + \limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} (\Phi(i+ct+Z(t)) - \Phi(i+ct)) \\
&\leq \delta.
\end{aligned}$$

Now we can take the absolute value of the difference $U_{i,j}(t) - \Phi(i+ct)$ implying

$$\limsup_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} |U_{i,j}(t) - \Phi(i+ct)| \leq \delta$$

for the supersolution and

$$\liminf_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} |U_{i,j}(t) - \Phi(i+ct)| \leq \delta$$

for the subsolution. Both limits together imply that

$$\lim_{t\to\infty} \sup_{(i,j)\in\mathbb{Z}^2} |U_{i,j}(t) - \Phi(i+ct)| \leq \delta.$$

Finally, we can let $\delta \to 0$ and the theorem is proven. $\square$

A. Appendix

The first part of the appendix contains lemmas for the estimates. We frequently need smaller estimates for functions of $\frac{j}{t}$ in the calculations. Therefore we prove the following preliminary lemmas first.

Lemma A.1. *For $x \geq 0$ the following inequality holds*

$$\sqrt{1+x^2} \geq 1 + \frac{x^2}{2} - \frac{x^4}{8}. \tag{A.1}$$

Proof. The function $\sqrt{1+x^2}$ is positive everywhere. Therefore, it suffices to prove the lemma for the positive part of $1 + \frac{x^2}{2} - \frac{x^4}{8}$. The latter function is positive on the interval $[-\sqrt{2(1+\sqrt{3})}, \sqrt{2(1+\sqrt{3})}]$, as

$$1 + \frac{x^2}{2} - \frac{x^4}{8} = 0$$

has solutions $x = \pm\sqrt{2(1+\sqrt{3})}$ and the function is 1 for $x = 0$.
The proof is completed if the quadrate of the claim holds, i.e.

$$1 + x^2 \geq \left(1 + \frac{x^2}{2} - \frac{x^4}{8}\right)^2 = 1 + x^2 - \frac{x^6}{8} + \frac{x^8}{64}.$$

The inequality clearly holds, because

$$-\frac{x^6}{8} + \frac{x^8}{64} \leq 0 \iff x^2 \leq 8$$

clearly holds for $x \in [-\sqrt{2(1+\sqrt{3})}, \sqrt{2(1+\sqrt{3})}]$, since $\sqrt{2(1+\sqrt{3})} \approx 5.46$. □

Lemma A.2. *For $x \geq -1$ we have the following inequality*

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x.$$

Proof. The functions $\sqrt{1+x}$ and $1 + \frac{1}{2}x$ intersect only in 0. We take the derivative on both sides of the inequality

$$\frac{1}{2\sqrt{1+x}} \leq \frac{1}{2}.$$

Since $\frac{1}{\sqrt{1+x}} \leq 1$ we see that the last inequality is true, so the function on the left hand side always stays below the one on the right. □

Lemma A.3. *For $x > 0$ we have the following inequality*

$$\sqrt{1+x^2} \leq x + \frac{1}{2x}. \tag{A.2}$$

Proof. The functions on both sides of the inequality are positive for positive x. Therefore the claim is proven if the quadrated inequality holds. Upon quadatrating both sides we get

$$1 + x^2 \leq \left(x + \frac{1}{2x}\right)^2 = x^2 + 1 + \frac{1}{4x^2},$$

which holds, because for positive x

$$0 \leq \frac{1}{4x^2}.$$

□

Lemma A.4. *For $x \in \mathbb{R}$ we have the following inequality*

$$ln(1 + x + \frac{1}{2}x^2) \leq x + \frac{1}{2}x^2.$$

Proof. The functions $ln(1 + x + \frac{1}{2}x^2)$ and $x + \frac{1}{2}x^2$ intersect only in 0 and -2. We take the derivative on both sides of the inequality

$$\frac{1+x}{1+x+\frac{1}{2}x^2} \leq 1 + x. \tag{A.3}$$

Clearly, $\frac{1}{1+x+\frac{1}{2}x^2} \leq 1$ for real x. Therefore (A.3) is true, so the function on the left hand side always stays below the one on the right. □

