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A Brief Introduction to Riemannian Geometry and Hamilton's Ricci Flow

with a focus on examples, visuals and intuition

Bachelor Thesis

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Preface

Before you lies the thesis: "A Brief Introduction to Riemannian Geometry and Hamilton's Ricci Flow: with a focus on examples, visuals and intuition", accessible to undergraduates of mathematics with at least two years of experience. It has been written in order to fulfil the graduation requirements of the Bachelor of Mathematics at Leiden University.

The subject matter studied was chosen together with my supervisor, dr. H. J. Hupkes. His goal was to get more acquainted with the Ricci flow via explicit examples. The Ricci flow required however also a lot of preliminary knowledge of Riemannian geometry. We deliberately chose the more abstract approach of smooth and Riemannian manifolds, because those who study the Ricci flow follow this approach too. This resulted into a thesis with as purpose to make introduced concepts as intuitive as possible.

I would like to thank my supervisor for his guidance and support during this project, which most importantly includes all his good advice with regard to my two presentations. We have also invested a lot of time together trying to accomplish section 5.3, for which I am really grateful. After the summer break, we have only worked on finishing this particular concise yet difficult section. Ultimately, I want to thank those who have helped me during this period.

I hope you enjoy your reading.

Mark van den Bosch

December 7, 2018.

Contents

Introduction

The purpose of this thesis is to provide a text that both introduces the theory of Riemannian geometry and introduces Hamilton's Ricci flow, such that it also contains numerous explicit examples and visualisations. Our aim is therefore to hand over as much intuition to the reader as possible with regard to the introduced abstract concepts. In particular, an adequate development of "curvature" for example requires a lot of technical machinery, which makes it easy to lose the intuition of the underlying geometric content.

Riemannian geometry

Throughout the beginning and the middle of the 18th century, one studied curves and surfaces lying in some Euclidean space. Considering a geometric object within a bigger space is what we call an *extrinsic* point of view. This approach is favourable in the sense that it is visually intuitive. Riemann on the other hand started to develop the *intrinsic* point of view, where one cannot speak of moving outside the geometric object since it is regarded as a space on its own.

The intrinsic point of view appears to be more flexible. In general relativity for example, one studies the geometry of space-time which cannot naturally be a part of a bigger space. However, the intrinsic approach increases the technical complexity, it requires a lot of machinery and it easily becomes in particular much less visually intuitive. This approach is nonetheless required in order to work with ideas like Hamilton's Ricci flow.

One main object of study in this thesis are Riemannian manifolds. Simply put, a Riemannian manifold is some kind of smooth geometric object M, such as a sphere or torus for example, that is equipped with a Riemannian metric g (a smoothly varying choice of inner products on its tangent spaces). A Riemannian metric allows us to measure geometric quantities such as distances, angles and curvature. This results for instance into Gauss's Theorema Egregium, a fundamental result in Riemannian geometry which states that the Gaussian curvature, a way of defining curvature extrinsically, can simply be determined intrinsically.

Hamilton's Ricci flow

Richard Hamilton introduced the Ricci flow in 1982 in his paper: "Three-manifold with positive Ricci curvature", see [Ham82]. The Ricci flow was utilised to gain more insight into Thurston's Geometrisation conjecture, a generalisation of the well-known Poincaré conjecture. These two conjectures are far beyond the scope of this thesis, but in relatively simple terms it is about the classification of certain three-dimensional spaces.

The Ricci flow is a geometric evolution of Riemannian metrics on M, where one starts with some initial metric g_0 and subsequently lets it evolve by Hamilton's Ricci flow equation

$$\frac{\partial}{\partial t}g(t) = -2Ric[g(t)].$$

As we will soon discover, the Ric operator simply measures some (intrinsic) curvature, meaning that $Ric[g(t)]$ is the curvature of M equipped with Riemannian metric $g(t)$ for some time t.

Hamilton's motive to define the Ricci flow was because he wanted to have some kind of non-linear diffusion equation, like the classical heat equation, that would evolve an initial metric towards a metric that is more even. Evolving some metric based on its curvature is precisely the reason why it evolves to a more even metric. For example, when interpreting things visually, we see that the initial geometric object in the figure below evolves towards a more even geometric object, namely a sphere.

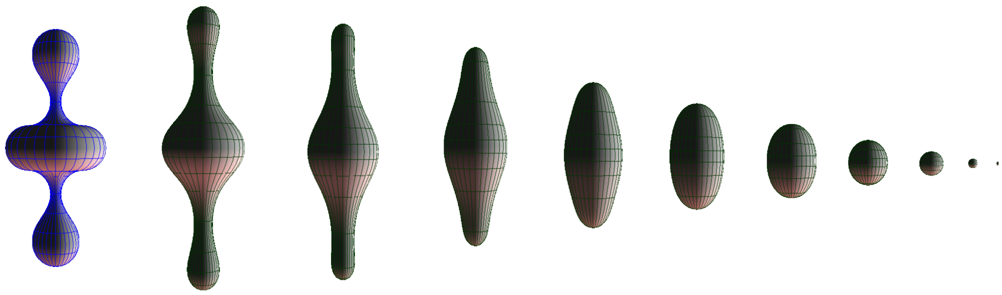

Figure 1: A visualisation of the Ricci flow of a 2-dimensional surface, see chapter 5.

For over two decades the Ricci flow was not very popular, until Grigori Perelman published a series of papers in 2002 and 2003 in which he proved Thurston's Geometrisation conjecture. Perelman essentially used the Ricci flow techniques as proposed by Hamilton together with several innovations of his own, most importantly the Ricci flow with surgery.

Outline of this thesis

In the first chapter we give an overview of preliminary notions, definitions and important results concerning smooth manifolds. The reader is not assumed to be familiar with smooth manifolds, hence we focus here a lot on the geometric interpretations of the required concepts.

In the second and third chapter we continue with Riemannian manifolds and discuss their geometric properties, most importantly curvature. We define intrinsic measures of curvature thoroughly and briefly discuss their link with Gaussian curvature and parallel transport.

In the fourth chapter we define the Ricci flow in full depth and consider three types of solutions: ancient, immortal and eternal solutions. We end this chapter with a brief discussion on short- and long-time existence and uniqueness of the Ricci flow, the link between manifold classification and the Ricci flow, and the Ricci flow with surgery.

Lastly, in the final chapter we consider the Ricci flow solution of a so-called surface of revolution. The first two sections of this chapter are fully inspired by the work of Rubinstein and Sinclair, see [RS08]. They provided us the `Ricci_rot` program, a publicly available code that visualises the Ricci flow of a surface of revolution. In the last section we give a quite detailed sketch of our proof on the short-time existence and uniqueness of the Ricci flow for a surface of revolution. In contrast to the general analysis in any textbook or paper, we tried to achieve the short-time properties without using parabolic PDE theory on manifolds.

The following link includes the original code and application (tested on a Mac OS X) as well as an executable file `Ricci_rot_windows` created by the author of this thesis in order to work with the `Ricci_rot` program on a Windows 10 computer:

`http://pub.math.leidenuniv.nl/~hupkeshj/ricci_simulations.zip`

This thesis is based on several textbooks, papers and lecture notes. The most important sources were [Lee97], [Lee13] and [Tho79] for the first three chapters, and [CK04] and [CLN06] for the last two chapters. Statements without a proof are provided with a source reference; various proofs within this thesis are from the author himself; and whenever a given proof originates from the literature, but some gaps have been filled in or adjustments have been made in order to be coherent with the rest of this thesis, we mention it as follows: *Proof.* (Based on [...]).

We also want to note that Riemannian geometry and the Ricci flow are strongly related to algebraic topology, as becomes clear by reading [Lee97] and [CK04] for example. In this thesis, we will omit all the group theory and focus on smooth geometric objects that can easily been seen as a subspace of some Euclidean space. Lastly, we will often consider two-dimensional examples in order to keep things concise and make visualising easier.

Discretisation of the Ricci flow

We like to end this introduction with a side note concerning applications of the Ricci flow besides being a tool to achieve the manifold classification. The Ricci flow became worldwide known since Perelman proved Thurston's Geometrisation conjecture in 2003. Numerous papers followed and mathematicians started to apply the Ricci flow directly in more tangible fields of mathematics. This is basically achieved by discretising Hamilton's Ricci flow and we like to refer to [ZG13, p. 59] for a complete course on the discretisation of the Ricci flow.

For instance, see [GWK+07], the discretised Ricci flow can convert all three-dimensional problems into two-dimensional domains and hence offers a general framework for surface analysis. Its applicability has been demonstrated through standard shape analysis problems, such as but not limited to three-dimensional shape matching and registration.

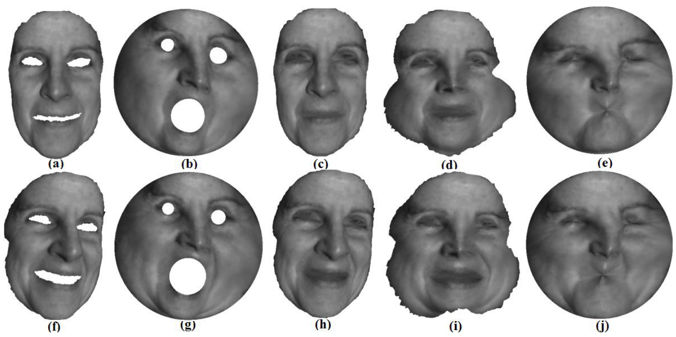

Figure 2: See [GWK+07, p. 7]. Comparison of Ricci flow with LSCM and harmonic maps. (a) and (f) are two surfaces to be registered. (b) and (g) are their Ricci flow maps. (c) and (h) are these two surfaces after hole-filling. (d) and (i) are their LSCMs. (e) and (j) are their harmonic maps.

Another application, see [WYZ+08], is within brain mapping research. In short, brain morphology in 21 patients with Williams Syndrome and 21 matched healthy control subjects with the discrete surface Ricci flow has been studied. Their results show that the discrete surface Ricci flow effectively detects group differences on cortical surfaces.

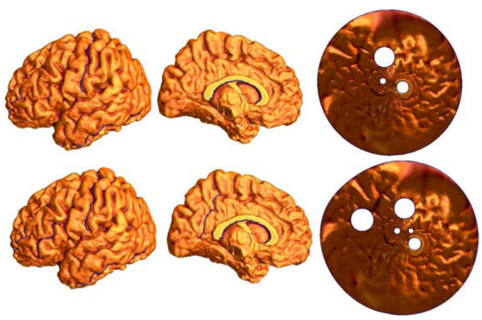

Figure 3: See [WYZ+08, p. 43]. Cortical surfaces with landmark curves and their conformal parameterization results. The first row shows a cortex with 4 landmarks and the second row shows a cortex with 7 landmarks (one landmark is not visible in this view).

Chapter 1

Preliminaries

In this chapter we will give a succinct overview of definitions and important results needed to construct Riemannian manifolds. We like to note that there are several ways to approach this construction since various concepts can be defined differently. Throughout this chapter we will focus a lot on the intuition behind definitions and their geometric interpretation in order to achieve a better understanding of the more abstract concepts and matter.

The reader is moreover assumed to be familiar with advanced calculus, linear algebra and the basics of topology. For a complete introduction on smooth manifolds we refer to either [Lee13] or [Lee00]. The latter is an early draft of [Lee13] that is publicly available.

1.1 Topological Manifolds

Let M be a topological space. To be precise, we consider a pair $(M, \mathcal{T})$ such that M is a set with a topology $\mathcal{T}$. We however omit this notation and simply write M since the topology will be clear from the context. It is often the Euclidean (subspace) topology. Consequently, when we say $S \subset M$ is a subset of M then we have to interpret S as the subspace of M with its induced subset topology. Now also recall that a homeomorphism is a continuous bijective map with a continuous inverse.

Definition 1.1. *Let M be a topological space. Then M is* **locally n-dimensional Euclidean** *if for every $p \in M$ there exits an open neighbourhood $U_p \subset M$ such that U_p is homeomorphic to an open subset of $\mathbb{R}^n$.*

Definition 1.2. *Let M be a second countable Hausdorff topological space. Then M is said to be a* **topological n-manifold** *when it is locally n-dimensional Euclidean.*

Requiring manifolds to be second countable and Hausdorff ensures us that the manifolds behave and look like $\mathbb{R}^n$ even more. Nonetheless, these technical properties are often left out by authors to generalise the concept, see for example [Sch08, p. 18].

Proposition 1.3. *Let M be a topological n-manifold. Every open subset of M is again a topological n-manifold.*

Note this proposition follows directly from the fact that topological subspaces inherit the second countable and Hausdorff properties. Very important though trivial examples of topological manifolds of dimension n are $\mathbb{R}^n$ itself and any open subset of $\mathbb{R}^n$.

Example 1.4. Continuous plane curves that do not self-intersect and circles are topological 1-manifolds. Spheres, tori, and hyperboloids are topological 2-manifolds. The n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ with $\|\cdot\|$ being the standard Euclidean norm and the graph of a continuous map $f : \mathbb{R}^n \to \mathbb{R}$ are topological n-manifolds. △

Remark 1.5. The examples given in example 1.4 can be thought as subsets of $\mathbb{R}^n$. In this thesis we will be limiting ourselves to such examples. Examples of non-Euclidean manifolds are the real projective space $\mathbb{RP}^n$ and the space of $m \times n$ matrices, see [Lee13, p. 19].

Definition 1.6. *Let M be a topological n-manifold. A* **coordinate chart** *on M is a pair (U, φ) with $U \subset M$ open and $\varphi : U \to \varphi(U)$ a homeomorphism onto an open subset of $\mathbb{R}^n$. Such a map φ is called a* **local coordinate map** *and its inverse φ^{-1} is a* **local parametrisation** *of M. The component functions of φ are called* **local coordinates** *on U.*

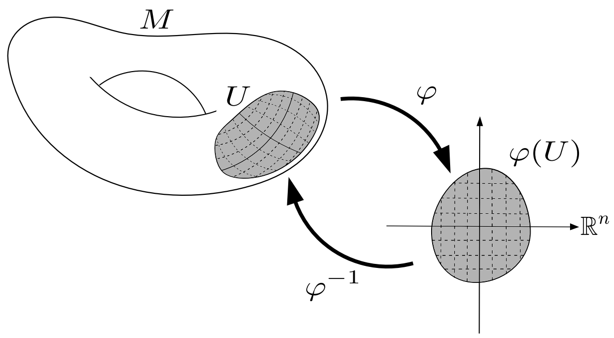

Figure 1.1: A visualisation of definition 1.6.

Local coordinates on U are often denoted as $(x^1, ..., x^n)$, where $x^i : U \to \mathbb{R}$ is the i-th component function of φ. Although local coordinates constitute formally a map from U to $\mathbb{R}^n$, it is more common to identify a point $p \in U$ with its coordinate representation in $\mathbb{R}^n$. Therefore, we will say $p = (x^1, ..., x^n)$ in local coordinates while actually we have to say $\widehat{p} = (x^1(p), ..., x^n(p))$ is a local coordinate representation of p.

Example 1.7. Consider $R = \mathbb{R}^2 \backslash \{(x, 0) : x \leq 0\}$ which is an open subset of $\mathbb{R}^2$ and therefore a topological 2-manifold. Note that (R, ψ) with $\psi : R \to R,\ (x, y) \mapsto (x, y)$ and (R, φ) with $\varphi : R \to \mathbb{R}_{>0} \times (-\pi, \pi),\ (x, y) \mapsto (\sqrt{x^2 + y^2}, \operatorname{arctan2}(y, x))$ are both coordinate charts.[1] The components functions (x^1, x^2) of ψ are called the **standard coordinates** on R and the **polar coordinates** (r, θ) are the components functions of φ. Note that the following well-known implicit formulae hold:

$$x^1 = r\cos\theta \quad \text{and} \quad x^2 = r\sin\theta. \tag{1.1}$$

For any $p \in R$ we now would say $p = (x^1, x^2)$ in standard coordinates or $p = (r, \theta)$ in polar coordinates. Ultimately, we deliberately looked at R instead of $\tilde{R} = \mathbb{R}^2 \backslash \{(0, 0)\}$ for example because $\psi(\tilde{R}) = \mathbb{R}_{>0} \times (-\pi, \pi]$ is not an open subset of $\mathbb{R}^2$. △

Since topological manifolds are locally Euclidean by definition, it follows immediately that each point $p \in M$ is contained in the domain of some chart (U, φ). This implies the existence of charts. Moreover, any topological manifold can be written as a union of chart domains. A collection of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_i$ such that the U_i cover M is called an **atlas**. Commonly, one defines a topological manifold to be a topological space that is able to admit an atlas, see for example [Sch08, p. 18]. Note this is equivalent with definition 1.2.

1.2 Smooth Manifolds

In simple terminology, smooth manifolds are objects that locally look like some Euclidean space and on which one can do calculus. With smooth we mean infinitely differentiable, thus for a

[1] The function $\operatorname{arctan2}(y, x)$ returns the angle of $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ within the interval $(-\pi, \pi]$.

map $f : U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open all the partial derivatives of f need to exist and need to be continuous on U. When f is smooth we use the common notation $f \in \mathcal{C}^\infty(U)$. In general, a function $F : U \to \mathbb{R}^k$ with $U \subset \mathbb{R}^l$ open is said to be smooth if each component function F_i of $F = (F_1, ..., F_k)$ is smooth. Our next focus will therefore be to define for a map $f : M \to \mathbb{R}^k$ what it means to be a smooth function.

Suppose M is a subspace of $\mathbb{R}^l$ for some $l > 0$. One way to achieve a smoothness criterion is to say that a map f is smooth when it can be extended to a smooth map $\tilde{f} : U \to \mathbb{R}^k$ with U open such that $\tilde{f}|_M = f$, see for example [Tho79, p. 23] and [Sch08, p. 21]. It would be a plausible approach, however there are two disadvantages. First, this cannot be done in general which follows from remark 1.5. Secondly, in order to do this we need to suppose M is surrounded by some ambient space (see also section 1.3). In later chapters this seems to be undesirable since we want to consider a manifold as a space on its own and not being part of some bigger space.

Definition 1.8. *Suppose $U \subset \mathbb{R}^l$ and $V \subset \mathbb{R}^k$ are open subsets. A map $F : U \to V$ is called a* **diffeomorphism** *if it is a bijective smooth map with a smooth inverse.*

Note that when F is a diffeomorphism, it is clearly a homeomorphism too. Since standard calculus is defined on $\mathbb{R}^k$, the following approach seems to be the most logical. In what will follow, M is assumed to be a topological n-manifold unless otherwise specified.

Definition 1.9. *Two charts (U, φ) and (V, ψ) on M are said to be* **compatible** *if either the intersection $U \cap V$ is disjoint or the transition map*

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \to \psi(U \cap V) \tag{1.2}$$

is a diffeomorphism.

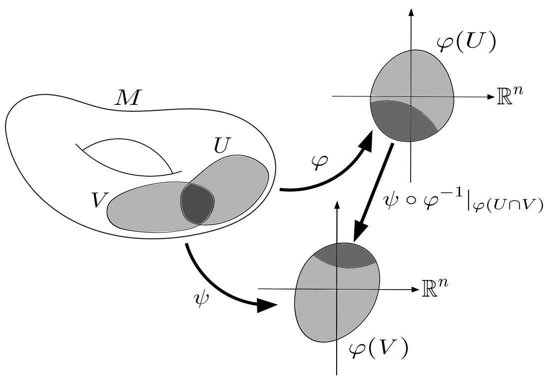

Figure 1.2: A visualisation of definition 1.9.

Definition 1.10. *A collection of charts $\mathcal{A}$ whose domains cover M is called a* **smooth atlas** *of M if and only if any two charts in the atlas $\mathcal{A}$ are compatible. Moreover, the atlas $\mathcal{A}$ is said to be* **complete** *whenever it is not contained in any other smooth atlas.*

Definition 1.11. *The pair $(M, \mathcal{A})$ is a* **smooth n-manifold** *whenever $\mathcal{A}$ is a complete atlas of M. Charts in the complete atlas $\mathcal{A}$ are called* **smooth***.*

Hence one might say that a complete atlas $\mathcal{A}$ gives M a **smooth structure**. Finding a complete atlas is difficult, but the following proposition shows how to induce such structure.

Proposition 1.12. [Lee13, p. 13] *Suppose $\mathcal{A}$ is a smooth atlas of M. Then there exists a unique complete atlas $\mathcal{A}'$ containing $\mathcal{A}$. We say $\mathcal{A}'$ is the complete atlas* **determined** *by $\mathcal{A}$.*

In conclusion, if there exists a smooth atlas then M can be extended to a smooth manifold. We must note that a smooth atlas might not exist and therefore M will not attain a smooth structure. Also M may admit different smooth structures. But luckily both cases do not occur for dimensions $n < 4$, see for example [Lee13, p. 40].

Example 1.13. We will show S^2 is a topological 2-manifold and that it can be extended to a smooth manifold. Recall $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Define the open sets:

$$S^2_+ = \{(x, y, z) \in S^2 : z > 0\} \quad \text{and} \quad B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

We call S^2_+ the **northern hemisphere**. Now note that the projection map

$$\varphi : S^2_+ \to B^2,\ (x, y, z) \mapsto (x, y) \tag{1.3}$$

is a homeomorphism since we have

$$\varphi^{-1} : B^2 \to S^2_+,\ (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2}) \tag{1.4}$$

and both φ and φ^{-1} are clearly continuous. Hence (S^2_+, φ) is a chart of S^2.

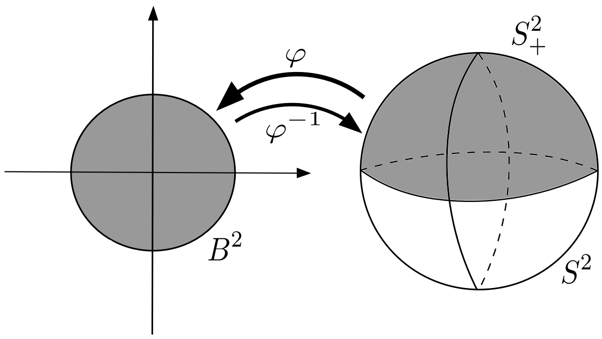

Figure 1.3: A visualisation of the given chart (S^2_+, φ).

Similarly, we can consider the **southern hemisphere** chart:

$$S^2_- = \{(x, y, z) \in S^2 : z < 0\}, \qquad \psi : S^2_- \to B^2,\ (x, y, z) \mapsto (x, y);$$

and the remaining four charts needed to cover the 2-dimensional sphere:

$$\begin{aligned}
U_+ &= \{(x, y, z) \in S^2 : x > 0\}, &\quad \pi_1 &: U_+ \to B^2,\ (x, y, z) \mapsto (y, z);\\
U_- &= \{(x, y, z) \in S^2 : x < 0\}, &\quad \pi_2 &: U_- \to B^2,\ (x, y, z) \mapsto (y, z);\\
V_+ &= \{(x, y, z) \in S^2 : y > 0\}, &\quad \pi_3 &: V_+ \to B^2,\ (x, y, z) \mapsto (x, z);\\
V_- &= \{(x, y, z) \in S^2 : y < 0\}, &\quad \pi_4 &: V_- \to B^2,\ (x, y, z) \mapsto (x, z).
\end{aligned}$$

Now let $\mathcal{A}$ be the collection of the six charts which is an atlas for S^2 and therefore the 2-dimensional sphere is a topological 2-manifold. Furthermore, $\mathcal{A}$ is a smooth atlas which can be easily verified. For example, consider the functions

$$(\pi_2 \circ \varphi^{-1})(x, y) = (y, \sqrt{1 - x^2 - y^2})$$

and

$$(\pi_2 \circ \varphi^{-1})^{-1}(y, z) = (\varphi \circ \pi_2^{-1})(y, z) = (\sqrt{1 - y^2 - z^2}, y)$$

which are clearly smooth when restricted to $\varphi(S^2_+ \cap U_-)$ and $\pi_2(S^2_+ \cap U_-)$ respectively, hence the maps φ and π_2 are compatible. According to proposition 1.12, the atlas $\mathcal{A}$ determines a smooth structure, which is said to be the **standard smooth structure** on S^2. △

Example 1.14. Let U be any open subset of $\mathbb{R}^n$. The chart (U, φ) with φ the identity map already defines an atlas $\{(U, \varphi)\}$ for U. It is automatically a smooth atlas, since φ is compatible with itself. Thus $\{(U, \varphi)\}$ determines a smooth structure on U. In general, every open subset V of a smooth manifold $(M, \mathcal{A})$ is a topological manifold due to proposition 1.3 which can be extended smoothly by considering the atlas $\mathcal{A}|_V = \{(U, \varphi) \in \mathcal{A} : U \subset V\}$. Then the smooth manifold $(V, \mathcal{A}|_V)$ is said to be an **open submanifold** of $(M, \mathcal{A})$. △

Generally, all examples listed in 1.4 can be extended to smooth manifolds with an analogous approach. The notion of a complete atlas and a smooth structure is extremely important, since this enables us to define smoothness on functions $f : M \to \mathbb{R}^k$ in an acceptable manner.

Definition 1.15. *Let $(M, \mathcal{A})$ be a smooth n-manifold. A map $f : M \to \mathbb{R}^k$ is* **smooth** *if for any smooth chart $(U, \varphi) \in \mathcal{A}$ the composition $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth.*

For $k = 1$ we usually denote $f \in \mathcal{C}^\infty(M)$. Checking whether a map f is smooth would be a lot of work if not impossible. It appears however that looking at a smooth atlas is sufficient.

Lemma 1.16. *Suppose $\mathcal{A} = \{(U_i, \varphi_i)\}_i$ is a smooth atlas of M. Then a map $f : M \to \mathbb{R}^k$ is smooth if and only if for all i the map $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}^k$ is smooth.*

Proof. (Based on [Lee00, p. 24]) Recall proposition 1.12 and let (U, φ) be any smooth chart from the complete atlas determined by $\mathcal{A}$. It suffices to show that $f \circ \varphi^{-1}|_{N_{\varphi(p)}}$ is smooth for some open neighbourhood $N_{\varphi(p)}$ of $\varphi(p)$ for any $p \in U$. Note there is a chart (U_j, φ_j) in the atlas $\mathcal{A}$ whose domain contains p. The charts (U, φ) and (U_j, φ_j) are compatible, hence the composition

$$f \circ \varphi^{-1}\big|_{N_{\varphi(p)}} = (f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi^{-1})\big|_{N_{\varphi(p)}} : N_{\varphi(p)} \to \mathbb{R}^k$$

is smooth because the maps $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}^k$ are smooth by assumption. Since this holds for all $p \in U$, we conclude that the map $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth. □

Suppose we have a smooth atlas $\mathcal{A}$ on M. Then by proposition 1.12 we know $\mathcal{A}$ determines the smooth structure on M. Furthermore, due to lemma 1.16, in order to check whether a map $f : M \to \mathbb{R}^k$ is smooth, one only considers the charts in $\mathcal{A}$. In practice, a smooth atlas is often clear from the context and thus one says M is a smooth manifold and omits the pair notation.

Remark 1.17. According to the lemma above, it seems to be somewhat redundant to define a complete atlas because one smooth atlas is sufficient. Nonetheless, defining $f : M \to \mathbb{R}^k$ to be smooth if it is smooth for some smooth atlas may seem as an atlas dependent definition.

Recall the discussion at the beginning of this section. According to [Tho79, p. 128], when M is an n-surface in $\mathbb{R}^{n+1}$ (see section 1.3) then definition 1.15 is equivalent with saying that the function f is smooth if some extended map $\tilde{f}$ is smooth. Additionally, a similar result is true in [Sch08, p. 22]. Hence this is another indication that it is a good definition. Note that this definition can be generalised to smooth maps between smooth manifolds.

Definition 1.18. *Let M and N be smooth manifolds. A map $F : M \to N$ is called* **smooth** *if for any smooth chart (U, φ) on M and any smooth chart (V, ψ) on N, the coordinate representation $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ of F is smooth.*

Conform to the previous lemma it suffices once again to show the smoothness for two specific smooth atlases $\mathcal{A}$ and $\mathcal{B}$ on M and N respectively. This enables us to see that definition 1.15 is a special case of the above, namely take $N = \mathbb{R}^k$ and the atlas defined in 1.14. Note that definition 1.18 makes it possible to speak about diffeomorphisms between smooth manifolds.

1.3 Submanifolds of Euclidean Space

As stated in remark 1.5, in our examples we will only be considering manifolds that can be thought as a subset of some Euclidean space. To be precise, in our examples we focus on what we call n-surfaces and more generally submanifolds of Euclidean space. For a much more abstract approach, see [Lee13, p. 98].

Definition 1.19. *Let $S \subset \mathbb{R}^{n+1}$ be a topological space. Then S is said to be an* n**-surface** *if there exists a map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $S = f^{-1}(0)$ and $\nabla f(p) \neq 0$ for all $p \in S$.*

Example 1.20. Consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R},\ x \to \|x\|^2 - 1$ with $\|\cdot\|$ being the standard Euclidean norm. Note the equality $S^n = f^{-1}(0)$. We have $\nabla f(p) = 2p \neq 0$ for all $p \in S^n$ hence S^n is by definition an n-surface. △

The definition of n-surfaces is used throughout the entire book [Tho79]. Note that we have the following generalisation of definition 1.19.

Definition 1.21. *Let $k \geq 0$ and $M \subset \mathbb{R}^{n+k}$ be a topological space such that for every $p \in M$ there exists an open subset $U_p \subset \mathbb{R}^{n+k}$ with $p \in U_p$ and a smooth map*

$$f = (f_1, ..., f_n) : U_p \to \mathbb{R}^k \tag{1.5}$$

with Jacobian Jf so that the following properties hold:

$$f^{-1}(0) = M \cap U_p \quad \textit{and} \quad \operatorname{rank} Jf(p) = k. \tag{1.6}$$

Then M is called an n-dimensional **submanifold of Euclidean space** *and $\mathbb{R}^{n+k}$ is said to be its* **ambient space**.

Thus n-surfaces are submanifolds of Euclidean space of dimension n with $\mathbb{R}^{n+1}$ as its ambient space. Note that for $k = 0$ we have $\mathbb{R}^0 = \{0\}$, hence every open set of $\mathbb{R}^n$ is an n-dimensional submanifold of Euclidean space.

Also, when a submanifold M has $\mathbb{R}^n$ as ambient space, then $\mathbb{R}^{n+l}$ for all $l \geq 0$ is an ambient space too for M when seen as a subset of $\mathbb{R}^{n+l}$. Let us illustrate this in the following example.

Example 1.22. Consider the topological space $M = \{(0, y, z) \in \mathbb{R}^3 : y^2 + 4z^2 = 1\}$. It is an ellipse in a 3-dimensional space, on the yz-plane to be precise. Let f be the following smooth function:

$$f : \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, y^2 + 4z^2 - 1).$$

Then we have

$$Jf(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2y & 8z \end{pmatrix}$$

which has clearly rank 2 for all $(x, y, z) \in M$. Moreover, we have $f^{-1}(0, 0) = M$. Therefore the ellipse is an 1-dimensional submanifold of Euclidean space in its ambient space $\mathbb{R}^3$ which is usually called a **space curve**. △

Definition 1.23. *Let M be a smooth n-manifold. A smooth map $F : M \to \mathbb{R}^{n+k}$ is said to be an* **embedding** *if it is a topological embedding, a homeomorphism onto its image, and for any smooth chart φ we have $\operatorname{rank} J(F \circ \varphi^{-1})(p) = n$ for all $p \in U$.*

Theorem 1.24. [And01, p. 18] *Let M be a subset of $\mathbb{R}^{n+k}$. Then M is an n-dimensional submanifold of Euclidean space if and only if M is a topological n-manifold which can be extended smoothly in such a way that the inclusion map $\iota : M \hookrightarrow \mathbb{R}^{n+k}$ is an embedding.*

We like to note that this proof relies highly on the Inverse Function Theorem. For a detailed proof of the Inverse Function Theorem, see for example [Lee13, p. 657].

Definition 1.25. *A smooth manifold $M \subset \mathbb{R}^k$ is an* **embedded submanifold** *of $\mathbb{R}^k$ when the inclusion map $\iota : M \hookrightarrow \mathbb{R}^k$ is an embedding.*

Example 1.26. Let $f : (a, b) \to \mathbb{R}$ be a continuous function and consider its graph:

$$M = \{(x, f(x)) : x \in (a, b)\}.$$

Note that both $\varphi : M \to (a, b),\ (x, y) \mapsto x$ and $\varphi^{-1} : (a, b) \to M,\ x \mapsto (x, f(x))$ are clearly continuous and each others inverse. This implies $\mathcal{A} = \{(M, \varphi)\}$ is an atlas which consists of a single chart, hence a smooth atlas like in example 1.14. This means M is a smooth manifold.

When considering for example $f(x) = |x|$ it seems quite counter intuitive. This is due to the fact M is no embedded submanifold of $\mathbb{R}^2$ since the inclusion map $\iota : M \hookrightarrow \mathbb{R}^2$ is not smooth because $\iota \circ \varphi^{-1} : (a, b) \to \mathbb{R}^2,\ x \to (x, |x|)$ is clearly not a smooth function. $\triangle$

From the previous examples, we deduce that the most intuitive (smooth) manifolds in $\mathbb{R}^n$ are the submanifolds of Euclidean space and especially n-surfaces. Thanks to theorem 1.24, we are able to forget its ambient space, which is favourable in chapters later on because we want to consider a manifold as a space on its own and not being part of some bigger space.

1.4 Tangent Vectors

First suppose M is an n-dimensional embedded submanifold of $\mathbb{R}^k$ for some $k \geq n$. Then one usually defines the tangent space of M at p to be the following set:

$$T_pM = \big\{\gamma'(0) : \text{a path } \gamma : (-\varepsilon, \varepsilon) \to M \text{ with } \gamma \in \mathcal{C}^\infty\big((-\varepsilon, \varepsilon)\big) \text{ and } \gamma(0) = p\big\} \subset \mathbb{R}^k. \tag{1.7}$$

Although this approach is favourable from a geometric point of view since a tangent vector, an element of T_pM, is in fact tangent to M, it requires M to be a subset of $\mathbb{R}^k$. Also note this interpretation needs an ambient space and it is an $\mathbb{R}$-vector space of dimension n. For arbitrary smooth (abstract) manifolds another definition is required.

There are various ways to define tangent vectors: one may generalise the above via equivalence classes, see [vdV12, p. 33]; one says a tangent vector is a quantity satisfying a specific transformation law, see [Ros03, p. 38]; one introduces derivations, see [Lee13, p. 54]; etcetera. In this thesis we consider derivations. Introducing this abstract concept of tangent vectors appears to have several advantages like decreasing the difficulty of definitions and proofs later on. For what will follow, let M be a smooth n-manifold unless otherwise specified.

Definition 1.27. *A linear map $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ is said to be a* **derivation** *of M at p when it satisfies the Leibniz condition*

$$X(fg) = f(p)Xg + g(p)Xf \tag{1.8}$$

for all $f, g \in \mathcal{C}^\infty(M)$. The **tangent space** *of M at p, denoted by T_pM, is the set of all derivations of M at p. An element of T_pM is also called a* **tangent vector** *of M at p.*

Clearly a tangent space is an $\mathbb{R}$-vector space. Furthermore note that the Leibniz condition is some kind of product rule, hence an essential example of a derivation is the **directional derivative** of a function along a smooth path. Particularly, let $\gamma : I \to M$ be some smooth path with the property $\gamma(0) = p$. Then the map X acts as

$$X(f) = \frac{d}{dt}\Big|_{t=0}(f \circ \gamma) \tag{1.9}$$

for all $f \in \mathcal{C}^\infty(M)$ defines a derivation at p. This follows quite directly by noting the useful equality $fg \circ \gamma = (f \circ \gamma)(g \circ \gamma)$. The converse is actually true as well.

Theorem 1.28. [And01, p. 25] *Every derivation of M at p can be written as the directional derivative of a function along some smooth path through p at time zero.*

Now suppose again $M \subset \mathbb{R}^k$ is an embedded submanifold and let $X \in T_pM$. Then the inclusion map $\iota : M \hookrightarrow \mathbb{R}^k$ is smooth by definition hence the **projection maps** $\pi^j : M \to \mathbb{R}$ are also smooth because π^j is the j-th component of ι. Thanks to the theorem above, we observe:

$$X(\pi^j) = \frac{d}{dt}\Big|_{t=0}(\pi^j \circ \gamma) = \frac{d}{dt}\Big|_{t=0}(\gamma^j) = (\gamma^j)'(0) \tag{1.10}$$

with $\gamma = (\gamma^1, ..., \gamma^k)$ a corresponding smooth path on M through p at time zero.

Hence we can interpret a derivation X geometrically as a real vector $X \in \mathbb{R}^k$, since we have

$$X(\iota) := \big(X(\pi^1), ..., X(\pi^k)\big) = \big((\gamma^1)'(0), ..., (\gamma^k)'(0)\big) = \gamma'(0) \tag{1.11}$$

which moreover coincides with geometric approach (1.7). Note that the above gives us a natural isomorphism $\eta : T_pM \to \eta(T_pM),\ X \mapsto X(\iota)$ with $\eta(T_pM) \subset \mathbb{R}^k$ an $\mathbb{R}$-vector space.

