Nonlinear stability of semidiscrete shocks for two-sided schemes

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October 30, 2009

Abstract

The nonlinear stability of travelling Lax shocks in semidiscrete conservation laws involving general spatial forward-backward discretization schemes is considered. It is shown that spectrally stable semidiscrete Lax shocks are nonlinearly stable. In addition, it is proved that weak semidiscrete Lax profiles satisfy the spectral stability hypotheses made here and are therefore nonlinearly stable. The nonlinear stability results are proved by constructing the resolvent kernel using exponential dichotomies, which have recently been developed in this setting, and then using the contour integral representation for the associated Green's function to derive pointwise bounds that are sufficient for proving nonlinear stability. Previous stability analyses for semidiscrete shocks relied primarily on Evans functions, which exist only for one-sided upwind schemes.

*Beck was partially supported under NSF grant DMS-0602891
†Hupkes acknowledges support from the Netherlands Organization for Scientific Research (NWO)
‡Sandstede was partially supported by the NSF through grant DMS-0907904
§Zumbrun was partially supported by the NSF through grant DMS-0801745

1 Introduction

Our starting point is semidiscrete systems of conservation laws, where semidiscreteness refers to the property that the equations are discrete in space and continuous in time. Thus, the underlying equations can be thought of as lattice dynamical systems that are posed on a spatial lattice $\mathbb{Z}$ of equidistant grid points with a time evolution in the continuous time variable $t \in \mathbb{R}$. Such equations arise naturally as spatial finite-difference approximations of systems of conservation laws, and it is therefore of interest to investigate the similarities and differences between the original continuous conservation law and its semidiscrete analogue.

Lax shocks are an important feature of both continuous and semidiscrete systems of conservation laws, and the issue addressed in this paper is the nonlinear stability of semidiscrete Lax shocks. More specifically, we consider semidiscrete Lax shocks that travel with nonzero speed through the lattice. Indeed, since the underlying lattice breaks translation symmetry, standing and travelling waves will have very different properties: standing shocks admit discrete profiles that satisfy difference equations; the profiles of travelling shocks, on the other hand, depend on a continuous spatial variable, and they satisfy a functional differential equation (FDE) of mixed type that contains both advanced and retarded terms if the finite-difference scheme uses both downwind and upwind terms. The presence of these terms complicates the analysis of the travelling-wave FDE dramatically as the associated initial-value problem is ill-posed; this is in sharp contrast to the continuous case where shocks of any speed satisfy an ordinary differential equation, which is well-posed. Nevertheless, the existence of weak Lax shocks for FDEs of mixed type was established in [3] using a center-manifold reduction.

Our goal is to prove that spectrally stable semidiscrete Lax shocks are nonlinearly stable. For upwind schemes, this result was proved in [4] for shocks of arbitrary strength. The approach taken there is to derive pointwise bounds for the resolvent kernel, which then generate, via the inverse Laplace transform, pointwise bounds for the Green's function. The latter can then be used to establish nonlinear stability by setting up an appropriate iteration argument via Duhamel's formula. We note that this approach is similar to the stability analysis of viscous conservation laws, and indeed one of the crucial assumptions in [4] is that the semidiscrete system exhibits appropriate dissipation. To obtain the pointwise bounds on the resolvent kernel, it is necessary to extend this operator meromorphically across the essential spectrum (as in the continuous case, the essential spectrum contains the origin due to transport along characteristics). The key difficulty that restricts the analysis presented in [4] to upwind schemes arises during the meromorphic extension: since the derivative of the Lax shock creates an eigenvalue at the origin that is embedded in the essential spectrum, the meromorphic extension of the resolvent kernel will have a pole at the origin. This pole is typically captured by an Evans function, which is essentially a Wronskian determinant of appropriate solutions of the linearization of the semidiscrete system about the shock. For semidiscrete shocks that travel with nonzero speed, the relevant linearization in the comoving frame is again a functional differential equation of mixed type for which Evans functions are not available as the initial-value problem is ill-posed.

In the context of time-periodic Lax shocks of continuous viscous conservation laws, similar difficulties were recently resolved in [1] by a combination of Lyapunov–Schmidt reduction and exponential dichotomies instead of Evans functions. Exponential dichotomies encode the property that the underlying phase space can be written as the direct sum of two subspaces such that the ill-posed equation can be solved in forward time[1] for initial data in the first subspace and in backward time on the second subspace. For functional differential equations of mixed type, the existence of exponential dichotomies was established in [9, 19], see also [6, 26], and our goal in this paper is to show that these results in conjunction with the techniques developed in [1, 4] provide the means to prove that semidiscrete Lax shocks are nonlinearly stable whenever they are spectrally stable.

When implementing the strategy outlined above, several nontrivial issues need to be addressed. First, we need to prove regularity of the Green's function in the spatial variables, which requires a careful examination of the

[1] "Time" refers to the evolution variable which, in the present context, is the spatial variable x.

existence proofs for exponential dichotomies given in [9, 19]. A second obstacle is the adjoint of the linearization about the shock, which is needed in the meromorphic extension of the resolvent kernel: the issue is that the FDE of mixed type that represents the linearization about the shock and the associated adjoint system are related via an inner product (commonly referred to as the Hale inner product) that is much weaker than the L^2-scalar product. Thus, the exponential dichotomies of the linearization and its adjoint, which are typically related by taking L^2 adjoints and solving backwards, are no longer related in a transparent fashion, and care is needed when using the adjoint system. Finally, a certain nondegeneracy condition on the coefficients of the travelling-wave FDE that implies a number of useful properties is not satisfied by the two-sided schemes most commonly employed for conservation laws. Not having these properties prevents us, for instance, from using variation-of-constants formulae, which further complicates the analysis.

We now give a more detailed technical account of our setting and the results we shall prove in this paper. We consider the lattice dynamical system

$$\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{1}{h}\left[f(v_{j-p+1}, \dots, v_{j+q}) - f(v_{j-p}, \dots, v_{j+q-1})\right] = 0, \qquad j \in \mathbb{Z}, \tag{1.1}$$

where $v_j(t) \in \mathbb{R}^N$ for all j. Throughout this paper, we assume that the discrete flux $f : \mathbb{R}^{N(p+q)} \to \mathbb{R}^N$ is C^2. The relation of (1.1) to spatial discretizations of conservation laws becomes clear once we define

$$\bar{f}(v) := f(v, \dots, v)$$

for $v \in \mathbb{R}^N$ and introduce the system

$$v_t + \bar{f}(v)_x = 0, \qquad x \in \mathbb{R} \tag{1.2}$$

of conservation laws. Indeed, applying a finite-difference scheme with spatial step size h to the flux in (1.2) gives an equation of the form (1.1), where the integers $p, q \geq 0$ correspond to the end points of the spatial discretization stencil, and $v_j(t)$ is meant to approximate a solution $v(x,t)$ of (1.2) evaluated at the grid points $x = jh$. We reiterate that we regard the step size $h > 0$ as fixed and investigate the dynamics of the semidiscrete system (1.1) in its own right.

Travelling waves of the semidiscrete system (1.1) are solutions of the form

$$v_j(t) = u_*\left(j - \frac{\sigma}{h}t\right),$$

where $u_* : \mathbb{R} \to \mathbb{R}^N$ characterizes the profile of the wave, and σ denotes its wave speed. Substituting this ansatz into (1.1), we find that the profile $u_*(x)$ needs to satisfy the equation

$$\sigma u_*'(x) = f\left(u_*(x-p+1), \dots, u_*(x+q)\right) - f\left(u_*(x-p), \dots, u_*(x+q-1)\right). \tag{1.3}$$

When $p, q > 0$ are both strictly positive, we see that (1.3) is a functional differential equation of mixed type as it contains both advanced and retarded terms. As mentioned above, such systems are difficult to analyse as the right-hand side depends on the future and the history of $u(x)$. We shall assume that $u_*(x)$ is a solution of (1.3) for some $\sigma > 0$ and that there are constants $u_\pm \in \mathbb{R}^N$ so that

$$u_*(x) \to u_\pm \qquad \text{as} \qquad x \to \pm\infty. \tag{1.4}$$

Furthermore, we assume that u_* is a Lax k-shock in the following sense:

Hypothesis (H1) *The ordered eigenvalues $a_1^\pm < \dots < a_N^\pm$ of $\bar{f}_u(u_\pm)$ are real and distinct, and there is a number $k \in \{1, \dots, N\}$ such that $a_{k-1}^- < \sigma < a_k^-$ and $a_k^+ < \sigma < a_{k+1}^+$.*

We denote by $l_n^\pm$ and $r_n^\pm$ the left and right eigenvectors of $\bar{f}_u(u_\pm)$ associated with the eigenvalues $a_n^\pm$ for $n = 1, \dots, N$. The eigenvalues $a_n^\pm$ are often referred to as characteristics, and the associated eigenvectors correspond to directions along which movement, measured relative to the shock speed σ, is directed in towards the shock or out away from the shock. Thus, we will often refer to them as incoming and outgoing characteristics and denote them by

$$\underbrace{a_1^- < \ldots < a_{k-1}^-}_{=:a_{n,\mathrm{out}}^-} < \sigma < \underbrace{a_k^- < \ldots < a_N^-}_{=:a_{n,\mathrm{in}}^-}, \qquad \underbrace{a_1^+ < \ldots < a_k^+}_{=:a_{n,\mathrm{in}}^+} < \sigma < \underbrace{a_{k+1}^+ < \ldots < a_N^+}_{=:a_{n,\mathrm{out}}^+}. \tag{1.5}$$

The following lemma, which will be proved in §4.2, states that a Lax k-shock converges exponentially towards its end states and that its speed is given by the Rankine–Hugoniot condition.

Lemma 1.1 *If $u_*(x)$ is a Lax k-shock whose end states $u_\pm$ and speed σ satisfy (H1) and the condition (S4) stated below in Definition 1.2, then $\sigma(u_+ - u_-) = \bar{f}(u_+) - \bar{f}(u_-)$, and there are positive constants κ and K so that $|u_*'(x)| \le K\mathrm{e}^{-\kappa|x|}$ for $x \in \mathbb{R}$.*

Throughout this paper, we need the following technical assumption on the semidiscrete scheme that guarantees uniqueness of an appropriate initial-value problem associated with (1.3).

Hypothesis (H2) *The derivative $u_*'(x)$ does not vanish identically on any interval of length at least $p+q$.*

If (H2) is not met, then the profile u_* is constant on an interval of the form $[y-p, y+q]$, $[y,\infty)$, or $(-\infty, y]$ for some y. In the first case, the profile can be broadened by extending the interval on which u_* is constant to any larger interval, and the scheme does not generate a locally unique Lax k-shock profile. We believe that our analysis can be extended to cover the remaining two cases, where u_* vanishes along one or both of its tails, though we will not discuss this further. We remark that a sufficient condition for (H2) that has many other useful implications is that the determinants of the coefficient matrices $A_{-p}(x)$ and $A_q(x)$ that we introduce in (1.7) below do not vanish on any open interval: this condition, however, is typically violated for the two-sided schemes used in practice; see [3] for examples.