Lemma A.5. *For $-\frac{4}{3} \leq x \leq \frac{4}{3}$ we have the following inequality*

$$\sqrt{1+x^2} \geq 1 + \frac{3}{8}x^2.$$

Proof. We calculate the intersections of the functions $\sqrt{1+x^2}$ and $1 + \frac{3}{8}x^2$

$$\begin{aligned}
\sqrt{1+x^2} &= 1 + \frac{3}{8}x^2 \\
1 + x^2 &= 1 + \frac{3}{4}x^2 + \frac{9}{64}x^4 \\
\frac{9}{64}x^4 - \frac{1}{4}x^2 &= 0 \\
x &= \mp\frac{4}{3} \quad \text{and} \quad x = 0.
\end{aligned}$$

Due to symmetry it suffices to check the inequality for a positive value smaller than $\frac{4}{3}$. For $x = \frac{1}{2}$ we indeed find that the inequality holds as $\frac{\sqrt{5}}{2} \approx 1.12$ and

$$\frac{\sqrt{5}}{2} \geq \frac{35}{32} \approx 1.09.$$

□

Lemma A.6. *For $x \geq 0$ we have the following inequality*

$$ln(1+x) \geq x - \frac{1}{2}x^2.$$

Proof. The functions $ln(1+x)$ and $x - \frac{1}{2}x^2$ intersect only in 0. Therefore it suffices to show the inequality for any positive value of x. For convenience we take $x = 1$ and find that indeed

$$\frac{1}{2} \leq ln(2).$$

□

Lemma A.7. *For $x \geq 0$ we have the following inequality*

$$\sqrt{1+x^2} \leq 1 + x.$$

Proof. We only consider nonnegative values of x, therefore it suffices to show the squared inequality

$$1 + x^2 \leq 1 + 2x + x^2,$$

which is trivially true for $x \geq 0$. □

In the second part we show formulations of two theorems taken from [12].

Theorem A.8. *The following two-sided bounds hold for the function $a_j(t) = \frac{t\mathcal{I}'_j(t)}{\mathcal{I}_j(t)}$*

$$-\frac{t^2}{2(t^2+j^2)^{\frac{3}{2}}} < a_j(t) - \sqrt{t^2+j^2} + \frac{t^2}{2(t^2+j^2)} < \frac{t^2}{2(t^2+j^2)^{\frac{3}{2}}},$$

where the upper bound holds for all $t > 0$ and $j \geq 0$ and the lower bound holds on the set

$$\{(t,j) : t > 0, j \geq \frac{1}{2}\} \bigcup \{(t,j) : t > 0, j \geq 0, \sqrt{j^2+t^2} \geq \frac{\sqrt{7}+2}{3}\}.$$

Theorem A.9. *The following two-sided bound hold for the modified Bessel functions $\mathcal{I}_j(t)$*

$$e^{-\frac{1}{2\sqrt{t^2+j^2}}} \leq \mathcal{I}_j(t)\sqrt{2\pi}(t^2+j^2)^{\frac{1}{4}} e^{-(\sqrt{t^2+j^2}+j ln(\frac{t}{j+\sqrt{t^2+j^2}}))} \leq e^{\frac{1}{2\sqrt{t^2+j^2}}},$$

where the upper bound holds for all $t > 0$ and $j \geq 0$ and the lower bound holds on the set

$$\{(t,j) : t > 0, j \geq \frac{1}{2}\} \bigcup \{(t,j) : t > 0, j \geq 0, \sqrt{j^2+t^2} \geq \frac{\sqrt{7}+2}{3}\}.$$

We also need the definition of the modified Bessel function and some of its properties taken from [15].

Definition A.10. The modified Bessel function of the first kind is given by

$$\mathcal{I}_j(t) = \frac{1}{\pi}\int_0^{\pi} e^{t\cos(\omega)}\cos(j\omega)d\omega - \frac{\sin(j\pi)}{\pi}\int_0^{\infty} e^{-t\cosh(t)-jt}dt.$$

For whole j we note that the factor $\frac{\sin(j\pi)}{\pi}$ is zero, so the expression becomes

$$\mathcal{I}_j(t) = \frac{1}{\pi}\int_0^{\pi} e^{t\cos(\omega)}\cos(j\omega)d\omega.$$

In this thesis we make use of the following properties of the modified Bessel function of the first kind

$$2\mathcal{I}_j'(t) = \mathcal{I}_{j+1}(t) + \mathcal{I}_{j-1}(t).$$
$$0 < \mathcal{I}_j < \mathcal{I}_{j+1}.$$

Furthermore, we need to define discrete Fourier transformation taken from [16].