Definition 1.29. *Let $p \in U$ with $U \subset \mathbb{R}^n$ an open submanifold and consider the standard coordinates $(x^1, ..., x^n)$ on U. Then the* **standard coordinate vectors** *of U at p are*

$$\frac{\partial}{\partial x^1}\Big|_p, ..., \frac{\partial}{\partial x^n}\Big|_p \tag{1.12}$$

which act on any $f \in \mathcal{C}^\infty(U)$ as just a partial derivative, that is $\frac{\partial}{\partial x^i}\big|_p f = \frac{\partial f}{\partial x^i}(p)$.

Taking partial derivatives of a product satisfies the Leibniz rule obviously, hence $\frac{\partial}{\partial x^i}\big|_p$ are derivations of U at p. The standard coordinate vectors are also linearly independent:

$$\sum_{i=1}^n \lambda^i \frac{\partial}{\partial x^i}\Big|_p f = 0$$

for all $f \in \mathcal{C}^\infty(U)$. This clearly implies $\lambda^i = 0$ for any i because we can take f to be simply the projection maps. Moreover, the standard coordinate vectors of U at p form a basis for T_pU because with the chain rule one get for any $X \in T_pU$ the following:

$$X(f) = \frac{d}{dt}\Big|_{t=0}(f \circ \gamma) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(0)) \cdot (\gamma^i)'(0) = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\Big|_p f$$

with $\gamma = (\gamma^1, ..., \gamma^k)$ a corresponding smooth path on U with $\gamma(0) = p$ and $X^i = (\gamma^i)'(0)$. In short, for $X \in T_pU$ we have $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}|_p$ for particular $X^i \in \mathbb{R}$. More specific, note that one has $x^j \in \mathcal{C}^\infty(U)$ and therefore $X(x^j) = \big(X^i \frac{\partial}{\partial x^i}\big|_p\big)(x^j) = X^i \frac{\partial x^j}{\partial x^i}\big|_p = X^j$.

Remark 1.30. (Einstein's summation convention) Whenever in any term the same index name appears twice, as both an upper and a lower index, that term is summed over all possible values of that index (often from 1 to n) unless stated otherwise. For example, writing the expression $X = X^i \frac{\partial}{\partial x^i}|_p$ is a shorthand notation for $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}|_p$. Note that we regard an upper index in the denominator as a lower index. Throughout this thesis, we will be using the summation convention.

Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$, then we can also write:

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{d}{dt}\Big|_{t=0} f(p + te_i). \tag{1.13}$$

Thus there is a natural identification, as mentioned in (1.11), between $\frac{\partial}{\partial x^i}|_p$ and e_i. From this observation, we conclude the isomorphic relation $T_pU \cong \mathbb{R}^n$ for any $U \subset \mathbb{R}^n$ open and $p \in U$.

Our next goal is to relate a tangent space T_pM with some **Euclidean tangent space**, which is a tangent space of an open subset of $\mathbb{R}^n$ as discussed above. This will be achieved by "pushing" tangent vectors "forward" from one manifold to another.

Definition 1.31. *Let M and N be smooth manifolds and $F : M \to N$ a smooth map. For each $p \in M$ the map $F_* : T_pM \to T_{F(p)}N$ is defined by*

$$F_*X(f) = X(f \circ F) \tag{1.14}$$

for all $f \in \mathcal{C}^\infty(N)$ and is said to be the **pushforward** *of F at p.*

Note that F_*X is indeed a derivation at $F(p)$ whenever X is derivation at p because the Leibniz condition is satisfied, that is:

$$(F_*X)(fg) = f(F(p))(F_*X)(g) + g(F(p))(F_*X)(f).$$

Also note that the notation F_* does not show the dependence of a particular point $p \in M$ since it should be clear from the context which pushforward we are dealing with.

Example 1.32. Consider the identity map $\mathbf{1}_M : M \to M$ which is clearly smooth. Then the pushforward of $\mathbf{1}_M$ at $p \in M$ is given by the identity map

$$\mathbf{1}_{T_pM} : T_pM \to T_pM,$$

since $(\mathbf{1}_M)_*X(f) = X(f \circ \mathbf{1}_M) = X(f)$ for any $X \in T_pM$ and all $f \in \mathcal{C}^\infty(M)$. △

Lemma 1.33. *Let M and N be smooth manifolds, $p \in M$ and $F : M \to N$ a diffeomorphism. Then the map $F_* : T_pM \to T_{F(p)}N$ is an isomorphism between vector spaces.*

Proof. It is clear that F_* is a linear map. Furthermore, F is a diffeomorphism hence there is a smooth map $G : N \to M$ such that we have $G \circ F = \mathbf{1}_M$ and $F \circ G = \mathbf{1}_N$. Now consider the pushforward $G_* : T_{F(p)}N \to T_pM$ of G at $F(p)$. Then by example 1.32 we have:

$$G_* \circ F_* = (G \circ F)_* = (\mathbf{1}_M)_* = \mathbf{1}_{T_pM} \quad \text{and} \quad F_* \circ G_* = (F \circ G)_* = (\mathbf{1}_N)_* = \mathbf{1}_{T_{F(p)}N}.$$

Hence G_* is the inverse of F_* and therefore it is an isomorphism. □

Now let (U, φ) be a smooth chart on M. Then we naturally extend U and $\varphi(U)$ to be the open submanifolds of M and $\mathbb{R}^n$ respectively. In particular $\varphi : U \to \varphi(U)$ is a diffeomorphism due to example 1.14. Consequently, according to lemma 1.33, the pushforward of φ^{-1} at p is an isomorphism. Lastly recall that in general isomorphisms send a basis to a basis.

Definition 1.34. *Let $(U, \varphi = (x^1, ..., x^n))$ be a smooth chart on M. Then the* **coordinate vectors** *of U at p associated with the given local coordinates are defined by:*

$$\frac{\partial}{\partial x^i}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}. \tag{1.15}$$

Together these n derivations of T_pU form a basis of T_pU.

Note that one uses the same notation for two different objects with on the right hand side the standard coordinate vectors of $\varphi(U)$ at $\varphi(p)$. This appears to be abuse of notation, however if we have (U, φ) with $U \subset \mathbb{R}^n$ open and φ the identity map, as in example 1.14, then the following holds according to example 1.32: $\frac{\partial}{\partial x^i}|_p = (\mathbf{1}_U^{-1})_* \frac{\partial}{\partial x^i}|_{\varphi(p)} = \frac{\partial}{\partial x^i}|_p$.

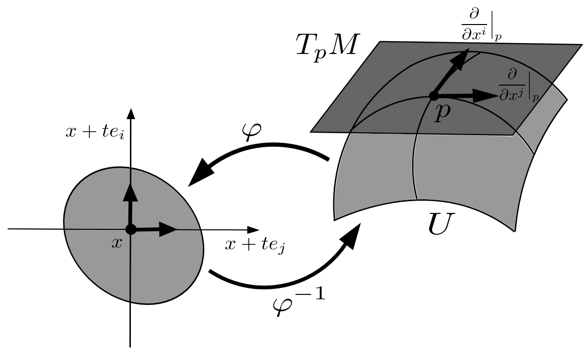

Figure 1.4: A visualisation of the geometric interpretation of definition 1.34. Note $x = \varphi(p)$.

Coordinate vectors act on smooth functions $f \in \mathcal{C}^\infty(U)$ as:

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} (f \circ \varphi^{-1}). \tag{1.16}$$

In other words, $\frac{\partial}{\partial x^i}|_p\, f$ is the i-th partial derivative of the coordinate representation $f \circ \varphi^{-1}$ of the map f at the coordinate representation $\varphi(p)$ of p. Again, when $X \in T_pU$ is known, we can write $X = X^i \frac{\partial}{\partial x^i}\big|_p$ with $X(x^j) = X^i \frac{\partial}{\partial x^i}\big|_p x^j = X^j$.

Now let $M \subset \mathbb{R}^k$ be an embedded submanifold again and recall (1.13). Then one can naturally identify $\frac{\partial}{\partial x^i}|_p$, as mentioned in (1.11), with the following real vector:

$$\frac{\partial}{\partial x^i}\Big|_p \iota = \left(\frac{\partial}{\partial x^i}\Big|_x (\pi^j \circ \varphi^{-1})\right)_j = \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(x + te_i) = J\varphi^{-1}(x)e_i \tag{1.17}$$

where $x = \varphi(p)$ as in figure 1.4. Visually, a smooth path $t \to x + te_i$ on $\mathbb{R}^n$ is being transported via φ^{-1} thus we get the map $t \to \varphi^{-1}(x + te_i)$ which is a smooth path on M. The best linear approximation of the last path gives us the tangent vector of M at p.

The above is also an explanation why the inclusion map is required to be an embedding, see definition 1.23 and theorem 1.24. The Jacobian of φ^{-1} needs to have rank n to ensure the fact that we have an n-dimensional tangent space whenever the dimension of the manifold is n.

Remark 1.35. For U an open submanifold of M we have $T_pU \cong T_pM$ for any $p \in U$ which is intuitively and geometrically speaking quite evident. The main idea is that one can extend any map $f \in \mathcal{C}^\infty(U)$ to an arbitrary smooth map $\widetilde{f} \in \mathcal{C}^\infty(M)$ such that $f = \widetilde{f}$ in a small neighbourhood of p. See [Lee13, p. 56] for a proof without using theorem 1.28. This means we can interpret elements of T_pU as elements of T_pM as done in figure 1.4.

Example 1.36. Note that one can write $S^2 = f^{-1}(0)$ with $f(x, y, z) = x^2 + y^2 + z^2 - 1$ and most importantly $Jf(x, y, z) = (2x, 2y, 2z) \neq 0$ for all $(x, y, z) \in S^2$ thus the 2-sphere is an embedded submanifold of $\mathbb{R}^3$ by theorem 1.24. Recall example 1.13 and note S^2_+ and B^2 are open submanifolds of S^2 and $\mathbb{R}^2$ respectively. Thus the following maps are diffeomorphisms:

$$\varphi : S^2_+ \to B^2,\ (x, y, z) \mapsto (x, y) \quad \text{and} \quad \varphi^{-1} : B^2 \to S^2_+,\ (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2}).$$

We consider the smooth chart $(S^2_+, \varphi = (x^1, x^2))$ and thus the coordinate vectors of S^2_+ at p, associated with the given local coordinates, are given by

$$\frac{\partial}{\partial x^1}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^1}\Big|_{\varphi(p)} \quad \text{and} \quad \frac{\partial}{\partial x^2}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^2}\Big|_{\varphi(p)}.$$

These two derivations form a basis for $T_pS^2_+$ and subsequently also for T_pS^2 due to the remark above. These derivations act on functions $f \in \mathcal{C}^\infty(S^2_+)$ as described in (1.16) and moreover, by writing $p = (x, y, z)$ the derivations can be geometrically interpreted as the real vectors

$$\frac{\partial}{\partial x^1}\Big|_p \iota = \left(1, 0, \frac{-x}{\sqrt{1 - x^2 - y^2}}\right)^\top \quad \text{and} \quad \frac{\partial}{\partial x^2}\Big|_p \iota = \left(0, 1, \frac{-y}{\sqrt{1 - x^2 - y^2}}\right)^\top \tag{1.18}$$

which follows from (1.17) and by calculating the Jacobian $J\varphi^{-1}(x, y)$. △

Ultimately, when two smooth charts $(U, \varphi = (x^1, ..., x^n))$ and $(V, \varphi = (y^1, ..., y^n))$ on M overlap in the region $U \cap V$ then a tangent vector at $p \in U \cap V$ can be written as a linear combination of both $(\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p)$ and $(\frac{\partial}{\partial y^1}|_p, ..., \frac{\partial}{\partial y^n}|_p)$. A significant question may be: how are these two different bases of coordinate vectors related to one another?

Lemma 1.37. (Change of Coordinates) *Let $(U, \varphi = (x^1, ..., x^n))$ and $(V, \psi = (y^1, ..., y^n))$ be two smooth charts on M with $U \cap V \neq \varnothing$. Then the coordinate vectors at $p \in U \cap V$ associated with the local coordinates $(x^1, ..., x^n)$ and $(y^1, ..., y^n)$ are related by the following transformation rule:*

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\Big|_p \tag{1.19}$$

where one writes the y^j in terms of the coordinate functions x^i for the expression $\frac{\partial y^j}{\partial x^i}$.

The proof can be found on page 14 and note that (1.19) is abbreviated according to Einstein's summation convention. Furthermore, as briefly mentioned in the beginning of this section, one may define a tangent vector at p as a quantity which relative to a coordinate chart around p is represented by an n-tuple. In addition, this definition requires that two n-tuples representing the same tangent vector relative to two different charts have to be related by a very similar transformation rule as seen above. This approach can be found in for example [Ros03, p.38]. Assuming such transformation rules was very common during the early days of differential geometry, as stated in [Lee00, p. 52], and in a lot of physics literature it still is. See [Car97] and [FG17] for example. Now consider the change of coordinates in practice.

Example 1.38. Recall example 1.7 and let $p \in R$ be arbitrarily given. Note that R is an open submanifold of $\mathbb{R}^2$ thus $(R, \psi = (x^1, x^2))$ is a smooth chart. A little verification shows that the chart $(R, \varphi = (r, \theta))$ is compatible with (R, ψ) and is therefore also contained in the atlas of R. Writing x^1 and x^2 in terms of the coordinate functions r and θ gives the formulae (1.1), hence we have the following relations:

$$\frac{\partial}{\partial r}\Big|_p = \frac{\partial x^1}{\partial r}(\varphi(p))\frac{\partial}{\partial x^1}\Big|_p + \frac{\partial x^2}{\partial r}(\varphi(p))\frac{\partial}{\partial x^2}\Big|_p = \cos\theta(p)\frac{\partial}{\partial x^1}\Big|_p + \sin\theta(p)\frac{\partial}{\partial x^2}\Big|_p \tag{1.20}$$

and similarly:

$$\frac{\partial}{\partial \theta}\Big|_p = -r(p)\sin\theta(p)\frac{\partial}{\partial x^1}\Big|_p + r(p)\cos\theta(p)\frac{\partial}{\partial x^2}\Big|_p. \tag{1.21}$$

Now suppose we have the tangent vector $X = \frac{\partial}{\partial x^1}|_p - \frac{\partial}{\partial x^2}|_p$ at $p = (2, \frac{\pi}{4})$ in polar coordinates. Then by the above, we have $\frac{\partial}{\partial r}|_p = \frac{1}{2}\sqrt{2}\frac{\partial}{\partial x^1}|_p + \frac{1}{2}\sqrt{2}\frac{\partial}{\partial x^2}|_p$ and $\frac{\partial}{\partial \theta}|_p = -\sqrt{2}\frac{\partial}{\partial x^1}|_p + \sqrt{2}\frac{\partial}{\partial x^2}|_p$. Consequently we can write $X = -\frac{1}{2}\sqrt{2}\frac{\partial}{\partial \theta}|_p$ in terms of the polar coordinates. △

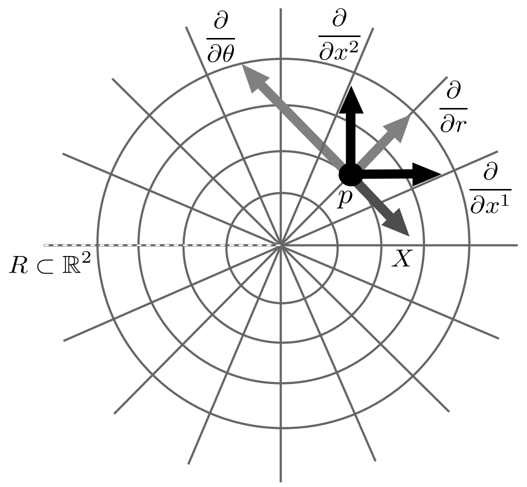

Figure 1.5: A visualisation of example 1.38 with the geometric interpretation of derivations.

Now let $F : U \to V$ be a smooth map with $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open and $p \in U$ arbitrary and denote $(x^1, ..., x^n)$ and $(\tilde{x}^1, ..., \tilde{x}^m)$ for the standard coordinates on U and V respectively. Consider the pushforward $F_* : T_pU \to T_{F(p)}V$ and by the chain rule we have the following:

$$\left(F_*\frac{\partial}{\partial x^i}\Big|_p\right)(f) = \frac{\partial(f\circ F)}{\partial x^i}(p) = \frac{\partial f}{\partial \tilde{x}^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial \tilde{x}^j}\Big|_{F(p)}\right)(f)$$

for all $f \in \mathcal{C}^\infty(U)$, hence we have

$$F_*\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial \tilde{x}^j}\Big|_{F(p)}. \tag{1.22}$$

Note again the presence of the Einstein summation convention. Moreover, the matrix of F_* in terms of the standard coordinate vector bases $(\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p)$ and $(\frac{\partial}{\partial y^1}|_p, ..., \frac{\partial}{\partial y^n}|_p)$ is simply the Jacobian $JF(p)$. This result comes in handy in the proof below.

Proof of lemma 1.37. (Based on [Lee13, p. 63]) By writing the y^j in terms of the coordinate functions x^i, we actually introduce the notation

$$(\psi^{-1} \circ \varphi)(x) = \big(y^1(x), ..., y^n(x)\big)$$

for $x \in \varphi(U \cap V)$. By definition of coordinate vectors and by using formula (1.22), we have:

$$\frac{\partial}{\partial x^i}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} = (\psi^{-1} \circ (\psi \circ \varphi^{-1}))_* \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} = (\psi^{-1})_* \left((\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} \right) =$$

$$(\psi^{-1})_* \left(\frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\Big|_{\psi(p)} \right) = \frac{\partial y^j}{\partial x^i}(\varphi(p))(\psi^{-1})_* \frac{\partial}{\partial y^j}\Big|_{\psi(p)} = \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\Big|_p.$$

Note that we have used the fact that a pushforward is a linear map. □

Lastly, for a smooth map $F : M \to N$ with M and N smooth manifolds the pushforward matrix representation can be easily generalised. For charts (U, φ) on M containing p and (V, ψ) on N containing $F(p)$, an analogous computation gives that the matrix of $F_* : T_pM \to T_{F(p)}N$ (note remark 1.35) in terms of the coordinate vector bases is precisely the Jacobian of the coordinate representation of F at p, in other words the Jacobian $J(\psi \circ F \circ \varphi^{-1})(p)$.

Now recall the previous definition of an embedding, see definition 1.23. Intuitively we want an embedding to be a topological embedding such that the tangent space structure of a smooth manifold will not be lost during the embed process.

Definition 1.39. *A smooth map $F : M \to N$ is said to be an* **immersion** *if the pushforward $F_* : T_pM \to T_{F(p)}N$ is injective for all $p \in M$. The map F is called an* **embedding** *if it is an immersion and a topological embedding.*

Note that the above is a coordinate independent definition, which becomes useful in the next chapters. In particular, a smooth map $F : M \to \mathbb{R}^{n+k}$ with M a smooth n-manifold is an immersion if and only if for any smooth chart (U, φ) we have rank $J(F \circ \varphi^{-1})(p) = n$ for all points $p \in U$. Thus definition 1.23 is indeed a special case of the above.

1.5 Vector Fields

Let M be an n-dimensional smooth manifold. Before we are able to construct vector fields we will first have to define the notion of a tangent bundle.

Definition 1.40. *The* **tangent bundle** *of M, denoted by TM, is the disjoint union of tangent spaces T_pM at all points $p \in M$. That is: $TM = \{(p, X) : p \in M$ and $X \in T_pM\}$.*

An important fact is that the tangent bundle TM has a natural topology and smooth structure for what it becomes a smooth $2n$-manifold. For any smooth chart $(U, \varphi = (x^1, ..., x^n))$ on M, one defines an open set $\widetilde{U} = \{(p, X) \in TM : p \in U\}$ and a map:

$$\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^{2n}, \ \left(p, X^i \tfrac{\partial}{\partial x^i}\big|_p\right) \mapsto (\varphi(p), X^1, ..., X^n). \tag{1.23}$$

Then the collection $\widetilde{\mathcal{A}}$ of all coordinate charts $(\widetilde{U}, \widetilde{\varphi})$ is a smooth atlas and hence it determines a smooth structure on TM which we call the **natural smooth structure** of TM. For more details, see [Lee13, p. 66].

Example 1.41. Recall example 1.36. Then we have:

$$TS^2_+ = \left\{(p, X) : p \in S^2_+ \text{ and } X = X^1 \tfrac{\partial}{\partial x^1}\big|_p + X^2 \tfrac{\partial}{\partial x^2}\big|_p \text{ for some } X^1, X^2 \in \mathbb{R}\right\}.$$

By interpreting $TS^2_+ \subset TS^2$, which can be done because $T_pS^2_+ \cong T_pS^2$ holds for all $p \in S^2_+$ according to remark 1.35, we could determine other tangent bundles like the tangent bundle of the southern hemisphere TS^2_- in order to construct TS^2 as a whole. △

Definition 1.42. *Let $U \subset M$ be open. A* **(tangent) vector field** *of M defined on U is a map $X : U \to TM$ such that for each point $p \in U$ we have $X|_p \in T_pM$.*

Note that we write $X|_p$ instead of $X(p)$ to avoid conflict with the notation for tangent vectors acting on smooth function. Formally speaking we should also write $X|_p \in \{p\} \times T_pM \subset TM$, but it is more usual to consider the notation as in the definition. Therefore we will often omit the pair notation (p, X) and write $X|_p \in T_pM$ and $X|_p \in TM$ interchangeably because the dependence of p is clear from the context and notation.

Let $(x^1, ..., x^n)$ be any local coordinates on some open subset $U \subset M$ and consider some global vector field $X : M \to TM$. Then we can express it locally as

$$X|_p = X^i(p) \frac{\partial}{\partial x^i}\Big|_p \tag{1.24}$$

for all $p \in U$ and specific functions $X^i : U \to \mathbb{R}$ that we call the **component functions** of X with respect to the given coordinates. Furthermore, since TM is a smooth $2n$-manifold one can speak of smooth vector fields. A vector field X is said to be **smooth** if it is a smooth map between smooth manifolds. We write $\mathcal{T}M$ for the set of all smooth vector fields defined on M.

Lemma 1.43. *Let $X : M \to TM$ be a vector field. Then X is smooth if and only if for any smooth chart (U, φ) (of a smooth atlas) the component functions $X^1, ..., X^n$ are smooth (as in definition 1.15 with the subset U interpreted as the smooth open submanifold of M).*

Proof. (Based on [Lee00, p. 61]) Now recall lemma 1.16 or actually the analogous variant for definition 1.18. In order to show that the vector field X is smooth, it suffices to look at all the charts of M (in some smooth atlas of M) and all the charts contained in $\widetilde{\mathcal{A}}$ of TM which is the collection of smooth charts defined in the beginning of this section.

Let (U, φ) and (V, φ) be arbitrary smooth charts on M and consider $(\widetilde{V}, \widetilde{\psi}) \in \widetilde{\mathcal{A}}$. Observe:

$$\begin{aligned} \widetilde{\psi} \circ X \circ \varphi^{-1} &: \varphi(U \cap X^{-1}(\widetilde{V})) \to \widetilde{\psi}(\widetilde{V}), \\ x &\mapsto \big((\psi \circ \varphi^{-1})(x), (X^1 \circ \varphi^{-1})(x), ..., (X^n \circ \varphi^{-1})(x)\big). \end{aligned}$$

Note $x \mapsto (\psi \circ \varphi^{-1})(x)$ is clearly smooth on its domain because the charts (U, φ) and (V, ψ) are compatible. Now assume the vector field X is smooth. Then $\widetilde{\psi} \circ X \circ \varphi^{-1}$ and consequently all $X^i \circ \varphi^{-1}$ are smooth for any smooth charts (U, φ) and $(\widetilde{V}, \widetilde{\psi})$. Hence by definition all the component functions X^i are smooth.

By assuming the contrary we have by definition that all $X^i \circ \varphi^{-1}$ are smooth hence $\widetilde{\psi} \circ X \circ \varphi^{-1}$ are smooth for all charts (U, φ) on M and $(V, \psi) \in \widetilde{\mathcal{A}}$. Consequently X is smooth. ☐

Corollary 1.44. Let (U, φ) be any smooth chart of M and interpret U as the open submanifold of M. Then we have that $X : U \to TM$ is a smooth vector field if and only if the component functions $X^1, ..., X^n$ with respect to (U, φ) are smooth.

Definition 1.45. *Let $(x^1, ..., x^n)$ be local coordinates on an open set $U \subset M$, then*

$$\frac{\partial}{\partial x^i} : U \to TM, \; p \mapsto \frac{\partial}{\partial x^i}\Big|_p \tag{1.25}$$

determines a smooth vector field on U said to be the i-th **coordinate vector field**.

Note that the coordinate vector fields are clearly smooth due to corollary 1.44 since the j-th component functions with $j \neq i$ are all identically zero and the i-th component function is constant since it equals 1 everywhere hence all component functions are smooth.

In general we will define objects globally, thus a smooth vector field from M to TM for example. In practice however, as we will see in the next chapters, one looks at these objects locally for some smooth chart since this enables computations and explicit examples. Importantly note that some smooth vector field of M defined on an open subset U cannot always be extended smoothly, consider for example $(0,1) \subset \mathbb{R}$ and the smooth vector field

$$X : (0,1) \to T\mathbb{R},\ p \mapsto p^{-1}(1-p)^{-1} \frac{\partial}{\partial x^1}\Big|_p.$$

Note that X is smooth because its component function $x \to x^{-1}(1-x)^{-1}$ is smooth thanks to corollary 1.44 since it is smooth with respect to the standard coordinate chart.

Example 1.46. Recall example 1.38. Then the coordinates vector fields on R associated with the polar coordinates (r, θ) in terms of the standard coordinates (x^1, x^2) are given by:

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2} \quad \text{and} \quad \frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x^1} + r\cos\theta \frac{\partial}{\partial x^2}. \tag{1.26}$$

Here one needs to interpret r and θ within the component functions of the coordinate vector fields as the coordinate maps from $R \to \mathbb{R}_{>0}$ and $R \to (-\pi, \pi)$ respectively as discussed in example 1.7.

Note that both the standard and polar coordinate vector fields are smooth which follows from the definition. Moreover, the smoothness of the polar coordinate vector fields can also be determined by looking at (1.26) and by noting that its component functions are smooth for the coordinate chart $(R, \psi = (x^1, x^2))$. Hence $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ are smooth because of corollary 1.44. △

Thanks to the observations within the example above, we see that the following lemma is an immediate consequence of lemma 1.37. An useful note is that when (U, φ) is a smooth chart of M, then we have that $(U', \varphi|_{U'})$ with $U' \subset U$ is a smooth chart of M as well.

Lemma 1.47. (Change of Coordinates) *Let $(U, \varphi = (x^1, ..., x^n))$ and $(V, \psi = (y^1, ..., y^n))$ be two smooth charts on M such that $U \cap V \neq \varnothing$. Then the coordinate vector fields on the open submanifold $U \cap V$ associated with the local coordinates $(x^1, ..., x^n)$ and $(y^1, ..., y^n)$ are related by the following transformation rule:*

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}. \tag{1.27}$$

Moreover the component functions $\frac{\partial y^j}{\partial x^i} : U \cap V \to \mathbb{R},\ p \mapsto \frac{\partial y^j}{\partial x^i}(\varphi(p))$ are smooth.

Note that the lemma above is an easy tool to perform a change of coordinates for general smooth vector fields, which is useful in later chapters. Also we now know that changing coordinates preserves the smoothness of the vector field which of course one would like to require.

1.6 Covector and Tensor Fields

Recall for some arbitrary vector space V we have a dual vector space, denoted by V^*, which is the space of linear maps $F : V \to \mathcal{F}$ with the set $\mathcal{F}$ being a field. We consider $\mathcal{F} = \mathbb{R}$ and our V will be finite dimensional. Two significant results are: $\dim(V) = \dim(V^*)$ and when the collection $(E_1, ..., E_n)$ is a basis of V then V^* has a unique basis $(\varepsilon^1, ..., \varepsilon^n)$ called the dual basis such that the relation $\varepsilon^j(E_i) = \delta_i^j$ holds with δ_i^j being the Kronecker delta.

Definition 1.48. *The n-dimensional* **cotangent space** *T_p^*M of M at p is the dual space of the tangent space T_pM and an element of T_p^*M is called a* **covector** *of M at p. The* **cotangent bundle** *of M, denoted by T^*M, is the disjoint union of cotangent spaces T_p^*M at all $p \in M$. In other words: $T^*M = \{(p, \omega) : p \in M \text{ and } \omega \in T_p^*M\}$.*

Again one can give T^*M a natural $2n$-dimensional smooth manifold structure similar to what we have done briefly in the previous section. For a detailed approach, see [Lee13, p. 276]. We will omit the pair notation again and write $\omega|_p \in T_pM$ and $\omega|_p \in T^*M$ interchangeably.

Definition 1.49. *Let $U \subset M$ be an open submanifold. A* **covector field** *of M defined on U is a map $\omega : U \to T^*M$ such that $\omega|_p \in T_p^*M$ for each point $p \in U$.*

Definition 1.50. *Let $(x^1, ..., x^n)$ be local coordinates on an open set $U \subset M$. Then the unique map $dx^j : U \to T^*M$ satisfying the properties $dx^j|_p \in T_p^*M$ and $dx^j|_p(\frac{\partial}{\partial x^i}|_p) = \delta_i^j$ for all points $p \in U$ is said to be the j-th* **coordinate covector field**.

Now let $(x^1, ..., x^n)$ be any local coordinates on some open subset $U \subset M$ and consider a global covector field $\omega : M \to T^*M$. Then we can express it locally as

$$\omega|_p = \omega_i(p)dx^i\big|_p \quad \text{or} \quad \omega = \omega_i dx^i \tag{1.28}$$

for all $p \in U$ and functions $\omega_i : U \to \mathbb{R}$ called the **component functions** of ω with respect to the given coordinates. An important observation is that we have $\omega_i(p) = \omega|_p(\frac{\partial}{\partial x^i}|_p)$. Similarly, a covector field ω is said to be **smooth** if it is a smooth map between smooth manifolds and we denote $\mathcal{T}_0^1 M$ as the set of all smooth covector fields defined on M. Completely analogous to the previous section, we get the following results upon giving the cotangent bundle T^*M its natural smooth structure.

Lemma 1.51. [Lee13, p. 278] *Let $\omega : M \to T^*M$ be a covector field. Then ω is smooth if and only if for any smooth chart (of a smooth atlas) the component functions $\omega_1, ..., \omega_n$ are smooth.*

Corollary 1.52. Let (U, φ) be any smooth chart of M. Then $\omega : U \to T^*M$ is a smooth covector field if and only if the component functions $\omega_1, ..., \omega_n$ with respect to (U, φ) are smooth.

Due to the corollary, one concludes that the coordinate covector fields associated to some local coordinates are always smooth because the component functions are all constant.

Definition 1.53. *A* **local frame** *for TM on an open set $U \subset M$ is a collection $(E_1, ..., E_n)$ of smooth vector fields defined on U such that $(E_1|_p, ..., E_n|_p)$ is a basis for T_pM for every $p \in U$. The corresponding* **local coframe** *for T^*M is the collection $(\varepsilon^1, ..., \varepsilon^n)$ that consists of the unique smooth covector fields defined on U satisfying $\varepsilon^i(E_j) = \delta_j^i$. Often the map ε^i is simply called the* **dual field** *of E_i.*

Example 1.54. Recall example 1.46. Let $X : \mathbb{R}^2 \to T\mathbb{R}^2$ be an arbitrary smooth vector field. Then $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$ and $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ are two local frames and note one can locally write X as

$$X = X^1\frac{\partial}{\partial r} + X^2\frac{\partial}{\partial \theta}$$

with $X^1, X^2 : R \to \mathbb{R}$ smooth component functions. Substituting the relations from (1.26) into the above gives the following:

$$X = (X^1\cos\theta - X^2 r\sin\theta)\frac{\partial}{\partial x^1} + (X^1\sin\theta - X^2 r\cos\theta)\frac{\partial}{\partial x^2}.$$

Consequently, since dx^1 is the dual field of $\frac{\partial}{\partial x^1}$ we have for all $p \in R$ the expression

$$dx^1|_p(X|_p) = X^1(p)\cos\theta(p) - X^2(p)r(p)\sin\theta(p) = \cos\theta(p)dr(X|_p) - r\sin\theta(p)d\theta(X|_p)$$

because dr and $d\theta$ are the dual fields of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ respectively. Thus we have

$$dx^1 = \cos\theta dr - r\sin\theta d\theta, \tag{1.29}$$

and conform to the above we can show the equality

$$dx^2 = \sin\theta dr + r\cos\theta d\theta. \tag{1.30}$$

Now the above shows the relation between the two local coframes (dx^1, dx^2) and $(dr, d\theta)$. $\triangle$

As we have done in the previous example, one can do the exact same computations for a general case. Thus the following lemma is a generalisation of the example above.

Lemma 1.55. (Change of Coordinates) *Let $(U, \varphi = (x^1, ..., x^n))$ and $(V, \psi = (y^1, ..., y^n))$ be two smooth charts on M such that $U \cap V \neq \varnothing$. Then the coordinate covector fields on the open submanifold $U \cap V$ associated with the local coordinates $(x^1, ..., x^n)$ and $(y^1, ..., y^n)$ are related by the following transformation rule:*

$$dy^i = \frac{\partial y^j}{\partial x^i} dx^j. \tag{1.31}$$

Moreover the component functions $\frac{\partial y^j}{\partial x^i} : U \cap V \to \mathbb{R},\ p \mapsto \frac{\partial y^j}{\partial x^i}(\varphi(p))$ are smooth.

Importantly note that there is a subtle difference between the transformation rule (1.27) of vector fields and the transformation rule (1.31) of covector fields. Ultimately, our goal is to generalise the concept of covector fields. Consider a general n-dimensional vector space V.

Definition 1.56. *A* **tensor** *on V of type $\binom{k}{l}$ is a multilinear map*

$$F : \underbrace{V \times ... \times V}_{k \text{ times}} \times \underbrace{V^* \times ... \times V^*}_{l \text{ times}} \to \mathbb{R}. \tag{1.32}$$

The set of all tensors on V of type $\binom{k}{l}$ is denoted by $T^k_l(V)$.

Note that $T^k_l(V)$ is an $\mathbb{R}$-linear vector space. Also we have $T^1_0(V) = V^*$ and $T^0_1(V) = V^{**} \cong V$ and by convention we have $T^0_0(V) = \mathbb{R}$.

Definition 1.57. *Let $F \in T^k_l(V)$ and $G \in T^p_q(V)$. The* **tensor product** *of F and G, denoted by $F \otimes G \in T^{k+p}_{l+q}(V)$, is defined by the natural product:*

$$\begin{gathered} F \otimes G(X_1, ..., X_{k+p}, \omega^1, ..., \omega^{l+q}) = \\ F(X_1, ..., X_k, \omega^1, ..., \omega^l) \cdot G(X_{k+1}, ..., X_{k+p}, \omega^{l+1}, ..., \omega^{l+q}). \end{gathered} \tag{1.33}$$

Clearly we have that the tensor product is an associative operator, thus we define:

$$F \otimes G \otimes H = (F \otimes G) \otimes H = F \otimes (G \otimes H).$$

For an arbitrary amount of products its definition is of course very much alike.

Proposition 1.58. *Let $(E_1, ..., E_n)$ be a basis for V and $(\varepsilon^1, ..., \varepsilon^n)$ its corresponding dual basis. Then the collection*

$$\mathcal{B} = \left\{\varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} \otimes E_{j_1} \otimes ... \otimes E_{j_l}\right\}_{1 \le i_1, ..., i_k, j_1, ... j_l \le n}$$

is a basis for $T^k_l(V)$.

Proof. (Based on [Lee97, p. 13]) Consider an arbitrary multilinear map $F \in T^k_l(V)$ and define its components:

$$F^{j_1 ... j_l}_{i_1 ... i_k} = F(\varepsilon^{i_1}, ..., \varepsilon^{i_k}, E_{j_1}, ..., E_{j_l}). \tag{1.34}$$

Furthermore let $1 \le s_1, ..., s_l, t_1, ..., t_k \le n$ and note that we have

$$\begin{gathered} F^{j_1 ... j_l}_{i_1 ... i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} \otimes E_{j_1} \otimes ... \otimes E_{j_l}(E_{t_1}, ..., E_{t_l}, \varepsilon^{s_1}, ..., \varepsilon^{s_k}) = \\ F^{j_1 ... j_l}_{i_1 ... i_k} \varepsilon^{i_1}(E_{t_1}) ... \varepsilon^{i_k}(E_{t_k}) E_{j_1}(\varepsilon^{s_1}) ... E_{j_l}(\varepsilon^{s_l}) = F^{j_1 ... j_l}_{i_1 ... i_k} \delta^{i_1}_{t_1} ... \delta^{i_k}_{t_k} \delta^{s_1}_{j_1} ... \delta^{s_l}_{j_l} = F^{s_1 ... s_l}_{t_1 ... t_k}. \end{gathered}$$

Importantly note the presence of Einstein's summation convention and the fact $V^{**} \cong V$. A tensor is by definition multilinear, therefore one can write

$$F = F^{j_1 ... j_l}_{i_1 ... i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} \otimes E_{j_1} \otimes ... \otimes E_{j_l}. \tag{1.35}$$

Due to the fact F was arbitrary it follows that $\mathcal{B}$ spans $T^k_l(V)$. Now assume

$$\lambda^{j_1 ... j_l}_{i_1 ... i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} \otimes E_{j_1} \otimes ... \otimes E_{j_l} = 0.$$

With the exact same computation as above, we deduce every $\lambda^{j_1 ... j_l}_{i_1 ... i_k}$ equals zero. Hence the elements in $\mathcal{B}$ are linearly independent which implies $\mathcal{B}$ is a basis for $T^k_l(V)$. $\square$

We note that the two equations (1.34) and (1.35) are of major importance in our future study. Suppose M and N are a smooth n- and m-manifold respectively until the end of this section and recall that the tangent space T_pM is an n-dimensional $\mathbb{R}$-linear vector space.