To describe our spectral stability assumptions on the Lax k-shock u_*, we use the notation

$$\partial_j f(v_{-p+1}, \ldots, v_q) := \frac{\partial f}{\partial v_j}(v_{-p+1}, \ldots, v_q), \qquad j \in \{-p+1, \ldots, q\}$$

to denote the derivative of f with respect to v_j and shall always set $\partial_{-p} f = \partial_{q+1} f = 0$. We can now introduce the operator

$$\mathcal{L}: \quad L^2(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N), \quad u \longmapsto \sigma u'(x) - \sum_{j=-p}^{q} A_j(x) u(x+j), \tag{1.6}$$

with domain $H^1(\mathbb{R}, \mathbb{C}^N)$, where

$$A_j(x) = \partial_j f\left(u_*(x-p+1), \ldots, u_*(x+q)\right) - \partial_{j+1} f\left(u_*(x-p), \ldots, u_*(x+q-1)\right). \tag{1.7}$$

Since $\mathcal{L}(\mathrm{e}^{2\pi\mathrm{i}x} u) = \mathrm{e}^{2\pi\mathrm{i}x}(2\pi\mathrm{i}\sigma + \mathcal{L})u$, the spectrum of $\mathcal{L}$ is invariant under shifts by $2\pi\mathrm{i}\sigma$, which reflects the fact that the original lattice equation (1.1) does not feel oscillations that occur on a scale smaller than the distance between consecutive elements in $\mathbb{Z}$. To avoid the resulting ambiguity, we consider only those elements in the spectrum of $\mathcal{L}$ that have imaginary parts of modulus smaller or equal to $\pi\sigma$. Lastly, we define the viscosity matrices

$$B^\pm = \frac{1}{2} \sum_{j=-p}^{q} (1-2j) \partial_j f(u_\pm, \ldots, u_\pm), \tag{1.8}$$

which, as we shall see below, encode whether diffusion is present in the semidiscrete system (1.1). The following definition encapsulates spectral stability of Lax k-shocks.

Definition 1.2 *A Lax k-shock u_* is said to be* spectrally stable *if*

(S1) *The operator $\mathcal{L}$, as given in (1.6) with domain $H^1(\mathbb{R}, \mathbb{C}^N)$, has no spectrum in $\{\mathrm{Re}\,\lambda \geq 0\} \setminus 2\pi\mathrm{i}\sigma\mathbb{Z}$.*

(S2) *The only nontrivial solutions $u \in H^1(\mathbb{R}, \mathbb{C}^N)$ of $\mathcal{L}u = 0$ are $u = u_*'$ and scalar multiples thereof.*

(S3) *The outgoing characteristics and the jump $[u_*] := u_+ - u_-$ across the shock u_* are linearly independent:*

$$\det[r_1^-, \ldots, r_{k-1}^-, [u_*], r_{k+1}^+, \ldots, r_N^+] \neq 0.$$

(S4) *The nonresonance condition*

$$\det\left[\sigma\nu - \sum_{j=-p}^{q} (\partial_j - \partial_{j+1}) f(u_\pm, \ldots, u_\pm) \mathrm{e}^{\nu j}\right] \neq 0$$

holds for all $\nu \in \mathrm{i}\mathbb{R} \setminus \{0\}$.

(S5) *The viscosity matrices $B^\pm$ are dissipative along characteristics: $\langle l_n^\pm, B^\pm r_n^\pm \rangle > 0$ for $n = 1, \ldots, N$.*

Before we state our results and comment on situations where our hypotheses are met, we briefly discuss our definition of spectral stability. First, (S2) assumes that the translation eigenvalue $\lambda = 0$ of $\mathcal{L}$ has minimal geometric multiplicity. Upon recalling that the spectrum of $\mathcal{L}$ is invariant under translations by $2\pi\mathrm{i}\sigma$, we see that (S1) requires that $\mathcal{L}$ has no unstable or marginally stable elements in the spectrum besides those enforced by translation symmetry. Condition (S3) is the same as for Lax shocks of (1.2), where it is known as the Liu–Majda condition: in our context, it guarantees that the translation eigenvalue $\lambda = 0$ has, in an appropriate sense, algebraic multiplicity one. Finally, (S4) reflects the assumption that the essential spectrum of $\mathcal{L}$ at $\lambda = 0$ is generated solely by the characteristics, while (S5) implies that the essential spectrum near $\lambda = 0$ consists of nondegenerate parabolas that open into the left half-plane and whose time evolution is given by Gaussians that propagate along characteristics.

Our main result, which is analogous to the theory in the setting of viscous shocks [1, 20, 28] and of semidiscrete shocks with either advanced or retarded terms [4], asserts that spectral stability implies nonlinear stability and gives detailed pointwise estimates for how perturbations decay as $t \to \infty$ in the spaces

$$L^\alpha(\mathbb{Z}) := \left\{ v : \mathbb{Z} \to \mathbb{R}^N : \ |v|_{L^\alpha(\mathbb{Z})} := \Big[\sum_{j\in\mathbb{Z}} |v_j|^\alpha\Big]^{1/\alpha} < \infty \right\},$$

where $\alpha \geq 1$ and $|v|_{L^\infty(\mathbb{Z})} := \sup_{j\in\mathbb{Z}} |v_j|$.

Theorem 1 *Assume that there are constants $u_\pm$, a speed σ, and a profile $u_*(x)$ that satisfy (1.3), (1.4), and Hypotheses (H1)-(H2) and (S1)-(S5), then there are constants $\epsilon > 0$ and $K > 0$ such that the following is true. For each initial condition $\{v_j(0)\}_{j\in\mathbb{Z}}$ with $|\{v_j(0) - u_*(j)\}_{j\in\mathbb{Z}}|_{L^1(\mathbb{Z})} \leq \epsilon$, there is a function $\rho : \mathbb{R}^+ \to \mathbb{R}$ with*

$$\sup_{t\geq 0} \Big(|\rho(t)| + (1+|t|)^{\frac{1}{2}} |\dot\rho(t)| \Big) \leq K\epsilon$$

such that the solution $\{v_j(t)\}_{j\in\mathbb{Z}}$ of (1.1) with initial condition $\{v_j(0)\}_{j\in\mathbb{Z}}$ satisfies

$$|\{v_j(t) - u_*(j + \rho(t) - \sigma t/h)\}_{j\in\mathbb{Z}}|_{L^\alpha(\mathbb{Z})} \leq \frac{K\epsilon}{(1+|t|)^{\frac{1}{2}(1-\frac{1}{\alpha})}}, \qquad t \geq 0$$

for each $\alpha \geq 1$. In other words, spectrally stable Lax k-shock solutions $u_*(j - \sigma t/h)$ of (1.1) are nonlinearly stable.

If the initial perturbation is sufficiently localized, the shock position $\rho(t)$ in the above theorem converges, and we obtain nonlinear stability with asymptotic phase:

Remark 1.3 *If the initial condition satisfies* $|v_j(0) - u_*(j)| \leq \epsilon(1 + jh)^{-3/2}$ *for* $j \in \mathbb{Z}$*, then it follows as in [10] that there exists a* ρ_∞ *with* $|\rho_\infty| \leq K\epsilon$ *for which*

$$|\rho(t) - \rho_\infty|\,(1 + |t|)^{1/2} + |\dot{\rho}(t)|\,(1 + |t|) \leq K\epsilon,$$

along with detailed pointwise bounds on the solution. Note that the pointwise bounds that we shall establish in Theorem 8 are all that were used in [10] to prove the refined result in the continuous case.

For upwind schemes, Theorem 1 was proved in [4]. Certain aspects of the analysis in [4] apply also to equations with both advanced and retarded terms, and we shall exploit this in our proof. What is new in our paper is the meromorphic extension of the resolvent kernel and the resulting pointwise bounds for arbitrary schemes. Nonlinear stability of travelling waves in general lattice differential equations was proved earlier in [5] under the assumption that the spectrum is contained strictly in the left half-plane except for a simple translation eigenvalue at the origin; as we shall see below, this hypothesis is necessarily violated for semidiscrete conservation laws.

Finally, we comment on existence and spectral stability results of semidiscrete Lax shocks. All results we are aware of in the semidiscrete context concern weak shocks for which $|u_+ - u_-| \ll 1$. As in the continuous case, weak shocks exist and are spectrally stable.

Theorem 2 ([2–4]) *Assume that* $u_0 \in \mathbb{R}^N$ *satisfies the following conditions: The eigenvalues* a_n^0 *of* $\bar{f}_u(u_0)$ *are real and distinct with left and right eigenvectors* l_n^0 *and* r_n^0*, respectively, and we have* $a_k^0(u_0) \neq 0$ *and* $(\partial_u a_k^0)(u_0) r_k^0 \neq 0$ *for some* k*. Furthermore, we assume that (S4)-(S5), with all terms evaluated at* u_0 *instead of* $u_\pm$*, are met, and that* $\partial_n f(u, \ldots, u)$ *and* $\bar{f}_u(u)$ *commute for* $n = 1, \ldots, N$ *and all* u *in a neighborhood of* u_0*. Under these hypotheses, there is a neighborhood* $\mathcal{U}$ *of* u_0 *in* $\mathbb{R}^N$ *so that any Lax* k*-shock* (u_+, u_-, σ) *of (1.2) with* $\sigma = a_k^0$ *and* $u_\pm \in \mathcal{U}$ *admits a semidiscrete Lax* k*-shock profile* $u_*(x)$ *of (1.3)-(1.4) that satisfies (H1)-(H2). Furthermore, each of these weak semidiscrete Lax* k*-shocks satisfies Hypotheses (S1)-(S5) and is therefore nonlinearly stable in the sense of Theorem 1.*

Proof. Under the assumptions of the theorem, the existence of weak Lax shocks that satisfy (H1) was shown in [2] for upwind schemes and in [3] for arbitrary schemes. In particular, the shock profile u_* was constructed inside a smooth $N + 1$-dimensional center manifold of (1.3), and (H2) follows from uniqueness of solutions of ordinary differential equations. It remains to address (S1)-(S5). Condition (S1) was proved in [4, Theorem 3.9] via energy estimates. Hypothesis (S2) follows from [4, Proof of Proposition 2.4], where it was shown that the shock profile decays exponentially: it is therefore constructed as the intersection of one-dimensional stable and unstable manifolds of the two equilibria $u_\pm$ that lie in an N-dimensional surface of equilibria within the $N+1$-dimensional center manifold of (1.3), which implies that the geometric multiplicity of the eigenvalue $\lambda = 0$ of $\mathcal{L}$ is one. Next, we verify (S3). Let ϵ denote the diameter of $\mathcal{U}$ in $\mathbb{R}^N$, then the hypothesis that $(u_+, u_-, \sigma = a_k^0)$ is a Lax k-shock of (1.2) in $\mathcal{U}$ gives $a_k^0(u_+ - u_-) = f(u_+) - f(u_-)$, and it follows easily that $r_k^0 = (u_+ - u_-)/|u_+ - u_-| + \mathrm{O}(\epsilon)$ and $r_n^\pm = r_n^0 + \mathrm{O}(\epsilon)$ for each n, which gives (S3). Condition (S5) holds by continuity of f. It remains to establish (S4) which follows essentially from continuity in u_0: we will prove this in more detail below in Remark 4.2. ■

The rest of the paper is organized as follows. In §2, we discuss the relationship between the resolvent kernel of $\mathcal{L}$ and the Green's function of the linearization of the semidiscrete system (1.1) about the shock. In §3, we provide the necessary results on exponential dichotomies for abstract FDEs of mixed type and their adjoints. In §4, these results are then used to construct the resolvent kernel of $\mathcal{L}$, to extend it meromorphically across the imaginary axis, and to derive the necessary pointwise bounds. Section 5 contains a brief summary of the resulting pointwise bounds for the Green's function, via its representation as a contour integral of the resolvent kernel, and the proof of nonlinear stability. In §6, we discuss open problems and future challenges.