Definition A.11. The discrete Fourier transformation is given by

$$(\theta_j(t))_j \in l^2(\mathbb{R}) \iff \hat{\theta}_\omega(t) \in L^2_{per}[-\pi,\pi]$$

and takes the form

$$\theta_j(t) = \int_{-\pi}^{\pi} e^{i\omega j}\hat{\theta}_\omega(t)d\omega.$$

Lastly, we need the comparison principle as seen in [1] (proposition 3.1). We formulate the comparison principle on the lattice in the absence of an obstacle first.

Theorem A.12. *If u^- is a subsolution, i.e.*

$$\dot{u}^-_{ij}(t) \leq \Delta^+ u^-_{i,j}(t) + g(u^-_{i,j}(t))$$

and $u^+_{i,j}$ is a supersolution i.e.

$$\dot{u}^+_{ij}(t) \geq \Delta^+ u^+_{i,j}(t) + g(u^+_{i,j}(t)),$$

and

$$u^-_{i,j}(0) \leq u^+_{i,j}(0)$$

holds on the whole lattice, then

$$u^-_{i,j}(t) \leq u^+_{i,j}(t) \quad \textit{for all} \quad t \geq 0$$

also holds on the whole lattice.

The comparison principle in the presence of an obstacle becomes

Theorem A.13. *Remove any obstacle $K \subset \mathbb{Z}^2$ from the lattice. Take a pair of functions $u, v \in C^1([0,\infty), l^\infty(\Lambda, \mathbb{R})$ that satisfy the initial inequality*

$$u_{i,j}(0) \geq v_{ij}(0).$$

Suppose furthermore that for any value of $t \geq 0$ and all $(i,j) \in \Lambda$ at least one of the following properties is satisfied.
(a) The differental inequalities

$$\dot{u}_{ij}(t) \geq \Delta_\Lambda u_{i,j}(t) + g(u_{i,j}(t)) \quad \textit{and} \quad \dot{v}_{ij}(t) \leq \Delta_\Lambda v^+_{ij}(t) + g(v^+_{ij}(t))$$

hold for t or
(b) we have the inequality $u_{i,j}(t) \geq v_{ij}(t)$.
then we in fact have

$$u^-(t) \leq u^+(t) \quad \textit{for all} \quad t \geq 0$$

for all $t \geq 0$ on the whole of Λ.

Populaire Samenvatting

Wij willen de volgende partiële differentiaalvergelijking op het rooster bekijken

$$u_t = \Delta u + u(u-a)(1-u), \tag{A.4}$$

waarbij we de constante $0 < a < 1$ vastleggen. Laat $g(u) = u(u-a)(1-u)$. De vergelijking (A.4) op een continue en eindig dimensionale ruimte is een voorbeeld van een reactie-diffusie-vergelijking. Hierbij zorgt de term Δu voor de diffusie, dat wil zeggen voor de verspreiding in de ruimte. De term $g(u)$ modelleert een reactie. Wij kunnen ook andere functies in u kiezen, belangrijk is dat er nulpunten zijn, hier zijn dat $u = 0, u = a$ en $u = 1$, die evenwichten zijn van (A.4). De benoeming reactie-diffusie-vergelijking geeft al weg, dat er legio toepassingen zijn. In feite is (A.4) zelfs het prototype voor de modellering van bijvoorbeeld scheikundige reacties, populatiemodellen of modellen voor de verspreiding van ziektes.
Maar niet elke ruimte waarop processen moeten worden gesimuleerd, heeft een continue structuur. Denk bijvoorbeeld aan kristallen of pixels op een scherm. Daarom willen wij (A.4) graag vertalen naar de discrete ruimte en dan naar oplossingen kijken. Uiteindelijk zijn we in deze scriptie geïnteresseerd in de stabiliteit van oplossingen van het discrete analoog van (A.4) op het rooster $\mathbb{Z}^2$. Eerst moeten wij een goede definitie zien te vinden. Het rooster $\mathbb{Z}^2$ is te zien in Fig. A.1. In tegenstelling tot de continue ruimte $\mathbb{R}^2$ maakt het uit vanuit welke richting wij naar het rooster kijken. Wij gaan zien dat dit consequenties voor de existentie en stabiliteit van oplossingen heeft. Laat u een functie zijn op het rooster. Dan is u goed gedefinieerd als wij aan elk roosterpunt (i, j) voor elk tijdstip t een reële waarde toewijzen, dit geeft $u_{i,j}(t) \in \mathbb{R}$ oftewel