Definition 1.59. *The $\binom{k}{l}$* **tensor bundle** *of M, denoted by T^k_lM, is the disjoint union of the spaces $T^k_l(T_pM)$ at all points $p \in M$. That is: $T^k_lM = \{(p, F) : p \in M \text{ and } F \in T^k_l(T_pM)\}$.*

Definition 1.60. *A map $\sigma : U \to T^k_lM$ with $U \subset M$ an open subset is called a* **tensor field** *of type $\binom{k}{l}$ defined on U whenever we have $\sigma|_p \in T^k_l(T_pM)$ for all $p \in U$.*

Once more we will often omit the pair notation and write $\sigma|_p \in T^l_k(T_pM)$ and $\sigma|_p \in T^k_lM$ interchangeably since its p dependence is clear from the context. Now let $(x^1, ..., x^n)$ be any local coordinates on some open subset $U \subset M$. According to the results above, we note that a tensor field $\sigma : M \to T^k_lM$ can be expressed locally on U as

$$\sigma = \sigma^{j_1...j_l}_{i_1...i_k} dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_l}} \tag{1.36}$$

where $\sigma^{j_1...j_l}_{i_1...i_k} : U \to \mathbb{R}$ are the **component functions** of σ with respect to the given local coordinates. Keep in mind we use the Einstein summation convention. Moreover, as the equations (1.34) and (1.35) already suggests, we will use the shorthand notation

$$\sigma^{j_1...j_l}_{i_1...i_k} = \sigma\left(\tfrac{\partial}{\partial x^{j_1}}, ..., \tfrac{\partial}{\partial x^{j_l}}, dx^{i_1}, ..., dx^{i_k}\right) \tag{1.37}$$

because

$$\sigma^{j_1...j_l}_{i_1...i_k} : U \to \mathbb{R},\ p \to \sigma|_p \left(\tfrac{\partial}{\partial x^{j_1}}\Big|_p, ..., \tfrac{\partial}{\partial x^{j_l}}\Big|_p, dx^{i_1}|_p, ..., dx^{i_k}|_p\right) \tag{1.38}$$

is what we have due to the multilinearity of $\sigma|_p$ for all $p \in M$. A tensor field is said to be **smooth** if it is a smooth map between smooth manifolds however it will suffice again to only look at the component functions, see lemma 1.61 and its corollary. Note that a tensor field of type $\binom{1}{0}$ is precisely a covector field hence the notation $\mathcal{T}^1_0M$ is self-explanatory. In general we denote $\mathcal{T}^k_lM$ as the set of all smooth $\binom{k}{l}$ tensor fields defined on M.

Lemma 1.61. [Lee97, p. 19] *Let $\sigma : M \to T^k_lM$ be a tensor field. Then σ is smooth if and only if for any smooth chart (of a smooth atlas) its component functions $\sigma^{j_1...j_l}_{i_1...i_k}$ are smooth.*

Corollary 1.62. Let (U, φ) be any smooth chart of M. Then $\sigma : U \to T^k_lM$ is a smooth tensor field if and only if the component functions $\sigma^{j_1...j_l}_{i_1...i_k}$ with respect to (U, φ) are smooth.

The proof is totally conform to that of the vector and covector fields and it requires the natural structure of tensor bundles. See the general notion of vector bundles [Lee13, p. 249] for more detail on the natural smooth structures.

Ultimately, recall the definition of a pushforward. Now we will define a similar concept, namely a pullback of smooth tensor fields. Intuitively it "pulls back" a smooth tensor field on N to a smooth tensor field on M.

Definition 1.63. *Let $F : M \to N$ be a smooth map and let σ be a $\binom{k}{0}$ tensor field. Then $F^*\sigma$ is the $\binom{k}{0}$ tensor field on M, called the* **pullback** *of σ by F, defined by*

$$(F^*\sigma)\big|_p(X_1, ..., X_k) = \sigma\big|_{F(p)}(F_*X_1, ..., F_*X_k) \tag{1.39}$$

for any vectors $X_1, ..., X_k \in T_pM$ and all $p \in M$.

We note that whenever σ is smooth, then the pullback $F^*\sigma$ is smooth. This will be deduced from the following proposition and lemma and is not really that evident.

Proposition 1.64. *Let $F : M \to N$ be a smooth map. Suppose $(x^1, ..., x^n)$ are local coordinates on $U \subset M$ and $(y^1, ..., y^m)$ local coordinates on $V \subset N$ such that $F(U) = V$. Then we have*

$$F^* dy^i = \frac{\partial}{\partial x^j}\Big|_p (y^i \circ F)\, dx^j. \tag{1.40}$$

Proof. As (1.36) suggests, we can write $F^* dy^i = \omega_i dx^i$ with ω_i the component functions of the covector field $F^* dy^i$. By unravelling the definitions, we find for all $p \in U$ the expression:

$$\omega_i(p) = (F^* dy^i)\big|_p \big(\tfrac{\partial}{\partial x^j}\big|_p\big) = dy^j\big|_{F(p)} (F_* \tfrac{\partial}{\partial x^j}\big|_p) = dy^j\big|_{F(p)} \Big(\big(F_* \tfrac{\partial}{\partial x^j}\big|_p\big)(y^i) \tfrac{\partial}{\partial y^i}\big|_{F(p)}\Big) =$$

$$\big(F_* \tfrac{\partial}{\partial x^j}\big|_p\big)(y^i) \cdot dy^j\big|_{F(p)} \big(\tfrac{\partial}{\partial y^i}\big|_{F(p)}\big) = F_* \big(\tfrac{\partial}{\partial x^j}\big|_p\big)(y^j) = \tfrac{\partial}{\partial x^j}\big|_p (y^i \circ F).$$

We used the linearity of the coordinate covectors and $F_* \frac{\partial}{\partial x^j}|_p = \big(F_* \frac{\partial}{\partial x^j}|_p\big)(y^i) \frac{\partial}{\partial y^i}|_{F(p)}$ holds, which follows from the observation stated below equation (1.16). Ultimately, the above implies that equation (1.40) holds. □

Lastly, we introduce the notation dF^i which is a shorthand notation for $\frac{\partial}{\partial x^j}|_p (y^i \circ F)\, dx^j$. The reason for this has some historical background which will be discussed in the next chapter and moreover it is conform to the concept of a differential of a function, see [Lee13, p. 280].

Furthermore, note that the pullback operator F^* is $\mathbb{R}$-linear and that we have the following two properties: $F^*(f\sigma) = (f \circ F) F^*\sigma$ and $F^*(\sigma \otimes \tau) = F^*\sigma \otimes F^*\tau$. Consequently, with the help of proposition 1.64, one gets the following lemma.

Lemma 1.65. [Lee13, p. 320] *Let $F : M \to N$ be a smooth map. Suppose $(x^1, ..., x^n)$ are local coordinates on $U \subset M$ and $(y^1, ..., y^m)$ local coordinates on $V \subset N$ such that $F(U) = V$. Then the following formula holds:*

$$F^*(\sigma_{j_1 \dots j_k} dy^{j_1} \otimes ... \otimes dy^{j_k}) = (\sigma_{j_1 \dots j_k} \circ F) dF^{j_1} \otimes ... \otimes dF^{j_k}. \tag{1.41}$$

Moreover $F^*\sigma$ is a smooth $\binom{k}{0}$ tensor field whenever we have $\sigma \in \mathcal{T}^k_0 N$.

Note that the smoothness of the pullback map $F^*\sigma$ follows from the fact that the composition functions $\sigma_{j_1 \dots j_k} \circ F$ are smooth because σ and F are smooth. We want to highlight the fact that proposition 1.64 and lemma 1.65 are of major significance in the following chapters.

1.7 Tensor Characterisation Lemma

It is often convenient to generalise the shorthand notation in (1.37). Suppose $\sigma \in \mathcal{T}^k_l M$ and let $X_1, ..., X_k$ and $\omega^1, ..., \omega^l$ be any smooth vector and covector fields of M defined on some open set $U \subset M$. Now we define the function $\sigma(X_1, ..., X_k, \omega^1, ..., \omega^l)$ on the subset U such that

$$\sigma(X_1, ..., X_k, \omega^1, ..., \omega^l)(p) = \sigma\big|_p (X_1\big|_p, ..., X_k\big|_p, \omega^1|_p, ..., \omega^l|_p) \tag{1.42}$$

for all $p \in U$. An important observation is the following: lemma 1.61 directly implies that σ is a smooth tensor field if and only if $\sigma(X_1, ..., X_k, \omega^1, ..., \omega^l) : U \to \mathbb{R}$ is a smooth function for every smooth vector fields $X_1, ..., X_k$ and covector fields $\omega^1, ..., \omega^l$ of M defined on U.

Due to the previous section and to keep things readable, we reduce ourselves to $\binom{k}{0}$ tensor fields. Given a smooth tensor field $\sigma \in \mathcal{T}^k_0 M$ and smooth vector fields $X_1, ..., X_k \in \mathcal{T}M$, we now know that $\sigma(X_1, ..., X_k) : M \to \mathbb{R}$ is a smooth map. Thus σ induces a map

$$\Sigma : \mathcal{T}M \times ... \times \mathcal{T}M \to \mathcal{C}^\infty(M), \tag{1.43}$$

defined by $\Sigma(X_1, ..., X_k) = \sigma(X_1, ..., X_k)$ of course.

Due to the fact that $\sigma|_p$ is a tensor on T_pM for all $p \in M$, we obtain that Σ is multilinear over the field $\mathcal{C}^\infty(M)$, that is: for any functions $f, g \in \mathcal{C}^\infty(M)$ and smooth vector fields X and Y of M, we have

$$\Sigma(..., fX + gY, ...) = f\Sigma(..., X, ...) + g\Sigma(..., Y, ...). \tag{1.44}$$

As the next lemma will show, the converse statement is true as well. The next lemma also includes a similar fact which will be very useful in chapter 3.

Lemma 1.66. (Tensor Characterisation Lemma) *A map*

$$T : \mathcal{T}M \times ... \times \mathcal{T}M \to \mathcal{C}^\infty(M) \tag{1.45}$$

is induced by a $\binom{k}{0}$ *smooth tensor field* τ *if and only if it is multilinear over* $\mathcal{C}^\infty(M)$*. A map*

$$T : \mathcal{T}M \times ... \times \mathcal{T}M \to \mathcal{T}M \tag{1.46}$$

is induced by a $\binom{k}{1}$ *smooth tensor field* τ *if and only if it is multilinear over* $\mathcal{C}^\infty(M)$*.*

Proof. (Based on [DeB15]) The observation above shows that if τ is a smooth $\binom{k}{0}$ tensor field, it induces a map T as in (1.45) that is multilinear over $\mathcal{C}^\infty(M)$. This proves one implication. The converse is completely conform to the proof we are going to give for $\binom{k}{1}$ tensor fields.

Now let $\tau \in T^k_1 M$ be a smooth tensor field on M. Similar to the above we note that τ induces a particular map T as in (1.46). It satisfies

$$\tau(X_1, ..., X_k, \omega) = (\omega \circ T)(X_1, ..., X_k)$$

for any $X_1, ..., X_k \in \mathcal{T}M$ and $\omega \in \mathcal{T}^1_0 M$. Again it is straightforward to check in local coordinates that the vector field $T(X_1, ..., X_n)$ so-defined is indeed smooth, see section 1.6. Since ω is arbitrary, we must have that T is multilinear over $\mathcal{C}^\infty(M)$ too.

Note that $\tau(X_1, ..., X_k, \cdot) = T(X_1, ..., X_k)$ holds, which explains the similarity between the induction of a $\binom{k}{0}$ tensor field, see (1.43), and the induction of a $\binom{k}{1}$ tensor field. The existence of map T is non-trivial, yet it follows quite naturally from the fact that there exists an isomorphism between $T^k_1(V)$ and the space of multilinear maps

$$A : \underbrace{V \times ... \times V}_{k \text{ times}} \to V$$

when V is a finite-dimensional vector space, see for example [Lee97, p. 12]. Denote the space above by $E(V)$, we obtain that the map $\Phi : E(V) \to T^k_1(V)$ with ΦA the $\binom{k}{1}$ tensor defined by $\Phi A(X_1, ..., X_k, \omega) = (\omega \circ A)(X_1, ..., X_k)$ is precisely an isomorphism.

Now suppose the contrary, thus let $T : \mathcal{T}M \times ... \times \mathcal{T}M \to \mathcal{T}M$ be a map that is multilinear over $\mathcal{C}^\infty(M)$. Now note that the following fact is key.

Fact. Suppose $p \in M$ is arbitrary. If the equality $X_i = Y_i$ holds for each i on some open neighbourhood $U \subset M$ of p, then we have $T(X_1, ..., X_k)(p) = T(Y_1, ..., Y_k)(p)$.

Proof of Fact. Let $f \in \mathcal{C}^\infty(M)$ satisfy $f(q) = 1$ for all $q \in V$ with $V \subset U$ some open subset and $\text{supp}(f) = \overline{\{p \in M : f(p) \neq 0\}} \subset U$; the topological closure of the set $\{p \in M : f(p) \neq 0\}$ is obtained in U. Importantly note that such a function is called a smooth **bump function**. Their existence is fully discussed in for example [Lee13, p. 40] and [Tho79, p. 150]. The three valid statements below prove the fact:

$$T(fX_1, ..., fX_k)(p) = f(p)^k T(X_1, ..., X_k)(p) = T(X_1, ..., X_k)(p);$$

$$T(fY_1, ..., fY_k)(p) = f(p)^k T(Y_1, ..., Y_k)(p) = T(Y_1, ..., Y_k)(p);$$

$$T(fX_1, ..., fX_k) = T(fY_1, ..., fY_k).$$

Note that the last equation holds because fX_i is identical to fY_i on the entire manifold M. For each i we have that it agrees on U since the smooth vector fields X_i and Y_i agree there and outside U the smooth vector fields fX_i and fY_i vanish identically. $\square$

Now consider a coordinate chart (U, φ) with $\varphi = (x^1, ..., x^n)$ and let f be again a bump function satisfying the properties mentioned in the proof of the fact. For any point $q \in V$ and $i, j_1, ..., j_k \in \{1, ..., n\}$, let us define $T^i_{j_1,...,j_k}$ by

$$T\left(f\frac{\partial}{\partial x^{j_1}}, ..., f\frac{\partial}{\partial x^{j_k}}\right)(q) = T^i_{j_1,...,j_k}(q)\frac{\partial}{\partial x^i}\Big|_p.$$

The reason for multiplying the function f to the coordinate vector fields is in order to have vector fields defined on the entire manifold M (which vanish identically outside U). Additionally, the functions $T^i_{j_1,...,j_k}$ on V are smooth since T sends k-tuples of smooth vector fields to smooth vector fields, see lemma 1.43.

Doing the procedure above for each $p \in U$ gives well-defined functions $T^i_{j_1,...,j_k}$ on U. Note that this statement follows directly from the fact. Moreover, for any smooth vector fields $X_1, ..., X_k$ on M we can write $X_i = X_i^j \frac{\partial}{\partial x^j}$ and consequently for any $p \in U$ we have

$$T(X_1, ..., X_k)(p) = T(fX_1, ..., fX_k)(p) = X_1^{j_1}\big|_p \cdot ... \cdot X_k^{j_k}\big|_p \cdot T\left(f\frac{\partial}{\partial x^{j_1}}, ..., f\frac{\partial}{\partial x^{j_k}}\right)(p).$$

The first equality follows from the fact, the latter from the multilinearity of T. We now deduce from the definition of the $T^i_{j_1,...,j_k}$ that in local coordinates on U we have

$$T = T^i_{j_1,...,j_k} dx^{j_1} \otimes ... \otimes dx^{j_k} \otimes \frac{\partial}{\partial x^i}.$$

More precisely, the formula above defines a smooth tensor field $\tau|_U$ on U which is unique with the property that for any smooth vector fields $X_1, ..., X_n$ on M we have

$$\tau|_U(X_1|_U, ..., X_k|_U, \cdot) = T(X_1, ..., X_k)|_U,$$

since the coefficient functions $T^i_{j_1,...,j_s}$ are uniquely determined. The fact that $\tau|_U$ is uniquely determined by T ensures that the smooth tensor field $\tau|_{\widetilde{U}}$ analogously determined by T on a different chart domain $\widetilde{U}$ agrees with $\tau|_U$ on the overlap $U \cap \widetilde{U}$. Hence the local smooth tensor fields $\tau|_U$ patch together to determine a well-defined smooth tensor field τ on M. $\square$

We note that this lemma is a very useful tool for showing that: if a map as in (1.45) or (1.46) is multilinear over $\mathcal{C}^\infty(M)$, it gives for any smooth vector fields X_i and Y_i with $X_i|_p = Y_i|_p$ at a point $p \in M$ the same output at p. This result is used for example in proposition 3.21, one of the most significant statements in Riemannian geometry.

Ultimately, the proof uses a smooth **bump function**. It is a function $f \in \mathcal{C}^\infty(M)$ satisfying $f(q) = 1$ for all $q \in V$ with $V \subset U$ some open subset of some open subset U and there needs to hold $\text{supp}(f) = \overline{\{p \in M : f(p) \neq 0\}} \subset U$. We say that such function f has support in U.

Proving their existence is quite technical but very intuitive. A construction may be based on the smooth function

$$f : \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases} \tag{1.47}$$

Given any real numbers $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, we subsequently define the smooth function

$$h : \mathbb{R} \to \mathbb{R},\ t \to \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}, \tag{1.48}$$

which satisfies $h(t) = 1$ for $|t| \leq r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) = 0$ for $|t| \geq r_2$.

Finally, we denote the open ball with radius r centred at 0 as $B(r) = \{x \in \mathbb{R}^n : |x| < r\}$ and define the smooth function

$$H : \mathbb{R}^n \to \mathbb{R},\ x \to h(|x|). \tag{1.49}$$

Note that H is a smooth function because it is smooth on $\mathbb{R}^n \backslash \{0\}$, since it is a composition of smooth functions there, and because H is identically equal to 1 on $B(r_1)$. It satisfies $H(x) = 1$ for all $x \in \bar{B}(r_1)$, $0 < H(x) < 1$ for all $x \in B(r_2) \backslash \bar{B}(r_1)$, and $H(x) = 0$ for all $\mathbb{R}^n \backslash B(r_2)$. Thus the function H is a bump function on $\mathbb{R}^n$ with $\mathrm{supp}(H) = \bar{B}(r_2)$.

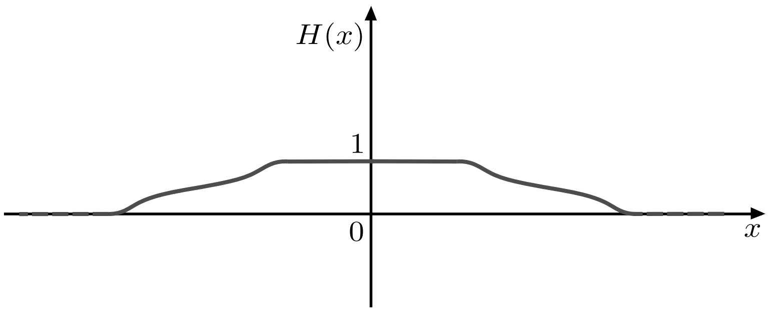

Figure 1.6: A one-dimensional visualisation of bump function H.

The construction of smooth bump functions on an arbitrary smooth manifold M is similar and requires the function H (or one of a similar kind) and the coordinate charts of M, see [Lee13, p. 44]. The existence of bump functions appears to be very useful in proofs concerning local properties, as we have seen in the proof of lemma 1.66 and as we will see in chapter 3.

Chapter 2

Riemannian Manifolds

A Riemannian metric on a smooth manifold M will define the lengths of tangent vectors and the lengths of paths. Throughout this chapter we will focus on some basic concepts of Riemannian manifolds with as purpose to show that the "curvature" of M is affected by giving M different metrics. For a complete introduction on (abstract) Riemannian manifolds we refer to [Lee97].

Also recall that we want to consider M as a space on its own and not being part of some bigger space. In [Tho79], the ambient space is always taken into account which appears to make calculations possible but difficult and limited. In particular, changing the metric on M cannot be done when the ambient space is present as well.

2.1 Riemannian Metrics

Let M be a smooth n-manifold. Riemann himself defined a **Riemannian metric** g on M as a collection of inner products $g|_p : T_pM \times T_pM \to \mathbb{R}$ for all $p \in M$ such that all the $g|_p$ vary smoothly, see [CLN06, p. 2]. Varying smoothly means that

$$g(X, Y) : M \to \mathbb{R},\ p \mapsto g|_p(X|_p, Y|_p) \tag{2.1}$$

is a smooth function for any smooth vector fields X and Y on M. This definition for g is still considered abundantly, for example in [And01] and [Tho79]. For notational simplicity, we also often write $g(X, Y)$ instead of $g|_p(X, Y)$ for any $X, Y \in T_pM$ if p is clear from the context.

Note that a Riemannian metric g is positive definite – that is $g(X, X) > 0$ if $X \neq 0$ – and symmetric – that is $g(X, Y) = g(Y, X)$ – for all $X, Y \in T_pM$ and $p \in M$ because the $g|_p$ are inner products according to the definition above.

Moreover, recall that the smoothness condition above is equivalent with saying that the component functions $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ for any local coordinate frame $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ are smooth, see section 1.7. Hence by lemma 1.61 we have the following equivalent definition.

Definition 2.1. *A* **Riemannian metric** *g on a smooth n-manifold M is a symmetric, positive definite smooth tensor field $g \in \mathcal{T}_0^2(M)$. We call the pair (M, g) a* **Riemannian manifold** *of dimension n.*

This definition can be found for example in [Lee97] and [CLN06] and is very common in the study of the Ricci flow. Now let (M, g) be a Riemannian n-manifold. Then we define, just as in Euclidean geometry, the **norm** of a tangent vector $X \in T_pM$ as:

$$|X|_g = \sqrt{g(X, X)}. \tag{2.2}$$

Similarly, the **angle** between two nonzero vectors $X, Y \in T_pM$ equals the unique $\theta \in [0, \pi]$ that satisfies the next formula:

$$\cos\theta = \frac{g(X,Y)}{|X|_g |Y|_g}. \tag{2.3}$$

Now consider any local coordinates $(x^1, ..., x^n)$ on $U \subset M$ open. As we discussed in section 1.6, we have that a metric g can be expressed locally on the subset U as

$$g = g_{ij} dx^i \otimes dx^j, \tag{2.4}$$

with $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ since it is a smooth tensor field. Note Einstein's summation convention. We will also often omit the sign $\otimes$ for practical reasons and because this is common in older and physics related literature. See again [Car97] and [FG17] for example. Hence the local expression in (2.4) simply becomes

$$g = g_{ij} dx^i dx^j. \tag{2.5}$$

Also it is useful to consider the matrix $G = (g_{ij})_{ij}$ associated to the given chart. When writing tangent vectors $X = X^i \frac{\partial}{\partial x^i}\big|_p$ and $Y = Y^i \frac{\partial}{\partial x^i}\big|_p$ of T_pM locally, one easily calculates

$$g(X,Y) = g_{ij} X^i Y^j = \begin{pmatrix} X^1 & \cdots & X^n \end{pmatrix} G(p) \begin{pmatrix} Y^1 \\ \vdots \\ Y^n \end{pmatrix} \tag{2.6}$$

via matrix multiplication. Note that the matrix G is positive definite and symmetric for any point $p \in U$. A Riemannian metric in a local frame is therefore sometimes also denoted by the matrix G associated to the same frame, see for example [Khu17, p. 7].

Example 2.2. (Euclidean space) Recall definition 1.29 and let $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ be the standard local frame for $T\mathbb{R}^n$ defined on $U \subset \mathbb{R}^n$ open. Then the pair $(U, \bar{g})$ with

$$\bar{g}(X,Y) = \bar{g}|_p\big(X^i \tfrac{\partial}{\partial x^i}\big|_p, Y^i \tfrac{\partial}{\partial x^i}\big|_p\big) = \textstyle\sum_{i=1}^n X^i Y^i \tag{2.7}$$

for any tangent vectors $X, Y \in T_p\mathbb{R}^n$ and $p \in U$ defines a Riemannian manifold. Note that the component functions $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$ with δ_{ij} the Kronecker delta are clearly smooth hence we have $\bar{g} \in \mathcal{T}_0^2 U$ due to corollary 1.62. It is moreover symmetric and positive definite, therefore $\bar{g}$ is a metric. The metric $\bar{g}$ can (locally) be written in several ways:

$$\bar{g} = \delta_{ij} dx^i \otimes dx^j = \textstyle\sum_{i=1}^n dx^i \otimes dx^i = \sum_{i=1}^n (dx^i)^2. \tag{2.8}$$

The Riemannian metric $\bar{g}$ on an open subset of $\mathbb{R}^n$ is known as the **canonical Euclidean metric** and $(\mathbb{R}^n, \bar{g})$ is said to be the **Euclidean space**. Note that the matrix $\bar{G}$ associated to the standard local frame is the identity matrix I_n and the norm of a vector $X \in T_p\mathbb{R}^n$ is precisely the Euclidean norm of the real vector $(X^1, ..., X^n)^\top$. △

Example 2.3. Recall example 1.54 and the two formulae

$$dx^1 = \cos\theta dr - r\sin\theta d\theta \quad \text{and} \quad dx^2 = \sin\theta dr + r\cos\theta d\theta.$$

Hence the canonical Euclidean metric can be expressed in terms of polar coordinates:

$$\begin{aligned} \bar{g} &= (dx^1)^2 + (dx^2)^2 = (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 \\ &= (\cos^2\theta + \sin^2\theta) dr^2 + (-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta) dr d\theta + r^2(\sin^2\theta + \cos^2\theta) d\theta^2 \\ &= dr^2 + r^2 d\theta^2. \end{aligned}$$

Note that we have $dr^2 = (dr)^2 = dr dr = dr \otimes dr$ and not $d(r^2)$. Ultimately, we have

$$\bar{g} = dr^2 + r^2 d\theta^2 \quad \text{and} \quad \bar{G}_{polar} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{2.9}$$

with $\bar{G}_{polar}$ the matrix associated to the polar coordinate frame. △

Example 2.4. Let g_1 and g_2 be two Riemannian metrics on M. A convex linear combination, thus $\lambda g_1 + (1-\lambda)g_2$ with $\lambda \in [0,1]$, defines again a Riemannian metric on M. △

In Riemannian geometry, one is mostly interested in whether two Riemannian manifolds have something in common. First we will define what it means to have conformal metrics.

Definition 2.5. *Two metrics g_1 and g_2 on a manifold M are said to be* **conformal** *to one another if and only if there exists a positive function $f \in \mathcal{C}^\infty(M)$ such that $g_2 = fg_1$.*

Example 2.6. Let (U^2, g) be the Riemannian manifold with $U^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ the open upper half plane and the metric g such that it has the following local expression

$$g = \frac{dx^2 + dy^2}{y^2} \tag{2.10}$$

in the standard coordinates (x, y) on U^2. Note that we do not write the standard coordinates as (x^1, x^2) in order to simplify the local expression of g above. Note that the norm of a tangent vector at $p = (x, y)$ goes to 0 when $y \to \infty$ and increases infinitely when $y \to 0$.

The Riemannian manifold (U^2, g) is known as the **Lobachevsky plane** and is a 2-dimensional version of the **Poincaré half-space model**. Clearly g is conformal to the canonical Euclidean metric $\bar{g}$ on U^2 since $f : U^2 \to \mathbb{R},\ (x, y) \mapsto y^{-2}$ is a smooth function due to lemma 1.16. The coordinates (x, y) are also so-called **isothermal coordinates** on (U^2, g) because we can write the given metric in these coordinates as $g = h(x, y)(dx^2 + dy^2)$ for some smooth map h.

Now introduce the local coordinates (u, v) on $V^2 = \mathbb{R}_{>0} \times \mathbb{R}_{>0} \subset U^2$ related to (x, y) as follows:

$$x = u^2 - v^2 \quad \text{and} \quad y = 2uv.$$

According to the change of coordinates lemma, see lemma 1.55, we find

$$dx = 2udu - 2vdv \quad \text{and} \quad dy = 2udv + 2vdu.$$

Hence the expression of g in terms of the local coordinates (u, v) is:

$$g = \frac{(2udu - 2vdv)^2 + (2udv + 2vdu)^2}{4u^2v^2} = \left(\frac{1}{u^2} + \frac{1}{v^2}\right)(du^2 + dv^2).$$

Note that (u, v) are isothermal coordinates too, since we can write $g = k(u, v)(du^2 + dv^2)$ with k a smooth map. In general, changing coordinates does not necessarily keep the isothermal property. The standard coordinates on $(R, \bar{g})$ are isothermal coordinates whereas the polar coordinates on $(R, \bar{g})$ are not isothermal, see example 2.3. △

The definition of conformal is independent of the choice of coordinates and hence this enables us to classify Riemannian manifolds with it. The following coordinate independent and much stronger notion provides (as we will see later) an equivalence between Riemannian manifolds.

Definition 2.7. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds and $F : M \to \widetilde{M}$ a diffeomorphism. The map F is called an* **isometry** *if we have $F^*\widetilde{g} = g$. Moreover, the Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ are said to be* **isometric** *under F when it is an isometry.*

Riemannian geometry is primarily concerned with properties that are preserved by isometries. Such properties are: angles between tangent vectors, lengths of paths and curvature. In simple terms, doing calculations on (M, g) corresponds to doing calculations on $(\widetilde{M}, \widetilde{g})$.

2.2 Induced Metrics

Intuitively, if two Riemannian manifolds are isometric then they are "the same". Our next goal is to link an arbitrary Riemannian manifold (M, g) isometrically to some subspace of $\mathbb{R}^n$ with a specific metric, since doing calculations on $\mathbb{R}^n$ is often much easier.

In order to do this, we first have to determine when the pullback of a metric preserves the metric properties. Let M and N be two smooth manifolds and recall definition 1.39.

Proposition 2.8. *Suppose $F : M \to N$ is a smooth map and let g be a Riemannian metric on N. Then F^*g is a Riemannian metric on M whenever F is a smooth immersion.*

Proof. Since we have $g \in \mathcal{T}_0^2 N$, it follows that $F^*g \in \mathcal{T}_0^2 M$ holds due to lemma 1.65. Now we have by definition

$$(F^*g)\big|_p(X, Y) = g|_{F(p)}(F_*X, F_*Y)$$

for any $X, Y \in T_pM$ and all $p \in M$. The metric g is symmetric, hence we have that F^*g is clearly symmetric too. Moreover F^*g is positive definite because g is positive definite and due to the fact F_* is injective, since it was given that F is a smooth immersion. In conclusion, we see that the pullback F^*g is a Riemannian metric on M. $\square$

Now recall definitions 1.25 and 1.39. In general $S \subset M$ is said to be an embedded submanifold of M if the inclusion map $\iota : S \hookrightarrow M$ is an embedding. In that case, the map $\iota_* : T_pS \to T_pM$ is injective. Therefore, for any $X \in T_pS$ we have a different vector $\iota_*X \in T_pM$ such that

$$(\iota_*X)(f) = X(f \circ \iota) = X(f|_S) \tag{2.11}$$

for any $f \in \mathcal{C}^\infty(M)$. Hence we can naturally identify T_pS as a certain linear subspace of T_pM and interpret X as ι_*X and vice versa. For more details, see [Lee13, p. 116].

Recall that, when S is an embedded submanifold of $\mathbb{R}^n$, we can just interpret the tangent vectors $X \in T_pS$ and $\iota_*X \in T_p\mathbb{R}^n$ as the real vector (1.11) discussed in section 1.4. Note that this identification will be used throughout the entire thesis.

Definition 2.9. *Let (M, g) be a Riemannian manifold and $S \subset M$ an embedded submanifold. Then the map $\iota : S \hookrightarrow M$ induces a pullback metric $g|_S := \iota^*g$ on S called the* **induced metric**. *The Riemannian manifold $(S, g|_S)$ is a* **Riemannian submanifold** *of (M, g).*

Note that the definition of the induced metric requires proposition 2.8. Also, by using the natural identifications described above, we have for any $X, Y \in T_pS$ the following:

$$g|_S(X, Y) = \iota^*g(X, Y) = g(\iota_*X, \iota_*Y) = g(X, Y). \tag{2.12}$$

Hence $g|_S$ is just the restriction of g to the tangent bundle TS.

Example 2.10. Consider the Euclidean space $(\mathbb{R}^{n+1}, \bar{g})$ and note that the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ is an embedded submanifold of the smooth manifold $\mathbb{R}^{n+1}$, as we have seen in examples 1.20 and 1.36. Then $(S^n, \bar{g}|_{S^n})$ is a Riemannian submanifold of the Euclidean space with $g|_{S^n} = \iota^*\bar{g}$ the induced metric, known as the **round metric** on S^n. $\triangle$

Now let (M, g) be a Riemannian n-manifold and $S \subset M$ some m-dimensional submanifold. Computations on an embedded submanifold are often most conveniently carried out in terms of a local parametrisation. Recall that a local coordinate map $\varphi : U \to \varphi(U)$ with $U \subset S$ open and the local parametrisation $\varphi^{-1} : V \to U$ with $V = \varphi(U) \subset \mathbb{R}^m$ open are diffeomorphisms due to example 1.14. Moreover, the maps φ and φ^{-1} are embeddings because of lemma 1.33. Now let us write $\rho = \varphi^{-1}$ for notational purposes. Since one has $\iota \circ \rho = \rho$ with $\iota : S \hookrightarrow M$ the inclusion map, we consequently have that

$$\rho^*(g|_S) = \rho^*(\iota^*g) = (\rho \circ \iota)^*g = \rho^*g \tag{2.13}$$

is the local coordinate representation of $g|_S$. Moreover, the map ρ^*g determines a metric on V due to proposition 2.8. Note that the Riemannian manifolds $(U, g|_U)$ and (V, ρ^*g) are isometric, with ρ the isometry, and therefore we can interpret (V, ρ^*g) as the **induced Riemannian manifold** of $(U, g|_U)$. See figure 2.1 for a visualisation of this procedure.

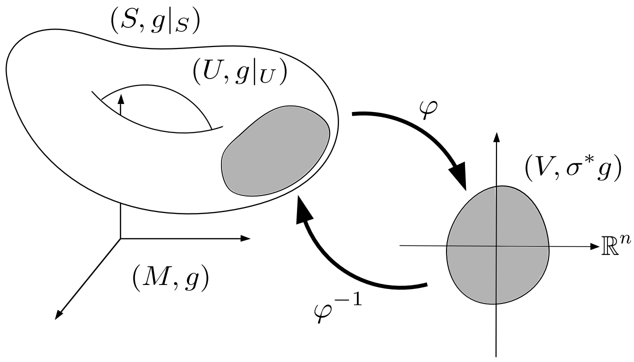

Figure 2.1: A visualisation of the previously discussed compute technique for the induced metric $g|_S$.

Most of the time, one just considers (M, g) to be the Euclidean space $(\mathbb{R}^n, \bar{g})$. For some local parametrisation $\rho : V \to U$ of $S \subset \mathbb{R}^n$, we have that the induced metric $\bar{g}|_S$ of S in standard coordinates $(x^1, ..., x^m)$ on V is given by

$$\rho^* \bar{g} = \delta_{ij} d\rho^i d\rho^j = \delta_{ij} \frac{\partial \rho^i}{\partial x^a} \frac{\partial \rho^j}{\partial x^b} dx^a dx^b = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{\partial \rho^i}{\partial x^j} dx^j \right)^2 \tag{2.14}$$

according to proposition 1.64 and lemma 1.65 with $\rho = (\rho^1, ..., \rho^n)$ in standard coordinates. Note that we have different dimensions within the summations. Now, while one usually does calculations on some surface in $\mathbb{R}^n$, one can also consider the induced Riemannian manifolds and do calculations on open subsets of $\mathbb{R}^m$. This approach appears to be quite useful as we will see in the next chapters. Now we will give some examples to illustrate this approach.

Example 2.11. Let $R > 0$ and define $S^1_R = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 = R^2\}$ as the circle centred at $(a, b) \in \mathbb{R}^2$ with radius R. Consider $M = S^1_R \backslash \{(a, b + R)\}$ and note that it is an embedded submanifold of $\mathbb{R}^2$ due to theorem 1.24, hence the pair $(M, \bar{g}|_M)$ is a Riemannian submanifold of the Euclidean space.