2 The Green's function via the resolvent kernel

Recall equation (1.1),

$$\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{1}{h}\left[f(v_{j-p+1},\ldots,v_{j+q}) - f(v_{j-p},\ldots,v_{j+q-1})\right] = 0, \qquad j \in \mathbb{Z},$$

which we rewrite more conveniently and concisely as

$$\dot{v}_j = F(v)_j, \qquad j \in \mathbb{Z}, \tag{2.1}$$

where $v = (v_j)_{j\in\mathbb{Z}}$. The profile u_* corresponds to the Lax k-shock solution v^* with

$$v_j^*(t) = u_*\left(j - \frac{\sigma}{h}t\right)$$

of (2.1). Note that

$$v_j^*\left(\frac{h}{\sigma}\right) = v_{j-1}^*(0), \qquad \forall j \in \mathbb{Z},$$

so that v^* is a relative periodic orbit with respect to the symmetry action generated by the shift on the underlying lattice $\mathbb{Z}$. The linearization of (2.1) about v^* is given by

$$\dot{v} = F_v(v^*(t))v. \tag{2.2}$$

The Green's function $\mathcal{G}(j,i,t,s)$ associated with (2.2) is given as the solution $v_j(t)$ of the initial-value problem

$$\dot{v}_j = \left(F_v(v^*(t))v\right)_j, \qquad v_j|_{t=s} = \delta_{ij}, \qquad j \in \mathbb{Z}$$

which then generates the general solution of

$$\dot{v} = F_v(v^*(t))v + h(t)$$

via

$$v_j(t) = \sum_{i\in\mathbb{Z}} \mathcal{G}(j,i,t,0)v_i(0) + \int_0^t \sum_{i\in\mathbb{Z}} \mathcal{G}(j,i,t,s)h_i(s)\,\mathrm{d}s.$$

On the other hand, we can consider the linearization

$$\mathcal{L}u = \sigma u_x(x) - \sum_{j=-p}^{q} A_j(x)u(x+j)$$

of the travelling-wave FDE about the profile u_*, where the coefficient matrices $A_j(x)$ are given in (1.7). We define the resolvent kernel $G(x,y,\lambda)$ associated with the operator $\mathcal{L}$ as the solution of the system

$$(\mathcal{L} - \lambda)u = \delta(\cdot - y).$$

The following theorem, quoted from [4], relates the Green's function $\mathcal{G}$ and the resolvent kernel G.

Theorem 3 ([4, Theorem 4.2]) *For each fixed γ with $\mathrm{Re}\,\gamma \gg 1$ sufficiently large, we have*

$$\mathcal{G}(j,i,t,s) = \mathcal{G}\left(j - \frac{\sigma t}{h}, i - \frac{\sigma s}{h}, t-s\right) \quad \text{with} \quad \mathcal{G}(x,y,\tau) = \frac{-1}{2\pi\mathrm{i}\sigma}\int_{\gamma-\mathrm{i}\pi\sigma}^{\gamma+\mathrm{i}\pi\sigma} \mathrm{e}^{\lambda\tau} G(x,y,\lambda)\,\mathrm{d}\lambda. \tag{2.3}$$

Proof. The proof in [4] carries over to the situation considered here because it does not rely on the joint reduction hypothesis (H3) stated at the beginning of that paper. ∎

Note that, due to the relative periodicity of the shock on the lattice, the integration in (2.3) takes place only on a bounded contour rather than on an unbounded one. This is similar to the case of time-periodic viscous shocks that was studied recently in [1].

Our strategy is now to extend the resolvent kernel $G(x,y,\lambda)$ meromorphically across $\operatorname{Re}\lambda = 0$ and to use appropriate pointwise bounds for the meromorphic extension to expand $\mathcal{G}(j,i,t,s)$ using the inverse Laplace transform formulation (2.3) of $\mathcal{G}$ in terms of G. Contour integral representations similar to (2.3) have been used successfully in a variety of contexts to prove both the linear and nonlinear stability of shocks under appropriate spectral stability assumptions on the linearized operator: we refer to [10, 20–22, 28] for early work on viscous shocks and to [4] for results on semidiscrete shocks for upwind schemes. In both those cases, the resolvent kernel was constructed using an Evans function, which is not available for operators with both advanced and retarded terms. To extend these ideas to arbitrary schemes, we will construct the resolvent kernel using exponential dichotomies, which were developed for forward-backward schemes in [9, 19].

3 Exponential dichotomies for functional differential equations

In this section, we prepare the ground for the meromorphic extension of the resolvent kernel associated with the operator $\mathcal{L}$ by proving several technical results. Specifically, we extend the exponential-dichotomy results in [9], which were proved only for the case $p = q$, to arbitrary values of p, q and discuss the regularity of these dichotomies. We also clarify the relation between exponential dichotomies of $\mathcal{L}$ and those of its L^2-adjoint; we remark that this is an intricate issue even for equations that contain only retarded terms.

3.1 Asymptotic hyperbolicity and exponential dichotomies

Since the results in this section do not depend on the discrete conservation-law structure of (1.1), we consider a general linear functional differential equation

$$u_x(x) = \sum_{j=-p}^{q} A_j(x) u(x+j) \tag{3.1}$$

of mixed type. We say that a function $u(x)$ is an H^1-solution of (3.1) on the interval $[x_0, x_1]$ if $u \in L^2([x_0 - p, x_1 + q]) \cap H^1([x_0, x_1])$ and u satisfies (3.1), viewed in L^2_{loc}, for $x \in [x_0, x_1]$. We then define the bounded operator

$$\mathcal{L}: \quad H^1(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N), \quad u \longmapsto u_x - \sum_{j=-p}^{q} A_j(\cdot) u(\cdot + j)$$

and its L^2 adjoint

$$\mathcal{L}^*: \quad H^1(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N), \quad \hat{u} \longmapsto -\hat{u}_x - \sum_{j=-p}^{q} A_j(\cdot - j)^* \hat{u}(\cdot - j).$$

We make the following assumptions on the coefficient matrices $A_j(x)$ and the operators $\mathcal{L}$ and $\mathcal{L}^*$:

Hypothesis (H3) *We have $A_j(x) \in C^1(\mathbb{R}, \mathbb{C}^{N\times N})$, and the following conditions are met:*

(i) There are matrices $A_j^\pm$ such that $A_j(x) \to A_j^\pm$ as $x \to \pm\infty$ for $j = -p, \ldots, q$.

(ii) The operator $\mathcal{L}$ is asymptotically hyperbolic, that is, the characteristic equations

$$\det \Delta_\pm(\nu) := \det \left[\nu - \sum_{j=-p}^{q} A_j^\pm \mathrm{e}^{\nu j} \right] = 0$$

do not have any purely imaginary roots $\nu \in \mathrm{i}\mathbb{R}$.

(iii) Elements in the null spaces $\mathrm{N}(\mathcal{L})$ *and* $\mathrm{N}(\mathcal{L}^*)$ *of* $\mathcal{L}$ *and* $\mathcal{L}^*$ *cannot vanish on any interval of length* $p+q$ *unless they vanish everywhere.*

The following result is elementary but will be used frequently.

Lemma 3.1 *Let* $A_j^0 \in \mathbb{C}^{N\times N}$. *The function* $\det\left[\nu - \sum_{j=-p}^{q} A_j^0 \mathrm{e}^{\nu j}\right]$ *has only finitely many roots* ν *in each fixed vertical strip* $\{\nu : |\operatorname{Re}\nu| \le \kappa\}$ *with* $\kappa \ge 0$, *and these roots depend continuously on the coefficient matrices* A_j^0.

Proof. Finiteness of roots follows since the sum is bounded on each vertical strip, while continuity of roots follows from analyticity in ν. ∎

Applying the preceding lemma to the functions $\det\Delta_\pm(\nu)$, we can find a number $\kappa > 0$ so that the functions $\det\Delta_\pm(\nu)$ have no roots in $\{\nu : |\operatorname{Re}\nu| \le 2\kappa\}$.

We now record various implications of Hypothesis (H3). First, it was proved in [18, Theorem A] that Hypothesis (H3)(i)-(ii) implies that $\mathcal{L}$ is a Fredholm operator[2]. We remark that if Hypothesis (H3)(i) holds and $\mathcal{L}$ is Fredholm, then Hypothesis (H3)(ii) holds, too: this follows, for instance, from [18, Theorem C] upon using exponential weights, and we refer to Lemma 4.7 below for related arguments. If $\mathcal{L}$ has a bounded inverse, we define its resolvent kernel $G(x,y)$ as the solution, in the sense of distributions, of

$$\mathcal{L}G(\cdot,y) = \delta(\cdot - y).$$

To obtain pointwise information on the resolvent kernel, it is convenient to work with a dynamical-systems formulation. Thus, we consider the equation

$$U_x(x) = \mathcal{A}(x)U(x) \tag{3.2}$$

posed on the space $Y = L^2([-p,q],\mathbb{C}^N)\times\mathbb{C}^N$, where the operator $\mathcal{A}(x)$ is defined on Y by

$$\mathcal{A}(x)\begin{pmatrix}\phi\\ \alpha\end{pmatrix} = \begin{pmatrix}\phi_z\\ A_0(x)\alpha + \sum_{j=-p,\, j\neq 0}^{q} A_j(x)\phi(j)\end{pmatrix}$$

with dense domain given by

$$Y^1 = \left\{(\phi,\alpha)\in H^1([-p,q],\mathbb{C}^N)\times\mathbb{C}^N : \phi(0)=\alpha\right\}.$$

We say that a function $U(x)$ is an H^1-solution of (3.2) on some open interval I if $U \in L^2(I,Y^1)\cap H^1(I,Y)$ and (3.2) is satisfied for $x\in I$ with values in $L^2(I,Y)$. It is known[3] that the initial-value problem associated with (3.2) is not well-posed in the sense that a solution in forward or backward time x may not exist for given initial data. Exponential dichotomies clarify for which initial data (3.2) can be solved and in which direction of the evolution variable x.

Definition 3.2 *Let* $I = \mathbb{R}^+$, $\mathbb{R}^-$, *or* $\mathbb{R}$. *We say that (3.2) has an exponential dichotomy on* I *if there exist constants* $K>0$ *and* $\kappa^{\mathrm{s}} < \kappa^{\mathrm{u}}$ *and two strongly continuous families of bounded operators* $\Phi^{\mathrm{s}}(x,y)$ *and* $\Phi^{\mathrm{u}}(x,y)$, *defined on* Y *respectively for* $x\ge y$ *and* $x \le y$, *such that the following is true: We have*

$$\sup_{x\ge y,\, x,y\in I} \mathrm{e}^{-\kappa^{\mathrm{s}}(x-y)}\|\Phi^{\mathrm{s}}(x,y)\|_{L(Y)} + \sup_{x\le y,\, x,y\in I} \mathrm{e}^{\kappa^{\mathrm{u}}(y-x)}\|\Phi^{\mathrm{u}}(x,y)\|_{L(Y)} \le K,$$

and the operators $P^{\mathrm{s}}(x) := \Phi^{\mathrm{s}}(x,x)$ *and* $P^{\mathrm{u}}(x) := \Phi^{\mathrm{u}}(x,x)$ *are complementary projections for all* $x\in I$. *Furthermore, the functions* $\Phi^{\mathrm{s}}(x,y)U_0$ *with* $x>y$ *in* I *and* $\Phi^{\mathrm{u}}(x,y)U_0$ *with* $x<y$ *in* I *are mild solutions of (3.2) for each fixed* $U_0 \in Y$.