$$u : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}.$$

Omdat wij de tijd dus wel continu laten verlopen, is de vertaling van de tijdsafgeleide u_t in (A.4) geen probleem. Maar voor de vertaling van de diffusieterm Δu moeten wij een definitie voor de discrete tweede afgeleide vinden. Wij herinneren ons aan de formele definitie van de eerste afgeleide van een gladde functie $f : \mathbb{R} \to \mathbb{R}$

$$f'(x) = \frac{f(x+h) - f(x)}{h} \quad \text{als} \quad h \to 0.$$

Wij bekijken dus twee waarden $f(x)$ en $f(x+h)$ gedeeld door hun afstand h en laten dan de afstand naar 0 gaan.
Wij willen de afgeleide van u ook op het rooster associëren met de afstanden tussen $u_{i,j}$ en zijn buren. De afstand tussen twee roosterpunten is 1. De tweede afgeleide wordt een som van verschillen van verschillen van $u_{i,j}$ met zijn vier buurpunten

$$(u_{i+1,j} - u_{i,j}) - (u_{i,j} - u_{i-1,j}) + (u_{i,j+1} - u_{i,j}) - (u_{i,j} - u_{i,j-1}) = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}.$$

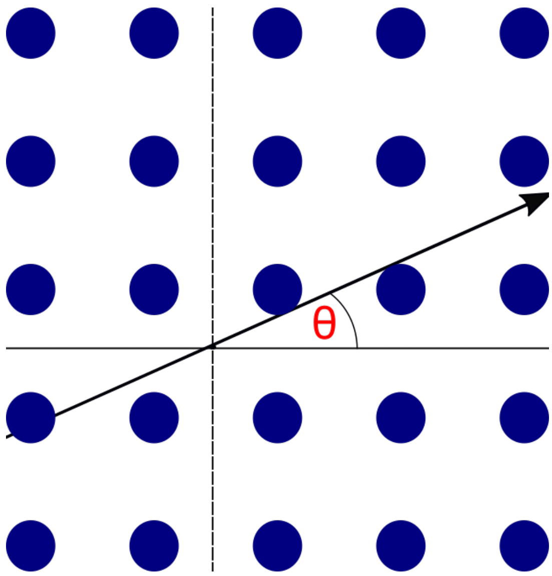

Figure A.1.: Lattice with direction denoted by θ

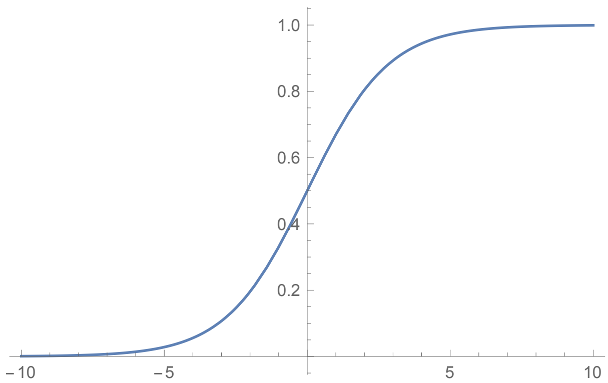

Figure A.2.: The wave profile Φ

Wij hebben het discrete analoog $[\Delta^+ u]_{i,j}$ van Δu gevonden en daarmee het analoog van (A.4), de zogenoemde *lattice differential equation* (LDE)

$$\dot{u}_{i,j} = [\Delta^+ u]_{i,j} + g(u_{i,j}). \tag{A.5}$$

Merk in het bijzonder op dat de definitie afhangt van de coördinaten (i,j), want dit heeft consequenties voor onze berekeningen en voor het karakter van oplossingen.
Om (A.5) te bestuderen, willen we zoveel mogelijk informatie over (A.4) gebruiken, want dit is een geval die wij goed begrijpen. Wij passen de Ansatz van een lopende golf toe op (A.4) op $\mathbb{R}^2$:

$$u(x,y,t) = \Phi(\sigma_h x + \sigma_v y + ct).$$

Hierbij is $c > 0$ de snelheid van de golf en (σ_h, σ_v) een richting op het vlak met $\sigma_h^2 + \sigma_v^2 = 1$.
Wij bekijken (A.4) in $\Phi = \Phi(\sigma_h x + \sigma_v y + ct)$

$$c\Phi' = (\sigma_h^2 + \sigma_v^2)\Phi'' + g(\Phi) = \Phi'' + g(\Phi).$$

Merk op dat de richting (σ_h, σ_v) uit de vergelijking is verdwenen. Dit is een gewone differentiaalvergelijking, die we expliciet kunnen oplossen, voor $\xi = \sigma_h x + \sigma_v y + ct$

$$\Phi(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\xi\right). \tag{A.6}$$