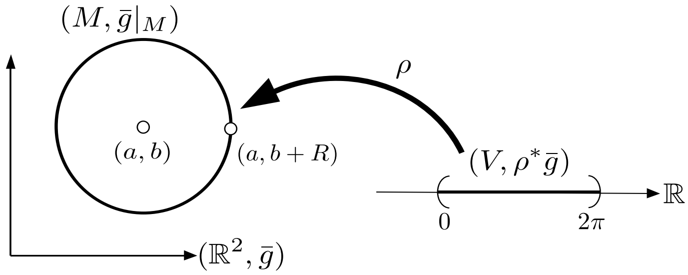

Figure 2.2: A visualisation of example 2.11.

Furthermore, $V = (0, 2\pi)$ is an open submanifold of $\mathbb{R}$ and let us denote θ for the (globally defined) standard coordinate on V. Now consider (V, g) to be the Riemannian manifold such that we have $g = R^2 d\theta^2$ expressed in the standard coordinate. Note that the map

$$\rho : V \to M,\ \theta \mapsto (a + R \cos \theta, b + R \sin \theta) \tag{2.15}$$

is a local (and actually in fact a global) parametrisation of M. Consequently, with the help of formula (2.14), we find the following:

$$\rho^* \bar{g} = (d\rho^1)^2 + (d\rho^2)^2 = (-R \sin \theta d\theta)^2 + (R \cos \theta d\theta)^2 = R^2 d\theta^2 \tag{2.16}$$

and note that $g = \rho^* \bar{g}$ holds. Thus we had given V precisely the coordinate representation of the induced metric $\bar{g}|_M$. In conclusion $(M, \bar{g}|_M)$ and (V, g) are isometric. Note that $\rho^* \bar{g}$ shares many properties with the polar metric $g = dr^2 + r^2 d\theta^2$. △

Example 2.12. Recall example 1.13 and note that S^2 and the northern hemisphere S^2_+ are embedded submanifolds of $\mathbb{R}^3$, which follows from example 1.36 and theorem 1.24. We now consider the Riemannian submanifolds $(S^2, \bar{g}|_{S^2})$ and $(S^2_+, \bar{g}|_{S^2_+})$ of the Euclidean space $(\mathbb{R}^3, \bar{g})$.

Also recall the local parametrisation

$$\rho : B^2 \to S^2_+,\ (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$$

of S^2 which is in fact a global parametrisation of S^2_+. This local parametrisation will give us a coordinate representation of $\bar{g}|_{S^2}$. Moreover, we can consider the induced Riemmannian manifold $(B^2, \rho^*\bar{g})$ that is isometric to $(S^2_+, g|_{S^2_+})$. Subsequently we will do some calculations on (B^2, g) with $g = \rho^*\bar{g}$. According to formula (2.14), we have

$$\rho^*\bar{g} = (d\rho^1)^2 + (d\rho^2)^2 + (d\rho^3)^2 =$$

$$dx^2 + dy^2 + \left(\frac{\partial\sqrt{1 - x^2 - y^2}}{\partial x}dx + \frac{\partial\sqrt{1 - x^2 - y^2}}{\partial y}dx\right)^2 = dx^2 + dy^2 + \frac{(xdx + ydy)^2}{1 - x^2 - y^2},$$

where we denote (x, y) again, as we did in example 2.6, for the standard coordinates on the open subset of $\mathbb{R}^2$. Expanding the brackets of the above gives us

$$g = \frac{1 - y^2}{1 - x^2 - y^2}dx^2 + \frac{2xy}{1 - x^2 - y^2}dxdy + \frac{1 - x^2}{1 - x^2 - y^2}dy^2. \tag{2.17}$$

Also it is useful to consider its matrix $G = (g_{ij})_{ij}$ associated to the standard coordinates:

$$G = \begin{pmatrix} \frac{1-y^2}{1-x^2-y^2} & \frac{xy}{1-x^2-y^2} \\ \frac{xy}{1-x^2-y^2} & \frac{1-x^2}{1-x^2-y^2} \end{pmatrix}. \tag{2.18}$$

Now let $0 < a < 1$ and consider the point $p = (0, a)$ on B^2 in standard coordinates for example and the tangent vectors $X = -\frac{\partial}{\partial y}|_p$ and $Y = \frac{\partial}{\partial x}|_p - \frac{\partial}{\partial y}|_p$ at p.

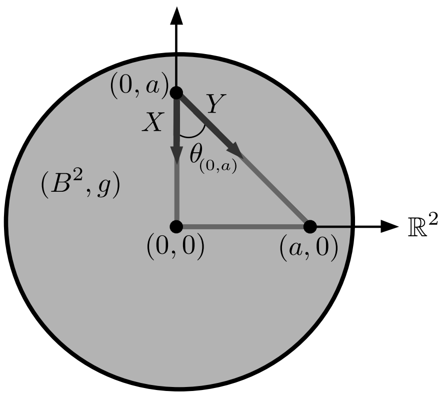

Figure 2.3: A visualisation of (B^2, g) including a triangle with its nodes at $(0, 0)$, $(a, 0)$ and $(0, a)$ in standard coordinates. Also a visual interpretation of the tangent vectors X and Y is given.

Note that the matrix G at point p is as follows:

$$G(p) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-a^2} \end{pmatrix}.$$

Now let us determine the angle $\theta_{(0,a)}$, see formula (2.3), between the vectors X and Y. In order to do this, we just need to calculate $g(X, X)$, $g(X, Y)$ and $g(Y, Y)$. For example, we have

$$g(Y, Y) = g(\tfrac{\partial}{\partial x}|_p, \tfrac{\partial}{\partial x}|_p) + g(\tfrac{\partial}{\partial y}|_p, \tfrac{\partial}{\partial y}|_p) - 2g(\tfrac{\partial}{\partial x}|_p, \tfrac{\partial}{\partial y}|_p) = 1 + \tfrac{1}{1-a^2} = \tfrac{2-a^2}{1-a^2}$$

due to the multilinearity and symmetry properties of the Riemannian metric.

These calculations can also be done by considering the matrix G, because we have

$$g(Y,Y) = \begin{pmatrix} Y^1 & Y^2 \end{pmatrix} G(p) \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-a^2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + \tfrac{1}{1-a^2} = \tfrac{2-a^2}{1-a^2}.$$

Via similar calculations we find $g(X,X) = g(X,Y) = \frac{1}{1-a^2}$, and consequently we have

$$\theta_{(0,a)} = \arccos \frac{1}{\sqrt{2-a^2}}. \tag{2.19}$$

Recall the triangle on B^2 we have drawn in figure 2.3. Note that we have already determined the angle at the node $(0,a)$ and doing so for the remaining two nodes, we get

$$\theta_{total} = \theta_{(0,0)} + \theta_{(a,0)} + \theta_{(0,a)} = \frac{\pi}{2} + 2\arccos \frac{1}{\sqrt{2-a^2}}. \tag{2.20}$$

In Euclidean geometry we have the well-known triangle postulate: the sum of the angles of a triangle equals π. On (B^2, g) however this is no longer true. Also, we have

$$\theta_{total} \to \tfrac{\pi}{2} \text{ as } a \to 1 \quad \text{and} \quad \theta_{total} \to \pi \text{ as } a \to 0. \tag{2.21}$$

Note that we can deduce from (2.21) that the triangle postulate is only satisfied on (B^2, g) for an infinitesimal small triangle. Intuitively this indeed needs to holds, since $(S^2_+, g|_{S^2_+})$ looks locally like the 2-dimensional Euclidean space.

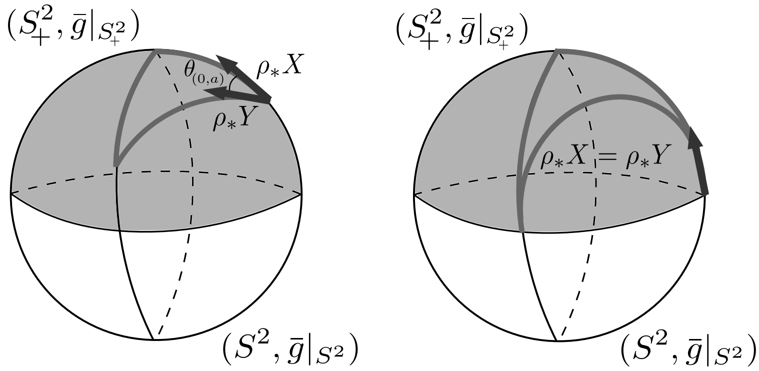

Figure 2.4: A visualisation of $(S^2_+, g|_{S^2_+})$ including the corresponding triangles and tangent vectors with some $0 < a < 1$ fixed (left illustration) and with $a \to 1$ (right illustration).

Note that we get a corresponding triangle on S^2_+ by transporting a triangle on B^2 using ρ. As we can see in figure 2.4, we have that $(S^2_+, g|_{S^2_+})$ indeed satisfies the fact $\theta_{total} \to \frac{\pi}{2}$ as $a \to 1$ because the angles at the nodes $(0,a)$ and $(0,a)$ in local coordinates tend to zero. △

The detailed example above shows that doing calculations on an induced Riemannian manifold is relatively easy, since coordinate vectors on an open subset of $\mathbb{R}^m$ can easily be visualised and interpreted as discussed in section 1.4. Therefore we preferably do calculations on some open subset of $\mathbb{R}^m$ instead of the surface in $\mathbb{R}^n$ itself.

Example 2.13. Consider the 2-dimensional sphere $S^2_R = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}$ with radius $R > 0$. An alternative way, compared to example 2.12, to express (an open subset of) S^2_R in local coordinates is by means of **geographical coordinates**. That is:

$$\rho : (0,\pi) \times (0, 2\pi) \to S^2_R \backslash \{N, S\}, \ (\phi, \theta) \mapsto (R \sin\phi \cos\theta, R\sin\phi\sin\theta, R\cos\phi) \tag{2.22}$$

is a local parametrisation of S^2_R with $N = (0,0,R)$ the north and $S = (0,0,-R)$ the south pole. The inverse map ρ^{-1} is the so-called **geographical coordinate map** and we denote (ϕ, θ) as its local coordinates, which is commonly known as the geographical coordinates.

Now let us consider the Riemannian submanifold $(S^2_R, \bar{g}|_{S^2_R})$ of the Euclidean space. The local parametrisation ρ gives us a coordinate representation of $\bar{g}|_{S^2_R}$, and according to (2.14) we have

$$
\begin{aligned}
\rho^*\bar{g} &= (d\rho^1)^2 + (d\rho^2)^2 + (d\rho^3)^2 \\
&= R^2(\cos\phi\cos\theta d\phi - \sin\phi\sin\theta d\theta)^2 + R^2(\cos\phi\sin\theta d\phi + \sin\phi\cos\theta d\theta)^2 + R^2(-\sin\phi d\phi)^2 \\
&= R^2(\cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta)d\phi^2 + R^2\big(\cos\phi\cos\theta\sin\phi\sin\theta \\
&\quad - \cos\phi\sin\theta\sin\phi\cos\theta\big)d\phi d\theta \; + R^2(\sin^2\phi\sin^2\theta + \sin^2\phi\cos^2\theta)d\theta^2 + R^2\sin^2\phi d\phi^2 \\
&= R^2 d\phi^2 + R^2\sin^2\phi\, d\theta^2
\end{aligned}
$$

where we also write (ϕ, θ) for the standard coordinates on $V = (0, \pi) \times (0, 2\pi)$. For most practical purposes, note that $(V, \rho^*\bar{g})$ is isometric with $(M, \bar{g}|_M)$ where $M = S^2_R \backslash \{N, S\}$. △

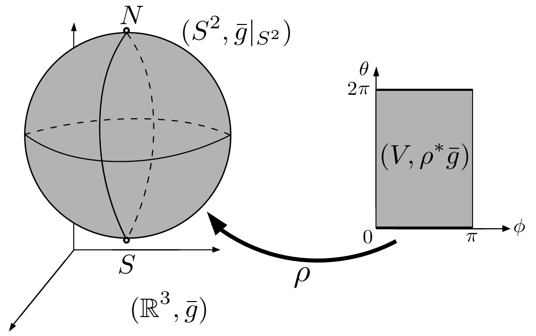

Figure 2.5: A visualisation of example 2.13.

Remark 2.14. The metric of some induced Riemannian manifold, say $(\varphi^{-1})^*(\bar{g}|_S) = (\varphi^{-1})^*\bar{g}$, does indeed give a local coordinate representation of $\bar{g}|_S$ in a direct way. When we determine with formula (2.14) the metric $(\varphi^{-1})^*\bar{g} = g_{ij}dx^i dx^j$ with $(x^1, ..., x^n)$ the standard coordinates on some open submanifold of $\mathbb{R}^n$, we also have

$$
g_{ij} = (\varphi^{-1})^*(\bar{g}|_S)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \bar{g}|_S\left((\varphi^{-1})_*\frac{\partial}{\partial x^i}, (\varphi^{-1})_*\frac{\partial}{\partial x^j}\right) \tag{2.23}
$$

by definition and note that $(\varphi^{-1})_*\frac{\partial}{\partial x^i}$ is the i-th coordinate vector field on S associated with the local coordinate map φ. Thus we can locally express $\bar{g}|_S = \tilde{g}_{ij}dx^i dx^j$ with $\tilde{g}_{ij}(p) = g_{ij}(\varphi(p))$ and $(x^1, ..., x^n)$ are the local coordinates of S associated with φ. For example, see the above, we locally have $\bar{g}|_{S^2_R} = R^2 d\phi^2 + R^2\sin^2\phi\, d\theta^2$ with (ϕ, θ) the geographical coordinates. Both interpretations can be used interchangeably, which is frequently done in this thesis.

The last brief example shows an isometry that has nothing to do with induced metrics.

Example 2.15. Recall the 2-dimensional Poincaré half-space model (U^2, g), see example 2.6. We claim that it is isometric with the 2-dimensional version of the **Poincaré ball model**; the Riemannian manifold (B^2, h) with the disk $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and

$$
h = \frac{4}{(1 - x^2 - y^2)^2}(dx^2 + dy^2) \tag{2.24}
$$

in standard coordinates (x, y) on B^2. Note that h is also conformal to the Euclidean metric. Moreover, by using proposition 1.64 and lemma 1.65, we find $\varphi^*g = h$ with the diffeomorphism

$$
\varphi : B^2 \to U^2, \; (x, y) \mapsto \left(\frac{2x}{x^2 + (y-1)^2}, \frac{1 - x^2 - y^2}{x^2 + (y-1)^2}\right).
$$

It is a long calculation but easily verified. Be aware that, in contrast to the previous examples, the component functions are not all constant. If one wants to use formula (2.14), one needs to multiply it with the function $\sigma(x, y) = (x^2 + (y-1)^2)^2(1 - x^2 - y^2)^{-2}$, which just follows from lemma 1.65. For more details on **hyperbolic spaces**, see [Lee97, p. 38]. △

Finally, let us recall the previous examples where we have visualised an induced Riemannian manifold $(V, \rho^* g)$. We note that the "curvature" of the original surface is "hidden" inside the metric $\rho^* g$. For example, when we look at figure 2.3, then one says intuitively that the sum of the angles of the given triangle equals π. The "curvature" of the original surface S^2_+ however, what causes the sum not to equal π, is "hidden" inside the metric of B^2.

2.3 Lengths of Paths

Let $\gamma : (a, b) \to M$ be a smooth path. At any time $t_0 \in (a, b)$ we define the **velocity** of the given path γ to be the push-forward $\gamma'(t_0) = \gamma_* \frac{d}{dt}\big|_{t_0}$ with $\frac{d}{dt}\big|_{t_0} \in T_{t_0}\mathbb{R}$ the standard coordinate vector of $\mathbb{R}$ at t_0. Hence by definition, it acts on functions $f \in \mathcal{C}^\infty(M)$ as

$$\gamma'(t_0)f = \gamma_* \frac{d}{dt}\Big|_{t_0} f = \frac{d}{dt}\Big|_{t_0} (f \circ \gamma) = \frac{d(f \circ \gamma)}{dt}(t_0). \tag{2.25}$$

Note that defining the velocity of a path in this way generalises what we have discussed in the beginning of section 1.4, or especially equations (1.9) and (1.11). Now let (U, φ) be a smooth chart on M and $(x^1, ..., x^n)$ the local coordinates. Suppose $\gamma(t_0) \in U$, then we can write the tangent vector $\gamma'(t_0)$ as a linear combination of coordinate vectors:

$$\gamma'(t_0) = (\gamma^i)'(t_0) \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)} \tag{2.26}$$

with $(\gamma^j)'(t_0) = \gamma'(t_0)x^j$, which is just the j-th coordinate representation of γ. Note the Einstein summation convention and recall that the above is possible because the coordinate vectors at $\gamma(t_0)$ form a basis for $T_{\gamma(t_0)}M$, see definition 1.34. Also remember that we have already seen formula (2.26) for an open submanifold of $\mathbb{R}^n$ in the beginning of section 1.4.

Definition 2.16. *A* **vector field along a smooth path** $\gamma : (a, b) \to M$ *is a (not necessarily smooth) map* $V : (a, b) \to TM$ *such that* $V(t) \in T_{\gamma(t)}M$ *for all* $t \in (a, b)$.

A vector field along a smooth path V is of course said to be **smooth** if it is a smooth map between smooth manifolds and we denote $\mathcal{T}(\gamma)$ as the set of all smooth vector fields along the smooth path γ. The most obvious and also most significant example of a smooth vector field along γ is the velocity vector field $\gamma' : (a, b) \to TM,\ t \to \gamma'(t)$. Suppose (U, φ) is a smooth chart on M such that $\gamma\big((a, b)\big) \subset U$, then we can write the vector field γ' globally as:

$$\gamma'(t) = (\gamma^i)'(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}. \tag{2.27}$$

Note that the smoothness of γ' follows from equation (2.27) and definition 1.18, where we need to consider the smooth structure of the tangent bundle, as discussed in section 1.5, and interpret the interval (a, b) as an open submanifold of $\mathbb{R}$.

Example 2.17. Consider the smooth manifold $M = \mathbb{R}^2$ and let $r : (a, b) \to \mathbb{R}$ be a smooth map. Then it defines a smooth path

$$\gamma : (a, b) \to M,\ t \mapsto (t, r(t)). \tag{2.28}$$

Let (x^1, x^2) be the standard coordinates on M, then we have due to formula (2.27) the following:

$$\gamma'(t) = \frac{d(x^1 \circ \gamma)}{dt}(t) \frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \frac{d(x^2 \circ \gamma)}{dt}(t) \frac{\partial}{\partial x^2}\Big|_{\gamma(t)} = \frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + r'(t) \frac{\partial}{\partial x^2}\Big|_{\gamma(t)}. \tag{2.29}$$

We can interpret $\gamma'(t)$ geometrically as the real vector $\gamma'(t)\iota = \big(1, r'(t)\big)$ for any $t \in (a, b)$. $\triangle$

Example 2.18. Recall example 1.36 and define the smooth path

$$\gamma : (0, \pi) \to S^2_+, \ t \mapsto \left(\tfrac{1}{2}\cos t, \tfrac{1}{2}\sin t, \tfrac{3}{4}\right).$$

In terms of the chart $(S^2_+, \varphi = (x^1, x^2))$, we get according to formula (2.27) the following:

$$\gamma'(t) = \frac{d(x^1 \circ \gamma)}{dt}(t)\frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \frac{d(x^2 \circ \gamma)}{dt}(t)\frac{\partial}{\partial x^2}\Big|_{\gamma(t)} = -\frac{1}{2}\sin t\frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \frac{1}{2}\cos t\frac{\partial}{\partial x^2}\Big|_{\gamma(t)}.$$

Consequently, with the help of the results in example 1.36, we can geometrically interpret the coordinate vectors of S^2_+ at $\gamma(t)$ as:

$$\frac{\partial}{\partial x^1}\Big|_p \iota = \left(1, 0, -\tfrac{1}{3}\sqrt{3}\cos t\right)^\top \quad \text{and} \quad \frac{\partial}{\partial x^2}\Big|_p \iota = \left(0, 1, -\tfrac{1}{3}\sqrt{3}\sin t\right)^\top.$$

Hence we can geometrically interpret $\gamma'(t)$ as the real vector field

$$\gamma'(t)\iota = \left(-\tfrac{1}{2}\sin t, \tfrac{1}{2}\cos t, 0\right)^\top$$

which follows by substitution of the two equation above into one another. Note that this result can also be determined immediately by just differentiating γ as a real function. △

In conclusion, when $M \subset \mathbb{R}^k$ is an embedded submanifold of $\mathbb{R}^k$, we have that the geometric interpretation of γ' is just differentiating γ as a real function (as we previously also did in section 1.4). This is clear from the two examples above.

The reason to define the velocity vector with a pushforward, is because it then works for abstract manifolds as well. But more importantly it is now a derivation and hence a tangent vector in our abstract sense. This enables us to define the following.

Definition 2.19. *Let (M, g) be a Riemannian manifold. Then one defines the* **length** *of a smooth path $\gamma : (a, b) \to M$ by the following formula:*

$$L_g(\gamma) = \int_\gamma ds = \int_a^b |\gamma'(t)|_g \, dt. \tag{2.30}$$

Note that the length of a smooth path, just as in ordinary and complex calculus, is well-defined because of the following proposition.

Proposition 2.20. [Lee97, p. 92] *The length of a path is independent of the parametrisation choice. In other words, we have $L(\gamma) = L(\tilde{\gamma})$ with $\tilde{\gamma}$ any reparametrisation of γ.*

In a lot of literature, see for example [Car97, p. 48] and [Ros03, p. 61], one writes ds^2 instead of g. Note that this does not mean $ds \otimes ds$. The reason behind this notion is for example the infinitesimal version of the Pythagorean theorem: $ds^2 = dx^2 + dy^2$.

Example 2.21. Recall example 2.17 and consider the Euclidean metric $\bar{g}$. We know

$$\gamma'(t) = \tfrac{\partial}{\partial x^1}\big|_{\gamma(t)} + r'(t)\tfrac{\partial}{\partial x^2}\big|_{\gamma(t)},$$

hence we have $|\gamma'(t)|^2_{\bar{g}} = \bar{g}(\gamma'(t), \gamma'(t)) = 1 + r'(t)^2$ and thus

$$L_{\bar{g}}(\gamma) = \int_a^b |\gamma'(t)|_{\bar{g}} dt = \int_a^b \sqrt{1 + r'(t)^2} \, dt. \tag{2.31}$$

Note that this is the well-known arc length formula of a function r. △

Ultimately, as we already claimed a several times, we have that isometric Riemannian manifolds are equivalent in some sense. Various properties are preserved under isometry and one of those properties is the lengths of smooth paths.

Proposition 2.22. *The lengths of smooth paths are isometry invariant. More precisely, let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds and $F : M \to \widetilde{M}$ an isometry. Then we have that the equality $L_g(\gamma) = L_{\widetilde{g}}(F \circ \gamma)$ holds for every smooth path γ on M.*

Proof. Let $\gamma : (a, b) \to M$ be an arbitrary smooth path. Then for all $t_0 \in (a, b)$ we have

$$(F \circ \gamma)'(t_0) = (F \circ \gamma)_* \frac{d}{dt}\Big|_{t_0} = (F_* \circ \gamma_*) \frac{d}{dt}\Big|_{t_0} = F_*(\gamma'(t_0)).$$

Since F is an isometry, we have $F^*\widetilde{g} = g$ and therefore the following holds:

$$|F_*X|^2_{\widetilde{g}} = \widetilde{g}(F_*X, F_*X) = F^*\widetilde{g}(X, X) = g(X, X) = |X|^2_g$$

and hence $|F_*X|_{\widetilde{g}} = |X|_g$ for any $X \in T_pM$ and all $p \in M$. Consequently, we have

$$L_{\widetilde{g}}(F \circ \gamma) = \int_a^b |(F \circ \gamma)'(t)|_{\widetilde{g}}\, dt = \int_a^b |F_*(\gamma'(t))|_{\widetilde{g}}\, dt = \int_a^b |\gamma'(t)|_g\, dt = L_g(\gamma). \tag{2.32}$$

We conclude that lengths of smooth paths are isometry invariant. ☐

Example 2.23. Let $N = (0, 1)$ be the north pole of $S^1_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}$, the circle centred at the origin with radius $R > 0$. Now define $M = S^1_R \backslash \{N\}$ and note that it is an embedded submanifold of the Euclidean space $(\mathbb{R}^2, \bar{g})$, see also example 2.11. Hence, we will consider the Riemannian manifold $(M, \bar{g}|_M)$.

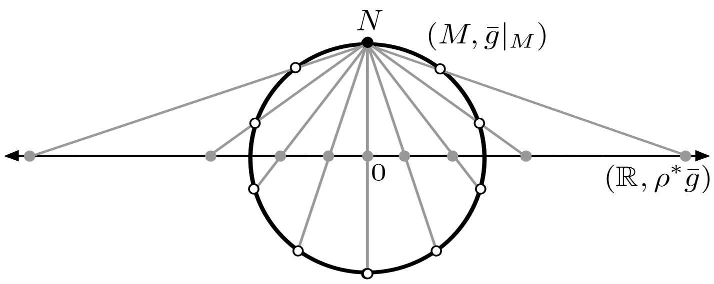

Figure 2.6: A visualisation of the stereographic projection with equidistant points on the circle.

Let φ be the stereographic projection of S^1_R, that is:

$$\varphi : M \to \mathbb{R},\ (x, y) \mapsto \frac{Rx}{R - y}. \tag{2.33}$$

Note that it is an local coordinate map of S^1_R and again a diffeomorphism between smooth manifolds with the smooth inverse

$$\rho : \mathbb{R} \to M,\ t \mapsto \left(\frac{2R^2t}{t^2 + R^2}, \frac{R(t^2 - R^2)}{t^2 + R^2} \right). \tag{2.34}$$

This local parametrisation ρ is in fact a global parametrisation of M and therefore we will consider the induced Riemmannian manifold $(\mathbb{R}, \rho^*g)$ that is isometric to $(M, g|_M)$. Let us write $\widetilde{g} = \rho^*g$. Now we get due to formula (2.14) the following:

$$\widetilde{g} = \rho^*g = (d\rho^1)^2 + (d\rho^2)^2 =$$

$$\left(\frac{2R^2(t^2 + R^2) - 4R^2t^2}{(t^2 + R^2)^2} dt \right)^2 + \left(\frac{2Rt(t^2 + R^2) - 2Rt(t^2 - R^2)}{(t^2 + R^2)^2} dt \right)^2 =$$

$$\frac{4R^2}{(t^2 + R^2)^2} \left[\left(R - \frac{2Rt^2}{t^2 + R^2} \right)^2 + \left(t - \frac{t(t^2 - R^2)}{t^2 + R^2} \right)^2 \right] dt^2 = \frac{4R^4}{(t^2 + R^2)^2} dt^2.$$

Note that we write t for our standard coordinate on $\mathbb{R}$. Moreover note that the component function of $\widetilde{g}$ vanishes as $t \to \pm\infty$ and attains it maximum at $t = 0$. As we see in figure 2.6: the equidistant points on the circle move on $\mathbb{R}$ more apart when going towards infinity.

Ultimately, consider the path

$$\gamma : (-\infty, \infty) \to \mathbb{R},\ s \to s.$$

Then the length of our path γ, which can be interpreted as the length of our real line $\mathbb{R}$, equals

$$L_{\widetilde{g}}(\gamma) = \int_{-\infty}^{\infty} |\gamma'(t)|_{\widetilde{g}}\, dt = \int_{-\infty}^{\infty} \sqrt{\widetilde{g}\big(\gamma'(t), \gamma'(t)\big)}dt = \int_{-\infty}^{\infty} \sqrt{\widetilde{g}\big(\tfrac{d}{dt}\big|_t, \tfrac{d}{dt}\big|_t\big)}dt =$$

$$\int_{-\infty}^{\infty} \sqrt{\frac{4R^2}{(t^2+R^2)^2}}dt = 2\int_{-\infty}^{\infty} \frac{R^2}{t^2+R^2}dt = 2\pi R$$

according to formula (2.27). This indeed corresponds, as we have proven in proposition 2.22, to the length of the circle S^1_R (without the north pole). $\triangle$

The above can also be done for arbitrary dimension. Consider the n-sphere $S^n_R \backslash \{N\}$ without its north pole with radius $R > 0$. Then $(S^n_R\backslash\{N\}, g|_{S^n_R\backslash\{N\}})$ is isometric to its induced Riemannian manifold $(\mathbb{R}^n, g)$ with

$$g = \frac{4R^4}{(\sum_{i=1}^n (x^i)^2 + R^2)^2} \sum_{i=1}^n (dx^i)^2 \tag{2.35}$$

in standard coordinates $(x^1, ..., x^n)$ on $\mathbb{R}^n$. Note that g is conformal to the canonical Euclidean metric and $(x^1, ..., x^n)$ are isothermal coordinates. See [Lee97, p. 36] for more details on g.

Finally, at the end of the previous section we noted that the "curvature" of the original surface is "hidden" inside the metric of the induced Riemannian manifold. Again, the previous example encourages this idea. Consider $(\mathbb{R}, \widetilde{g})$ from the previous example, then one would intuitively say that the length of $\mathbb{R}$ is infinitely large. The "curvature" of the original surface S^1_R however, what causes the length to equal $2\pi R$, is "hidden" inside the metric $\widetilde{g}$.

Chapter 3

How to measure Curvature

We, as inhabitants of the earth, were only able to deduce the geometry of the earth, before the invention of airspace technology, by measurements made on earth itself. Although the earth is within a bigger space, our universe, we did not have an observer in space to do measurements for us. Nonetheless, restricting ourselves with measurements we can do on earth, we could still measure the length of a trajectory travelled for example and via some calculations we are able to determine the velocity and speed along a path too.

Mathematically though, one often considers the **extrinsic** point of view: paths and surfaces are lying in some Euclidean space and the length of a path on a surface is calculated by just interpreting it as a path through Euclidean space. As discussed in the previous paragraph, the presence of some ambient space for calculating lengths of paths is unnecessary, which we have also seen in section 2.3. Hence all we need is a Riemannian metric defined on some smooth manifold M of our interest. All the geometry which can be derived from just a given metric, and thus without an observer in an ambient space, is what we call **intrinsic** geometry.

The intrinsic point of view, in which one cannot speak of moving outside the geometric object, is way more flexible. In this chapter we will work towards an intrinsic definition of curvature. In general relativity for example, see [Car97] or [FG17], one considers the curvature of space-time which cannot naturally be taken as extrinsic, since what would be the outside of our universe? For our purpose, the intrinsic approach enables us to define the Ricci flow because changing a metric on M is the main key behind the Ricci flow, which cannot be done when the ambient space is present as we discussed previously. Lastly, a fundamental result in intrinsic geometry is **Gauss's Theorema Egregium**. In short, see also section 3.4, it shows that the Gaussian curvature, a way of defining curvature extrinsically, can also be calculated intrinsically.

Now some technicalities, if $(x^1, ..., x^n)$ are local coordinates of M and there can be no confusion about which local coordinate system is meant, we abbreviate a coordinate vector $\frac{\partial}{\partial x^i}$ with ∂_i. As usual, we use the Einstein summation convention and note that writing ∂_i is more common then $\frac{\partial}{\partial x^i}$ in the literature. This also explains the upper and lower index rule, see remark 1.30.

3.1 Connections

In order to define curvature, it first requires more knowledge concerning Riemannian manifolds. To gain more information, we would like for example to differentiate (in some sense) vector fields on M. Let us implement this idea in the following example.

Example 3.1. Consider the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with radius one and its ambient space: the Euclidean space $(\mathbb{R}^3, \bar{g})$ where $\bar{g}$ is the canonical Euclidean metric.

Recall the geographical coordinate map φ and the corresponding local parametrisation

$$\varphi^{-1} : (0,\pi) \times (0,2\pi) \to S^2 \backslash \{N,S\},\ (\phi,\theta) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

as discussed in example 2.13. For the geographical coordinates (ϕ,θ) on $S^2\backslash\{N,S\}$, we have that the coordinate vectors at $p \in S^2\backslash\{N,S\}$ are given by:

$$\frac{\partial}{\partial\phi}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial\phi}\Big|_{\varphi(p)} \quad \text{and} \quad \frac{\partial}{\partial\theta}\Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial\theta}\Big|_{\varphi(p)}.$$

As suggested by formula (1.17) we calculate the Jacobian of φ^{-1}, which gives:

$$J\varphi^{-1} = \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{pmatrix}.$$

Consequently, the natural identification of the coordinate vector fields $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\theta}$ are given by the real vector fields $\hat{\phi}$ and $\hat{\theta}$ respectively:

$$\hat{\phi} = \big(\cos\phi\cos\theta,\, \cos\phi\sin\theta,\, -\sin\phi\big)^\top \quad \text{and} \quad \hat{\theta} = \big(-\sin\phi\sin\theta,\, \sin\phi\cos\theta,\, 0\big)^\top. \tag{3.1}$$

Our goal is now to differentiate the tangent vector fields $\hat{\phi}$ and $\hat{\theta}$ with respect to ϕ and θ.

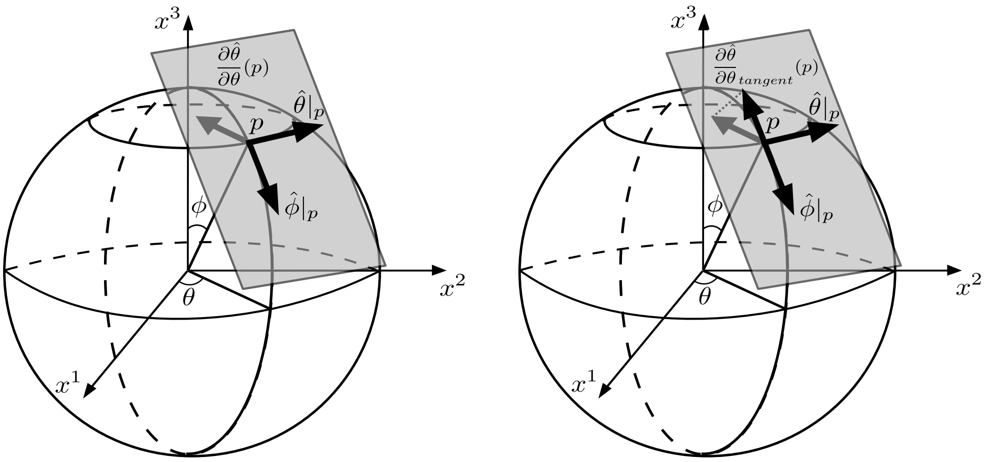

Figure 3.1: A visualisation of example 3.1 with $\frac{\partial\hat{\theta}}{\partial\theta}$ (on the left) the partial derivative of $\hat{\theta}$ with respect to θ and $\frac{\partial\hat{\theta}}{\partial\theta}_{tangent}$ (on the right) its orthogonal projection on the tangent space T_pS^2.

As suggested by the figure above, we determine the derivative of $\hat{\theta}$ with respect to θ:

$$\frac{\partial\hat{\theta}}{\partial\theta} = \big(-\sin\phi\cos\theta,\, -\sin\phi\sin\theta,\, 0\big)^\top \tag{3.2}$$

Unfortunately we encounter a problem as we can see from (3.2) and in figure 3.1. Differentiation of a coordinate vector field with respect to some coordinate does not (always) give us a tangent vector field in return. A solution to this problem is quite straightforward: orthogonally project the derivative of the (real) vector field at any p on T_pS. Doing this gives us:

$$\frac{\partial\hat{\theta}}{\partial\theta}_{tangent} = \frac{\partial\hat{\theta}}{\partial\theta} - \hat{N}\langle\frac{\partial\hat{\theta}}{\partial\theta}, \hat{N}\rangle \tag{3.3}$$

with $\langle\cdot,\cdot\rangle$ the standard inproduct (since we consider the Euclidean space) and $\hat{N}$ the normal unit (real) vector field, that is: for any $p \in S^2\backslash\{N,S\}$ we have $\hat{N}|_p \in \mathbb{R}^3$ such that $\langle\hat{N}_p, \hat{N}_p\rangle = 1$, which makes it a unit vector, and more importantly $\langle \hat{N}|_p, \hat{\theta}|_p \rangle = 0$ and $\langle \hat{N}|_p, \hat{\phi}|_p \rangle = 0$ because then it is orthogonal to any real identification of our tangent vectors at p. Note that we have

$$\hat{N} = \big(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta\big)^\top. \tag{3.4}$$

Hence, by the above and formula (3.3) we get the following expression:

$$\frac{\partial\hat{\theta}}{\partial\theta}_{tangent} = 0\cdot\hat{\theta} - \sin\phi\cos\phi\cdot\hat{\phi} = -\sin\phi\cos\phi\cdot\hat{\phi}, \tag{3.5}$$

as we already illustrated in figure 3.1. Note that the component functions associated to the geographical coordinates (ϕ, θ) of vector field (3.5) are smooth maps. Therefore, the corresponding (abstract) vector field defined on $S^2\backslash\{N, S\}$ is according to corollary 1.44 smooth. Conversely, the expression of (3.5) in our abstract sense is written as

$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = -\sin\phi\cos\phi\frac{\partial}{\partial\phi} \tag{3.6}$$

with ∇ the so-called **tangential connection**. Similarly, one can deduce

$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} = \frac{\cos\phi}{\sin\phi}\frac{\partial}{\partial\theta} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = 0. \tag{3.7}$$

Again these are smooth vector fields on $S^2\backslash\{N, S\}$. For more details concerning the (other) calculations, consult [Khu17, p. 52] for example. $\triangle$

As suggested by the above, we want to define an operator on M which enables us to differentiate smooth vector fields in some sense which gives us a smooth vector field in return. Note that the above is a local approach while the following definition is globally. Let M be an n-dimensional smooth manifold unless otherwise specified.