[2] The operator $\mathcal{L}$ is Fredholm if it has finite-dimensional null space $\mathrm{N}(\mathcal{L})$ and its range $\mathrm{Rg}(\mathcal{L})$ is closed and has finite codimension. Its Fredholm index is then given by $\dim\mathrm{N}(\mathcal{L}) - \operatorname{codim}\mathrm{Rg}(\mathcal{L})$.

[3] See, for instance, [9, (1.3)].

Often, we will have $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$, but this is not required in the definition. Our first result, stated below and proved later in §3.3, shows that mild solutions of the dynamical system (3.2) that are defined via exponential dichotomies yield H^1-solutions of the functional differential equation (3.1).

Theorem 4 *Assume that Hypothesis (H3) is met and let $\kappa > 0$ be as indicated after Lemma 3.1, then (3.2) has exponential dichotomies $\Phi_{\pm}^{\mathrm{s,u}}(x,y)$ on $\mathbb{R}^{\pm}$ with rates $\kappa^{\mathrm{s}} = -\kappa$ and $\kappa^{\mathrm{u}} = \kappa$. Writing $\Phi_+^{\mathrm{s,u}}(x,y)U_0 = (\phi_+^{\mathrm{s,u}}, \alpha_+^{\mathrm{s,u}})(x,y) \in Y$ for each $U_0 \in Y$, the following is true:*

- *(i) For $z \in [-p, 0)$, define $\alpha_+^{\mathrm{s}}(y+z, y) := \phi_+^{\mathrm{s}}(y,y)(z)$, then $\alpha_+^{\mathrm{s}}(\cdot, y)$ is an H^1-solution of (3.1) on (y, ∞). An analogous statement holds for $\alpha_+^{\mathrm{u}}(\cdot, y)$ on the interval $(0, y)$.*
- *(ii) If $U_0 = (0, \alpha_0)$, then $\alpha_+^{\mathrm{s}}(\cdot, y) \in H^2((y,\infty) \setminus \{y+1, \dots, y+q\})$ and $\alpha_+^{\mathrm{u}}(\cdot, y) \in H^2((0,y) \setminus \{y-p, \dots, y-1\})$.*
- *(iii) If $U_0 \in Y$, then $\Phi_+^{j}(x,y)U_0 \in Y^1$ are H^1-solutions of (3.2) for $x > y+p$ when $j = \mathrm{s}$ and for $0 \le x < y-q$ when $j = \mathrm{u}$.*

Properties analogous to (i)-(iii) hold for the dichotomy on $\mathbb{R}^-$.

To relate solutions of (3.1) and (3.2), we introduce the embedding and projection operators

$$\iota_2: \quad \mathbb{C}^N \longrightarrow Y, \quad \alpha \longmapsto (0, \alpha), \qquad\qquad \pi_2: \quad Y \longrightarrow \mathbb{C}^N, \quad (\phi, \alpha) \longmapsto \alpha. \tag{3.3}$$

The next result, which we will prove in §3.4, relates exponential dichotomies of (3.2) on $\mathbb{R}$ to the resolvent kernel of $\mathcal{L}$. Throughout, we use the notation $J = \{-p, \dots, q\}$.

Theorem 5 *Assume that $\mathcal{L}$ satisfies Hypothesis (H3) and is invertible. Let $\kappa > 0$ be as indicated after Lemma 3.1, then there is a constant K such that the following is true: Equation (3.2) has an exponential dichotomy $\Phi^{\mathrm{s,u}}(x,y)$ on $\mathbb{R}$ with rates $\kappa^{\mathrm{s,u}} = \mp\kappa$, and the resolvent kernel $G(x,y)$ of $\mathcal{L}$ is given by*

$$G(x,y) = \begin{cases} \pi_2 \Phi^{\mathrm{s}}(x,y)\iota_2 & \text{for } x > y, \\ -\pi_2 \Phi^{\mathrm{u}}(x,y)\iota_2 & \text{for } x < y. \end{cases} \tag{3.4}$$

Furthermore, $G(\cdot, y) \in H^1(\mathbb{R} \setminus \{y\}) \cap H^2(\mathbb{R} \setminus (\{y\} + J))$ and $G_y(\cdot, y) \in H^1(\mathbb{R} \setminus (\{y\} + J))$ pointwise in y with

$$|G(x,y)| + |\partial_y G(x,y)| \le K \mathrm{e}^{-\kappa|x-y|}. \tag{3.5}$$

If $\mathcal{L} - \lambda$ is invertible for all λ in an open subset Λ of $\mathbb{C}$, then the Green's function $G(x,y,\lambda)$ associated with $\mathcal{L} - \lambda$ is analytic in $\lambda \in \Lambda$, and its derivative with respect to λ satisfies the bound (3.5).

Our next result, which we will also prove in §3.4, asserts that the exponential dichotomies constructed above can be differentiated with respect to the initial time y and depend analytically on an eigenvalue parameter λ.

Theorem 6 *Assume that $\mathcal{L}$ satisfies Hypothesis (H3) and is onto. Let $\kappa > 0$ be as before, then there is a constant K such that the following is true:*

- *(i) Equation (3.2) has exponential dichotomies $\Phi_{\pm}^{\mathrm{s,u}}(x,y)$ on $\mathbb{R}^{\pm}$, and these dichotomies satisfy $\mathrm{Rg}(P_+^{\mathrm{u}}(0)) \subset \mathrm{Rg}(P_-^{\mathrm{u}}(0))$ and $\mathrm{Rg}(P_-^{\mathrm{s}}(0)) \subset \mathrm{Rg}(P_+^{\mathrm{s}}(0))$.*
- *(ii) If $u(x)$ is a bounded H^1-solution of (3.1) on $[y, \infty)$ for some $y \ge 0$, then $(u(x+\cdot), u(x)) \in \mathrm{Rg}(P_+^{\mathrm{s}}(x))$ for all $x \ge y$; an analogous statement holds for $\mathrm{Rg}(P_-^{\mathrm{u}}(x))$ on $\mathbb{R}^-$.*

(iii) The dichotomies $\Phi^{\rm s}_+(x,y)U_0$ and $\Phi^{\rm u}_+(x,y)U_0$ are differentiable in y with values in Y for $x > y+p$ and $0 < x < y-q$, respectively, for each fixed $U_0 \in Y$, and we have

$$\|\Phi^j_+(x,y)\|_{L(Y)} + \|\partial_y\Phi^j_+(x,y)\|_{L(Y)} \le K\mathrm{e}^{-\kappa|x-y|} \tag{3.6}$$

valid for $x > y+p$ when $j = \mathrm{s}$ and for $0 \le x < y-q$ when $j = \mathrm{u}$. Furthermore, $\partial_y\pi_2\Phi^{\rm s}_+(\cdot,y)\iota_2 \in H^1((y,\infty)\setminus(\{y\}+J))$ and $\partial_y\pi_2\Phi^{\rm u}_+(\cdot,y)\iota_2 \in H^1((0,y)\setminus(\{y\}+J))$ exist pointwise in y with the pointwise bounds

$$|\partial_y\pi_2\Phi^j_+(x,y)\iota_2| \le K\mathrm{e}^{-\kappa|x-y|}$$

valid for $x > y \ge 0$ when $j = \mathrm{s}$ and for $y > x \ge 0$ when $j = \mathrm{u}$. The same properties hold for the dichotomy on $\mathbb{R}^-$.

(iv) The conclusions of (ii)-(iii) above apply to $\mathcal{L}-\lambda$ in place of $\mathcal{L}$ for all λ sufficiently close to zero: the resulting dichotomies $\Phi^{\rm s,u}_\pm(x,y,\lambda)$ on $\mathbb{R}^\pm$ belonging to $\mathcal{L}-\lambda$ can be chosen to be analytic in λ near zero, and their derivatives with respect to λ satisfy (3.6), possibly for a different constant K.

Note that [18, Theorem A] implies that $\mathrm{N}(\mathcal{L}^*) = \{0\}$ if $\mathcal{L}$ is onto, and (H3)(iii) needs to be checked only for $\mathrm{N}(\mathcal{L})$. We remark that assertion (ii) in the preceding theorem holds without the assumption that $\mathcal{L}$ is onto.

The results we discussed so far pertain to exponential dichotomies for which $\kappa^{\rm s} < 0 < \kappa^{\rm u}$ so that solutions in the stable (unstable) subspaces decay for increasing (decreasing) values of x. We now briefly discuss exponential weights and how they can be used to define exponential dichotomies that are associated with other gaps in the spectra of $\mathcal{A}_\pm$. For each $\eta \in \mathbb{R}$, define the bounded operator

$$\mathcal{L}^\eta := \mathrm{e}^{-\eta x}\mathcal{L}\mathrm{e}^{\eta x}: \quad H^1(\mathbb{R},\mathbb{C}^N) \longrightarrow L^2(\mathbb{R},\mathbb{C}^N), \quad v \longmapsto v_x + \eta v - \sum_{j=-p}^{q} \mathrm{e}^{\eta j} A_j(\cdot)v(\cdot+j). \tag{3.7}$$

and the associated dynamical system

$$V_x = \mathcal{A}^\eta(x)V. \tag{3.8}$$

Given an $L^2_{\rm loc}$-function h, we see that v is an H^1-solution of

$$v_x(x) = -\eta v(x) + \sum_{j=-p}^{q} \mathrm{e}^{\eta j}A_j(x)v(x+j) + \mathrm{e}^{-\eta x}h(x) \tag{3.9}$$

if, and only if, $u(x) = \mathrm{e}^{\eta x}v(x)$ is an H^1-solution of

$$u_x(x) = \sum_{j=-p}^{q} A_j(x)u(x+j) + h(x). \tag{3.10}$$

Similarly, (3.8) has an exponential dichotomy $\tilde\Phi^{\rm s,u}_\pm(x,y)$ with rates $\tilde\kappa^{\rm s} < 0 < \tilde\kappa^{\rm u}$ on $\mathbb{R}^\pm$ if, and only if, (3.2) has an exponential dichotomy $\Phi^{\rm s,u}_\pm(x,y)$ with rates $\kappa^{\rm s} = \tilde\kappa^{\rm s} + \eta$ and $\kappa^{\rm u} = \tilde\kappa^{\rm u} + \eta$ on $\mathbb{R}^\pm$, and these dichotomies are related via $\Phi^{\rm s,u}_\pm(x,y) = \mathrm{e}^{\eta(x-y)}\tilde\Phi^{\rm s,u}_\pm(x,y)$. Such dichotomies exist for $\mathcal{L}^\eta$ provided it satisfies (H3)(iii) and its characteristic function $\det\Delta^\eta_\pm(\nu)$ does not vanish for $\nu \in \mathrm{i}\mathbb{R}$. Inspecting the coefficients of $\mathcal{L}^\eta$, we see that $\det\Delta^\eta_\pm(\nu) = \det\Delta_\pm(\nu+\eta)$, and $\mathcal{L}^\eta$ therefore satisfies (H3)(ii) if, and only if, $\det\Delta_\pm(\nu)$ does not have any roots ν with $\mathrm{Re}\,\nu = \eta$.