Dit geeft reeds existentie en uniciteit van lopende golven. De grafiek van Φ zien we in Fig. 1.3.
Daarom willen wij de Ansatz van een lopende golf ook voor (A.5) gebruiken. Op $\mathbb{Z}^2$ wordt dit

$$u_{i,j}(t) = \Phi(\sigma_h i + \sigma_v j + ct)$$

en (A.5) met $\sigma_h i + \sigma_v j + ct = \xi_{ij}$ wordt

$$c\Phi'(\xi_{i,j}) = \Phi(\xi_{i+1,j}) + \Phi(\xi_{i-1,j}) + \Phi(\xi_{i,j+1}) + \Phi(\xi_{i,j-1}) - 4\Phi(\xi_{i,j}) + g(\Phi(\xi_{i,j})).$$

Dit is een ander type vergelijking, omdat er naast een tijdsafgeleide ook lokale verschillen in zitten. Daarnaast raken wij de richtingsafhankelijkheid niet kwijt. Dit maakt existentie en uniciteit van oplossingen voor (A.5) voor alle richtingen (σ_h, σ_v) moeilijker te bewijzen, maar dit is wel al gedaan.
Daarom kunnen we ons concentreren op de stabiliteit van de lopende golven onder grote lokale verstoringen. Specifieker willen wij het volgende bewijzen. Elke oplossing van (A.5) tussen 0 en 1, wiens beginwaarde voor ver weg gelegen roosterpunten op $t = 0$ naar die van Φ op $t = 0$ convergeert, convergeert op den duur op het hele rooster naar Φ.
Wij maken hiervoor gebruik van het vergelijkingsprincipe. Wij noemen $u_{i,j}^-$ een onderoplossing en $u_{i,j}^+$ een bovenoplossing als

$$\dot{u}_{ij}^-(t) \leq [\Delta^+ u]_{i,j}^-(t) + g(u_{i,j}^-(t)) \text{ en } \dot{u}_{ij}^+(t) \geq [\Delta^+ u]_{i,j}^+(t) + g(u_{i,j}^+(t)).$$

Het vergelijkingsprincipe zegt dat als $u_{i,j}^-$ onder $u_{i,j}^+$ blijft op tijdstip $t = 0$ dan geldt dit ook voor alle tijdstippen t. Hiervoor moeten we dus wel een onderoplossing en een

bovenoplossing vinden. Gelukkig kunnen wij hen alleen in teken laten verschillen en het is voldoende om één van die twee te vinden. Wij concentreren ons op het vinden van een onderoplossing, die zo groot mogelijke verstoringen op kan vangen.
Wij leggen de richting op het rooster vast en concentreren ons op de horizontale richting $(\sigma_h, \sigma_v) = (1, 0)$. We geven de hoofdstelling van deze scriptie, de stelling over de stabiliteit van golven nog een keer formeel voor $(1, 0)$.

Theorem A.14. *Als* $U : [0, \infty) \to l^\infty(\mathbb{Z}^2, \mathbb{R})$ *een differentieerbare oplossing is van* (A.5) *voor alle* $t \geq 0$ *en*

$$|U_{i,j}(0) - \Phi(i)| \to 0 \quad \textit{as} \quad |i| + |j| \to \infty,$$

waarbij $0 \leq U_{i,j}(0) \leq 1$, *dan hebben wij de uniforme convergentie*

$$\sup_{(i,j)\in\mathbb{Z}^2} |U_{i,j}(t) - \Phi(i + ct)| \to 0 \quad \textit{for} \quad t \to \infty.$$