Definition 3.2. *A* **(linear) connection** *on M is a map*

$$\nabla : \mathcal{T}M \times \mathcal{T}M \to \mathcal{T}M$$

denoted as $\nabla(X, Y) = \nabla_X Y$ that satisfies the following three properties:

- *it is $\mathcal{C}^\infty(M)$-linear in the first argument:*

$$\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y \text{ for all } f, g \in \mathcal{C}^\infty(M); \tag{3.8}$$

- *it is $\mathbb{R}$-linear in the second argument:*

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \text{ for all } a, b \in \mathbb{R}; \tag{3.9}$$

- *it satisfies the Leibniz rule:*

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \text{ for all } f \in \mathcal{C}^\infty(M). \tag{3.10}$$

We moreover call $\nabla_X Y$ the **covariant derivative** *of Y in the direction of X.*

Note that for $X \in \mathcal{T}M$ and $f \in \mathcal{C}^\infty(U)$ with $U \subset M$ open, the smooth function $Xf \in \mathcal{C}^\infty(U)$ is defined for all $p \in U$ as the point-wise action $Xf(p) = X|_p f$. If we have $g \in \mathcal{C}^\infty(U)$, then the product rule holds:

$$X(fg) = fXg + gXf. \tag{3.11}$$

See [Lee13, p. 180] for more details. Moreover, even though a connection is defined as a global operator, it is actually a local operator since we have $\nabla_{\widetilde{X}}\widetilde{Y}|_p = \nabla_X Y|_p$ whenever $X(p) = \widetilde{X}(p)$ and $Y = \widetilde{Y}$ in an arbitrarily small neighbourhood of the point $p \in M$, see [Lee97, p. 50]. Recall the end of section 1.7 and note that the proof of the statement above is highly reliant on the existence of bump functions.

Therefore, let $(x^1, ..., x^n)$ be any local coordinates on some open submanifold $U \subset M$ and let us consider the corresponding local coordinate frame $(\partial_1, ..., \partial_n)$ for TM. Assume that ∇ is a connection on U, then for any i and j we can write the covariant derivative $\nabla_{\partial_i}\partial_j$ in terms of the same local frame. In other words, we have

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k \tag{3.12}$$

for some particular functions $\Gamma^k_{ij} : U \to \mathbb{R}$. Equivalently, one can write $\Gamma^k_{ij} = dx^k(\nabla_{\partial_i}\partial_j)$.

Definition 3.3. *The n^3 smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$ are the* **Christoffel symbols** *of the connection ∇ on U with respect to the local coordinate frame $(\partial_1, ..., \partial_n)$.*

Note that the smoothness of the Christoffel symbols follows immediately from corollary 1.44. Moreover, any smooth vector fields $X, Y \in \mathcal{T}U$ can be expressed in terms of the coordinate vector fields by $X = X^i\partial_i$ and $Y = Y^j\partial_j$, hence we have

$$\begin{gathered}\nabla_X Y = \nabla_X(Y^j\partial_j) = Y^j\nabla_X\partial_j + (XY^j)\partial_j = (XY^j)\partial_j + Y^j\nabla_{X^i\partial_i}\partial_j = \\ (XY^j)\partial_j + X^iY^j\nabla_{\partial_i}\partial_j = (XY^j)\partial_j + X^iY^j\Gamma^k_{ij}\partial_k = \left[XY^k + X^iY^j\Gamma^k_{ij}\right]\partial_k\end{gathered} \tag{3.13}$$

in terms of the local coordinates. Note that we have used properties (3.10) and (3.8) respectively. Observe that the component functions of $\nabla_X Y$ contain $X(Y^k)$, the directional derivative of a component function of Y in the direction of X (see the beginning of section 1.4), and some extra term which is the main reason why $\nabla_X Y$ is said to be the "covariant" derivative of Y in the direction of X.

Example 3.4. Let $U \subset \mathbb{R}^n$ be an open submanifold and consider the **Euclidean connection** on U, which is defined by

$$\overline{\nabla}_X Y = \overline{\nabla}_X(Y^j\partial_j) = (XY^j)\partial_j \tag{3.14}$$

in standard coordinates. The three properties of a connection are clearly satisfied. In particular,

$$\overline{\nabla}_{\partial_i} Y = (\partial_i Y^j)\partial_j = \frac{\partial Y^j}{\partial x^i}\partial_j$$

is the smooth vector field whose component functions are just the i-th partial derivative of the components functions of Y defined on U (which can be whole $\mathbb{R}^n$). Also note that all the Christoffel symbols are identically zero. △

Recall the problem we encountered in example 3.1, just differentiating tangent vector fields will not give us a tangent vector field in return, and note that this problem does not appear when we consider (an open subset of) the Euclidean space.

Definition 3.5. *Let X and Y be two smooth vector fields on M. The* **Lie bracket** $[X, Y]$ *of X and Y is the smooth vector field defined on M by*

$$[X, Y]\big|_p f = X|_p(Yf) - Y|_p(Xf) \tag{3.15}$$

for any smooth function $f \in \mathcal{C}^\infty(M)$ and any point $p \in M$.

Note that $[X, Y]$ is a vector field on M because for every $p \in M$ we have $[X, Y]\big|_p \in T_pM$ since

$$[X, Y]\big|_p(fg) = f(p)[X, Y]\big|_p g + g(p)[X, Y]\big|_p f,$$

which follows from (3.11). Note that we also can write $[X, Y]f = XYf - YXf$. The smoothness of a Lie bracket, thus $[X, Y] \in \mathcal{T}M$, follows from lemma 1.43 and the local expressions:

$$[X, Y] = [X^j\partial_j, Y^j\partial_j] = \left(XY^j - YX^j\right)\partial_j = \left(X^i\partial_i Y^j - Y^i\partial_i X^j\right)\partial_j. \tag{3.16}$$

Expression (3.16) follows from a direct computation and relies on the fact that mixed partial derivatives of smooth functions commute, see also [Lee13, p. 187]. Hence, the Lie bracket of two coordinate vector fields always vanishes identically, in other words: $[\partial_i, \partial_j] = 0$.

Definition 3.6. *Suppose (M, g) is a Riemannian manifold. A connection ∇ on M is said to be* **compatible with the metric** *g if*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{3.17}$$

holds and ∇ is said to be **torsion-free** *if we have*

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{3.18}$$

By definition we have for any $X, Y \in \mathcal{T}M$ that $\nabla_X Y$ is a smooth vector field too and recall that the function $g(X, Y) : M \to \mathbb{R}$ is a smooth function, see section 1.7. Consequently, we have the smooth function $Xg(Y, Z) \in \mathcal{C}^\infty(M)$. Moreover, the two properties (3.17) and (3.18) are of major significance as we will see and they arise in the following example.

Example 3.7. Recall the Euclidean connection $\overline{\nabla}$. Note that $\overline{\nabla}$ is clearly torsion-free due to equation (3.16). On can also easily deduce that the connection is compatible with $\bar{g}$, since the standard coordinate frame $(\partial_1, ..., \partial_n)$ is global. We have

$$\begin{gathered} X\bar{g}(Y, Z) = X(Y^i Z^i) = Y^i(XZ^i) + Z^i(XY^i) = \\ \bar{g}((XZ^i)\partial_i, Y^i\partial_i) + \bar{g}((XY^i)\partial_i, Z^i\partial_i) = \bar{g}(\nabla_X Z, Y) + \bar{g}(\nabla_X Y, Z). \end{gathered} \tag{3.19}$$

Therefore $\overline{\nabla}$ is the unique Levi-Civita connection on $(U, \bar{g})$ with $U \subset \mathbb{R}^n$ open. Note that $\overline{\nabla}$ is also one of the main reasons why one would consider these properties in the first place. △

Theorem 3.8. (Fundamental Theorem of Riemannian Geometry) [Lee97, p. 68] *Let us consider a Riemannian manifold (M, g). There exists a unique connection ∇ on M that is compatible with g and torsion-free. This connection is called the* **Levi-Civita** *connection.*

We like to note that studying the proof of this theorem is quite worthwhile because it implies the following corollary, which is one of the most important results in Riemannian geometry.

Corollary 3.9. Let (M, g) be a Riemannian manifold and $G = (g_{ij})_{ij}$ the matrix associated with the local frame $(\partial_1, ..., \partial_n)$ defined on $U \subset M$ open. Consider the inverse $G^{-1} = (g^{ij})_{ij}$. Then from the proof one obtains an explicit formula for the Christoffel symbols of the Levi-Civita connection on $(U, g|_U)$, which is:

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{3.20}$$

Be aware of the Einstein summation convention. Note that ∂_k is a vector field and acts on the functions $g_{ij} : U \to \mathbb{R}$ as follows: $\partial_k g_{ij}$ it is the k-th partial derivative of the coordinate representation of the function g_{ij} on U, see the discussion in section 1.4 below equation (1.16). Lastly it is important to note the symmetry within the Christoffel symbols: $\Gamma^k_{ij} = \Gamma^k_{ji}$.

As the proof of theorem 3.8 in [Lee97, p. 68] claims, we have that the Levi-Civita connection on $(U, g|_U)$ fixes the Levi-Civita connection on (M, g) for all points in U. In [Tu17, p. 77] one can find the following more detailed explanation. We define $\nabla^U : \mathcal{T}U \times \mathcal{T}U \to \mathcal{T}U$ such that for any point $p \in U$ we have

$$\nabla^U_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p \tag{3.21}$$

where $\widetilde{X}$ and $\widetilde{Y}$ are smooth global vector fields on M that agree with X and Y in a neighbourhood of p. Because ∇ is a local operator, it is independent of the choices of $\widetilde{X}$ and $\widetilde{Y}$. One easily show that ∇^U satisfies the properties of a connection and moreover, when ∇ is the Levi-Civita connection on (M, g) then ∇^U is the Levi-Civita connection on $(U, g|_U)$. Thus due to (3.21) we can speak of the Christoffel symbols of the Levi-Civita connection on (M, g). Note that the existence of bump functions play a crucial part in the procedure above.

In conclusion, covariant differentiation with the Levi-Civita connection is instrinsic, since it can be fully determined with the given metric only: apply formulae (3.13) and (3.20). Now let us consider some explicit examples.

Example 3.10. Again consider the Euclidean connection on $(\mathbb{R}^n, \bar{g})$. Alternatively, we deduce with formula (3.20) that all Christoffel symbols are identically zero. By applying (3.13), we can also conclude that $\overline{\nabla}$ is precisely the Levi-Civita connection. △

Example 3.11. Recall example 2.3 and the fact that (r, θ) are local polar coordinates of $\mathbb{R}^2$, defined on the open submanifold R (see example 1.7), with

$$G_{polar} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad G_{polar}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

the matrix associated to the canonical Euclidean metric $\bar{g}$ in polar coordinates and its inverse. According to (3.20), the Christoffel symbols of the Levi-Civita connection on $(\mathbb{R}^2, \bar{g})$ associated to the polar coordinates are:

$$\Gamma_{22}^1 = \tfrac{1}{2} g^{11} \left(0 + 0 - \frac{\partial g_{22}}{\partial r} \right) + 0 = \frac{1}{2}(-2r) = -r; \tag{3.22}$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = 0 + \tfrac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial r} + 0 - 0 \right) = \frac{1}{2r^2}(2r) = \tfrac{1}{r}; \tag{3.23}$$

since differentiating with respect to θ results into zero elements and we claim that all the other Christoffel symbols are identically zero. This is easily verified with equation (3.20). △

The following example is a very useful generalisation of the above.

Example 3.12. Let (M, g) be a Riemannian manifold such that $g = p\, du^2 + q\, dv^2$ in local coordinates (u, v) where the coordinate representations $p = p(u, v)$ and $q = q(u, v)$ are smooth functions and nowhere zero. Then

$$G = \begin{pmatrix} p(u,v) & 0 \\ 0 & q(u,v) \end{pmatrix} \quad \text{and} \quad G^{-1} = \begin{pmatrix} \frac{1}{p(u,v)} & 0 \\ 0 & \frac{1}{q(u,v)} \end{pmatrix}$$

are the matrix associated with g in (u, v) coordinates and its inverse. According to (3.20), we have that the Christoffel symbols of the Levi-Civita connection on (M, g) are:

$$\begin{aligned} \Gamma_{11}^1 &= \tfrac{1}{2} g^{11} \partial_1 g_{11} = \frac{p_u}{2p}; & \Gamma_{12}^1 &= \Gamma_{21}^1 = \tfrac{1}{2} g^{11} \partial_2 g_{11} = \frac{p_v}{2p}; \\ \Gamma_{22}^1 &= -\tfrac{1}{2} g^{11} \partial_1 g_{22} = -\frac{q_u}{2p}; & \Gamma_{11}^2 &= -\tfrac{1}{2} g^{22} \partial_2 g_{11} = -\frac{p_v}{2q}; \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \tfrac{1}{2} g^{22} \partial_1 g_{22} = \frac{q_u}{2q}; & \Gamma_{22}^2 &= \tfrac{1}{2} g^{22} \partial_2 g_{22} = \frac{q_v}{2q}. \end{aligned} \tag{3.24}$$

Note that we write p_u and q_u to indicate the differentiation with respect to u. In other words, they represent $\frac{\partial p(u,v)}{\partial u}$ and $\frac{\partial p(u,v)}{\partial u}$ respectively. Also, the results found in example 3.11 can be calculated immediately with the formulae above. △

Now recall example 2.13 and the significant remark 2.14. Let (ϕ, θ) be the geographical coordinates and note that

$$\bar{g}|_{S^2} = d\phi^2 + \sin^2 \phi \, d\theta^2$$

is a local coordinate representation of the induced metric on S^2 with the component functions nowhere zero. With the help of the previous example, we deduce that the Christoffel symbols of the Levi-Civita connection on $(S^2, g|_{S^2})$ with respect to the geographical coordinates are:

$$\Gamma_{22}^1 = -\sin \phi \cos \phi \quad \text{and} \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\cos \phi}{\sin \phi} \tag{3.25}$$

and the others are identically zero. Importantly note that these components correspond to what we have seen in example 3.1 for the component functions of (3.6) and (3.7). In general, the Levi-Civita connection on an embedded submanifold S of Euclidean space $(\mathbb{R}^n, \bar{g})$ has precisely the interpretation as discussed in example 3.1.

The **tangential connection** (as we have introduced in example 3.1) orthogonally projects the Euclidean connection on $\mathbb{R}^n$ onto S, see [Lee97, p. 66] for the precise definition. It indeed satisfies the properties of a connection and is moreover torsion-free and compatible with the induced metric, see [Lee97, p. 67] and [Lee97, p. 68]. This makes the tangential connection the unique Levi-Civita connection and therefore the Levi-Civita connection on S can be interpreted as the operator that orthogonally projects differentiated vector fields as if they were vector fields in $\mathbb{R}^n$ onto the submanifold S.

We end this section with a lemma needed for the next chapter, which is moreover directly related to the Levi-Civita connection.

Lemma 3.13. *Let (M, g) and $(M, \lambda g)$ with $\lambda > 0$ be two Riemannian manifolds and denote their unique Levi-Cevita connections with ∇^g and $\nabla^{\lambda g}$ respectively. We have $\nabla^{\lambda g} = \nabla^g$.*

Proof. Let $X, Y, Z \in \mathcal{T}M$ be arbitrary smooth vector fields. By definition we have that a tangent vector satisfies the Leibniz condition, see (1.8). Consequently, the following holds:

$$X(\lambda g(Y, Z)) = \lambda X(g(Y, Z)).$$

Since $\nabla^{\lambda g}$ is compatible with λg we have $X(\lambda g(Y,Z)) = \lambda g(\nabla^{\lambda g}_X Y, Z) + \lambda g(Y, \nabla^{\lambda g}_X Z)$, and by applying the equality above and subsequently dividing both sides with λ we get the expression:

$$X(g(Y, Z)) = g(\nabla^{\lambda g}_X Y, Z) + g(Y, \nabla^{\lambda g}_X Z).$$

In conclusion, the connection $\nabla^{\lambda g}$ is compatible with g, hence by theorem 3.8 we have that the equality $\nabla^{\lambda g} = \nabla^g$ holds due to the uniqueness of the Levi-Cevita connection. □

3.2 Flat Riemannian Manifolds

Recall the definition of an isometry, see definition 2.7, and the fact: when two Riemannian manifolds are isometric, they are said to be equivalent because many properties, as discussed in the previous chapter, are preserved. Precisely for this reason we are also interested in whenever a Riemannian manifold is locally equivalent to some other Riemannian manifold.

Definition 3.14. *A Riemannian manifold (M, g) is called* **locally isometric** *to a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ if the following holds: each point $p \in M$ has a neighbourhood that is isometric to an open subset of $\widetilde{M}$.*

An open subset of a Riemannian manifold (M, g) obtains the metric which is just the restriction of the metric g on its tangent bundle, which is discusses in section 2.2. Also note that every isometry is a local isometry. Furthermore, in Riemannian geometry one is often interested in whether a Riemannian manifold is locally isometric to the Euclidean space.

Definition 3.15. *An n-dimensional Riemannian manifold (M, g) is said to be* **flat** *if it is locally isometric to the Euclidean space $(\mathbb{R}^n, \bar{g})$.*

Suppose we have a Riemannian m-submanifold $(S, \bar{g}|_S)$ of the Euclidean space $(\mathbb{R}^n, \bar{g})$. Then the above is equivalent with saying that for any point $p \in S$ there exists a local parametrisation $\rho : V \to U$ with $V \subset \mathbb{R}^m$ and $p \in U \subset M$ open such that the induced Riemannian manifold has $\rho^* \bar{g} = \delta_{ij} dx^i dx^j$ in standard coordinates of V.

Note that the observation above does not actually need to require the Riemannian manifold to be a Riemannian submanifold at all. Moreover, according to remark 2.14, an equivalent statement is that for any $p \in S$ there exists local coordinates $(x^1, ..., x^m)$ around p such that its metric can be locally expressed as $g = \delta_{ij} dx^i dx^j$.

Example 3.16. Recall example 2.11 and now consider a small adjustment of (2.15) which gives us the following local parametrisation of the circle S^1_R:

$$\rho : V \to \rho(V),\ \theta \mapsto (a + R\cos\tfrac{\theta}{R}, b + R\sin\tfrac{\theta}{R}) \tag{3.26}$$

with $V = (\theta_1, \theta_2)$ an open interval such that $\theta_2 - \theta_1 \leq 2\pi R$. Let $p \in S^1_R$ be arbitrary, then there clearly exists a set V such that ρ contains the point p in its image. A simple calculation, conform to example 2.11 and using formula (2.14), will show that the open subset $\rho(V)$ of the circle S^1_R with its induced metric is isometric to $(V, d\theta^2)$, where we also denote θ as the standard coordinate on $\mathbb{R}$. In conclusion, any circle is flat. △

In fact, one can show that every 1-dimensional Riemannian manifold is flat, see [Lee97, p. 116]. Therefore we might say that Riemannian 1-manifolds are not that interesting since they are all the same in some sense. For this reason, we now look at 2-dimensional Riemannian manifolds.

Example 3.17. Consider the cone $K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - k^2z^2 = 0,\ z > 0\}$ with $k > 0$ which is an embedded submanifold of $\mathbb{R}^3$, according to theorem 1.24. Now write the Euclidean space as $(\mathbb{R}^3, \bar{g}_3)$ and let us give K the induced metric $\bar{g}_3|_K$. We claim that $(K, \bar{g}_3|_K)$ is locally isometric to the Euclidean plane $(\mathbb{R}^2, \bar{g}_2)$, hence the cone is flat. We note that

$$\rho_\alpha : V \to \rho_\alpha(V),\ (u, v) \mapsto \begin{cases} x = k\sqrt{\frac{u^2+v^2}{k^2+1}}\cos\left(\frac{\sqrt{k^2+1}}{k}\operatorname{arctan2}(v, u) + \alpha\right) \\ y = k\sqrt{\frac{u^2+v^2}{k^2+1}}\sin\left(\frac{\sqrt{k^2+1}}{k}\operatorname{arctan2}(v, u) + \alpha\right) \\ z = \sqrt{\frac{u^2+v^2}{k^2+1}} \end{cases} \tag{3.27}$$

with

$$V = \{(x, y) \in \mathbb{R}^2 : -\tfrac{\pi k}{\sqrt{k^2+1}} < \operatorname{arctan2}(y, x) < \tfrac{\pi k}{\sqrt{k^2+1}}\} \tag{3.28}$$

are local parametrisations of K such that $\{\rho_\alpha(V)\}_\alpha$ covers K entirely.[1] More importantly, the local parametrisations ρ_α are isometries, thus we have $(\rho_\alpha)^*\,\bar{g}_3 = \bar{g}_2$ on V.

Showing that ρ_α is an isometry can be done with brute force by using formula (2.14), however the following approach also shows how we even came up with these local parametrisations.

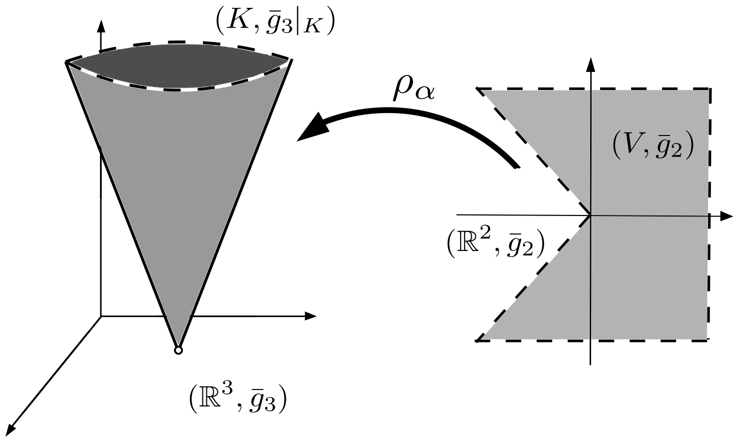

Figure 3.2: A visualisation of example 3.17.

First let us consider the smooth chart (U, φ) such that its local parametrisation is

$$\varphi^{-1} : W \to U,\ (t, \theta) \mapsto (kt\cos\theta, kt\sin\theta, t) \tag{3.29}$$

with $W = \mathbb{R}_{>0} \times (-\pi, \pi)$ open. Carefully note that the equality $U = \rho_0(V)$ holds.

[1] The function $\operatorname{arctan2}(y, x)$ returns the angle of $(x, y) \in \mathbb{R}^2\backslash\{(0, 0)\}$ within the interval $(-\pi, \pi]$.

According to formula (2.14) and remark 2.14, we obtain the local expression of

$$g|_K = (k\cos\theta dt - kt\sin\theta d\theta)^2 + (k\sin\theta dt + kt\cos\theta d\theta)^2 + dt^2 = (k^2+1)dt^2 + k^2t^2d\theta^2$$

where we also denote (t, θ) for the local coordinates on $U = \rho_0(V)$ associated to the coordinate map φ. We observe that the local representation of $g|_K$ looks a lot like the polar representation of the Euclidean metric, see example 2.3.

Therefore we introduce, analogous to example 1.7, the implicit formulae with $a, b \in \mathbb{R}$ constants:

$$u = at\cos(b\theta) \quad \text{and} \quad v = at\sin(b\theta) \tag{3.30}$$

with (u, v) the local coordinates defined on U that satisfy the above. Changing coordinates, according to lemma 1.55, gives us the following expression:

$$du^2 + dv^2 = (a\cos(b\theta)dt - abt\sin(b\theta)d\theta)^2 + (a\sin(b\theta)dt + abt\cos(b\theta)d\theta)^2 =$$

$$a^2dt^2 + (ab)^2t^2d\theta^2.$$

Hence we fix the constants $a = \sqrt{k^2+1}$ and $b = \frac{k}{\sqrt{k^2+1}}$ in order to correspond with the induced metric on K, which implies that (u, v) are local coordinates on K such that $g|_K = du^2 + dv^2$. We like to note, as discussed previously, that this observation is enough to conclude that the cone is flat (since we can easily modify the (co)domains).

Ultimately, formula (3.27) with $\alpha = 0$ is obtained by writing the implicit formulae given in equation (3.30) as, see also example 1.7, the following equivalent implicit formulae:

$$t = a^{-1}\sqrt{u^2+v^2} = \sqrt{\frac{u^2+v^2}{k^2+1}} \quad \text{and} \quad \theta = b^{-1}\,\mathrm{arctan2}(v, u) = \frac{\sqrt{k^2+1}}{k}\,\mathrm{arctan2}(v, u). \tag{3.31}$$

The fact that ρ_α is an isometry follows from the above. Thus the cone is indeed flat. $\triangle$

Nonetheless, not all the Riemannian 2-manifolds are locally isometric with the Euclidean plane.

Proposition 3.18. *The 2-dimensional sphere S^2 is not flat. Due to fact that S^2 is highly symmetric, we have the following: for any point $p \in S^2$ there exists no neighbourhood of p that is isometric to an open subset of the Euclidean plane $(\mathbb{R}^2, \bar{g})$.*

Note that the second part of the proposition above is an immediate consequence of the first part. Moreover, showing that the 2-dimensional sphere is not locally isometric to the Euclidean plane can be done in various ways. See for example problem 5.4 of [Lee97, p. 87]. Another way to prove this proposition, as we will do in the next section, is by defining an intrinsic concept that we call curvature.

3.3 Curvature Tensor Fields

Throughout this section, when we consider a connection ∇ then it is assumed to be the unique Levi-Civita connection. Also recall the fact that covariant differentiation is then intrinsic. Now note that for the Euclidean space $(\mathbb{R}^n, \bar{g})$, see example 3.4, we have the expression:

$$\nabla_X\nabla_Y Z = \nabla_X(\nabla_Y Z) = \nabla_X(Y(Z^k)\partial_k) = X(Y(Z^k))\partial_k$$

and similarly we deduce $\nabla_Y\nabla_X Z = Y(X(Z^k))\partial_k$. Consequently, we obtain the formula

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = \nabla_{[X,Y]}Z \tag{3.32}$$

which follows from the local expression (3.16). Equation (3.32) is said to be the **flatness criterion** because the Euclidean space is flat. It also motivates the following definition.

Definition 3.19. *Let (M, g) be a Riemannian manifold. Then the* **Riemann curvature endomorphism** *is the map $R : \mathcal{T}M \times \mathcal{T}M \times \mathcal{T}M \to \mathcal{T}M$ defined by*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.33}$$

where one denotes $R(X, Y, Z)$ as $R(X, Y)Z$.

We first observe that the map R is skew-symmetric in the first two arguments if $[X, Y] = 0$, in other words: $R(X, Y)Z = -R(Y, X)Z$ for all $Z \in \mathcal{T}M$ if $[X, Y] = 0$.

Example 3.20. Consider the Euclidean space $(\mathbb{R}^n, \bar{g})$. Its connection satisfies the flatness criterion and therefore the Riemann curvature endomorphism R vanishes identically. △

Recall that a connection is defined as a global operator, but it is actually a local operator: we have $\nabla_{\widetilde{X}} \widetilde{Y}|_p = \nabla_X Y|_p$ if $X(p) = \widetilde{X}(p)$ and $Y = \widetilde{Y}$ holds in an arbitrarily small neighbourhood of any point $p \in M$. For the Riemann curvature endomorphism we have a similar but much stronger result.

Proposition 3.21. *For any $X, Y, Z \in \mathcal{T}M$ and any smooth function $f \in C^\infty(M)$, we have*

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z. \tag{3.34}$$

Moreover, we have that $(R(X,Y)Z)|_p \in T_pM$ depends only on $X|_p, Y|_p, Z|_p \in T_pM$. Thus we may consider the Riemann curvature endomorphism as the collection of multilinear maps

$$R|_p : T_pM \times T_pM \times T_pM \to T_pM$$

for each $p \in M$.

Proof. (Based on [Lee97, p. 118]) We will first show (3.34). For any $f \in \mathcal{C}^\infty(M)$ we compute

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f\nabla_X \nabla_Y Z - (f\nabla_Y \nabla_X Z + (Yf)\nabla_X Z) - \nabla_{f[X,Y]-(Yf)X} Z \\ &= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - (Yf)\nabla_X Z - \nabla_{f[X,Y]} Z + (Yf)\nabla_X Z \\ &= fR(X,Y)Z, \end{aligned}$$

where we have used in the second equality the following:

$$[fX,Y]g = fX(Yg) - Y(fXg) = fX(Yg) - fY(Xg) - (Xg)Yf = f[X,Y](g) - (Yf)Xg$$

for any $g \in \mathcal{C}^\infty(M)$. Also, we have mentioned earlier that $R(X, Y)Z$ is skew-symmetric in the first two arguments. This implies that $R(X, fY)Z = fR(X, Y)Z$ holds. Lastly, we compute

$$\begin{aligned} R(X,Y)fZ &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X((Yf)Z + f\nabla_Y Z) - \nabla_Y((Xf)Z + f\nabla_X Z) - ([X,Y]f)Z - f\nabla_{[X,Y]} Z \\ &= (Yf)\nabla_X Z + X(Yf)Z + f\nabla_X \nabla_Y Z + (Xf)\nabla_Y Z - (Xf)\nabla_Y Z - Y(Xf)Z \\ &\quad - f\nabla_Y \nabla_X Z - (Yf)\nabla_X Z - ([X,Y]f)Z - f\nabla_{[X,Y]} Z \\ &= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - \nabla_{f[X,Y]} Z + X(Yf)Z - Y(Xf)Z - ([X,Y]f)Z \\ &= fR(X,Y)Z, \end{aligned}$$

because we can simply write the Lie bracket of X and Y as $[X, Y]f = X(Yf) - Y(Xf)$.

Ultimately, the map R is clearly multilinear over $\mathbb{R}$ and hence (3.34) implies that R is multilinear over $\mathcal{C}^\infty(M)$. By the Tensor Characterization Lemma, see 1.66, we conclude that R is induced by a $\binom{3}{1}$ tensor field. Hence the statement above follows. □

Now let $(\partial_1, ..., \partial_n)$ be any local coordinate frame for TM defined on $U \subset M$ open. Due to proposition 3.21, we can consider the local expressions

$$R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l \tag{3.35}$$

with smooth functions $R^l_{ijk} : U \to \mathbb{R}$, see lemma 1.61 and the proof above, which are said to be the **component functions** of the Riemann curvature endomorphism associated to the given local frame. Equivalently, one can write $R^l_{ijk} = dx^l(R(\partial_i, \partial_j)\partial_k)$.

The following lemma implies that the Riemann curvature endomorphism is intrinsic.

Lemma 3.22. *In local coordinates we have*

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}. \tag{3.36}$$

Proof. Recall the observation done in (3.21). It therefore suffices to do calculations on an open submanifold $(U, g|_U)$ of (M, g) with $(\partial_1, ..., \partial_n)$ a local coordinate frame on U. Since the Lie bracket of two coordinate vector fields always vanishes identically, thus $[\partial_i, \partial_j] = 0$, we only have to look at the following:

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_k = \nabla_{\partial_i}(\Gamma^m_{jk}\partial_m) = \Gamma^m_{jk}\nabla_{\partial_i}\partial_m + (\partial_i\Gamma^m_{jk})\partial_m =$$

$$\Gamma^m_{jk}\Gamma^l_{im}\partial_l + (\partial_i\Gamma^m_{jk})\partial_m = (\partial_i\Gamma^l_{jk} + \Gamma^m_{jk}\Gamma^l_{im})\partial_l.$$

Note that in the first and third equality we have applied the definition of Christoffel symbols, see formula (3.20). The second equality uses the Leibniz property of a connection and the last equality is just a change of index. By doing the same for $\nabla_{\partial_j}\nabla_{\partial_i}\partial_k$, and subsequently using that $R^l_{ijk} = dx^l(R(\partial_i, \partial_j)\partial_k)$ holds, we have finished the proof. $\square$

Example 3.23. Recall example 3.17 and that

$$\bar{g}|_K = (k^2 + 1)\, dt^2 + k^2 t^2 d\theta^2$$

is a local coordinate representation of the induced metric on K with the component functions nowhere zero. With the help of example 3.11, we deduce that the Christoffel symbols of the Levi-Civita connection on $(K, \bar{g}|_K)$ with respect to the local coordinates (t, θ) are:

$$\Gamma^1_{22} = -\frac{k^2}{k^2+1}t \quad \text{and} \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{t} \tag{3.37}$$

and the others are identically zero. According to lemma 3.22, we have for example

$$\begin{aligned} R^2_{121} &= \frac{\partial\Gamma^2_{21}}{\partial t} - \frac{\partial\Gamma^2_{11}}{\partial\theta} + \Gamma^1_{21}\Gamma^2_{11} + \Gamma^2_{21}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} \\ &= -\frac{1}{t^2} - 0 + 0 + \frac{1}{t^2} - 0 - 0 = 0 \end{aligned}$$

and

$$\begin{aligned} R^1_{122} &= \frac{\partial\Gamma^1_{22}}{\partial t} - \frac{\partial\Gamma^1_{12}}{\partial\theta} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} \\ &= -\frac{k^2}{k^2+1} - 0 + 0 + 0 - 0 + \frac{k^2}{k^2+1} = 0 \end{aligned}$$

and similarly we obtain that all the component functions of R are identically zero. This implies, since we can vary the domains of the local coordinates (t, θ) such that it covers K, that the Riemann curvature tensor vanishes identically on K. In other words, the cone $(K, \bar{g}|_K)$ satisfies the flatness criterion, see formula (3.32). $\triangle$

From example 3.17 and the example above we now know that the cone is both flat and it satisfies the flatness criterion. The following theorem shows that it is (obviously) no coincidence.

Theorem 3.24. [Lee97, p. 119] *A Riemannian manifold is flat if and only if it satisfies the flatness criterion: its Riemann curvature endomorphism vanishes identically.*

Instead of working with the Riemann curvature endomorphism R explicitly, one often considers the Riemann curvature tensor. This is done because it is directly related to a $\binom{4}{0}$ tensor field.

Definition 3.25. *The* **Riemann curvature tensor** *is defined by*

$$Rm(X, Y, Z, W) = g(R(X, Y)Z, W) \tag{3.38}$$

for all $X, Y, Z, W \in \mathcal{T}M$.

Note that Rm is a smooth $\binom{4}{0}$ tensor field due to the Tensor Characterisation Lemma, see 1.66, because the map Rm is multilinear over $\mathcal{C}^\infty(M)$. This basically follows from (3.34) in proposition 3.21 and because g is a $\binom{2}{0}$ tensor field. In local coordinates we can write

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \tag{3.39}$$

with smooth component functions $R_{ijkl} : U \to \mathbb{R}$. Note that $R_{ijkl} = g_{lm} R^m_{ijk}$ holds.

Neither a $\binom{4}{0}$ nor a $\binom{3}{1}$ tensor field is very convenient to work with. Therefore the question arises: is there a way to simplify the above definitions without loosing too much information?

Definition 3.26. *The* **Ricci curvature tensor** *is a smooth $\binom{2}{0}$ tensor field defined by*

$$Ric|_p(Y, Z) = tr\big(T_pM \to T_pM,\ X \mapsto R|_p(X, Y)Z\big) \tag{3.40}$$

for all $Y, Z \in T_pM$ and all $p \in M$.

Note that Ric is well-defined thanks to proposition 3.21 and because of the known fact that the trace of an finite dimensional endomorphism is basis independent. Now let $(\partial_1, ..., \partial_n)$ be any local frame of TM defined on U. Then we have

$$Ric|_p(\partial_i|_p, \partial_j|_p) = tr\big(T_pM \to T_pM,\ X^k(p)\partial_k|_p \to X^k(p) R^l_{kij}(p)\partial_l|_p\big) = R^k_{kij}(p)$$

for any $p \in U$ because $(R^l_{kij}(p))_{kl}$ is the matrix associated to the linear map above with respect to the basis $(\partial_1|_p, ..., \partial_n|_p)$ of T_pM whose diagonal elements are $R^1_{1ij}(p), ..., R^n_{nij}(p)$. To summarise, in local coordinates we have:

$$Ric = Ric_{ij} dx^i dx^j = R^k_{kij} dx^i dx^j. \tag{3.41}$$

Note that the smoothness of Ric follows from the fact that its components in any local frame are smooth, see lemma 1.61. Since we have $g_{jk} g^{ik} = \delta^k_j$, we also have $Ric_{ij} = g^{kl} R_{kijl}$. The Ricci curvature tensor is also a symmetric tensor field, because the following holds:

$$Ric_{ij} = g^{kl} R_{kijl} = g^{kl} R_{jlki} = g^{kl} R_{ljik} = g^{lk} R_{ljik} = Ric_{ji},$$

where we have used in the second-last equality the fact that a Riemannian metric is symmetric and in the second and third equality we have used the symmetries of the Riemann curvature tensor: $R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}$. For a proof see [Lee97, p. 121].