3.2 The adjoint operator and its associated dynamical system

Recall the L^2 adjoint operator

$$\mathcal{L}^*: \quad H^1(\mathbb{R},\mathbb{C}^N) \longrightarrow L^2(\mathbb{R},\mathbb{C}^N), \quad \hat u \longmapsto -\hat u_x - \sum_{j=-p}^{q} A_j(\cdot-j)^*\hat u(\cdot-j),$$

which corresponds to the adjoint functional differential equation

$$\hat{u}_x(x) = -\sum_{j=-p}^{q} A_j(x-j)^* \hat{u}(x-j) =: \sum_{j=-q}^{p} \hat{A}_j(x)\hat{u}(x+j), \tag{3.11}$$

where we set $\hat{A}_j(x) := -A_{-j}(x+j)^*$ with $A^* := \overline{A}^T$ for a matrix A. This system is of the same form as (3.1) except that the set $J = \{-p, \dots, q\}$ that gives the relative location of advanced and retarded grid points now becomes $\hat{J} = \{-q, \dots, p\}$. The dynamical system

$$\hat{U}_x(x) = \hat{\mathcal{A}}(x)\hat{U}(x) \tag{3.12}$$

associated with $\mathcal{L}^*$ is defined through the operator

$$\hat{\mathcal{A}}(x)\begin{pmatrix}\psi\\ \beta\end{pmatrix} = \begin{pmatrix}\psi_z\\ -A_0(x)^*\beta - \sum_{j=-p,\, j\neq 0}^{q} A_j(x-j)^*\psi(-j)\end{pmatrix},$$

which is posed on $\hat{Y}$ with domain $\hat{Y}^1$ given by

$$\hat{Y} = L^2([-q,p],\mathbb{C}^N)\times\mathbb{C}^N, \qquad \hat{Y}^1 = \left\{(\psi,\beta)\in H^1([-q,p],\mathbb{C}^N)\times\mathbb{C}^N : \psi(0)=\beta\right\}.$$

Note that (3.1) and (3.11) are symmetric counterparts: if we start with (3.11) and take its adjoint, we arrive back at (3.1) so that $\hat{\hat{A}}_j(x) = A_j(x)$ and consequently

$$\hat{\hat{\mathcal{A}}}(x) = \mathcal{A}(x).$$

Note also that the coefficient matrices $\hat{A}_j(x)$ in (3.11) satisfy Hypothesis (H3) if, and only if, this assumption is met for the coefficients $A_j(x)$ of (3.1). Thus, if we assume Hypothesis (H3), then we can apply Theorem 4 also to (3.12) and conclude that this system has exponential dichotomies $\hat{\Phi}^{\mathrm{s,u}}_{\pm}(x,y)$ on $\mathbb{R}^{\pm}$. It is now natural to investigate the relationship between solutions of (3.2) and (3.12). To this end, we use the Hale inner product of elements $\hat{U} = (\psi,\beta)\in\hat{Y}$ and $U = (\phi,\alpha)\in Y$ that was introduced in [8] and is defined by

$$\langle \hat{U}, U\rangle_{(x)} := \langle\beta,\alpha\rangle - \sum_{j=-p}^{q}\int_0^j \langle \psi(z-j), A_j(x+z-j)\phi(z)\rangle\,\mathrm{d}z,$$

where $\langle\beta,\alpha\rangle := \bar{\beta}^T\cdot\alpha$, and the subscript (x) indicates that the Hale inner product depends explicitly on x through the coefficient matrices that appear on the right-hand side. Given a subspace E of Y, we define its annihilator $E^\perp\subset\hat{Y}$ by

$$E^\perp = \{\hat{U}\in\hat{Y} :\ \langle\hat{U},U\rangle_{(0)} = 0\ \forall U\in E\}.$$

Note that $E^\perp$ is a closed subspace of $\hat{Y}$ since the Hale inner product is a continuous bilinear form on $\hat{Y}\times Y$. We let $\hat{P}^{\mathrm{s,u}}_{\pm}(x) := \hat{\Phi}^{\mathrm{s,u}}_{\pm}(x,x)$ be the projections associated with the dichotomies of the adjoint system (3.12).

Lemma 3.3 *Assume that $A_j\in C^0(\mathbb{R},\mathbb{R}^N)$, then the following is true:*

(i) If $\hat{u}(x)$ and $u(x)$ are H^1-solutions of (3.1) and (3.11), respectively, on some common interval $I\subset\mathbb{R}$, then $\langle\hat{U}(x),U(x)\rangle_{(x)}$ is independent of x on I, where $\hat{U}(x) = (\hat{u}(x+\cdot),\hat{u}(x))\in\hat{Y}$ and $U(x) = (u(x+\cdot),u(x))\in Y$.

(ii) Assume that Hypothesis (H3) holds. If $\hat{U}(x)$ and $U(x)$ are of the form $\hat{U}(x) = \hat{\Phi}^{\mathrm{s}}(x,y_0)\hat{U}_0 + \hat{\Phi}^{\mathrm{u}}(x,y_1)\hat{U}_1$ and $U(x) = \Phi^{\mathrm{s}}(x,y_0)U_0 + \Phi^{\mathrm{u}}(x,y_1)U_1$, where $\hat{\Phi}^{\mathrm{s,u}}(x,y)$ and $\Phi^{\mathrm{s,u}}(x,y)$ are dichotomies of (3.12) and (3.2), respectively, on $\mathbb{R}^+$ or $\mathbb{R}^-$, then $\langle\hat{U}(x),U(x)\rangle_{(x)}$ is independent of $x\in[y_0,y_1]$.

(iii) If Hypothesis (H3) holds, then $\mathrm{Rg}(\hat{P}^{\mathrm{s}}_{+}(0))\subset\mathrm{Rg}(P^{\mathrm{s}}_{+}(0))^\perp$ and $\mathrm{Rg}(\hat{P}^{\mathrm{u}}_{-}(0))\subset\mathrm{Rg}(P^{\mathrm{u}}_{-}(0))^\perp$.

Proof. The first statement follows from differentiating the Hale inner product $\langle \hat{U}(x), U(x)\rangle_{(x)}$ with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\langle \hat{U}(x), U(x)\rangle_{(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\langle \hat{u}(x), u(x)\rangle - \sum_{j=-p}^{q}\int_0^j \langle \hat{u}(x+z-j), A_j(x+z-j)u(x+z)\rangle\,\mathrm{d}z\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\langle \hat{u}(x), u(x)\rangle - \sum_{j=-p}^{q}\int_x^{x+j} \langle \hat{u}(z-j), A_j(z-j)u(z)\rangle\,\mathrm{d}z\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\langle \hat{u}(x), u(x)\rangle - \sum_{j=-p}^{q}\langle \hat{u}(z-j), A_j(z-j)u(z)\rangle\Big|_{z=x}^{z=x+j},
\end{aligned}$$

which is easily seen to vanish upon using the equations satisfied by the functions $\hat{u}(x)$ and $u(x)$. Lemma 3.3(ii) is now a consequence of Theorem 4(i). To prove Lemma 3.3(iii), note that solutions to (3.2) and (3.12) associated with initial data in $\mathrm{Rg}(P_+^{\mathrm{s}}(0))$ and $\mathrm{Rg}(\hat{P}_+^{\mathrm{s}}(0))$, respectively, exist for $x \geq 0$ and decay exponentially in x. Hence, their Hale inner product, which does not depend on x by Lemma 3.3(ii), must vanish, and we conclude $\mathrm{Rg}(\hat{P}_+^{\mathrm{s}}(0)) \subset \mathrm{Rg}(P_+^{\mathrm{s}}(0))^{\perp}$. The same argument applies to the unstable subspaces. ∎

We comment briefly on some of the difficulties related to the adjoint operator and the Hale inner product. Since the Hale inner product is a continuous bilinear form, Lax–Milgram implies that there is a bounded operator $S(x) : \hat{Y} \to Y$ so that $\langle \hat{U}, U\rangle_{(x)} = \langle S(x)\hat{U}, U\rangle_Y$ for all $(\hat{U}, U) \in \hat{Y} \times Y$. However, $S(x)$ is invertible only when $\det A_{-p}(x)$ and $\det A_q(x)$ are nonzero. Thus, it is not clear that the Hale inner product behaves like a genuine dual product: for instance, if $\det A_q(z)$ vanishes on a set of open measure in $(x, x+q)$, we can find nonzero elements $\hat{U} \in \hat{Y}$ for which $\langle \hat{U}, U\rangle_{(x)} = 0$ for all $U \in Y$, and, in particular, we cannot expect that equality holds in the statements asserted in Lemma 3.3(iv)[4]. An alternative way to proceed is to work with the dynamical system

$$W_x(x) = -\mathcal{A}(x)^* W(x), \qquad W \in Y, \tag{3.13}$$

where $\mathcal{A}(x)^*$ is the Y-adjoint of $\mathcal{A}(x)$. Proceeding as in [9, Proof of Lemma 2.3], we find[5] that this operator is given by

$$\mathcal{A}(x)^* \begin{pmatrix} \tilde{\psi} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} -\tilde{\psi}_z \\ A_0(x)^*\tilde{\beta} + \tilde{\psi}(0^-) - \tilde{\psi}(0^+) \end{pmatrix}$$

with domain

$$\left\{(\tilde{\psi}, \tilde{\beta}) \in L^2([-p,q], \mathbb{C}^N) \times \mathbb{C}^N :\ \tilde{\psi} \in H^1((j, j+1), \mathbb{C}^N)\ \forall j \text{ and } \tilde{\psi}(j^+) - \tilde{\psi}(j^-) = A_j(x)^*\tilde{\beta}\ \forall j \neq 0\right\},$$

where $\tilde{\psi}(j^+) := \lim_{z\downarrow j}\psi(z)$ and $\tilde{\psi}(j^-) := \lim_{z\uparrow j}\psi(z)$. We may expect that (3.13) has exponential dichotomies given by the adjoints $\Phi_{\pm}^{\mathrm{s,u}}(y,x)^*$ of the dichotomies of (3.2). This is not clear, however, since (3.13) is not of the same form as (3.2) and the domain of $\mathcal{A}(x)^*$ depends on x, so that we cannot apply Theorem 4 to (3.13). For this reason, we work primarily with the dynamical-system formulation (3.12) associated with the adjoint of $\mathcal{L}$ instead of the adjoint (3.13) of the dynamical system (3.2). For later use, we state the following result which shows that solutions of (3.12) yield solutions of (3.13).

Lemma 3.4 *Let $\hat{U}(x) = (\psi, \beta)(x)$ be an H^1-solution of (3.12), then*

$$W(x) := \begin{pmatrix} \tilde{\psi} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} -\sum_{j=-p}^{q}\chi_{(0,j)}(\cdot)A_j(x+\cdot-j)^*\psi(x,\cdot-j) \\ \beta(x) \end{pmatrix} \tag{3.14}$$

satisfies (3.13) with values in Y, and we have $\langle \hat{U}(x), U\rangle_{(x)} = \langle W(x), U\rangle_Y$ for all $U \in Y$. Here, $\chi_I(y)$ is the indicator function of the interval I, and we set $\chi_{(0,j)}(y) := -\chi_{(j,0)}(y)$ for $j < 0$.