De resonantie met het rooster in de horizontale richting is samen met de verticale richting het sterkst, dit betekent dat wij met de hoogst mogelijke frequentie roosterpunten tegenkomen in deze richting. Bovendien vallen veel complicaties weg met $\sigma_v = 0$. Het voordeel is dat wij sterke en directe afschattingen kunnen maken voor de constructie van de onderoplossing. We willen dus een u_{ij}^- vinden zodat de restterm van (A.5)

$$J = \dot{u}_{ij}^-(t) - [\Delta^+ u]_{i,j}^-(t) - g(u_{i,j}^-(t))$$

negatief is. Om dit te bewerkstelligen kijken wij opnieuw naar (A.4), waar wij stabiliteit van de lopende golf met de continue versie van het vergelijkingsprincipe kunnen aantonen. De continue onderoplossing is een modificatie van de lopende golf gegeven door

$$u^-(x, y, t) = \Phi(x + ct - \theta(y, t) - Z(t)) - z(t),$$

waarbij wij de functies $\theta(y, t), z(t)$ en $Z(t)$ moeten bepalen. Het blijkt dat het nuttig is om de restterm op te splitsen in drie delen, die wij vervolgens apart gaan bekijken

$$\begin{aligned} J_{global} &= -z'(t) - Z'(t)\Phi'(\xi(t)) + g(\Phi(\xi)) - g(\Phi(\xi) - z(t)), \\ J_{nl} &= -\Phi''(\xi(t))\theta_y(y, t)^2, \\ J_{heat} &= -\Phi'(\xi(t))(\theta_{yy}(y, t) - \theta_t(y, t)). \end{aligned}$$

Het globale gedeelte hangt alleen af van $z(t)$ en $Z(t)$, daarom kunnen wij J_{global} negatief houden door $z(t)$ en $Z(t)$ goed te bepalen. Voor de andere twee gedeeltes willen wij gebruik maken van wat we al weten van Φ uit (A.6). We kunnen de afgeleides Φ' en Φ'' rechtstreeks gaan bepalen en het blijkt dat Φ' positief is, maar, dat Φ'' geen globaal teken heeft. Daarom willen wij de term

$$\theta_{yy}(y, t) - \theta_t(y, t)$$

zo positief mogelijk praten om de absolute waarde van het niet-lineaire gedeelte van de restterm J_{nl} te kunnen domineren met J_{heat}.
Daarom is het zaak om goed naar de restterm van de warmtevergelijking in $\theta(y, t)$ te

kijken. Nadat wij $\theta(y,t)$ goed hebben bepaald, is de onderoplossing gevonden.
Wij maken gebruik van het continue geval om de discrete onderoplossing

$$u^-_{i,j}(t) = \Phi(i + ct - \theta_j(t) - Z(t)) - z(t)$$

te vinden. Merk op dat de functie $\theta_j(t)$ ook hier als enige een transversale afhankelijkheid heeft, in het continue geval is dat de $y-$as en in het discrete geval de $j-$coördinaat. Met een beetje meer moeite en voorbereiding vinden wij analoge resttermen. Hier geven wij alleen de twee meest interessante delen van de restterm

$$\begin{aligned} J_{nl} &= -\frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu^+_{i,j}(t))(\theta_j - \theta_{j+1})^2 - \frac{1}{2}\Phi''(\xi_{i,j}(t) + \nu^-_{i,j}(t))(\theta_j - \theta_{j-1})^2 \\ J_{heat} &= \Phi'(\xi_{i,j}(t))(\theta_{j+1}(t) + \theta_{j-1}(t) - 2\theta_j(t) - \theta'_j(t)). \end{aligned}$$

Wij kunnen bewijzen dat de discrete lopende golf $\Phi = \Phi(\xi_{ij}(t))$ hetzelfde gedrag vertoont als de analoge continue lopende golf. Daarom willen wij hetzelfde principe gebruiken als hierboven en de term

$$\theta_{j+1}(t) + \theta_{j-1}(t) - 2\theta_j(t) - \theta'_j(t)$$

zo positief mogelijk praten om uiteindelijk J_{nl} met J_{heat} te domineren. Maar de oplossing van de discrete warmtevergelijking is niet zo makkelijk te modificeren als die van de continue warmtevergelijking. Wij krijgen te maken met de gemodificeerde Besselfuncties van de eerste orde en moeten afschattingen vinden voor zowel de afgeleide $\theta'_j(t)$ als van de verschillen $\theta_{j\pm1}(t) - \theta_j(t)$ voor de oplossing van de discrete warmtevergelijking.
Een groot deel van deze scriptie is gewijd aan het vinden van deze afschattingen. Telkens moeten wij de afschattingen in vier aparte gevallen, afhankelijk van de verhouding tussen tijd t en de transversale richting in de ruimte j, bekijken. Daarna bewijzen wij $\theta_j(t)$ zo kunnen modificeren, dat de restterm negatief wordt. Daarmee is de onderoplossing gevonden en kunnen wij het bewijs van de hoofdstelling voor de stabiliteit van de lopende golven Φ voltooien.