Ultimately, a useful observation is that whenever the Ricci curvature tensor does not vanish, we have that the Riemann curvature endomorphism does not vanish as well. Therefore it follows immediately from theorem 3.24 that the Riemannian manifold is not flat. Note that from lemma 3.22 we have in local coordinates the following expression:

$$Ric_{ij} = R^k_{kij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{km} - \Gamma^m_{kj}\Gamma^k_{im}. \tag{3.42}$$

From the above we also deduce that the Ricci curvature tensor is intrinsic too. Completely written out we have that a 2-dimensional Riemannian manifold has the following expression for the first component of the Ricci curvature tensor:

$$\begin{aligned}
Ric_{11} &= \partial_1\Gamma^1_{11} + \partial_2\Gamma^2_{11} - \partial_1\Gamma^1_{11} - \partial_1\Gamma^2_{21} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^2_{21} \\
&\quad + \Gamma^2_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^1_{11} - \Gamma^2_{11}\Gamma^1_{12} - \Gamma^1_{21}\Gamma^2_{11} - \Gamma^2_{21}\Gamma^2_{12} \\
&= \partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{21} + \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^2_{11} - \Gamma^2_{21}\Gamma^2_{12}.
\end{aligned}$$

Example 3.27. Recall example 2.13 and the fact that in geographical coordinates (ϕ, θ) we have the following local coordinate representation of the induced metric on the 2-sphere:

$$\bar{g}|_{S^2} = d\phi^2 + \sin^2\phi \, d\theta^2.$$

Recall that this followed from remark 2.14. Earlier we also deduced with example 3.12 that the Christoffel symbols of $(S^2, \bar{g}|_{S^2})$ with respect to the geographical coordinates are

$$\Gamma^1_{22} = -\sin\phi\cos\phi \quad \text{and} \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{\cos\phi}{\sin\phi} \tag{3.43}$$

and the others are identically zero. Via formula (3.42) we obtain the component functions of the Ricci curvature tensor:

$$Ric_{11} = -\frac{\partial\Gamma^2_{21}}{\partial\phi} - \Gamma^2_{21}\Gamma^2_{12} = \frac{1}{\sin^2\phi} - \frac{\cos^2\phi}{\sin^2\phi} = 1;$$

$$Ric_{22} = \frac{\partial\Gamma^1_{22}}{\partial\phi} - \Gamma^1_{22}\Gamma^2_{21} = (-\cos^2\phi + \sin^2\phi) + \cos^2\phi = \sin^2\phi;$$

and similarly we deduce that $Ric_{12} = Ric_{21} = 0$ holds. So in local coordinates we have the coordinate expression $Ric = d\phi^2 + \sin^2\phi d\theta^2$. Note that $Ric = \bar{g}|_{S^2}$ on $S^2\backslash\{N, S\}$ holds and because of the fact that S^2 is highly symmetric, we conclude that the equality $Ric = \bar{g}|_{S^2}$ holds globally.

Ultimately, the Ricci curvature tensor does not vanish and therefore the Riemann curvature endomorphism does not vanish as well. Hence the 2-dimensional sphere is not flat according to theorem 3.24, which proves proposition 3.18 as promised.

In general, for any $n > 0$ we will write the Ricci curvature tensor on $(S^n, \bar{g}|_{S^n})$ as $Ric[\bar{g}|_{S^n}]$ and similar to the above one can show that $Ric[\bar{g}|_{S^n}] = (n-1)\bar{g}|_{S^n}$ holds. We conclude that all spheres of dimension bigger than 1 are not locally isometric to the Euclidean space. △

Example 3.28. Let us now consider $T = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 - r^2 = 0\}$, the torus of revolution with $R > r > 0$. Note that it is an embedded submanifold of $\mathbb{R}^3$, according to theorem 1.24. Moreover, we have the local parametrisation

$$\rho : V \to \rho(V), \; (\phi, \theta) \mapsto \big((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi\big) \tag{3.44}$$

with for example $V = (0, 2\pi) \times (0, 2\pi)$. Let us also denote (ϕ, θ) as the local coordinates associated to the coordinate map ρ^{-1}.

Conform to many previous examples, we find with the help of equation (2.14) the following local representation of the induced metric on the torus:

$$\bar{g}|_T = r^2 d\phi^2 + (R + r\cos\phi)^2 d\theta^2. \tag{3.45}$$

We deduce with example 3.12 that the Christoffel symbols on $(T, \bar{g}|_T)$ are

$$\Gamma^1_{22} = \frac{1}{r}\sin\theta\phi(R + r\cos\phi) \quad \text{and} \quad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{r\sin\phi}{R + r\cos\phi} \tag{3.46}$$

and any other Christoffel symbol vanishes identically. Via lemma 3.22 we conclude

$$Ric_{\bar{g}|_T} = \frac{r\cos\phi}{R + r\cos\phi} d\phi^2 + \frac{1}{r}\cos\phi(R + r\cos\phi)d\theta^2 \tag{3.47}$$

in local coordinates (ϕ, θ). Note that torus T is a typical example of a surface of revolution, see chapter 5. Therefore one can verify the local expression of the Ricci curvature tensor via equation (5.10). In conclusion, the torus is not flat either. $\triangle$

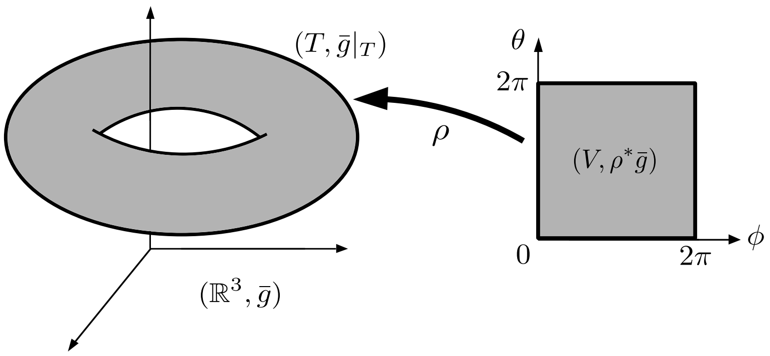

Figure 3.3: A visualisation of example 3.28

In the beginning of this section we said that the Riemann curvature endomorphism is skew-symmetric in the first two arguments when $[X, Y] = 0$. From this we conclude, which can also be seen in lemma 3.22, that we have $R^k_{kij} = -R^k_{ikj}$. Therefore we could have defined the Ricci curvature tensor by taking the trace over the second argument, because it would have given us the same tensor field up to only a sign. Taking the trace of the Riemann curvature endomorphism over the third argument though results into losing all information.

Proposition 3.29. *We have $R^k_{ijk} = 0$ for all indices $1 \leq i, j \leq n$ in any local frame.*

Proof. According to lemma 3.22, we have

$$R^k_{ijk} = \partial_i \Gamma^k_{jk} - \partial_j \Gamma^k_{ik} + \Gamma^m_{jk}\Gamma^k_{im} - \Gamma^m_{ik}\Gamma^k_{jm} = \partial_i \Gamma^k_{jk} - \partial_j \Gamma^k_{ik},$$

because $\Gamma^m_{jk}\Gamma^k_{im} - \Gamma^m_{ik}\Gamma^k_{jm} = 0$ holds due to the presence of the Einstein summation convention. Similarly, we deduce with (3.20) the following:

$$2\Gamma^k_{ik} = g^{kl}(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}) = g^{kl}\partial_i g_{jl}.$$

Consequently, substituting the two expression of the above into one another gives

$$2R^k_{ijk} = \partial_i g^{kl}\partial_j g_{kl} + g^{kl}\partial_i\partial_j g_{kl} - \partial_j g^{kl}\partial_i g_{kl} - g^{kl}\partial_j\partial_i g_{kl} = \partial_i g^{kl}\partial_j g_{kl} - \partial_j g^{kl}\partial_i g_{kl}.$$

This is the result of the product rule and because all component functions g_{kl} are smooth we can exchange the order of differentiation. Our goal is now to show the following:

$$\partial_i g^{kl}\partial_j g_{kl} - \partial_j g^{kl}\partial_i g_{kl} = 0.$$

This encourages us to look at the derivative of the inverse matrix G^{-1}. Suppose $A = (a_{ij})_{ij}$ is an $n \times n$ matrix with $a_{ij} = a_{ij}(z)$ functions dependent on the variable z. Then we can write $I = AA^{-1}$ and subsequently by differentiating both sides, we get $0 = (\frac{d}{dz}A)A^{-1} + A(\frac{d}{dz}A^{-1})$. Consequently, we have

$$\frac{d}{dz}A^{-1} = -A^{-1}\left(\frac{d}{dz}A\right)A^{-1}.$$

In components, the above reduces to

$$\left(\frac{d}{dz}A^{-1}\right)_{ij} = -\sum_m\sum_n a_{im}\left(\frac{d}{dz}a_{mn}\right)a_{nj} = -\sum_m\sum_n a_{im}a_{nj}\left(\frac{d}{dz}a_{mn}\right).$$

Hence we conclude that $\partial_i g^{kl} = -g^{km}g^{nl}\partial_i g_{mn}$ holds. Finally, we obtain

$$2R^k_{ijk} = -g^{km}g^{nl}\partial_i g_{mn}\partial_j g_{kl} + g^{km}g^{nl}\partial_j g_{mn}\partial_i g_{kl} =$$

$$-g^{km}g^{nl}\partial_i g_{mn}\partial_j g_{kl} + g^{nl}g^{km}\partial_j g_{lk}\partial_i g_{nm} = 0,$$

where we have only changed the index names in the second equality and used the fact that the Riemannian metric g is symmetric. In conclusion, all components vanish identically. □

We end this section with two important propositions needed for the next two chapters.

Proposition 3.30. *Let (M, g) and $(M, \lambda g)$ with $\lambda > 0$ be two Riemannian manifolds. By denoting their unique Riemann curvature endomorphism, tensor and Ricci curvature tensor with their metric, we have $R[\lambda g] = R[g]$, $Rm[\lambda g] = Rm[g]$ and $Ric[\lambda g] = Ric[g]$.*

This is clearly an immediate consequence of proposition 3.13. Lastly, as we have claimed a several times in the previous chapter: curvature is preserved under isometries.

Proposition 3.31. [Lee97, p. 119] *The Riemann curvature tensor and Ricci curvature tensor are isometry invariant. More precisely, let $F : (M, g) \to (\widetilde{M}, \tilde{g})$ be an isometry. Then we have*

$$F^*\widetilde{Rm} = Rm \quad \textit{and} \quad F^*\widetilde{Ric} = Ric. \tag{3.48}$$

The proof basically follows from [Lee97, p. 70] and it is quite similar to the proof of the fact that lengths of paths are isometry invariant, see proposition 2.22. Note that the above clarifies the fact that deducing the curvature of a Riemannian submanifold of Euclidean space can either be done on the surface itself, as we now have done a several times, or on the induced Riemannian manifold. The latter will be very useful when considering the Ricci flow.

3.4 Gaussian Curvature

In the beginning of this chapter we said that we want an intrinsic way of measuring curvature. We have shown for example that the Ricci curvature tensor measures curvature in some sense, because when a Riemannian manifold is flat then *Ric* vanishes identically, and when *Ric* does not vanish it implies that is not a flat. More importantly, it is an intrinsic way of measuring curvature since we can write its component functions as

$$Ric_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{km} - \Gamma^m_{kj}\Gamma^k_{im}.$$

Thus *Ric* can be fully determined with only the knowledge of the given metric g. Alternatively, one can also measure curvature in some sense by not doing measurements on the object itself but by considering its ambient space. The upcoming text is quite succinct, thus for a more complete approach of the Gaussian curvature we refer to [Tho79, p. 82] and [Lee97, p. 131].

Suppose we have a 2-dimensional Riemannian submanifold $(S, \bar{g}|_S)$ of Euclidean space $(\mathbb{R}^3, \bar{g})$. One defines the **Weingarten map** at some point $p \in S$ to be the map L_p that measures the turning of a unit normal vector N as one moves in S through p in the direction of $X \in T_pS$. Note that the derivation N is nothing else then the abstract version of $\hat{N}$, see example 3.1. Mathematically, the Weingarten map is just:

$$L_p : T_pS \to T_pS,\ X \to -\overline{\nabla}_X N. \tag{3.49}$$

The minus sign is a choice made for the upcoming definition. One defines the map

$$k_p : T_pS \to \mathbb{R},\ X \mapsto \frac{\bar{g}(L_p(X), X)}{\bar{g}(X, X)}, \tag{3.50}$$

where $k_p(X)$ is said to be the **normal curvature** of S at p in the direction of X. Note that if we have $k(X) > 0$ then S bends towards the unit normal N in the direction of X, and similarly if $k(X) < 0$ then S bends away from N in the direction of X. See also figure 3.4.

Ultimately, for any $p \in S$ we denote $\kappa_1(p)$ and $\kappa_2(p)$ for the minimum and maximum value of the normal curvature k_p, which are known as the **principal curvatures** of S at p.

Definition 3.32. *The* **Gaussian curvature** *of $(S, \bar{g}|_S)$ is the smooth map $K : S \to \mathbb{R}$ defined by the product of the principal curvatures. In other words: $K(p) = \kappa_1(p)\kappa_2(p)$ for all $p \in S$.*

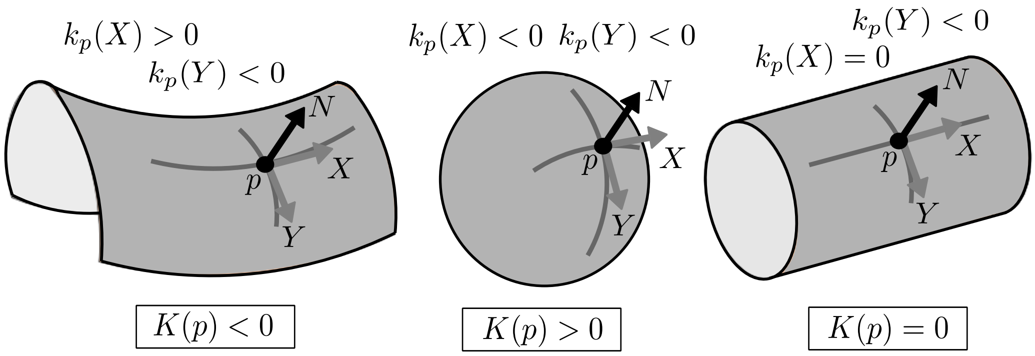

Figure 3.4: A visualisation of the definitions above with $\kappa_1(p) = k_p(X)$ and $\kappa_2(p) = k_p(Y)$.

For any $p \in S$ we have either $K(p) < 0$, $K(p) > 0$ or $K(p) = 0$ and we say that S has negative, positive or zero curvature at p respectively. Note that K is independent of the direction of the unit normal vector N.

Gauss himself proved almost 200 years ago that the Gaussian curvature of S (with its induced metric) is an intrinsic property of S. In other words, we have that K is independent of its isometric embedding in the Euclidean space, even though one determines the value of K at any point p via its ambient space.

Theorem 3.33. (Gauss's Theorema Egregium, 1828) [Lee97, p. 143] *Suppose $(S, \bar{g}|_S)$ is a 2-dimensional Riemannian submanifold of $(R^3, \bar{g})$. For all $p \in M$ we have*

$$K(p) = \frac{Rm(X, Y, Y, X)}{\bar{g}|_S(X, X)\bar{g}|_S(Y, Y) - \bar{g}|_S(X, Y)^2} \tag{3.51}$$

for any $X, Y \in T_pM$ that form a basis of T_pM.

From formula (3.51) one deduces that the Gaussian curvature is an intrinsic measure of curvature, which is moreover strongly related to the Riemann curvature endomorphism. The theorem above also implies, as discussed in [Lee97, p. 144], that in local coordinates we have

$$Rm_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl}), \tag{3.52}$$

where we write $\bar{g}|_S = g_{ij}dx^i dx^j$. Consequently we have

$$Ric_{ij} = g^{kl}R_{kijl} = K(g^{kl}g_{kl}g_{ij} - g^{kl}g_{kj}g_{il}) = K(2g_{ij} - g_{ij}) = Kg_{ij}, \tag{3.53}$$

since we have $g_{jk}g^{ik} = \delta^i_j$. In conclusion, the equality $Ric = K\,\bar{g}|_S$ holds. In other words, we have that Ric is conformal to $\bar{g}|_S$ with the smooth function being the Gaussian curvature K.

Recall examples 3.23, 3.27 and 3.28. We obtain (by looking at the Ricci curvature tensor) that

$$K[\bar{g}|_C] = 0, \quad K[\bar{g}|_{S^2}] = 1 \quad \text{and} \quad K[\bar{g}|_T] = \frac{\cos\phi}{r(R + r\cos\phi)} \tag{3.54}$$

are the Gaussian curvature's of the cone, 2-sphere and torus respectively.

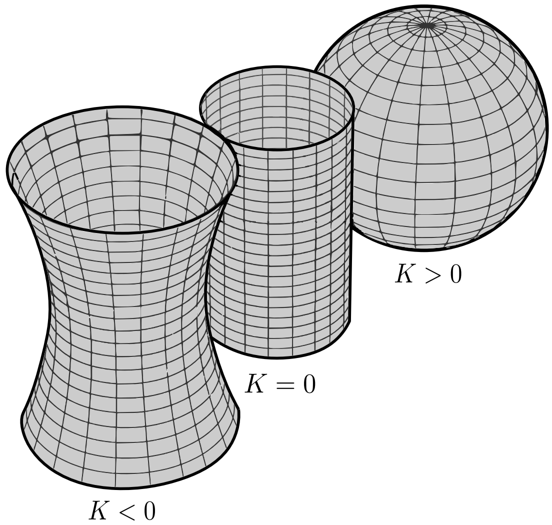

Figure 3.5: A visualisation of three 2-dimensional Riemannian submanifolds of Euclidean space with negative, zero and positive Gaussian curvature everywhere.

For a 2-dimensional submanifold of Euclidean space we have that the intrinsic and extrinsic way of measuring curvature coincide. Hence, the observation above again motivates us to use the Ricci curvature tensor, since we are left with enough significant information concerning the "curvature" of a surface. Also note that we now define the Gaussian curvature for an arbitrary 2-dimensional Riemannian manifold by formula (3.51), see section 4.5.

3.5 Curvature and Parallel Transport

Another important concept in Riemannian geometry is parallel transport. In this thesis we will not discuss parallel transport in great depth but we will mention it briefly for a complete overview of the matter. We refer to [Lee97, p. 59] for the complete abstract approach and [Tho79, p. 46] for a more complete intuitive approach.

Let (M, g) be a Riemannian manifold, which we will interpret as a Riemannian submanifold of Euclidean space, and $\gamma : I \to M$ a smooth path. Moreover recall section 2.3. A smooth vector field V along γ is said to be **Levi-Civita parallel** if V is a constant vector field along the path γ as seen from M. Visually, when a human points his arm into a direction, he fixes his arm and keeps on pointing into that direction while moving over the surface along γ.

Mathematically, see [Lee97, p. 57] or [Tho79, p. 45], one needs to define the **covariant derivative along a path** which needs to be identically zero for all t (in order to experience no change). Note that we will consider the Levi-Civita connection on M and hence the parallel transport depends on the (induced) metric g.

Given a smooth path $\gamma : I \to M$, $t_0 \in I$ and a tangent vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field V along γ such that $V(t_0) = V_0$. Thus we can speak of parallel transporting a vector $X \in T_{\gamma(t_0)}S$ along the path γ. For a proof, see either [Lee97, p. 60] or [Tho79, p. 47].

A piecewise smooth path $\gamma : [a, b] \to M$ is a continuous map such that the restrictions $\gamma|_{(t_i,t_{i+1})}$ are smooth for each $i \in \{0, 1, ..., k\}$ with $a = t_0 < t_1 < ... < t_{k+1} = b$. The **parallel transport** of a tangent vector $X \in T_{\gamma(a)}S$ along a (piecewise) smooth path γ is obtained by transporting the vector X along γ to $\gamma(t_1)$ to get the tangent vector $X_1 \in T_{\gamma(t_1)}S$, then transporting X_1 along γ to $\gamma(t_2)$ to get $X_2 \in T_{\gamma(t_1)}S$, and so on.

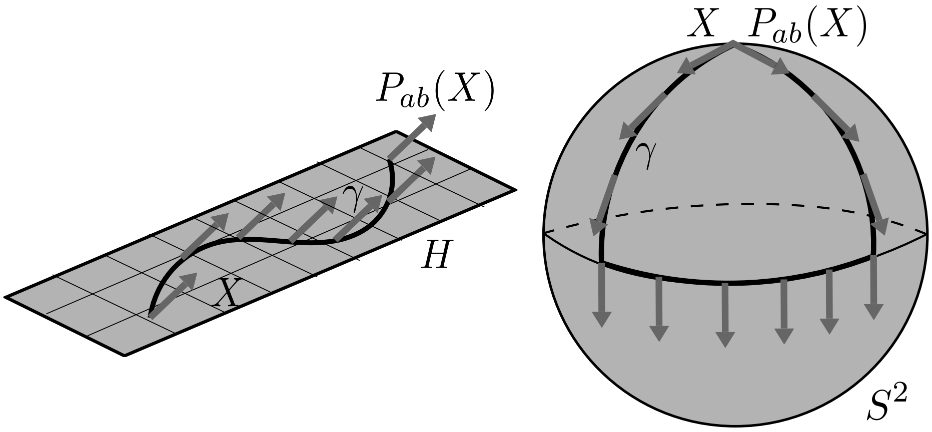

Figure 3.6: A visualisation of parallel transporting a tangent vector $X \in T_{\gamma(a)}M$ along a smooth path $\gamma : [a, b] \to M$ with M being a 2-dimensional plane H (left) and sphere S^2 (right).

As the figure above suggests, when $\gamma : I \to M$ is a (piecewise) smooth path and $t_0, t_1 \in I$, we denote the parallel transport operator from t_0 towards t_1 as follows:

$$P_{t_0t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M. \tag{3.55}$$

Importantly note that $P_{t_0t_1}$ is a linear isomorphism, see [Lee97, p. 62] or [Tho79, p. 51].

Now consider figure 3.6. Note that $P_{ab}(X) = X$ holds in the case of M being the plane H, where we use the real identification given in section 1.4. This is because Levi-Civita parallel corresponds with Euclidean parallel on a plane. Moreover, when γ is a closed smooth path on the plane H, we obtain that the parallel transport operator P_{ab} is the identity map.

Proposition 3.34. [Lee97, p. 129] *Parallel transport around any sufficiently small closed smooth path is the identity whenever (M, g) is flat. More precisely, for any $p \in M$ there exists an open neighbourhood $U \subset M$ of p such that if $\gamma : [a, b] \to U$ is a piecewise smooth path starting and ending at p, then we have that $P_{ab} : T_pM \to T_pM$ is the identity map.*

On the other hand, we deduce from figure 3.6 that we cannot obtain the identity map for a sufficiently small closed smooth path with some kind of triangular shape on the 2-sphere. Due to this observation and the proposition above, we conclude that parallel transport depends on the "curvature" of the manifold. The following theorem is a much stronger statement and gives a geometric interpretation of the Riemann curvature endomorphism.

Theorem 3.35. [Ros03, p. 106] *Consider a 2-dimensional Riemannian submanifold $(S, \bar{g}|_S)$ of Euclidean space and suppose (U, φ) is a chart containing the point $p = \varphi^{-1}(x, y)$. Additionally write $\varphi = (x^1, x^2)$ and suppose $X, Y, Z \in T_pM$ are three tangent vectors such that $X = \frac{\partial}{\partial x^1}\big|_p$ and $Y = \frac{\partial}{\partial x^2}\big|_p$ holds. Now define a linear isomorphism*

$$P = P_{(x,y;\Delta x,\Delta y)} : T_pS \to T_pS, \tag{3.56}$$

with Δx and Δy sufficiently small, by $P(Z) =$ "the parallel transport of Z along the boundary of the rectangle $Rec = \{\varphi^{-1}(a, b) : x \le a \le x + \Delta x,\ y \le b \le y + \Delta y\}$ via the counter-clockwise way". Then the following holds:

$$R(X, Y, Z)\big|_p = \lim_{\Delta x,\Delta y\to 0} \frac{1}{\Delta x\Delta y}\big(Z - P(Z)\big). \tag{3.57}$$

The theorem above is well-formulated due to proposition 3.21. For a more abstract formulation of theorem 3.35, which is in fact a generalisation, see [Lee97, p. 174].

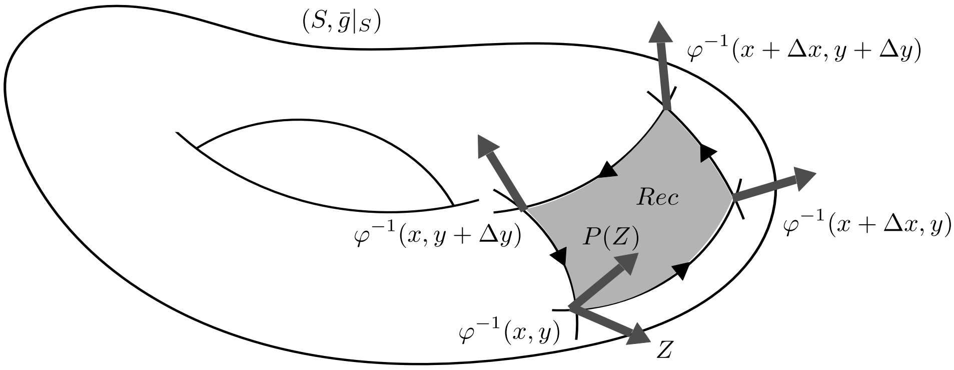

Figure 3.7: A visualisation of theorem 3.35.

In conclusion, the Riemann curvature endomorphism does not only determine whether a Riemannian manifold is flat or not, it also tells us how much it deviates from being flat. This again illustrates that both R and Ric are justifiable ways of measuring "curvature".

Chapter 4

Initiation of the Ricci Flow

In the early 1980's, the Ricci flow was first introduced by Hamilton in his paper: "Three-manifold with positive Ricci curvature", see [Ham82]. The Ricci flow equation is inspired by the classical heat equation from physics. Since heat tends to spread through a solid body until it reaches an equilibrium state of temperature, the Ricci flow was hoped to produce an "equilibrium geometry" for a smooth manifold. Hamilton's idea was to define a kind of non-linear diffusion equation which would smooth out irregularities in a given metric.

The Ricci flow is a geometric evolution of Riemannian metrics where one starts with a Riemannian manifold (M, g_0) and evolves its metric by **Hamilton's Ricci flow equation**

$$\frac{\partial}{\partial t} g(t) = -2Ric[g(t)]. \tag{4.1}$$

This equation was proposed as a strategy for proving the Thurston's Geometrisation conjecture, which concerns the topological classification of 3-dimensional smooth manifolds. A corollary of the Geometrisation conjecture is the well-known Poincaré conjecture. For about 20 years the Ricci flow was quite unknown, until Perelman sketched in 2003 a proof of the Geometrisation conjecture using (an adaptation of) the Ricci flow, see also section 4.5.

Throughout this chapter we will first rigorously define the meaning of $\frac{\partial}{\partial t} g(t)$ in equation (4.1). Subsequently, we consider three types of Ricci flow solutions and examine possible singularities. In the last section, short- and long-term existence and uniqueness of the Ricci flow will be discussed, some intuition behind the link between the classification of manifolds and the Ricci flow will be given, and we will briefly look at the Ricci flow with surgery.

In the last chapter, we will consider a more visual approach of the Ricci flow.

4.1 Time Derivative of Tensor Fields

Most definitions below are are based on the notions in [AH11, p. 69].

Let $\{F(t)\}_{t \in I}$ be a one-parameter family of smooth $\binom{k}{l}$ tensor fields on a smooth manifold M defined on a not necessarily open **time interval** $I \subset \mathbb{R}$. The family $\{F(t)\}_{t \in I}$ is said to be a **sufficiently smooth family** of $\binom{k}{l}$ tensor fields if for all points $p \in M$ we have that

$$F\big|_p(X_1, ..., X_k, \omega_1, ..., \omega_l) : I \to \mathbb{R}, \ t \mapsto F(t)\big|_p(X_1, ..., X_k, \omega_1, ..., \omega_l) \tag{4.2}$$

is a continuously differentiable function for any tangent vectors $X_1, ..., X_k \in T_pM$ and cotangent vectors $\omega_1, ..., \omega_l \in T_p^*M$. In other words, we require $F\big|_p(X_1, ..., X_k, \omega_1, ..., \omega_l) \in \mathcal{C}^1(I)$.

Now let $\xi : I \times U \to \mathbb{R}$ be any function with $U \subset M$ such that $t \to \xi(t,p)$ is continuously differentiable for every $p \in U$. The **time derivative** of ξ is defined by the following usual function:

$$\frac{\partial}{\partial t}\xi : I \times M \to \mathbb{R},\ (s,p) \mapsto \frac{\partial \xi(t,p)}{\partial t}\Big|_{t=s}. \tag{4.3}$$

We will also consider for any $t \in I$ the functions $\frac{\partial}{\partial t}\xi(t) : M \to \mathbb{R},\ p \to \left(\frac{\partial}{\partial t}\xi\right)(t,p)$ which is the time derivative evaluated at time t. Furthermore, note that when $\{F(t)\}_{t\in I}$ is a sufficiently smooth family of $\binom{k}{l}$ tensor fields, we can consider the time derivative of the function

$$F(X_1,...,X_k,\omega_1,...,\omega_l) : I \times M \to \mathbb{R},\ (t,p) \mapsto F(t)\big|_p(X_1|_p,...,X_k|_p,\omega_1|_p,...,\omega_l|_p) \tag{4.4}$$

with $X_1,...,X_k \in \mathcal{T}M$ and $\omega_1,...,\omega_k \in \mathcal{T}^1_0M$ arbitrary.

Definition 4.1. *The* **time derivative** *of a sufficiently smooth family $\{F(t)\}_{t\in I}$ of $\binom{k}{l}$ tensor fields is a one-parameter family, denoted by $\{\frac{\partial}{\partial t}F(t)\}_{t\in I}$, of tensor fields of the same type such that for all time $t \in I$ we have*

$$\left(\frac{\partial}{\partial t}F(t)\right)(X_1,...,X_k,\omega_1,...,\omega_l) = \frac{\partial}{\partial t}F(X_1,...,X_k,\omega_1,...,\omega_l)(t) \tag{4.5}$$

for any smooth vector fields $X_1,...,X_k \in \mathcal{T}M$ and covector fields $\omega_1,...,\omega_k \in \mathcal{T}^1_0M$.

Note that for every $t \in I$ we have that $\frac{\partial}{\partial t}F(t)$ is a multilinear map over $\mathcal{C}^\infty(M)$. Therefore, according to the Tensor Characterisation Lemma it indeed defines a family of $\binom{k}{l}$ tensor fields.

Example 4.2. Let $F \in \mathcal{T}^k_lM$ be any smooth tensor field. Consider $\{G(t)\}_{t\in I}$ to be a one-parameter family with $G(t) = \sigma(t)F$ for all $t \in I$. Suppose $\sigma \in \mathcal{C}^1(I)$, then we have a sufficiently smooth family of tensor fields, and moreover the equality $\frac{\partial}{\partial t}G(t) = \sigma'(t)F$ holds. △

In any local coordinates, we have $F(t) = F^{i_1...i_k}_{j_1...j_l}(t)\,dx^{j_1} \otimes ... \otimes dx^{j_l} \otimes \partial_{i_1} \otimes ... \otimes \partial_{i_k}$ and hence

$$\frac{\partial}{\partial t}F(t) = \frac{\partial}{\partial t}F^{i_1...i_k}_{j_1...j_l}(t)\,dx^{j_1} \otimes ... \otimes dx^{j_l} \otimes \partial_{i_1} \otimes ... \otimes \partial_{i_k}. \tag{4.6}$$

Now we have enough tools to define the Ricci flow properly. Note that when the pair (M,g) is a Riemannian manifold, we denote its Ricci curvature tensor with $Ric[g]$.

Definition 4.3. *A* **Ricci flow** *is a sufficiently smooth family of metrics $\{g(t)\}_{t\in I}$ on a smooth manifold M defined on a time interval I such that it satisfies the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t}g(t) &= -2Ric[g(t)] \\ g(0) &= g_0. \end{cases} \tag{4.7}$$

The metric g_0 is said to be the **initial metric** *on M and (M,g_0) the* **initial state**.

Thus a Ricci flow is some geometric evolution where one starts with a smooth Riemannian manifold (M,g_0) and where its geometry gets altered by changing the metric via the Hamilton's Ricci flow equation. From now one, we will simply say that $g(t)$ is a **Ricci flow solution**.

In any local coordinates, we have $g(t) = g_{ij}(t)\,dx^i \otimes dx^j$ and we can write

$$\frac{\partial}{\partial t}g = \frac{\partial}{\partial t}g_{ij}\,dx^i \otimes dx^j. \tag{4.8}$$

Recall that a Riemannian metric and the Ricci curvature tensor are symmetric, and that Ric is an operator which involves second order derivatives of the metric, see formula (3.42). Therefore the initial value problem (4.7) induces in any local coordinates a system of $\frac{1}{2}n(n+1)$ second order non-linear partial differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} &= -2Ric_{ij}[g] \\ g_{ij}(0) &= (g_0)_{ij}. \end{cases} \tag{4.9}$$

Whenever the local coordinates are in fact global, we have that the initial value problem is equivalent to system (4.9). Otherwise, multiple systems are required to obtain a global solution. Note that we write $\frac{\partial}{\partial t} g(t)$ and $\frac{\partial}{\partial t} g$ interchangeably. The reason for writing $g(t)$ instead of g in definition 4.3 is to emphasize the fact that the operator on the right hand side changes over time too.

A Ricci flow solution may have very different behaviours. In the study of differential equations, one usually asks the questions: what happens if we let $t \to \pm\infty$; on what time interval does the solution exist; is the given solution unique; and does there exist a solution at all? We will discuss these questions in the following sections.

4.2 Ancient Solutions

A solution $g(t)$ of Hamilton's Ricci flow equation is said to be an **ancient solution** if it exists on a maximal time interval $-\infty < t < T$ for a specific $T < \infty$. A well-known ancient solution is the Ricci flow solution of an n-dimensional sphere.

Example 4.4. Recall example 2.13. Consider the n-sphere S^n_R with radius $R > 0$ and its induced metric $g|_{S^n_R}$, and let (S^n, g_0) be the Riemannian manifold with $g_0 = R^2 \bar{g}|_{S^n}$. Importantly note that the Riemannian manifolds $(S^n_R, \bar{g}|_{S^n_R})$ and (S^n, g_0) are isometric, since

$$F : S^n \to S^n_R, \ (x^1, ..., x^n) \to (Rx^1, ..., Rx^n)$$

defines an isometry between them, which we will show at the end of this example. Therefore we can interpret (S^n, g_0) as the n-sphere with radius R embedded in Euclidean space.

Now we are interested in a Ricci flow solution $g(t)$ with (S^n, g_0) the initial state. Since the sphere is highly symmetric, we assume that the solution is of the following form:

$$g(t) = r(t)^2 g|_{S^n} \tag{4.10}$$

with $r : I \to \mathbb{R}$ a scalar function such that $r(0) = R$ holds. Note that (4.10) satisfies initial condition $g(0) = g_0$. As we have discussed in example 4.2, we have

$$\frac{\partial}{\partial t} g(t) = 2r(t)r'(t)g|_{S^n}, \tag{4.11}$$

since the only term that evolves in time is the scalar function r. In example 3.27 we deduced that $(S^n, \bar{g}|_{S^n})$ has Ricci curvature $Ric[\bar{g}|_{S^n}] = (n-1)\bar{g}|_{S^n}$. When (4.10) is indeed a Ricci flow solution, we obtain via proposition 3.30 that $(S^n, g(t))$ has the same Ricci curvature tensor for all time t. In short, we have

$$Ric[g(t)] = Ric[\bar{g}|_{S^n}] = (n-1)g|_{S^n}. \tag{4.12}$$

Substituting the results (4.11) and (4.12) into Hamilton's Ricci flow equation gives us

$$2r(t)r'(t)g|_{S^n} = -2(n-1)g|_{S^n},$$

which reduces to the following ordinary differential equation:

$$r(t)r'(t) = -(n-1). \tag{4.13}$$

Solving the ordinary differential equation above, that satisfies the initial condition $r(0) = R$ as well, gives us the following solution:

$$r(t) = \sqrt{R^2 - 2(n-1)t}. \tag{4.14}$$

Thus we obtain that

$$g(t) = (R^2 - 2(n-1)t)g|_{S^n}. \tag{4.15}$$

is a Ricci flow solution. By uniqueness, see section 4.5, we conclude that this is the solution to Hamilton's Ricci flow equation with initial state (S^n, g_0). Note that $g(t)$ is an ancient solution that implodes in finite time

$$T = \frac{R^2}{2(n-1)}$$

and blows up as $t \to -\infty$. Similar to what we discussed in the beginning of this example, we can interpret $(S^n, g(t))$ as the n-dimensional sphere with radius $r(t)$ embedded in Euclidean space. Hence the sphere shrinks with constant speed under the Ricci flow.