[4] We note that it is easy to see that equality holds in Lemma 3.3(iv) whenever $\det[A_{-p}(x)A_q(x)] \neq 0$ for all x.

[5] The formulae given here and in [9] differ by a minus sign which is due to a sign mistake in [9, Middle of p1091] that does not change any of the results in that paper.

Note that the converse of the above statement may not be true if $\det[A_q(x)A_{-p}(x)]$ has roots as we need to invert both $A_{-p}(x)$ and $A_q(x)$ to construct $\hat{U}$ from W: pick, for instance, $y \in (q-1, q)$, then (3.14) becomes

$$\tilde{\psi}(x,y) = -\sum_{j=-p}^{q} \chi_{(0,j)}(y) A_j(x+y-j)^* \psi(x, y-j) = -A_q(x+y-q)^* \psi(x, y-q),$$

and $\psi(x,\cdot)$ cannot be in L^2_{loc} with respect to the y-variable for appropriate choices of $\tilde{\psi}$ if $\det A_q(x)$ has a root of sufficiently high order in the interval $(x, x-1)$.

Proof. We show first that $W(x) = (\tilde{\psi}, \tilde{\beta})(x)$ lies in the domain of $\mathcal{A}(x)^*$. We have $\tilde{\psi}(x,\cdot) \in H^1(j, j+1)$ for all j since $\psi(x,\cdot) \in H^1(-q,p)$. Next, pick any integer k with $0 < k \leq q$, then

$$\tilde{\psi}(x,k^+) = -\sum_{j=k+1}^{q} A_j(x+k-j)^* \psi(x, k-j), \qquad \tilde{\psi}(x,k^-) = -\sum_{j=k}^{q} A_j(x+k-j)^* \psi(x, k-j)$$

and therefore

$$\tilde{\psi}(x,k^+) - \tilde{\psi}(x,k^-) = A_k(x)^* \psi(x,0) \stackrel{(3.12)}{=} A_k(x)^* \beta(x) = A_k(x)^* \tilde{\beta}(x).$$

Analogous statements hold for $-p \leq k < 0$, and we conclude that $W(x)$ lies in the domain of $\mathcal{A}(x)^*$.

Next, we show that $W(x)$ satisfies (3.13). First, note that $\tilde{\psi}_x(x,y) = \tilde{\psi}_y(x,y)$ for $y \in [-p,q] \setminus \{-p,\ldots,q\}$ as required. Next, we have

$$\tilde{\psi}(x,0^+) = -\sum_{j=1}^{q} A_j(x-j)^* \psi(x,-j), \qquad \tilde{\psi}(x,0^-) = \sum_{j=-p}^{-1} A_j(x-j)^* \psi(x,-j),$$

where we recall the definition $\chi_{(0,-j)} = -\chi_{(0,j)}$ of the indicator function for $j > 0$. Thus,

$$\tilde{\psi}(x,0^+) - \tilde{\psi}(x,0^-) = -\sum_{j \in \mathring{J}} A_j(x-j)^* \psi(x,-j) \stackrel{(3.12)}{=} \beta_x(x) + A_0(x)^* \beta(x) = \tilde{\beta}_x(x) + A_0(x)^* \tilde{\beta}(x),$$

which shows that $W(x)$ satisfies (3.13).

Finally, with $U = (\phi, \alpha)$, we have

$$\begin{aligned}
\langle \hat{U}(x), U \rangle_{(x)} &= \langle \beta(x), \alpha \rangle - \sum_{j=-p}^{q} \int_0^j \langle \psi(x, z-j), A_j(x+z-j)\phi(z) \rangle \, \mathrm{d}z \\
&= \langle \beta(x), \alpha \rangle - \sum_{j=-p}^{q} \int_{-p}^{q} \langle \chi_{(0,j)}(z) \psi(x, z-j), A_j(x+z-j)\phi(z) \rangle \, \mathrm{d}z \\
&= \langle \beta(x), \alpha \rangle - \int_{-p}^{q} \left\langle \sum_{j=-p}^{q} \chi_{(0,j)}(z) A_j(x+z-j)^* \psi(x, z-j), \phi(z) \right\rangle \mathrm{d}z \\
&= \langle \tilde{\beta}(x), \alpha \rangle + \int_{-p}^{q} \langle \tilde{\psi}(x,z), \phi(z) \rangle \, \mathrm{d}z = \langle W(x), U \rangle_Y,
\end{aligned}$$

which completes the proof. ∎

3.3 Proof of Theorem 4

We first consider exponential dichotomies of (3.2) on $\mathbb{R}^\pm$ in the space Y, whose existence has been proved in [9, Theorem 1.2] for the special case $p = q$. Fortunately, the only place where this restriction has been used is in [9, Proof of Proposition 3.1], where exponential dichotomies were obtained for an autonomous hyperbolic functional differential equation

$$u_x(x) = \sum_{j=-p}^{q} A_j u(x+j) \tag{3.15}$$

on the real line $\mathbb{R}$. Thus, we prove here the existence of exponential dichotomies of the associated dynamical system

$$U_x = \mathcal{A}_0 U, \qquad \mathcal{A}_0 \begin{pmatrix} \phi \\ \alpha \end{pmatrix} = \begin{pmatrix} \phi_z \\ A_0\alpha + \sum_{j=-p,\, j\neq 0}^{q} A_j \phi(j) \end{pmatrix} \tag{3.16}$$

on Y. The spectrum of $\mathcal{A}_0$ on Y consists entirely of eigenvalues, and these (and their multiplicity) are in 1:1 correspondence with roots (and their order) of the characteristic equation

$$\det \Delta_0(\nu) := \det \left[\nu - \sum_{j=-p}^{q} A_j \mathrm{e}^{\nu j} \right] = 0;$$

see [9, Lemma 3.1]. We assume that $\mathcal{A}_0$ is hyperbolic so that $\det \Delta_0(\nu)$ does not have any purely imaginary roots $\nu \in \mathrm{i}\mathbb{R}$. Denote by $\mathcal{L}_0$ the constant-coefficient operator

$$\mathcal{L}_0 : \quad L^2(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N), \quad u \longmapsto u_x - \sum_{j=-p}^{q} A_j u(\cdot + j) \tag{3.17}$$

with domain $H^1(\mathbb{R}, \mathbb{C}^N)$, then hyperbolicity of (3.15) implies that $\mathcal{L}_0$ has a bounded inverse by [18, Theorem A]. Finally, define the spaces

$$\begin{aligned} \mathcal{E}_0^{\mathrm{u}} &:= \{u \in L^2((-\infty, q], \mathbb{C}^N) \cap H^1((-\infty, 0], \mathbb{C}^N) : \text{ (3.15) is met on } \mathbb{R}^-\}, \\ \mathcal{E}_0^{\mathrm{s}} &:= \{u \in L^2([-p, \infty), \mathbb{C}^N) \cap H^1([0, \infty), \mathbb{C}^N) : \text{ (3.15) is met on } \mathbb{R}^+\}. \end{aligned}$$

Our first result translates [19, Theorem 3.1], which is formulated for C^0-spaces, to the current L^2-setting.

Lemma 3.5 *Assume that (3.15) is hyperbolic, then there exist strictly positive constants K and κ such that*

$$\|u(x + \cdot)\|_{L^2([-p,q])} \leq K \mathrm{e}^{-\kappa x} \|u(\cdot)\|_{L^2([-p,q])}$$

for each $u \in \mathcal{E}_0^{\mathrm{s}}$ uniformly in $x \geq 0$.

Proof. We first assert that the statement in the lemma follows from the following two claims:

(i) There exists an $x_0 \geq 0$ such that $\|u(x + \cdot)\|_{L^2([-p,q])} \leq \frac{1}{2} \sup_{y \geq 0} \|u(y + \cdot)\|_{L^2([-p,q])}$ for all $x \geq x_0$.

(ii) There exists a constant $K > 0$ such that $\|u(x + \cdot)\|_{L^2([-p,q])} \leq K \|u\|_{L^2([-p,q])}$ for all $x \geq 0$.

Indeed, these two claims imply that

$$\|u(x + \cdot)\|_{L^2([-p,q])} \overset{\text{(i)}}{\leq} \frac{1}{2^m} \sup_{y \geq 0} \|u(y + \cdot)\|_{L^2([-p,q])} \overset{\text{(ii)}}{\leq} \frac{K}{2^m} \|u(\cdot)\|_{L^2([-p,q])}$$

for all $x \geq m x_0$ by applying (i) recursively to appropriate translates of u. This establishes exponential decay of u, and it therefore suffices to prove the claims (i) and (ii).

To see (i), we argue by contradiction and assume that there are sequences $x_n \to \infty$ and $u_n \in \mathcal{E}_0^{\mathrm{s}}$ with $\sup_{y \geq 0} \|u_n(y + \cdot)\|_{L^2([-p,q])} = 1$ and $\|u_n(x_n + \cdot)\|_{L^2([-p,q])} \geq \frac{1}{2}$ for all integers $n \geq 0$. For $x \geq -x_n$, we define the translate $v_n(x) := u_n(x_n + x)$ and note that $\|v_n(\cdot)\|_{L^2([-p,q])} \geq \frac{1}{2}$. It is then easy to see that there exists a constant $K_1 > 0$ such that, for each $m \in \mathbb{Z}$, there exists an integer $n_m \geq 0$ with

$$\|v_n(\cdot)\|_{L^2([m-2p, m+1+2q])} \leq K_1$$

for all $n > n_m$. Using the differential equation (3.15), which v_n satisfies, we see that $\|v_n\|_{H^1([m-p,m+1+q])} \leq K_2$ for some K_2 that depends only on K_1 and the coefficient matrices in (3.15). Using the embedding $H^1([m-p, m+1+$

$q]) \subset C^0([m-p, m+1+q])$, we conclude that $\sup_{y\in[m-p,m+1+q]} |v_n(y)| < K_3$ and therefore $\|v_n\|_{C^1([m,m+1])} \leq K_4$, where we exploited again (3.15). We can now apply Arzela–Ascoli to infer that there is a continuous function v so that $v_n \to v$ on each compact subset of $\mathbb{R}$ as $n \to \infty$. Taking the limit in the integral form of the differential equation (3.15), we find that v is a differentiable and bounded solution of (3.15) on $\mathbb{R}$. Since $\|v\|_{L^2([-p,q])} \geq \frac{1}{2}$, the solution v is not zero, which contradicts the fact that the null space of $\mathcal{L}_0$ is trivial as $\mathcal{L}_0$ has a bounded inverse. This establishes (i).