Bibliography

[1] A. Hoffman, H.J. Hupkes, E.S. Van Vleck. Entire Solutions for Bistable Lattice Differential Equations with Obstacles *arXiv preprint, arXiv:1310.4978* 2014.

[2] H. Berestycki, F. Hamel, H. Matano. Bistable Traveling Waves around an Obstacle *Communications on Pure and Applied Mathematics, pp. 729-788, Vol. LXII*, 2009.

[3] A. Hoffman, H.J. Hupkes, E.S. Van Vleck. Multi-Dimensional Stability of Waves Travelling through Rectangular Lattices in Rational Directions *arXiv preprint, arXiv:1209.6416*, 2014.

[4] A. N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov. A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem *Bjul. Moskovskogo Gos. Univ. 1:7, 1 26*, 1937.

[5] T. Kapitula. Multidimensional Stability of Planar Travelling Waves *AMS, Vol. 349 (No. 1),pp. 257-269*, 1997.

[6] P. C. Fife, J. B. McLeod. The Approach of Solutions of Nonlinear Diffusion Equations to Travelling Front Solutions *Archive for Rational Mechanics and Analysis, Vol. 65 (Issue 4), pp. 333-361*, 1977.

[7] J. X. Xin. Multidimensional Stability of Travelling Waves in a Bistable-Reaction-Diffusion Equation *I. Communications in Partial Differential Equations, Vol. 17, pp. 1889-1900*, 1992.

[8] J. W. Cahn, J. Mallet-Paret, E. S. Van Vleck. Travelling Wave Solutions for Systems of ODEs on a Two-dimensional Spatial Lattice *SIAM Journal Applied Mathematics, Vol. 59 (No.2), pp. 455-493*, 1998.

[9] Shui-Nee Chow, J. Mallet-Paret, Wenxian Shen. Travelling Waves in Lattice Dynamical Systems *Journal of Partial Differential Equations, Vol. 149, pp. 248-291*, 1998.

[10] D. G. Aronson, H. F. Weinberger. Multidimensional Nonlinear Diffusion Arising in Population Genetics *Advances in Mathematics, Vol. 30(1), pp. 33-76*, 1978.

[11] P. W. Bates, Fengxin Chen. Spectral Analysis and multidimensional Stability of travelling waves for nonlocal Allen-Cahn equations *Journal Mathematical Analysis Applications, Vol. 273, pp. 45-57*, 2002.

[12] B. V. Pal'tsev. Two-sided Bounds Uniform in the Real Argument and the Index for Modified Bessel Functions *Mathematical Notes, Vol. 65 (No.5), pp. 681-692*, 1999.

[13] A. Hoffman, J. Mallet-Paret. Universality of Chrystallographic Pinning *arXiv:0811.0093v1*, 2008.

[14] J. Mallet-Paret. Chrystallographic Pinning: Direction Dependent Pinning in Lattice Differential Equations *Lefschetz Center for Dynamical Systems and Center for Control Sciences, Divison of Applied Mathematics, Brown University, pp. 92-137*, 2001.

[15] M. Abramowitz, I. A. Stegun (Eds). Modified Bessel Functions I and K *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, pp. 374-377 9th printing New York: Dover*, 1972.

[16] E. Oran Brigham. Discrete Fourier Transform Properties *The Fast Fourier Transform, pp. 91-109, Prentice Hall, New Jersey*, 1974.

[17] H. J. Hupkes, S. M. Verduyn Lunel. Analysis of Newton's Method to Compute Travelling Waves in Discrete Media *Journal of Dynamics and Differential Equations, Vol. 17, pp. 523-572*, 2005.

[18] J. Mallet-Paret. The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems *Journal of Dynamics and Differential Equations, Vol. 11, pp. 49-128*, 1999.