As promised, we will show that the map F is indeed an isometry and we reduce to the $n = 2$ case. Write (ϕ, θ) and (ϕ_R, θ_R) for the geographical coordinates on S^2 and S^2_R respectively. Then we have the local expressions

$$\bar{g}|_{S^2_R} = R^2 d\phi_R^2 + R^2 \sin^2 \phi_R d\theta_R^2 \quad \text{and} \quad g_0 = R^2 d\phi^2 + R^2 \sin^2 \phi d\theta^2.$$

With the help of proposition 1.64 and lemma 1.65, we locally get

$$F^*\bar{g}|_{S^n_R} = R^2(dF^1)^2 + R^2 \sin^2 \phi (dF^2)^2 = R^2 d\phi^2 + R^2 \sin^2 \phi d\theta^2 = g_0,$$

because $dF^1 = \frac{\partial}{\partial \phi}(\phi_R \circ F \circ \rho)d\phi + \frac{\partial}{\partial \theta}(\phi_R \circ F \circ \rho)d\theta = 1 \cdot d\phi + 0 \cdot d\theta = d\phi$ and $dF^2 = d\theta$ holds where ρ denotes the local parametrisation associated to (ϕ_R, θ_R). Since S^2 is highly symmetric, we obtain $F^*\bar{g}|_{S^2} = g_0$ globally. △

Due to the example above, we note that the Ricci flow is relatively easily deduced for generalised Riemannian manifolds with a so-called Einstein metric.

Definition 4.5. *A metric g on a smooth manifold M is said to be an* **Einstein metric** *if we have $Ric[g] = \lambda g$ for some $\lambda \in \mathbb{R}$, and (M, g) is then called an* **Einstein manifold**.

Note that the n-dimensional sphere is an Einstein manifold with $\lambda = n - 1$. The Euclidean space has $\lambda = 0$ and in general a Riemannian manifold is said to be **Ricci-flat** if $\lambda = 0$. Also note that the Ricci flow is a stationary flow if we have a Ricci-flat manifold.

As we recall the Lobachevsky plane and the Poincaré disk in examples 2.6 and 2.15, we note that it has constant Gaussian curvature $K = \lambda = -1$. We can for example obtain, with the help of example 3.12, the Christoffel symbols on the 2-dimensional version of the Poincaré model associated to the standard coordinates:

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{2x}{1 - x^2 - y^2} \quad \text{and} \quad \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} = \frac{2y}{1 - x^2 - y^2}.$$

Subsequently, one deduces with formula (3.42) that the following holds:

$$Ric_{11} = \frac{\partial \Gamma^1_{22}}{\partial y} - \frac{\Gamma^2_{21}}{\partial x} = -\frac{4}{(1 - x^2 - y^2)^2} = -g_{11}, \tag{4.16}$$

since the other terms cancel one another and similarly we deduce that $Ric_{12} = Ric_{21} = 0$ and $Ric_{22} = -g_{22}$ holds. Therefore we have indeed $\lambda = -1$ for both Riemannian manifolds, since they are isometric and because of proposition 3.31 or theorem 3.33.

Example 4.6. (Einstein solutions) Suppose that the initial state (M, g_0) is an Einstein manifold, thus we have $Ric[g_0] = \lambda g_0$ for some $\lambda \in \mathbb{R}$. Completely conform to example 4.4, we reduce the Ricci flow equation to the ordinary differential equation $r'(t)r(t) = -\lambda$ and conclude that

$$g(t) = (1 - 2\lambda t)g_0 \tag{4.17}$$

is a Ricci flow solution. In conclusion, when $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ holds we have that the initial metric g_0 expands, is steady or shrinks respectively under the Ricci flow. △

Recall that we examined in example 4.4 an isometry to interpret the Ricci flow visually. Hence a shrinking or expanding metric can be interpreted as a shrinking or expanding manifold via the same approach.

Remark 4.7. When we consider a Riemannian submanifold $(S, \bar{g}|_S)$ of Euclidean space, we can either look at the Riemannian manifold locally or at its induced Riemannian manifold with the differential equations given by (4.9). Proposition 3.31 tells us that we will get the same Ricci flow (under local isometry). In the last chapter we will work with the induced Riemannian manifold again to enable a visualisation, as in example 4.4, of the Ricci flow.

4.3 Immortal Solutions

A solution $g(t)$ of Hamilton's Ricci flow equation is said to be an **immortal solution** if it exists on a maximal time interval $T < t < \infty$ for a specific $T > -\infty$. Inspired by the analysis in [CK04, p. 34], we consider the following simplified example.

Example 4.8. Suppose $g(t)$ is a Ricci flow solution on $\mathbb{R}^2$ defined for all $t > T$ for some fixed time $-\infty < T < 0$. In polar coordinates, we assume $g(t)$ to have the local representation

$$g(t) = (t - T)(f(r)^2 dr^2 + r^2 d\theta^2), \tag{4.18}$$

with $f : \mathbb{R}^2 \to \mathbb{R}_{>0}$ some positive smooth function yet to be determined and $f(r)$ the polar coordinate representation of f independent of θ. Again with example 3.12, we obtain

$$\Gamma^1_{11} = \frac{f'(r)}{f(r)}, \quad \Gamma^1_{22} = -\frac{r}{f(r)^2} \quad \text{and} \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r} \tag{4.19}$$

for all t and the remaining Christoffel symbols on $(\mathbb{R}^2, g(t))$ vanish identically. Furthermore, with formula (3.42) we deduce that in polar coordinates we have

$$Ric_{11}[g(t)] = \frac{f'(r)}{rf(r)} \quad \text{and} \quad Ric_{22}[g(t)] = \frac{rf'(r)}{f(r)^3} \tag{4.20}$$

and the other two components are zero. Moreover, we clearly have

$$\frac{\partial}{\partial t} g(t) = f(r)^2 dr^2 + r^2 d\theta^2. \tag{4.21}$$

Substituting equations (4.20) and (4.21) into the local version of Hamilton's Ricci flow equation, see also (4.9), results into the following equations that need to hold:

$$f(r)^2 = -\frac{2f'(r)}{rf(r)} \quad \text{and} \quad r^2 = -\frac{2rf'(r)}{f(r)^3}. \tag{4.22}$$

Surprisingly, the two equations above are equivalent statements. In order to determine f, we therefore need to solve the following ordinary differential equation:

$$2f' + rf^3 = 0. \tag{4.23}$$

Since f must be positive, we obtain the following solutions of the above:

$$f(r) = \sqrt{\frac{2}{c + r^2}} \tag{4.24}$$

for any $c > 0$. Because the function f is required to be smooth on $\mathbb{R}^2$ we need to extend it smoothly, hence we conclude that

$$f : \mathbb{R}^2 \to \mathbb{R}_{>0}, \ (x, y) \mapsto \sqrt{\frac{2}{c + x^2 + y^2}} \tag{4.25}$$

for any $c > 0$ are the possible functions in order to let $g(t)$ be a Ricci flow solution. For more information on this Ricci flow, we refer to [CK04, p. 34]. △

Via this procedure, we see that we are able to find infinitely many explicit Ricci flow solutions. Also note that by replacing $(t - T)$ with $(t - T)^2$, we would get two equations like in (4.22) that are unfortunately not equivalent statements, hence no such function f exists. This procedure can therefore be used to check whether solutions of some sort exist.

4.4 Eternal Solutions

A solution $g(t)$ of Hamilton's Ricci flow equation is said to be an **eternal solution** if it exists for all time $-\infty < t < \infty$. These are the solutions that do note attain a singularity at any time $T \in \mathbb{R}$. Before we look at an explicit example of an eternal solution, we first consider a generalisation of the Einstein solutions.

Definition 4.9. *A Ricci flow solution $g(t)$ on M is called a* **Ricci soliton** *if there exists a one-parameter family of diffeomorphisms $\{F_t : M \to M\}_{t \in I}$ with F_0 the identity map such that*

$$g(t) = \sigma(t) F_t^* g(0) \tag{4.26}$$

holds for any $t \in I$ and $\sigma : I \to \mathbb{R}_{>0}$ a continuously differentiable scalar function with $\sigma(0) = 1$. A Ricci soliton $g(t)$ is said to be **shrinking**, **steady** *or* **expanding** *at time $t_0 \in I$ if we have respectively $0 < \sigma(t_0) < 1$, $\sigma(t_0) = 1$ or $\sigma(t_0) > 1$.*

As we recall the Einstein solutions in example 4.6, we note that they are Ricci solitons with the scalar function $\sigma(t) = 1 - 2\lambda t$ and with $\varphi_t = \varphi_0$ for all $t \in I$.

Note that the ancient and immortal solutions we have discussed in the previous sections are all Ricci solitons with a trivial one-parameter family of diffeomorphisms, that is $\varphi_t = \varphi_0$ for any $t \in I$. The upcoming example is a very well-known Ricci flow solution with a non-trivial family of diffeomorphisms.

Example 4.10. (Cigar soliton) Denote (x, y) for the standard coordinates on $\mathbb{R}^2$. Let us now consider the one-parameter family of metrics $\{g(t)\}_{t \in \mathbb{R}}$ defined on $\mathbb{R}^2$ by

$$g(t) = \frac{1}{e^{4t} + x^2 + y^2}(dx^2 + dy^2). \tag{4.27}$$

We note that this is a Ricci flow solution for the initial state $(\mathbb{R}^2, g_0)$ with $g(0) = g_0$. Showing this is analogous to the previous examples. We compute, via example 3.12, that the Christoffel symbols on $(\mathbb{R}^2, g(t))$ associated to the standard coordinates are

$$\begin{aligned} \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} &= -\frac{x}{e^{4t} + x^2 + y^2}; \\ \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} &= -\frac{y}{e^{4t} + x^2 + y^2}. \end{aligned} \tag{4.28}$$

With formula (3.42) we deduce

$$Ric_{11}[g(t)] = \partial_2\Gamma_{11}^2 - \partial_1\Gamma_{21}^2 = \frac{e^{4t}+x^2-y^2}{(1+x^2+y^2)^2} + \frac{e^{4t}-x^2+y^2}{(1+x^2+y^2)^2} = \frac{2e^{4t}}{(e^{4t}+x^2+y^2)^2},$$

since many terms cancel each other out. Similarly we find $Ric_{12}[g(t)] = Ric_{21}[g(t)] = 0$ and ultimately $Ric_{22}[g(t)] = Ric_{11}[g(t)]$. In conclusion we have

$$\mathrm{Ric}[g(t)] = \frac{2e^{4t}}{(e^{4t}+x^2+y^2)^2}(dx^2+dy^2). \tag{4.29}$$

According to (4.8), by differentiating metric (4.27) with respect to time we find

$$\frac{\partial}{\partial t}g(t) = \frac{-4e^{4t}}{(e^{4t}+x^2+y^2)^2}(dx^2+dy^2) \tag{4.30}$$

as desired because the above now shows that $g(t)$ indeed satisfies the Ricci flow equation. This solution is called the **Cigar soliton** because it is a steady Ricci soliton:

$$F_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \mapsto (e^{-2t}x, e^{-2t}y) \tag{4.31}$$

defines a one-parameter family of diffeomorphisms that satisfies $g(t) = F_t^* g_0$. We show this with the help of proposition 1.64 and lemma 1.65. We have

$$F_t^* g_0 = \frac{1}{1+e^{-4t}x^2+e^{-4t}y^2}\big((dF^1)^2+(dF^2)^2\big) = \frac{e^{-4t}}{1+e^{-4t}x^2+e^{-4t}y^2}(dx^2+dy^2) = g(t).$$

Note that the above is just a change of coordinates with $\widetilde{x} = e^{-2t}x$ and $\widetilde{y} = e^{-2t}y$. For more information on this Ricci flow, we refer to [CK04, p. 24] and [CLN06, p. 159]. △

Lastly, note that for all the examples of this chapter we found Ricci flow solutions of the form

$$g(t) = f(t)g_0 \tag{4.32}$$

with $\{f(t)\}_{t\in I}$ a family of smooth functions, thus $f(t) \in \mathcal{C}^\infty(M)$ for all $t \in I$. Equivalently, for all time $t \in I$ we have that $g(t)$ and g_0 are conformal. In examples 4.4 and 4.8 we have for any $t \in I$ that $f(t)$ is some positive constant and hence smooth. Moreover, note that the cigar soliton can be written as the above with

$$f(t) = \frac{e^{4t}+x^2+y^2}{1+x^2+y^2}.$$

Clearly we have $0 < f(t) \in \mathcal{C}^\infty(M)$ for all $t \in \mathbb{R}$, in other words: the cigar solution $g(t)$ is conformal to $g(0)$ for any time $t \in \mathbb{R}$. The following proposition shows that this is no coincidence for 2-dimensional Riemannian manifolds.

Proposition 4.11. *Let (M, g_0) be a 2-dimensional Riemannian manifold. Suppose $g(t)$ is a Ricci flow solution with g_0 as initial metric. Then $g(t)$ is conformal to g_0 for all $t \in I$.*

Proof. Recall section 3.4 where we have deduced that $Ric[g] = Kg$ holds with K the Gaussian curvature as defined in theorem 3.33. Note that K is a smooth function on M. The Ricci flow initial value problem for a 2-dimensional Riemannian manifold is hence equivalent to

$$\begin{cases} \frac{\partial}{\partial t}g(t) & = -2K[g(t)]g(t) \\ g(0) & = g_0. \end{cases}$$

Without loss of generality, we assume $I = [0, T)$ for some $T > 0$. Note that a solution of the above needs to be a sufficiently smooth family of metrics, hence $t \to K[g(t)](p)$ is continuously differentiable for all $p \in M$. A solution must therefore be of the following implicit form:

$$g(t) = g_0 \exp\left(-2\int_0^t K[g(s)]ds\right) = f(t)g_0.$$

Since $K[g(t)] \in \mathcal{C}^\infty(M)$ holds for all $t \in I$, we have $0 < f(t) \in \mathcal{C}^\infty(M)$ for all $t \in I$ because the exponential map is positive. The metrics $g(t)$ and g_0 on M are thus conformal for all $t \in I$. $\square$

We note that the proposition above is very useful for chapter 5. Moreover, it gives rise to the idea that we can classify Riemannian 2-manifold with the conformal property, as we will discuss in the next section. The fact above is unfortunately not true for higher dimensions.

4.5 Main Results of the Ricci Flow

In this section we will succinctly discuss some highlights of the Ricci flow equation. First, an essential step in the study of differential equations is to investigate whether the given initial problem is well-posed and thus has a unique solution. Note that a manifold is said to be **closed** if it is compact in the topological sense (and has moreover no boundary; we excluded manifolds with boundary, see [Lee13, p. 23] for the definition, throughout this entire thesis).

Theorem 4.12. (Short-Time Existence and Uniqueness) [CK04, p. 67] *Any Riemannian metric g_0 on a closed smooth n-manifold M admits a unique Ricci flow solution $g(t)$ on some positive time interval $[0, \varepsilon)$ such that $g(0) = g_0$.*

In 1982 Hamilton originally proved the short-time existence and uniqueness of the Ricci flow, which relied on difficult machinery of the Nash-Moser inverse function theorem, see [Ham82]. Shortly thereafter, in 1983, DeTurck suggested a simplified proof for the short-time existence and uniqueness of the Ricci flow by considering an "equivalent" non-linear system. This is known as **DeTurck's trick**, which will be discussed in detail in section 5.3. A more brief overview of this proof can also be found in [CLN06, p. 113].

In the two dimensional case we know that the Ricci curvature tensor can be written in terms of the Gaussian curvature K as $Ric = Kg$. Examining the Ricci flow equation (4.1) directly, we observe that regions tend to expand when $K < 0$ holds, and similarly regions tend to shrink when we have $K > 0$.

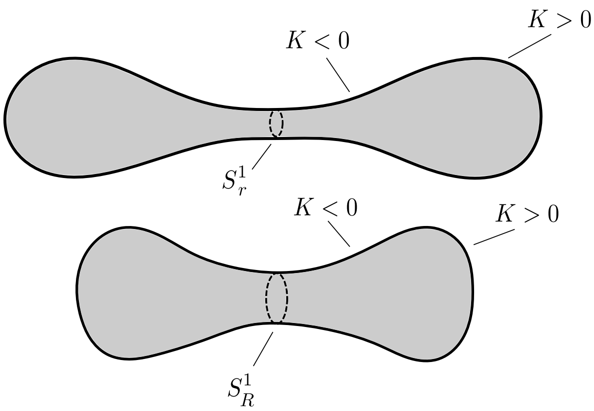

Figure 4.1: A visualisation of the observation discussed above. Here $R > r > 0$ holds.

Now one might guess that the Ricci flow tries to make a 2-sphere even more round. This is indeed the case because in example 4.4 we have deduced that a 2-sphere shrinks evenly to a point under the Ricci flow.

Due to the existence of ancient solutions, we may obtain at some finite time T a singularity and hence we are confronted with the problem that a given manifold may shrink to a point. It is indeed a problem since then we will lose all information concerning the metric. This problem can be solved by considering an adaptation of the Ricci flow. Let us consider the **normalised Ricci flow** initial value problem:

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2Ric[g(t)] + Vol[g(t)] \\ g(0) = g_0. \end{cases} \tag{4.33}$$

The operator Vol is added to the Ricci flow equation in order to preserve the volume of the manifold during the flow. See [CK04, p. 105] or [She06, p. 65] for more detail. Note that integration on smooth manifolds has not been examined throughout this thesis, hence we refer to [Lee13, p. 388] for the Riemannian volume form and [Lee13, p. 400] for a complete course on integration on smooth manifolds.

By preserving the volume of the manifold during the flow, the problem of the manifold shrinking to a point in finite time is eliminated. Consequently, the normalised Ricci solution $g(t)$ cannot obtain any singularities, which can suggest long-time existence and uniqueness.

Theorem 4.13. (Long-Time Existence and Uniqueness) [CK04, p. 105] *Let (M, g_0) be a closed 2-dimensional Riemannian manifold. There exists a unique solution $g(t)$ on $[0, \infty)$ of the normalised Ricci flow equation such that $g(0) = g_0$. Furthermore, the solution $g(t)$ converges to a metric g_∞ that is conformal to g_0 with (M, g_∞) having constant Gaussian curvature.*

In other words, solution $g(t)$ convergences conformally to an Einstein metric g_∞. Note that the above results into a proof of the **Uniformisation Theorem**: every closed Riemannian manifold (M, g_0) admits a metric g_∞ conformal to g_0 such that (M, g_∞) has constant Gaussian curvature. We also must note that the Uniformisation Theorem is not given in its full generality and was originally not proven with the Ricci flow.

The proof given in [CK04] does not prove the Uniformisation Theorem since this classical fact is used within the proof of theorem 4.13. According to [CLT06] however, one can make small adjustments in [CK04] that removes any reliance on the fact and hence the Ricci flow itself proves the Uniformisation Theorem.

Theorem 4.13 now implies that we can categorise all closed Riemannian 2-manifolds into conformal classes. Suppose g_0 is conformal to g_1, then there exists a positive function $f \in \mathcal{C}^\infty(M)$ such that $g_0 = fg_1$ holds. We also have that $g_2 = \lambda g_1$ with $\lambda > 0$ is conformal to g_0. Now recall proposition 3.30, we know that $Ric[g_1] = Ric[g_2]$ holds and therefore we have the following equality: $K[g_1] = \lambda^{-1}K[g_2]$. Since we can simply rescale, we conclude thanks to the Uniformisation Theorem that every closed 2-dimensional Riemannian manifold admits a conformal metric with constant Gaussian curvature $K = -1, 0$ or 1.

Due to the observation above, any closed Riemannian 2-manifold can be classified by precisely one out of the following three spaces:

- the Lobachevsky plane (U^2, g) / Poincaré disk (B^2, h) (recall examples 2.6 and 2.15);
- the Euclidean plane $(\mathbb{R}^2, \bar{g})$;
- and the 2-sphere $(S^2, \bar{g}|_{S^2})$,

because these spaces have constant Gaussian curvature $K = -1, 0$ and 1 respectively, which is stated in (3.54) and (4.16). The precise formulations of geometric structures and the manifold classification above require algebraic topology which is far beyond the scope of this thesis. We like to refer to [Zha14, p. 10, 14] for a detailed overview of the subject matter.

The three dimensional equivalent of the Uniformisation Theorem is **Thurston's Geometrisation Conjecture**, a generalisation of the well-known **Poincaré conjecture**. It shows that any closed Riemannian 3-manifold can be classified by precisely one out of eight different spaces. See [Zha14, p. 16] for more information.

Perelman sketched a proof of the full Geometrisation conjecture in 2003 using (an adjustment of) the Ricci flow. In the two dimensional case we know that the Ricci flow equation, once it is suitably renormalised, let arbitrary metrics flow to Einstein metrics. Unfortunately, renormalisation alone does not work in the three dimensional case since singularities as given in figure 4.2 can occur. He deals with these singularities by implementing the **Ricci flow with surgery**: very roughly, the manifold under the Ricci flow gets cut along the singularities, the manifold then splits into several pieces which are subsequently made smooth again, and then the Ricci flow continues on each of these pieces.

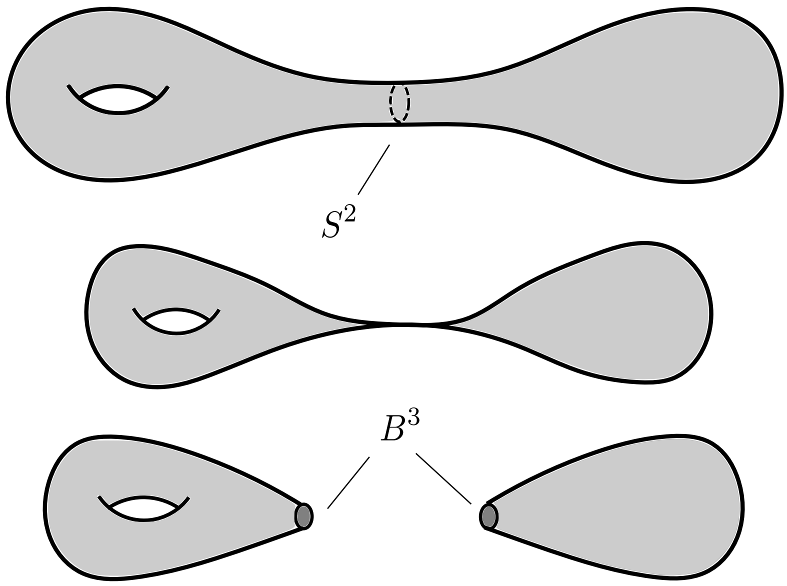

Figure 4.2: A visualisation of the Ricci flow with surgery.

A singularity like in the figure above is commonly known as a **neck pinch**. For a more detailed introduction on surgeries, we refer to [Top06, p. 12].

Chapter 5

Visualisation of the Ricci Flow

The first two sections of this chapter are fully inspired by the work of Rubinstein and Sinclair titled: "Visualizing Ricci Flow of Manifolds of Revolution", see [RS08]. Recall the purpose of this thesis is to establish as much intuition concerning the Ricci flow as possible, which can be achieved best by visualising the geometric evolution equation.

In the article above important results are given without any calculations, therefore we have worked out some of these results in section 5.1 thoroughly. In the next section we demonstrate some visualisations of the Ricci flow thanks to the publicly available code from [RS08], which had to be adjusted in order to work on a Windows computer. In the last section we explain DeTurck's trick in detail and use this for our analysis on the short-time existence and uniqueness of the Ricci flow for a surface of revolution. We tried to approach this problem without using parabolic PDE theory on manifolds and this approach seems to have succeeded. There are only a few parts left which need to be verified in more detail.

Lastly we work with the following mindset: start with a Riemannian submanifold of Euclidean space, consider the Ricci flow on its induced Riemannian manifold(s) and additionally embed for all time t the manifold back into Euclidean space, see also remark 4.7. We are going to examine 2-dimensional manifolds of revolution. That these surfaces tend to remain isometrically embedded in $\mathbb{R}^3$ is discussed and also proven in [RS08, p. 2], and this is what makes direct visualisation possible. As result we are just changing the metric of the initial state but obtain a visual interpretation of the Ricci flow where the manifold itself evolves.

5.1 Surface of Revolution

Let us consider a smooth injective curve $\gamma(\phi) = (a(\phi), b(\phi))$ in the xz-plane of $\mathbb{R}^3$ defined on some (not necessarily open) interval I with the following two properties: we require $\dot{\gamma}(\phi) \neq 0$ for all $\phi \in I$, and $a > 0$ on the interior and $a = 0$ at the possible end points of I.

Subsequently, let $S \subset \mathbb{R}^3$ be the **surface of revolution** obtained by revolving the image of the curve γ about the z-axis. Hence a local parametrisation of S would for example be

$$\rho(\phi, \theta) = (a(\phi)\cos\theta, a(\phi)\sin\theta, b(\phi)), \tag{5.1}$$

with a domain $V = (\phi_1, \phi_2) \times (\theta_1, \theta_2)$ where $(\phi_1, \phi_2) \subset I$ and $0 < \theta_2 - \theta_1 \leq 2\pi$. Recall (ϕ, θ) may be interpreted as standard coordinates on V or as local coordinates on S, see remark 2.14.

Furthermore, note that we are deliberately going to use the same notation for variables and components as in [RS08]. The only difference is that we write ϕ instead of ρ in order to be consistent with previous chapters.

Obviously a surface of revolution naturally extends to a smooth manifold. Examples of surfaces of revolutions are: the cone, see example 3.23; the sphere, see example 2.13; and the torus, see example 3.28. Recall that a manifold is said to be closed if it is compact in the topological sense. The cone is not closed and would be obtained by a generating curve that has an open interval I. On the other hand, the sphere and torus are closed and would be obtained by a generating curve that has a closed interval I.

Now we will equip the surface of revolution S with its induced metric $\bar{g}|_S$, which in local coordinates (ϕ, θ) has the following expression according to formula (2.14):

$$\begin{aligned}
\rho^*\bar{g} &= d(\rho^1)^2 + d(\rho^2)^2 + d(\rho^3)^2 \\
&= \left[\left(\frac{da}{d\phi}\right)^2 \cos^2\theta + \left(\frac{da}{d\phi}\right)^2 \sin^2\theta + \left(\frac{db}{d\phi}\right)^2\right] d\phi^2 + \big((-a\sin\theta)^2 + (a\cos\theta)^2\big)d\theta^2 \\
&= \left[\left(\frac{da}{d\phi}\right)^2 + \left(\frac{db}{d\phi}\right)^2\right] d\phi^2 + a^2 d\theta^2.
\end{aligned}$$

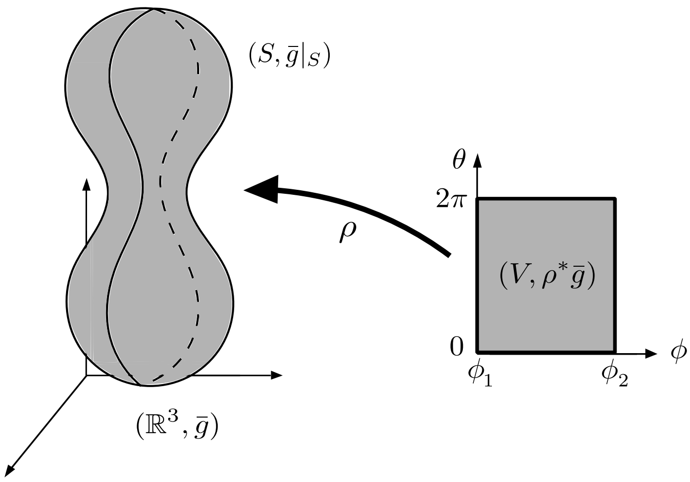

Figure 5.1: A visualisation of a closed surface of revolution with a local parametrisation ρ and where we have taken $\theta_1 = 0$ and $\theta_2 = 2\pi$.

So we can locally write $\bar{g}|_S = h(\phi)d\phi^2 + m(\phi)^2 d\theta^2$ with h and m yet to be determined. Note their independence of θ. For our convenience we will interpret (ϕ, θ) as the standard coordinates on the induced Riemannian manifold $(V, \rho^*\bar{g})$ and consider the matrix notation

$$G = \begin{pmatrix} h(\phi) & 0 \\ 0 & m(\phi) \end{pmatrix}. \tag{5.2}$$

From the derivation above, we deduce the following relations:

$$a(\phi) = \sqrt{m(\phi)} \tag{5.3}$$

and

$$\frac{db}{d\phi} = \sqrt{h(\phi) - \left(\frac{d\sqrt{m(\phi)}}{d\phi}\right)^2}$$

and hence

$$b(\phi) = \int_C^\phi \sqrt{h(s) - \left(\frac{d\sqrt{m(s)}}{ds}\right)^2}\, ds, \tag{5.4}$$

with $C \in I$ some arbitrary constant.

From now on, we will consider the closed interval $I = [0, \pi]$ and a closed surface S. Note that in [RS08, p. 4] they have chosen for $C = 0$, however we like to point out that for $C = \frac{\pi}{2}$ we obtain the sphere with the centre of mass at the origin. More precisely, consider

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \phi \end{pmatrix}.$$

Then by (5.3) and (5.4) we get $a(\phi) = \sin \phi$ and

$$b(\phi) = \int_C^{\phi} \sqrt{1 - \cos^2 s} ds = \cos^2 \phi - \cos^2 C,$$

which equals $\cos^2 \phi - 1$ for $C = 0$ and $\cos^2 \phi$ for $C = \frac{\pi}{2}$. The latter implies the 2-sphere S^2. We note that the only difference is the precise embedding into the Euclidean space, as we already knew from equation (5.1).

Before we determine a Ricci flow of S we first extend m and h at the poles. Since S is assumed to be a closed surface, we require

$$m(\phi_{pole}) = 0 \tag{5.5}$$

for $\phi_{pole} \in \{0, \pi\}$. Note that this follows from the fact $a(\phi_{pole}) = 0$. Intuitively, m should tend to zero as $\phi \to \phi_{pole}$ because θ is undefined at the poles. See also [RS08, p. 4]. Similarly, since smoothness at the poles are demanded, we also require

$$\frac{\partial \sqrt{m(\phi)}}{\partial \phi}\Big|_{\phi = \phi_{pole}} = \sqrt{h(\phi_{pole})} \tag{5.6}$$

for $\phi_{pole} \in \{0, \pi\}$. This relation is obtain by considering the equations (5.1) and (5.4).

Now let us determine the Ricci curvature tensor for the surface of revolution $(S, \bar{g}|_S)$. As we recall example 3.12, we deduce that the Christoffel symbols of $(S, \bar{g}|_S)$ with respect to the local coordinates (ϕ, θ) are

$$\begin{aligned} \Gamma^1_{11} &= \frac{h_\phi}{2h}; & \Gamma^1_{12} &= \Gamma^1_{21} = 0; \\ \Gamma^1_{22} &= -\frac{m_\phi}{2h}; & \Gamma^2_{11} &= 0; \\ \Gamma^2_{12} &= \Gamma^2_{21} = \frac{m_\phi}{2m}; & \Gamma^2_{22} &= 0. \end{aligned} \tag{5.7}$$

Via formula (3.42) we obtain the component functions of the Ricci curvature tensor:

$$\begin{aligned} Ric_{11} = -\partial_1 \Gamma^2_{21} + \Gamma^1_{11}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^2_{12} &= -\frac{d}{d\phi}\left(\frac{m_\phi}{2m}\right) + \frac{h_\phi m_\phi}{4hm} - \frac{m_\phi^2}{4m^2} = \\ \frac{m_\phi^2}{2m^2} - \frac{m_{\phi\phi}}{2m} + \frac{h_\phi m_\phi}{4hm} - \frac{m_\phi^2}{4m^2} &= \frac{m_\phi^2}{4m^2} - \frac{m_{\phi\phi}}{2m} + \frac{h_\phi m_\phi}{4hm}, \end{aligned} \tag{5.8}$$

and analogously one shows that $Ric_{12} = Ric_{21} = 0$ holds and

$$Ric_{22} = \frac{m_\phi^2}{4mh} - \frac{m_{\phi\phi}}{2h} + \frac{h_\phi m_\phi}{4h^2}. \tag{5.9}$$

Hence we have obtained the Ricci curvature tensor on the induced manifold $(V, \rho^* g)$ too with the local coordinates (ϕ, θ) being the standard coordinates. Also importantly note the following antisymmetry:

$$Ric_{11} = \frac{Q}{g_{22}} \quad \text{and} \quad Ric_{22} = \frac{Q}{g_{11}}, \tag{5.10}$$

where $Q = Q(\phi) = \frac{m_\phi^2}{4m} - \frac{m_{\phi\phi}}{2} + \frac{h_\phi m_\phi}{4h}$. This is because the object is rotationally symmetric.

Now let us consider a Ricci flow of S with its induced metric. Thanks to proposition 4.11, we know that a Ricci flow solution is conformal for all t, as long the solution exists, to the initial metric $\bar{g}|_S$. The Ricci flow initial value problem induces therefore a system of just two instead of three second order non-linear partial differential equations on the induced Riemannian manifold $(V, \rho^*\bar{g})$, see system (4.9). Moreover, we obtain that the local solution

$$G(t) = \begin{pmatrix} h(t;\phi) & 0 \\ 0 & m(t;\phi) \end{pmatrix} \tag{5.11}$$

remains independent of the local coordinate θ due to its rotational symmetry.

Since h and m are independent of the local coordinate θ, we can just equivalently consider the partial differential equations on the interior of I. Hence, let us suggest the following initial boundary problem:

$$\begin{cases} h_t = \dfrac{m_{\phi\phi}}{m} - \dfrac{m_\phi^2}{2m^2} - \dfrac{h_\phi m_\phi}{2hm}, & \phi \in (0,\pi), \\ m_t = \dfrac{m_{\phi\phi}}{h} - \dfrac{m_\phi^2}{2mh} - \dfrac{h_\phi m_\phi}{2h^2}, & \phi \in (0,\pi), \\ m(t;\phi_{pole}) = 0, & \phi_{pole} \in \{0,\pi\}, \\ \dfrac{\partial\sqrt{m(t;\phi)}}{\partial\phi}\Big|_{\phi=\phi_{pole}} = \sqrt{h(t;\phi_{pole})}, & \phi_{pole} \in \{0,\pi\}, \\ h(0;\phi) = h_0(\phi) \quad \text{and} \quad m(0;\phi) = m_0(\phi), & \phi \in [0,\pi]. \end{cases} \tag{5.12}$$

The boundary conditions above ensure us that as time flows onward, the embedded manifold in Euclidean space remains closed and remains smooth at the poles, see (5.5) and (5.6). Therefore, problem (5.12) is an equivalent formulation (under local isometry) of the Ricci flow equation with initial state $(S, \bar{g}|_S)$ due to the main results above. Even more precisely, the equivalence follows from the fact that the surface of revolution remains isometrically embedded so long as the Ricci flow equation gives a sufficiently smooth solution, see [RS08, p. 3].

The system above can be solved numerically, as is done in the article. We note that the authors approach it a bit differently and define reparametrisations in order to establish a numerical stable program. Also note that by solving it numerically, we use the fact that there exists a unique (short-time) Ricci flow solution, see theorem 4.12.

By solving the system above, one needs to choose h_0 and m_0 properly because not all functions define a surface of revolution, see equation (5.3) and (5.4). As suggested by [RS08, p. 5], the initial metric in the code is chosen to be based on

$$h_0(\phi) = 1 \quad \text{and} \quad m_0(\phi) = \left(\frac{\sin\phi + c_3\sin(3\phi) + c_5\sin(5\phi)}{1 + 3c_3 + 5c_5}\right)^2, \quad \phi \in [0,\pi] \tag{5.13}$$

with $-\varepsilon_i < c_i < \xi_i$ and $\varepsilon_i, \xi_i > 0$ sufficiently small. This metric is inspired by the metric of the 2-sphere: $c_3 = c_5 = 0$. Observe that h_0 and m_0 are chosen such that it satisfies the boundary conditions. Indeed we have

$$m_0(\phi_{pole}) = 0 \quad \text{and} \quad \frac{\partial\sqrt{m_0(\phi)}}{\partial\phi}\Big|_{\phi=\phi_{pole}} = \sqrt{h_0(\phi_{pole})}. \tag{5.14}$$

5.2 Visuals and Code

The `Ricci_rot` program is a publicly available code written in C and visualises the Ricci flow of a surface of revolution with the induced Riemannian metric based on formulae (5.13).

As is mentioned in the article [RS08, p. 6], it has only been tested on a Mac OS X. To get it to work on a computer with Windows 10, we have to:

- replace `#include <GLUT/glut.h>` by `#include <GL/glut.h>`;
- add the line `#include <windows.h>` at the beginning of the code;
- make sure that OpenGL Utility Toolkit (GLUT) is installed and added into the library.

We compiled the (modified) code with the free platform Code::Blocks. The original code and application can be found in the `ricci_src` file in following link:

`http://pub.math.leidenuniv.nl/~hupkeshj/ricci_simulations.zip`

This link moreover includes an executable file `Ricci_rot_windows` of the `Ricci_rot` program that should work on a Windows 10 computer instantly. Note that, when opening this file, the files `glut32.dll` and `libgcc_s_dw2-1.dll` should both be contained in the same folder as the executable file.