To prove (ii), we argue again by contradiction and assume that there exist a sequence $u_n \in \mathcal{E}_0^{\mathrm{s}}$ and constants $K_n > 1$ such that $K_n \to \infty$ and

$$1 = \sup_{y\geq 0} \|u_n(y+\cdot)\|_{L^2([-p,q])} = K_n \|u_n(\cdot)\|_{L^2([-p,q])}. \tag{3.18}$$

We write y_n for the point y at which the supremum in the preceding expression is obtained: note that y_n is well-defined because $y \mapsto \|u(y+\cdot)\|_{L^2([-p,q])}$ is continuous, and that $y_n \in [0, x_0]$ because of (i). Thus, after passing to a subsequence, we can arrange that $y_n \to \tilde{y} \in [0, x_0]$ for an appropriate $\tilde{y}$. Using (3.18) and the differential equation (3.15), we infer that the norm $\|u_n\|_{H^1([m,m+1])}$ is bounded uniformly in $m \geq 0$ and in n. Thus, we see that $u_n \to u$ in L^2 on compact subintervals of $\mathbb{R}^+$. Furthermore, since $\|u_n(\cdot)\|_{L^2([-p,q])} \to 0$ by (3.18), we infer that $u_n \to u$ in L^2 on compact subsets of $[-p, \infty)$, where $u|_{[-p,q]} = 0$. Taking again the limit in the integrated form of the differential equation (3.15), we see that u satisfies (3.15) on $\mathbb{R}^+$. If we write $\tilde{u}$ for the function that vanishes for $\mathbb{R}^-$ and coincides with u on $\mathbb{R}^+$, then u is a bounded nonzero solution of (3.15) on $\mathbb{R}$, which is again a contradiction. This establishes (ii) and therefore the statement of the lemma. ∎

Define

$$\begin{aligned} E_0^{\mathrm{u}} &:= \{(\phi,\alpha) \in Y : (\phi,\alpha) = (u|_{[-p,q]}, u(0^-)) \text{ for some } u \in \mathcal{E}_0^{\mathrm{u}}\}, \\ E_0^{\mathrm{s}} &:= \{(\phi,\alpha) \in Y : (\phi,\alpha) = (u|_{[-p,q]}, u(0^+)) \text{ for some } u \in \mathcal{E}_0^{\mathrm{s}}\}, \end{aligned}$$

and note that these definitions make sense due to the definition of the spaces $\mathcal{E}^{\mathrm{s,u}}$.

Lemma 3.6 *If (3.15) is hyperbolic, then the spaces E_0^{s} and E_0^{u} are closed subspaces of Y with $E_0^{\mathrm{s}} \oplus E_0^{\mathrm{u}} = Y$.*

Proof. Using the underlying equation, it is not difficult to see that the spaces $E_0^{\mathrm{s,u}}$ are closed in Y. Furthermore, it follows from the construction of the Green's function for $\mathcal{L}_0$ in [18, Proof of Theorem 4.1] that $\{0\} \times \mathbb{C}^N$ is contained in the sum $E_0^{\mathrm{s}} + E_0^{\mathrm{u}}$. Using this property, together with arguments as in [19, Proof of Theorem 3.2], we can show that $H^1([-p,q], \mathbb{C}^N) \times \mathbb{C}^N$ is contained in the sum $E_0^{\mathrm{s}} + E_0^{\mathrm{u}}$, which implies that $E_0^{\mathrm{s}} + E_0^{\mathrm{u}} = Y$.

It remains to show that the intersection $E_0^{\mathrm{s}} \cap E_0^{\mathrm{u}}$ is trivial. Thus, assume that $(\phi^{\mathrm{s}}, \phi^{\mathrm{s}}(0^+)) = (\phi^{\mathrm{u}}, \phi^{\mathrm{u}}(0^-))$ lies in the intersection, then, using the definitions of the spaces $E_0^{\mathrm{s,u}}$ and $\mathcal{E}_0^{\mathrm{s,u}}$, we see that each such element (ϕ, α) can be extended to $\mathbb{R}^\pm$ so that $\phi|_{\mathbb{R}^\pm} \in H^1(\mathbb{R}^\pm)$ with $\phi(0^\pm) = \alpha$. We see that ϕ is therefore continuous on $\mathbb{R}$ which, taken together with the fact that $\phi|_{(-\infty,q]}$ and $\phi|_{[-p,\infty)}$ satisfy (3.15) on $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively, shows that ϕ is, in fact, C^1 on $\mathbb{R}$. Thus, ϕ is a bounded solution of (3.15) on $\mathbb{R}$ and therefore $\phi = 0$ by hyperbolicity. This shows that $E_0^{\mathrm{s}} \cap E_0^{\mathrm{u}} = \{0\}$ as desired. ∎

The preceding lemma allows us to define the bounded projection $P_0^{\mathrm{s}} : Y \to Y$ via $\mathrm{Rg}(P_0^{\mathrm{s}}) = E_0^{\mathrm{s}}$ and $\mathrm{N}(P_0^{\mathrm{s}}) = E_0^{\mathrm{u}}$ and the complementary projection $P_0^{\mathrm{u}} := 1 - P_0^{\mathrm{s}}$.

Lemma 3.7 *Assume that (3.15) is hyperbolic, then there are constants K and $\kappa > 0$ such that the following is true. The spaces $E_0^j \cap Y^1$ are dense in E_0^j, we have $\mathcal{A}_0 : E_0^j \cap Y^1 \to E_0^j$ for $j = \mathrm{s}, \mathrm{u}$, and the spectra of $\mathcal{A}_0|_{E_0^{\mathrm{s}}}$ and $\mathcal{A}_0|_{E_0^{\mathrm{u}}}$ are equal to the intersections of the spectrum of $\mathcal{A}_0$ with the left and right open complex half-planes, respectively. Furthermore, the operators*

$$\Phi^{\mathrm{s}}(x)U_0 := (u|_{[x-p,x+q]}, u(x)), \qquad \Phi^{\mathrm{u}}(x)U_0 := (u|_{[x-p,x+q]}, u(x))$$

with $P^{\rm s}_0 U_0 =: (u|_{[-p,q]}, u(0^+))$ *and* $P^{\rm u}_0 U_0 =: (u|_{[-p,q]}, u(0^-))$ *define strongly continuous semigroups on* $E^{\rm s,u}_0$ *for* $x \gtrless 0$*, respectively, with* $\|\Phi^{\rm s}(x)\| + \|\Phi^{\rm u}(-x)\| \leq K{\rm e}^{-\kappa|x|}$ *for all* $x \geq 0$*.*

Proof. The preceding discussion shows that $\Phi^{\rm s}(x)$ is a strongly continuous semigroup on $E^{\rm s}_0$. Using the definition of $\mathrm{E}^{\rm s}_0$, it follows that the generator of $\Phi^{\rm s}(x)$, which is automatically closed and densely defined, has domain given by $E^{\rm s}_0 \cap Y^1$ and is, in fact, given by the restriction of $\mathcal{A}_0$ to $E^{\rm s}_0$. The claim about the spectrum of the restriction of $\mathcal{A}_0$ to $E^{\rm s}_0$ follows from the fact that $\mathcal{A}_0$ has only point spectrum. ∎

Thus, the constant-coefficient system (3.16) has an exponential dichotomy $\Phi^{\rm s,u}(x-y)$ on $\mathbb{R}$ on the space Y. Throughout this paper, we will therefore refer to the projections $P^{\rm s,u}_0$ as the stable and unstable projections of $\mathcal{A}_0$ and, similarly, to their ranges $E^{\rm s,u}_0$ as the associated stable and unstable subspaces. Note though that these projections are not spectral projections in the usual sense. Indeed, the restrictions $\mathcal{A}_0|_{E^{\rm s,u}_0} : E^{\rm s,u}_0 \cap Y^1 \to E^{\rm s,u}_0$ are, in general, not invertible, and the projections $P^{\rm s,u}_0$ will therefore not, in general, commute with $\mathcal{A}_0$. The following lemma gives conditions under which these projections are spectral projections, which, in turn, implies stronger regularity for solutions with initial data in Y^1.

Lemma 3.8 *If* $\mathcal{A}_0$ *is hyperbolic and* $\det[A_{-p}A_q] \neq 0$*, then the spaces* $\mathrm{E}^{\rm s}_0$ *and* $\mathrm{E}^{\rm u}_0$ *are given, respectively, as the closures in* Y *of the sums of the generalized stable and unstable eigenspaces of* $\mathcal{A}_0$ *in* Y*. Furthermore,* $P^{\rm s,u}_0$ *maps* Y^1 *into* $E^{\rm s,u}_0 \cap Y^1$*, and, for each* $U_0 \in Y^1$*, the functions* $\Phi^{\rm s}(x-y)U_0$ *and* $\Phi^{\rm u}(x-y)U_0$ *are strong* C^1*-solutions of (3.16) for* $x > y$ *and* $x < y$*, respectively.*

Proof. If $\det[A_{-p}A_q] \neq 0$, then the completeness result [9, Theorem 3.1] shows that the sum of the closures of the generalized stable and unstable eigenspaces of $\mathcal{A}_0$ gives Y. This result together with the definition of $E^{\rm s,u}_0$ immediately yields the characterization of the spaces $E^{\rm s,u}_0$ as the closures of generalized stable and unstable eigenspaces of $\mathcal{A}_0$. Using the denseness and regularity of stable and unstable eigenfunctions of $\mathcal{A}_0$ in $E^{\rm s,u}_0$ and the property that $\mathcal{A}_0 : Y^1 \to Y$ is invertible, it follows that $\mathcal{A}_0 : E^{\rm s,u}_0 \cap Y^1 \to E^{\rm s,u}_0$ is invertible. Thus, the spaces $E^{\rm s,u}_1 := E^{\rm s,u}_0 \cap Y^1 = \mathcal{A}_0^{-1} E^{\rm s,u}$ are invariant under the stable and unstable projections associated with $\mathcal{A}_0$, and they satisfy $E^{\rm s}_1 \oplus E^{\rm u}_1 = Y^1$. The claimed regularity is now a consequence of these statements. ∎

We now return to the non-autonomous, asymptotically hyperbolic system (3.2). As already mentioned, Lemmas 3.5-3.7 together with the results in [9, §4] or [14] imply the existence of exponential dichotomies $\Phi^{\rm s,u}_\pm(x,y)$ of (3.2) on $\mathbb{R}^\pm$.

Next, we discuss the regularity of the mild solutions $\Phi^{\rm s,u}_\pm(x,y)U_0$ with $U_0 \in Y$ or $U_0 \in Y^1$ that we obtain from the exponential dichotomies of (3.2). We pick an arbitrary constant-coefficient operator $\mathcal{A}_{\rm ref}$ of the form (3.16) for which $\det[A_{-p}A_q] \neq 0$ and denote the associated exponential dichotomy on $\mathbb{R}$ by $\Phi^{\rm s,u}(x-y)$. We now write the coefficient operator $\mathcal{A}(x)$ in (3.2) as

$$\mathcal{A}(x) = \mathcal{A}_{\rm ref} + \mathcal{B}(x).$$

The following lemma summarizes some of the findings in [9] and shows that the regularity of solutions is essentially determined by those of the constant-coefficient problem associated with $\mathcal{A}_{\rm ref}$.