How to use the program: Upon launch, a window is opened with the image of a 2-sphere. This initial state can be altered by holding down the left mouse button and dragging the mouse in any direction. The possible initial states are based on (5.13), where c_3 is varied by horizontal and c_5 by vertical motion of the mouse.

Pressing `f` will put the program into flow mode. At this moment, dragging the mouse will rotate the surface. When one has chosen a particular surface, press the up-arrow key. Holding it results into seeing the Ricci flow continuously. Pressing `n` will put the program back into its initial state.

Once the program meets numerical instability, the flow will stop. Any ripples which may appear on the surface at this stage are a result of numerical instability. The down-arrow key flows the surface backwards in time. This evolution however is highly unstable.

Pressing `m` at any moment will change the display mode into showing the components of the induced metric: $h(t;\phi)$ in green and $m(t;\phi)$ in blue. Pressing `s` brings us back to the surface in $\mathbb{R}^3$. See [RS08, p. 6] for a more detailed description of the `Ricci_rot` program.

On the next two pages, one finds three visualisations of possible Ricci flow solutions. The pictures have been taken at equal time intervals Δt and are drawn to the same scale. In order to track the elapsed time of the Ricci flow, two small pieces were added into the code. After lines 860 and 934 of the original code, the following were added:

```
861 /* Addition */
862 printf("Elapsed time: ");
863 printf("%f\n", tm);
864 /* Addition */
```

and respectively

```
935 /* Addition */
936 printf("New time: ");
937 tm = 0.0;
938 printf("%f\n");
939 /* Addition */
```

We briefly note that the `Ricci_rot` program is a very nice visualisation tool which is endowed with not that large numerical errors. For example, the 2-sphere S^2 with radius 1 (the simplest case) shrinks theoretically to a point at exactly $T = 0.5$, see formula (4.4). Numerically however it shrinks within less then 0.2, where time steps are taken of $dt = 0.0001$.

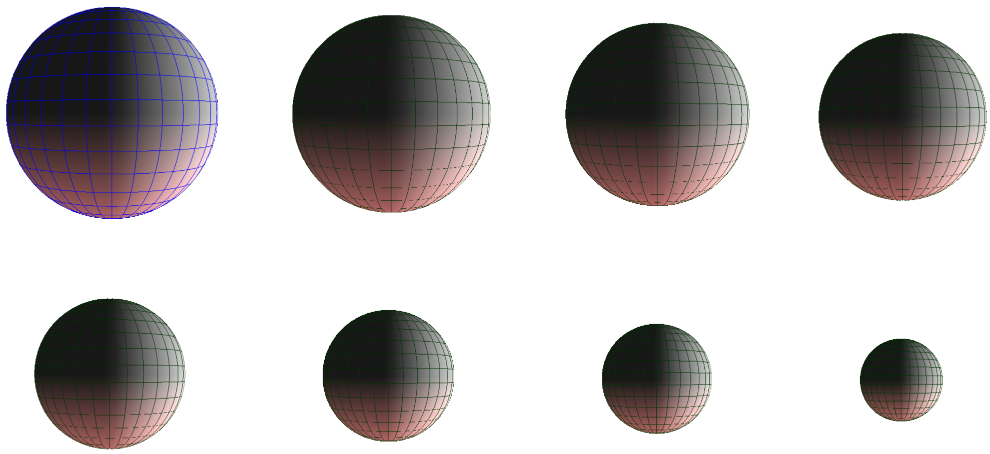

Figure 5.2: Evolution of the Ricci flow with $c_3 = c_5 = 0$, see equation (5.13). The pictures have been taken at equal time intervals $\Delta t = 0.0150$ and are drawn to the same scale.

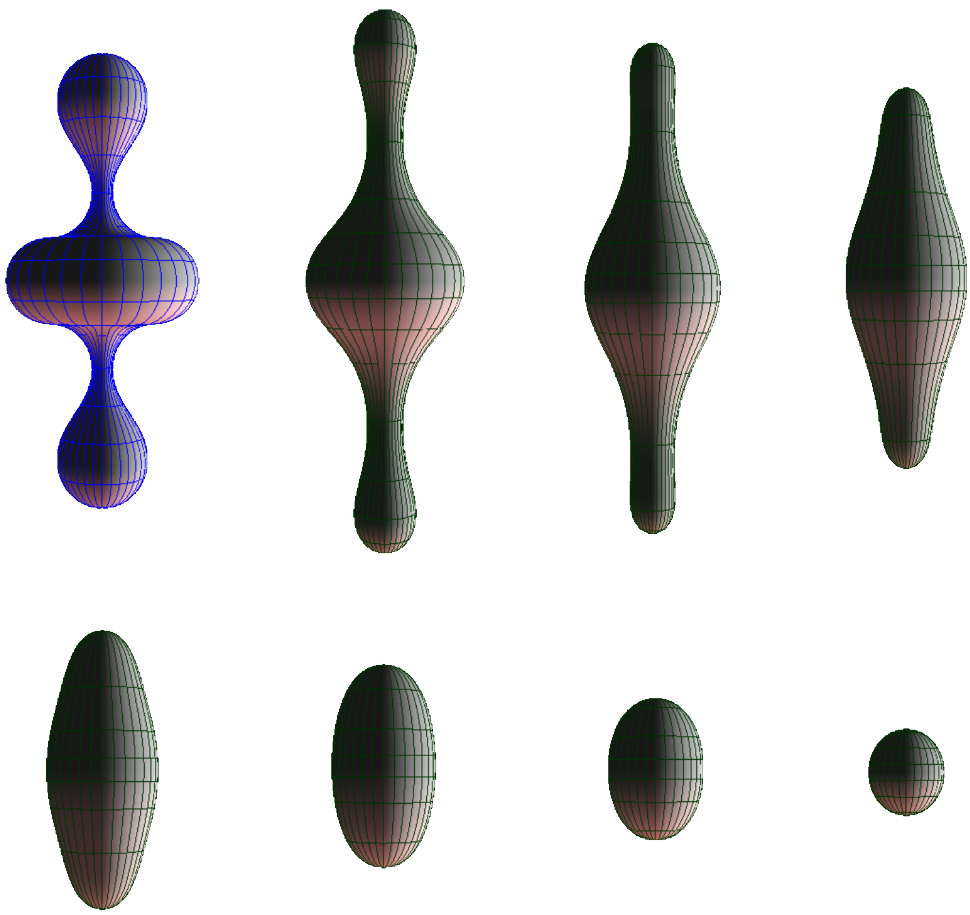

Figure 5.3: Evolution of the Ricci flow with $c_3 = -0.12$ and $c_5 = 0.55$, see equation (5.13). The pictures have been taken at equal time intervals $\Delta t = 0.0050$ and are drawn to the same scale.

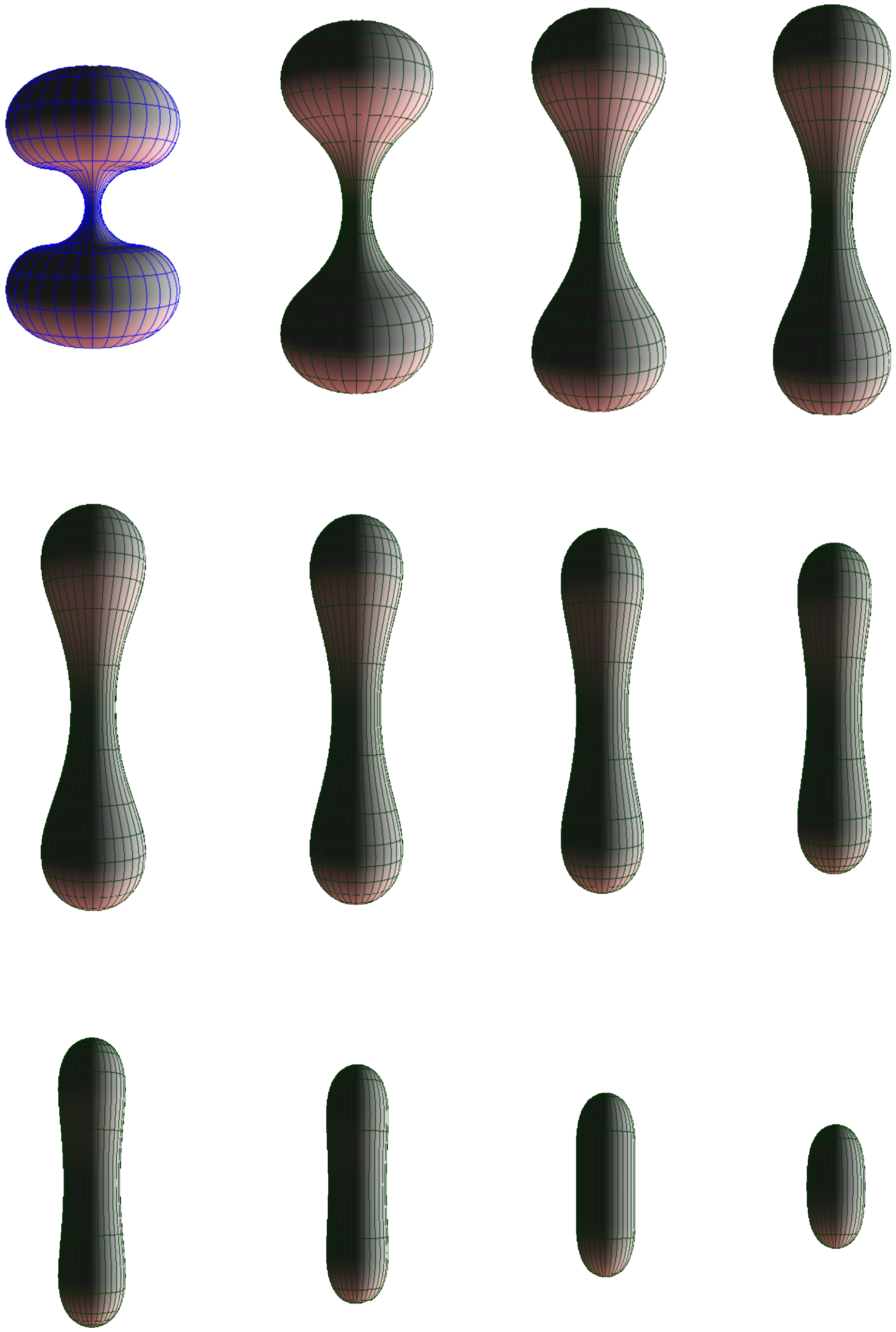

Figure 5.4: Evolution of the Ricci flow with $c_3 = 0.7$ and $c_5 = -0.15$, see equation (5.13). The pictures have been taken at equal time intervals $\Delta t = 0.0050$ and are drawn to the same scale.

Observe that the behaviour of the Ricci flow solutions of surfaces of revolution based on (5.13) is just like one would expect, recall the discussion in section 4.5. The surfaces tend to go to a surface with positive constant Gaussian curvature, namely a sphere.

5.3 Short-Time Existence and Uniqueness

Theorem 4.12 was first proven by Hamilton and shortly thereafter a more elegant proof was given by DeTurck. The short-time existence of the Ricci flow can established by considering DeTurck's trick, which will be discussed in detail soon. We hence consider this trick for our explicit initial boundary problem (5.12) too. We note that our upcoming approach will be a lot more explicit and also quite different in comparison to the general approach.

In order to explain DeTurck's trick, we first must define the Lie derivative of a metric.

Definition 5.1. *Let (M, g) be a Riemannian manifold. The* **Lie derivative** *of the metric g along a vector field $X \in \mathcal{T}M$, denoted by $\mathcal{L}_X g$, is defined at any point $p \in M$ by*

$$(\mathcal{L}_X g)|_p = \frac{d}{dt}\Big|_{t=0} (F_t^* g)\big|_p, \tag{5.15}$$

where $\{F_t\}_{t \in I}$ is a one-parameter family of diffeomorphisms from a neighbourhood in M to another neighbourhood in M with $F_0 : M \to M$ the identity map.

Intuitively, if you have a metric g and a vector field X, then $\mathcal{L}_X g$ is the infinitesimal change you would experience when we flow g using the vector field X. To make the Lie derivative more manageable, we will now state an important result. Note that the upcoming requires the Tensor Characterisation Lemma and other observations from section 1.7.

Proposition 5.2. [Lee13, p. 321] *Let (M, g) be a Riemannian manifold. The Lie derivative of the metric g along a vector field $X \in \mathcal{T}M$ is a smooth $\binom{2}{0}$ tensor field such that*

$$(\mathcal{L}_X g)(Y, Z) = Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \tag{5.16}$$

holds for any vector fields $Y, Z \in \mathcal{T}M$.

Now suppose $(\partial_1, ..., \partial_n)$ is any local frame. In these local coordinates, we can locally write the smooth vector fields as $X = X^k \partial_k$ and obtain the local expression

$$(\mathcal{L}_X g)_{ij} = X^k \partial_k g_{ij} + g_{ik} \partial_j X^k + g_{jk} \partial_i X^k. \tag{5.17}$$

Note that the above follows from the local expression of the Lie bracket, see equation (3.16), and the fact that a metric g is bilinear. Now **DeTurck's trick** is basically that we are encouraged to consider a **Ricci-DeTurck flow** initial value problem, which is defined as

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2Ric[g(t)] + \mathcal{L}_{W(t)}[g(t)] \\ g(0) = g_0 \end{cases} \tag{5.18}$$

with $\{W(t)\}_{t \in I}$ a one-parameter family of global vector fields, often said to be a time-dependent vector field $W(t)$, such that in any local coordinates its components satisfy

$$W^k = g^{ij}\big(\Gamma_{ij}^k - \widetilde{\Gamma}_{ij}^k\big). \tag{5.19}$$

These functions $\widetilde{\Gamma}_{ij}^k$ denote the Christoffel symbols of the Levi-Civita connection on $(M, \widetilde{g})$ with $\widetilde{g}$ an arbitrary Riemannian metric on M. Fix a metric $\widetilde{g}$ and call it our **background metric**. Because the difference of two connections becomes a smooth tensor field, see [Lee97, p. 63], it follows that the smooth vector field W is well-defined, that is: $W(t)$ is independent of the choice of local coordinates.

Note that in any local coordinates a Ricci-DeTurck flow is written as

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2Ric_{ij}[g] + (\mathcal{L}_W g)_{ij} \\ g_{ij}(0) = (g_0)_{ij}. \end{cases} \tag{5.20}$$

Now one might wonder why we would look at the Ricci-DeTurck flow equation (5.18) instead of the original Ricci flow equation. More importantly, why would this even help at all? We will answer this question very briefly and refer to [CK04, p. 81] for more details. Whenever $g(t)$ is a Ricci-DeTurck flow solution, one can define a one-parameter family of diffeomorphisms $\{F_t : M \to M\}_{t\in I}$ such that $F_0 : M \to M$ is the identity map and we have

$$\frac{\partial}{\partial t} F_t = -W(t). \tag{5.21}$$

Because M is a closed smooth manifold, there exists such a one-parameter family. Subsequently, with the help of definition 5.1, we obtain that $F_t^* g(t)$ is a Ricci flow solution:

$$\begin{aligned}
\frac{d}{dt}\big(F_t^* g(t)\big) &= \frac{d}{ds}\Big|_{s=0}\big(F_{t+s}^* g(t+s)\big) \\
&= F_t^*\left(\frac{d}{dt} g(t)\right) + \frac{d}{ds}\Big|_{s=0}\big(F_{t+s}^* g(t)\big) \\
&= F_t^*\big(-2Ric[g(t)]) + \mathcal{L}_{W(t)}[g(t)]\big) + \frac{d}{ds}\Big|_{s=0}\big((F_t^{-1}\circ F_{t+s})^* F_t^* g(t)\big) \\
&= -2Ric[F_t^* g(t)] + F_t^*\big(\mathcal{L}_{W(t)}[g(t)]\big) - \mathcal{L}_{(F_t^{-1})_* W(t)}\big(F_t^* g(t)\big) \\
&= -2Ric[F_t^* g(t)].
\end{aligned}$$

In conclusion, if we can proof the short-time existence and uniqueness for the Ricci-DeTurck flow, we have a unique solution $g(t)$ of (5.18) and therefore we have obtained the short-time existence of the Ricci flow. The uniqueness of the Ricci flow follows from the fact that the one-parameter family $\{F_t\}_{t\in I}$ is unique, see [CK04, p. 89].

The reason why we can "easily" proof that the Ricci-DeTurck flow admits short-time existence and uniqueness is because it is a **parabolic equation**. In other words, we have $\frac{\partial}{\partial t} g(t) = L[g(t)]$ with $L[g] = -2Ric[g] + \mathcal{L}_{W(t)}[g]$ which is an **elliptic operator**, see [CK04, p. 71] for the precise definition. A crucial example of an elliptic operator L is the standard Laplacian Δ.

As mentioned earlier, our aim is to proof short-time existence and uniqueness of the initial boundary problem (5.12) without using this parabolic PDE theory on manifolds. Thanks to the above, our approach became to consider an explicit Ricci-DeTurck flow and show that it admits short-time existence and uniqueness. Consequently, we tried to show the short-time properties with the help of standard parabolic PDE theory on the interval $I = [0, \pi]$.

Consider a Ricci-DeTurck flow on a surface of revolution (S, g_0) with $g_0 = g|_S$. In the original proof one takes $\widetilde{g}$ to be arbitrary, but we will let $\widetilde{g} = g_0$ be our background metric in order to simply the problem. Recall the Christoffel symbols of the surface (S, g_0) associated to the local coordinates (ϕ, θ), see equation (5.7). Hence the time-dependent component functions of the global time-dependent vector field $W(t)$ are

$$W^1 = g^{11}(\Gamma_{11}^1 - \widetilde{\Gamma}_{11}^1) + g^{22}(\Gamma_{22}^1 - \widetilde{\Gamma}_{22}^1) = \frac{h_\phi}{2h^2} - \frac{m_\phi}{2hm} + \Psi \quad \text{and} \quad W^2 = 0, \tag{5.22}$$

and due to the initial conditions (5.13) we have

$$\Psi = \frac{m_{0_\phi}}{2h_0 m} - \frac{h_{0_\phi}}{2h_0 h} = \frac{m_{0_\phi}}{2m}. \tag{5.23}$$

The Lie derivative along W associated to the coordinates (ϕ, θ), see formula (5.17), is given by

$$\begin{aligned}
(\mathcal{L}_W g)_{11} &= W^1 \partial_1 g_{11} + 2 g_{11} \partial_1 W^1 \\
&= \left(\frac{h_\phi}{2h^2} - \frac{m_\phi}{2hm}\right) h_\phi + 2h\left(\frac{h_{\phi\phi}}{h^2} + \frac{h_\phi m_\phi}{h^2 m} - \frac{2h_\phi^2}{h^3} - \frac{m_{\phi\phi}}{hm} + \frac{m_\phi^2}{hm^2}\right) + \Psi_1
\end{aligned} \tag{5.24}$$

$$= \frac{h_{\phi\phi}}{h} - \frac{m_{\phi\phi}}{m} + \frac{h_\phi m_\phi}{2hm} + \frac{m_\phi^2}{m^2} - \frac{3h_\phi^2}{2h^2} + \Psi_1,$$

with

$$\Psi_1 = \Psi \cdot h_\phi + 2h \cdot \frac{\partial \Psi}{\partial \phi} = \frac{hm_{0_{\phi\phi}}}{m} - \frac{hm_\phi m_{0_\phi}}{m^2} + \frac{m_{0_\phi} h_\phi}{2m}. \tag{5.25}$$

Similarly we obtain $(\mathcal{L}_W g)_{12} = (\mathcal{L}_W g)_{21} = 0$ and

$$(\mathcal{L}_W g)_{22} = W^1 \partial_1 g_{22} + 0 = \left(\frac{h_\phi}{2h^2} - \frac{m_\phi}{2hm}\right) m_\phi = \frac{h_\phi m_\phi}{2h^2} - \frac{m_\phi^2}{2hm} + \Psi_2, \tag{5.26}$$

with

$$\Psi_2 = \Psi \cdot m_\phi = \frac{m_{0_\phi} m_\phi}{2h_0 m}. \tag{5.27}$$

Subsequently, we can now look at the Ricci-DeTurck flow in local coordinates (ϕ, θ), see also system (5.20). Note that the above is again independent of the coordinate θ. With an analogous observation as in the end of section 5.1, we obtain that the initial boundary problem

$$\begin{cases} h_t = \dfrac{h_{\phi\phi}}{h} - \dfrac{3h_\phi^2}{2h^2} + \dfrac{m_\phi^2}{2m^2} + \Psi_1, & \phi \in (0, \pi), \\ m_t = \dfrac{m_{\phi\phi}}{h} - \dfrac{m_\phi^2}{mh} + \Psi_2, & \phi \in (0, \pi), \\ m(t; \phi_{pole}) = 0, & \phi_{pole} \in \{0, \pi\}, \\ \dfrac{\partial \sqrt{m(t;\phi)}}{\partial \phi}\Big|_{\phi = \phi_{pole}} = \sqrt{h(t; \phi_{pole})}, & \phi_{pole} \in \{0, \pi\}, \\ h(0; \phi) = h_0 \text{ and } m(0; \phi) = m_0, & \phi \in [0, \pi], \end{cases} \tag{5.28}$$

is an equivalent formulation (under local isometry) of the Ricci-DeTurck flow equation with initial state (S, g_0). It now suffices to show that this initial boundary problem admits short-time existence and uniqueness.

As we mentioned previously, the Ricci-DeTurck flow should give rise to the intuition that the original Ricci flow problem has a unique short-time solution. We will give a brief discussion on why system (5.28) should have the short-time properties at first glance. We want to note that the below is a typical approach with regard to standard parabolic PDE theory.

Let us start by writing the coupled differential equations in (5.28) as

$$\begin{pmatrix} h_t \\ m_t \end{pmatrix} = \mathcal{F}(h, h_\phi, h_{\phi\phi}, m, m_\phi, m_{\phi\phi}) = \mathcal{F}_2(h, m),$$

where we denote $\mathcal{F}_2$ to indicate that it is a function that also depends on the first and second derivative of the smooth input functions. Linearising the system gives us

$$\mathcal{F}_2(\bar{h} + h, \bar{m} + m) = \mathcal{F}_2(\bar{h}, \bar{m}) + D\mathcal{F}_2(\bar{h}, \bar{m})[h, m] + \mathcal{O}(h^2 + m^2),$$

where $\bar{h}$ and $\bar{m}$ represent the evaluation of a solution at some time $t \geq 0$, and h and m represent small distortions. Via a direct computation, we obtain

$$D\mathcal{F}_2(\bar{h}, \bar{m})[h, m] = \begin{pmatrix} \dfrac{1}{\bar{h}} h_{\phi\phi} & 0 \\ 0 & \dfrac{1}{\bar{h}} m_{\phi\phi} \end{pmatrix} + \text{ lower order derivatives}$$

$$= \begin{pmatrix} \frac{1}{\bar{h}}\Delta h & 0 \\ 0 & \frac{1}{\bar{h}}\Delta m \end{pmatrix} + \text{ lower order derivatives,}$$

with Δ the 1-dimensional Laplacian. We importantly note that the factor $\frac{1}{h}$ in front of the Laplacian behaves well for small $t \geq 0$ because $h = 1$ on $[0, \pi]$ at $t = 0$.

The "lower order derivatives" contain the remaining terms of the linearisation. Note that it also contains the linearisation about the evaluations $\bar{h}_\phi, \bar{h}_{\phi\phi}, \bar{m}_\phi$ and $\bar{m}_{\phi\phi}$ at some $t \geq 0$ near the initial time $t = 0$. It follows from standard parabolic PDE theory that the "lower order derivatives" are typically not of interest, since they cause a "limited distortion" compared to the highest order derivatives $h_{\phi\phi}$ and $m_{\phi\phi}$.

From the analysis above, one can conclude that system (5.28) admits short-time existence and uniqueness, since the system appears to be well-behaved about $t = 0$. However, the "lower order derivatives" may not behave as nice as we would like. Suggested by [LSU68, p. 449], we can write the differential equations of system (5.28) in divergence form:

$$\begin{cases} h_t = \dfrac{\partial}{\partial \phi}\left[\dfrac{h_\phi}{h}\right] + A(h, m, h_\phi, m_\phi) \\ m_t = \dfrac{\partial}{\partial \phi}\left[\dfrac{m_\phi}{h}\right] + B(h, m, h_\phi, m_\phi). \end{cases} \tag{5.29}$$

We basically want to show that that the above satisfies certain elliptic operator conditions, see equation (6.9) in [LSU68, p. 449]. The only problem now is that A and B may not be totally bounded, since for $\phi \to \phi_{pole}$ we may have that $A, B \to \pm\infty$ holds because $m = 0$ at the poles due to the given boundary conditions and dividing by m happens multiple times in $\mathcal{F}_2(h, m)$.

Specifically from this point, the problem seemed to be more difficult than expected in the first place. At the end, we tried to show that A and B behave well as ϕ tends to the poles via several substitutions. Eventually we considered the substitution $n(t;\phi) = \frac{m(t;\phi)}{m_0(\phi)}$ and determined the equivalent initial boundary problem in h and n. Recall

$$h_0(\phi) = 1 \quad \text{and} \quad m_0(\phi) = \left(\frac{\sin\phi + c_3\sin(3\phi) + c_5\sin(5\phi)}{1 + 3c_3 + 5c_5}\right)^2, \quad \phi \in [0, \pi]. \tag{5.30}$$

The substitution is as follows. Substituting within the second boundary condition gives us

$$\sqrt{h} = \frac{\partial\sqrt{m}}{\partial\phi} = \frac{1}{2\sqrt{nm_0}}(n_\phi m_0 + nm_{0_\phi}) = \frac{n_\phi\sqrt{m_0}}{2\sqrt{n}}.$$

Taking limits $\phi \to \phi_{pole}$ gives

$$n\Big|_{\phi=\phi_{pole}} = \frac{4hm_0}{m_{0_\phi}^2}\Big|_{\phi=\phi_{pole}} = h\Big|_{\phi=\phi_{pole}}, \tag{5.31}$$

since $m_{0_\phi}^2 = 4m_0$ for any valid c_3 and c_5. Importantly note that we are left with one degree of freedom with regard to the boundary conditions. We hence simply take

$$\frac{\partial n}{\partial\phi}\Big|_{\phi=\phi_{pole}} = 0. \tag{5.32}$$

This choice can easily be justified by visualising $n(t;\phi)$ for small $t \geq 0$. Furthermore, we have

$$h_t = \frac{h_{\phi\phi}}{h} - \frac{3h_\phi^2}{2h^2} + \frac{[nm_0]_\phi^2}{2(nm_0)^2} + \frac{hm_{0_{\phi\phi}}}{nm_0} - \frac{h[nm_0]_\phi m_{0_\phi}}{(nm_0)^2} + \frac{m_{0_\phi}h_\phi}{2nm_0}$$

$$
\begin{aligned}
&= \frac{h_{\phi\phi}}{h} - \frac{3h_\phi^2}{2h^2} + \frac{[n_\phi m_0 + n m_{0_\phi}]^2}{2(nm_0)^2} + \frac{h m_{0_{\phi\phi}}}{nm_0} - \frac{h[n_\phi m_0 + n m_{0_\phi}] m_{0_\phi}}{(nm_0)^2} + \frac{m_{0_\phi} h_\phi}{2nm_0} \\
&= \frac{h_{\phi\phi}}{h} - \frac{3h_\phi^2}{2h^2} + \frac{n_\phi^2}{2n^2} + \frac{n_\phi m_{0_\phi}}{nm_0} + \frac{m_{0_\phi}^2}{2m_0^2} + \frac{h m_{0_{\phi\phi}}}{nm_0} - \frac{h n_\phi m_{0_\phi}}{n^2 m_0} + \frac{h m_{0_\phi}^2}{n m_0^2} + \frac{m_{0_\phi} h_\phi}{2nm_0} \\
&= \frac{\partial}{\partial\phi}\left[\frac{h_\phi}{h}\right] + \widetilde{A}(h, n, h_\phi, n_\phi),
\end{aligned}
$$

and

$$
\begin{aligned}
n_t &= \left[\frac{m}{m_0}\right]_t = \frac{m_t}{m_0} \\
&= \frac{[nm_0]_{\phi\phi}}{hm_0} - \frac{[nm_0]_\phi^2}{nm_0^2 h} + \frac{[nm_0]_\phi \cdot m_{0_\phi}}{2nm_0^2} \\
&= \frac{n_{\phi\phi}}{h} + \frac{2n_\phi m_{0_\phi}}{m_0 h} + \frac{n m_{0_{\phi\phi}}}{m_0 h} - \left(\frac{n_\phi^2}{nh} + \frac{2n_\phi m_{0_\phi}}{m_0 h} + \frac{n m_{0_\phi}^2}{m_0^2 h}\right) + \frac{n_\phi m_{0_\phi}}{2nm_0} + \frac{m_{0_\phi}^2}{2m_0^2} \\
&= \frac{n_{\phi\phi}}{h} + \frac{n m_{0_{\phi\phi}}}{m_0 h} - \frac{n_\phi^2}{nh} - \frac{n m_{0_\phi}^2}{m_0^2 h} + \frac{n_\phi m_{0_\phi}}{2nm_0} + \frac{m_{0_\phi}^2}{2m_0^2} \\
&= \frac{\partial}{\partial\phi}\left[\frac{n_\phi}{h}\right] + \widetilde{B}(h, n, h_\phi, n_\phi).
\end{aligned}
$$

From the detailed calculations above, we obtain the equivalent transformed initial boundary problem

$$
\begin{cases}
h_t = \dfrac{\partial}{\partial\phi}\left[\dfrac{h_\phi}{h}\right] + \widetilde{A}(h, n, h_\phi, n_\phi), & \phi \in (0, \pi), \\
n_t = \dfrac{\partial}{\partial\phi}\left[\dfrac{n_\phi}{h}\right] + \widetilde{B}(h, n, h_\phi, n_\phi), & \phi \in (0, \pi), \\
n(t; \phi_{pole}) = h(t; \phi_{pole}), & \phi_{pole} \in \{0, \pi\}, \\
\dfrac{\partial n(t;\phi)}{\partial\phi}\Big|_{\phi=\phi_{pole}} = 0, & \phi_{pole} \in \{0, \pi\}, \\
h(0; \phi) = h_0 \text{ and } n(0; \phi) = 1, & \phi \in [0, \pi].
\end{cases}
\tag{5.33}
$$

A solution of (5.33) hence needs to satisfy $h = n$ at $t = 0$ because $h_0 = 1$ and we have $h = n$ at the poles for all time t. Exceptionally remarkable, we observe that

$$
\widetilde{A}(h, h, h_\phi, h_\phi) = \widetilde{B}(h, h, h_\phi, h_\phi) \tag{5.34}
$$

holds. This indicates that $h = n$ everywhere is a solution of (5.33) if and only if $h_0 = 1$, which is indeed the case, see the initial conditions (5.13).

We can now either look at $\widetilde{A}$ and $\widetilde{B}$ in (5.33) as ϕ tends to its poles or consider a one-dimensional initial value problem that would imply the short-time existence of system (5.33). We like to note that in both cases our upcoming analysis would be essentially the same.

Finally consider the one-dimensional initial value Neumann boundary problem

$$
\begin{cases}
h_t = \dfrac{\partial}{\partial\phi}\left[\dfrac{h_\phi}{h}\right] + \widetilde{A}(h, h, h_\phi, h_\phi), & \phi \in (0, \pi), \\
\dfrac{\partial h(t;\phi)}{\partial\phi}\Big|_{\phi=\phi_{pole}} = 0, & \phi_{pole} \in \{0, \pi\}, \\
h(0; \phi) = 1, & \phi \in [0, \pi].
\end{cases}
\tag{5.35}
$$

Now we want to show that $\widetilde{A}(h,h,h_\phi,h_\phi)$ is a well-behaving function as h changes slightly. We deduce from the analysis above the following:

$$\begin{aligned}\widetilde{A}(h,h,h_\phi,h_\phi) &= \widetilde{B}(h,h,h_\phi,h_\phi)\\ &= \frac{h_\phi^2}{h^2} + \frac{m_{0_{\phi\phi}}}{m_0} - \frac{h_\phi^2}{h^2} - \frac{m_{0_\phi}^2}{m_0^2} + \frac{n_\phi m_0}{2hm_0} + \frac{m_{0_\phi}^2}{2m_0^2}\\ &= \frac{m_{0_{\phi\phi}}}{m_0} - \frac{m_{0_\phi}^2}{2m_0^2} + \frac{h_\phi m_{0_\phi}}{2hm_0}. \end{aligned} \tag{5.36}$$

Let us consider the first two of the three terms in (5.36). Importantly note that the terms

$$\frac{m_{0_{\phi\phi}}}{m_0} \quad \text{and} \quad \frac{m_{0_\phi}^2}{2m_0^2}$$

diverge individually as $\phi \to \phi_{pole}$. Combined however we have

$$\frac{m_{0_{\phi\phi}}}{m_0} - \frac{m_{0_\phi}^2}{2m_0^2} = -2 \cdot \frac{\sin\phi + 9c_3\sin(3\phi) + 25c_5\sin(5\phi)}{\sin\phi + c_3\sin(3\phi) + c_5\sin(5\phi)} \tag{5.37}$$

$$= -2 \cdot \frac{1 + 27c_3 + 125c_5}{1 + 3c_3 + 5c_5} + \mathcal{O}\big((\phi - \phi_{pole})^2\big), \qquad \phi \to \phi_{pole}. \tag{5.38}$$

We note that equation (5.38) follows from a Taylor expansion about the poles. We moreover deduce, from (5.37), that the two terms combined also behave well on the interior $(0,\pi)$, since the constants c_3 and c_5 need to be chosen small enough in the first place, recall section 5.1. Thus the two terms combined does not diverge and hence is fully defined on $[0,\pi]$.

We conclude that $\widetilde{A}(h,h,h_\phi,h_\phi)$ is a well-behaving function as h changes slightly whenever the third term in (5.36) behaves well too. Again, we observe that $\frac{m_{0_\phi}}{m_0}$ diverges as ϕ tends to the poles and that $h_\phi \to 0$ holds as $\phi \to \phi_{poles}$. Note that the latter follows from the second boundary condition of system (5.35).

Our aim is to show that h_ϕ tends to 0 fast enough so that the divergence of $\frac{m_{0_\phi}}{m_0}$ gets cancelled. This is intuitively true when one interprets and visualises $n(t;\phi)$ for small $t \geq 0$. By again considering Taylor expansions about the poles, we more formally have

$$\frac{m_{0_\phi}}{m_0} = 2(\phi - \phi_{pole})^{-1} + \mathcal{O}(\phi - \phi_{pole}), \qquad \phi \to \phi_{pole}. \tag{5.39}$$

$$h = \text{constant} + \mathcal{O}\big((\phi - \phi_{pole})^2\big), \qquad \phi \to \phi_{pole}. \tag{5.40}$$

Note that approximation (5.40) follows from the boundary condition and because we want the solution to be sufficiently smooth for small $t \geq 0$. Thanks to results (5.39) and (5.40) we obtain

$$\frac{h_\phi m_{0_\phi}}{2hm_0} = \text{constant} + \mathcal{O}\big((\phi - \phi_{pole})^2\big), \qquad \phi \to \phi_{pole}. \tag{5.41}$$

Observe that the convergence is even quadratic. From this approximation we conclude that the third term behaves well at the poles. Consequently $\widetilde{A}(h,h,h_\phi,h_\phi)$ is a well-behaving function.

In conclusion, based on the analysis in [LSU68, p. 475] which discusses a general initial value Neumann boundary problem and conform to theorem 6.1 in [LSU68, p. 452], we obtain short-time existence and uniqueness of system (5.35). Subsequently, this unique solution h implies that system (5.33) admits short-time existence because $h = n$ is a solution of (5.33) thanks to the observation made in (5.34).

We want to point out that $h = n$ being a solution is a remarkable result, which can be explained by the fact that the initial state is rotationally symmetric. Using this fact in particular makes our approach very different compared to the original proofs, because in the general case this symmetry is no longer true.

We end our sketch by simply claiming that the short-time solution of system (5.33) found above is unique. Recall that we were not really able to deduce the short-time properties by standard parabolic PDE theory immediately, since the functions $\widetilde{A}(h, n, h_\phi, n_\phi)$ and $\widetilde{B}(h, n, h_\phi, n_\phi)$ do not seem to behave well when we both change h or n slightly.

Ultimately, thanks to the above we are confident of the short-time existence and uniqueness of system (5.33), which then implies the short-time existence and uniqueness of a Ricci-DeTurck flow, and therefore we have achieved the short-time properties of the Ricci flow of a surface of revolution.

Discussion

We like to stretch out once more that this final section is just a sketch and not a full proof for the short-time properties of the Ricci flow of a surface of revolution. Our explicit approach appeared to be way more difficult than expected. We have tried numerous approaches but we deliberately did not try using the parabolic PDE theory on manifolds, because we wanted to keep things as explicit and intuitive as possible for the reader and ourselves.

Even though our thorough analysis was often quite cumbersome and there are still a few gaps that need to be filled in, this part of this thesis was very educative for the author. Therefore we also like to end this chapter with the comment that it is a fascinating fact that the Ricci flow is a well-posed problem.

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