Lemma 3.9 *Assume that Hypothesis (H3) is met. For* $U_0 \in Y$ *and* $x > y \geq 0$*, define* $U(x) = \Phi^{\rm s}_+(x,y)U_0$*, then* $U(x) = U_{\rm ref}(x) + \tilde{U}(x)$*, where* $U_{\rm ref}(x) = \Phi^{\rm s}(x-y)U_0$ *and* $\tilde{U} \in L^2(\mathbb{R}, Y^1) \cap H^1(\mathbb{R}, Y)$ *satisfies* $\tilde{U}_x = \mathcal{A}(x)\tilde{U} + \mathcal{B}(x)U_{\rm ref}(x)$*. Furthermore, we have* $\tilde{U} = (\tilde{\phi}, \tilde{\alpha})$ *with* $\tilde{\alpha}(x+z) = \tilde{\phi}(x,z)$ *for all* $x > y$*,* $z \in [-p,q]$ *and* $\tilde{\alpha} \in H^1(\mathbb{R}, \mathbb{C}^N)$*. An analogous statement holds for* $\Phi^{\rm u}_+(x,y)U_0$ *for* $0 \leq x < y$ *and for the dichotomies on* $\mathbb{R}^-$*.*

Proof. For operators $\mathcal{L}$ that have a bounded inverse, the statements were proved in [9, Proof of Lemma 4.2 on p 1106]. If $\mathcal{L}$ is Fredholm, these arguments can be extended easily using the construction in [27, §5.3.2]; alternatively, details can be found in §3.4 below for the case where $\mathcal{L}$ is onto. ∎

The claims in Theorem 4(i) follow now directly from Lemma 3.9 and the properties of the spaces $\mathcal{E}_0^{\mathrm{s,u}}$ and $E_0^{\mathrm{s,u}}$ established above. Theorem 4(iii) follows from Theorem 4(i) upon using (3.1).

It remains to prove Theorem 4(ii). Given $U_0 = (0, \alpha_0) \in Y$, we note that we can write such elements uniquely as

$$U_0 = \begin{pmatrix} 0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} \phi^{\mathrm{s}} \\ \alpha^{\mathrm{s}} \end{pmatrix} - \begin{pmatrix} \phi^{\mathrm{u}} \\ \alpha^{\mathrm{u}} \end{pmatrix} \in E_0^{\mathrm{s}} \oplus E_0^{\mathrm{u}},$$

where $E_0^{\mathrm{s,u}}$ are associated with the reference operator $\mathcal{A}_{\mathrm{ref}}$ that we introduced above. In particular, $\phi^{\mathrm{s}} = \phi^{\mathrm{u}} =: \phi$ satisfies $\phi|_{(-p,0)} \in H^1((-p,0),\mathbb{C}^N)$ and $\phi|_{(0,q)} \in H^1((0,q),\mathbb{C}^N)$ due to the definitions of $E_0^{\mathrm{s,u}}$ and $\mathcal{E}_0^{\mathrm{s,u}}$. Bootstrapping shows that the solutions $\Phi^{\mathrm{s}}(x-y)U_0$ and $\Phi^{\mathrm{u}}(x-y)U_0$ of the equation $U_x = \mathcal{A}_{\mathrm{ref}}U$ have second components $\alpha^{\mathrm{s,u}}$ that satisfy $\alpha^{\mathrm{s}} \in H^2((y,\infty) \setminus \{y+1,\ldots,y+q\}$ and $\alpha^{\mathrm{u}} \in H^2((-\infty,y) \setminus \{y-1,\ldots,y-p\}$, respectively, when interpreted as elements of $\mathcal{E}^{\mathrm{s,u}}$. Using the equation for $\tilde{U}$ from Lemma 3.9, we arrive at the regularity stated in Theorem 4(ii). This completes the proof of Theorem 4.

3.4 Proof of Theorems 5 and 6

We begin with the proof of Theorem 5. With the exception of the regularity in y, the claims in Theorem 5, including analyticity in λ, follow directly from [9, Lemma 4.2] and from Theorem 4(ii). Thus, it suffices to consider regularity in y and pointwise bounds for the y-derivative of the exponential dichotomies. Since the proof of regularity in y for invertible $\mathcal{L}$ is similar to, and in fact much simpler than, the proof for the case where $\mathcal{L}$ is merely onto, we omit it and focus instead on the proof of Theorem 6.

Thus, assume from now on that $\mathcal{L}$ is onto. To prove the assertions in Theorem 6, we will repeat the construction of exponential dichotomies from [9, 27] and show how it implies our claims. For simplicity, we assume that the null space $\mathrm{N}(\mathcal{L})$ of $\mathcal{L}$ is one-dimensional, which implies that the Fredholm index of $\mathcal{L}$ is one: the proof for invertible $\mathcal{L}$ is much easier, and a higher-dimensional null space does not introduce any additional difficulties.

Denote by $V_*(x)$ the nonzero smooth solution of $U_x = \mathcal{A}(x)U$ that corresponds to a nonzero element in $\mathrm{N}(\mathcal{L})$. It follows as in [27, Lemma 5.7] that $V_*(x)$ decays exponentially to zero as $|x| \to \infty$. Furthermore, Hypothesis (H3)(iii) implies that $V_*(x) \neq 0$ for all x.

For $y \geq 0$, define the spaces

$$\begin{aligned} X &= X^0 = L^2(\mathbb{R}, Y) \\ X^1 &= \{(\phi,\alpha) \in L^2(\mathbb{R},Y) :\ (\partial_x - \partial_z)\phi \in L^2(\mathbb{R}\times[-p,q],\mathbb{C}^N),\ \alpha \in H^1(\mathbb{R},\mathbb{C}^N),\ [\phi(x)](0) = \alpha(x)\ \forall x\} \\ X_y &= \left[C^0(\mathbb{R}_y^+, Y) \cap L^2(\mathbb{R}_y^+, Y)\right] \oplus \left[C^0(\mathbb{R}_y^-, Y) \cap L^2(\mathbb{R}_y^-, Y)\right], \end{aligned}$$

where $\mathbb{R}_y^\pm = \{x :\ x \gtrless y\}$. Furthermore, define the bounded linear functional

$$[V_*, \cdot] : \quad X \longrightarrow \mathbb{R}, \quad U \longmapsto [V_*, U] := \int_{-\infty}^{-2q} \langle V_*(x), U(x) \rangle \,\mathrm{d}x,$$

and let $\mathcal{T}$ be the bounded operator defined as

$$\mathcal{T} : \quad X^1 \longrightarrow X, \quad U \longmapsto U_x - \mathcal{A}(\cdot)U.$$

It was shown in [9, §2] that the Fredholm indices of $\mathcal{T}$ and $\mathcal{L}$ and the dimensions of their null spaces coincide. In particular, $\mathcal{T}$ is Fredholm with index one, and its null space is spanned by V_*. As shown in [9, §4.1], $\mathcal{T}$ extends to a bounded operator $\mathcal{T}_{\mathrm{e}} : X \to [X^1]^*$ whose Fredholm index and null space coincide with those of $\mathcal{T}$. These results together with a bordering lemma imply that the operators

$$\tilde{\mathcal{T}} :\ X^1 \to X \times \mathbb{R},\ U \mapsto (\mathcal{T}U, [V_*, U]), \qquad \tilde{\mathcal{T}}_{\mathrm{e}} :\ X \to [X^1]^* \times \mathbb{R},\ U \mapsto (\mathcal{T}_{\mathrm{e}}U, [V_*, U])$$

are bounded and invertible. For each $y \geq 0$, define

$$\mathcal{S}_y : \quad Y \longrightarrow X_y, \quad U_0 \longmapsto (U^+, U^-) = \tilde{\mathcal{T}}_{\mathrm{e}}^{-1}(\delta(\cdot - y)U_0, 0)$$

and

$$\mathcal{J}_y : \quad Y \longrightarrow Y \times Y, \quad U_0 \longmapsto (U^+(0), U^-(0)) = ([\pi_1 \mathcal{S}_y U_0](0), [\pi_2 \mathcal{S}_y U_0](0)),$$

where π_j projects onto the jth component. It follows from [9, Lemma 4.2] that these operators are well defined and bounded uniformly in $y \geq 0$. Note that $(U^+, U^-) = \mathcal{S}_y U_0$ is the unique bounded weak solution of

$$\begin{aligned} U_x^\pm &= \mathcal{A}(x)U^\pm, \qquad x \gtrless y \\ U^+(y) - U^-(y) &= U_0 \\ \int_{-\infty}^{-2q} \langle V_*(x), U^-(x) \rangle \, \mathrm{d}x &= 0, \end{aligned}$$

which will allow us to use the operators $\mathcal{S}_y$ and $\mathcal{J}_y$ to construct exponential dichotomies of $U_x = \mathcal{A}(x)U$ on $\mathbb{R}^+$.

Lemma 3.10 *Let $y \geq 0$, then the spaces*

$$E_+^{\mathrm{s}}(y) := \mathrm{Rg}(\pi_1 \mathcal{J}_y), \qquad E_+^{\mathrm{u}}(y) := \{U_0 \in \mathrm{Rg}(\pi_2 \mathcal{J}_y) : \ \mathcal{J}_y U_0 = (0, -U_0)\}$$

are closed with $E_+^{\mathrm{s}}(y) \oplus E_+^{\mathrm{u}}(y) = Y$, and the operators

$$\begin{aligned} \Phi_+^{\mathrm{s}}(x,y)U_0 &:= \quad [\pi_1 \mathcal{S}_y U_0](x), \qquad x \geq y \geq 0 \\ \Phi_+^{\mathrm{u}}(x,y)U_0 &:= -[\pi_2 \mathcal{S}_y U_0](x), \qquad y \geq x \end{aligned}$$

define an exponential dichotomy of (3.2) on $\mathbb{R}^+$.

Proof. First, we have

$$\mathcal{S}_y V_*(y) = (V_*, 0), \qquad \mathcal{J}_y V_*(y) = (V_*(y), 0). \tag{3.19}$$

Since the bounded map $Y \times Y \to Y$, $(U_+, U_-) \mapsto U_+ - U_-$ is a left inverse of $\mathcal{J}_y$, we see that $\mathcal{J}_y$ is injective and that the spaces $E_+^{\mathrm{s}}(y) := \mathrm{Rg}(\pi_1 \mathcal{J}_y)$ and $E_-^{\mathrm{u}}(y) := \mathrm{Rg}(\pi_2 \mathcal{J}_y)$ are closed. It now follows as in [27, §5.3.2] that their intersection is spanned by $V_*(0)$ and their sum is Y.

For each $U_0 \in E_+^{\mathrm{s}}(y)$, we have

$$\mathcal{J}_y U_0 = (U_0 + \beta V_*(y), \beta V_*(y))$$

for some $\beta \in \mathbb{R}$ that depends on U_0. Thus, setting $(U_+, U_-) = \mathcal{S}_y U_0$, we find that $U^- = \beta V_*$, and the constraint

$$0 = [V_*, \mathcal{S}_y U_0] = \beta \int_{-\infty}^{-2q} |V_*(x)|^2 \, \mathrm{d}x$$

shows that β vanishes for each $U_0 \in E_+^{\mathrm{s}}(y)$. Thus, we have

$$\mathcal{J}_y U_0 = (U_0, 0), \qquad \mathcal{S}_y U_0 = (U^+, 0), \qquad \forall U_0 \in E_+^{\mathrm{s}}(y) = \mathrm{Rg}(\pi_1 \mathcal{J}_y). \tag{3.20}$$

In particular, for each $U_0 \in E_+^{\mathrm{s}}(y)$, there exists a unique bounded weak solution $U^+(x)$ of $U_x = \mathcal{A}(x)U$ for $x > y$ with $U^+(y) = U_0$.

For $U_0 \in E_-^{\mathrm{u}}(y)$, we have similarly

$$\mathcal{J}_y U_0 = ([\beta_y U_0] V_*(y), -U_0 + [\beta_y U_0] V_*(y))$$

for some linear bounded functional $\beta_y : Y \to \mathbb{R}$. Using (3.19), we see that

$$\mathcal{J}_y (U_0 - [\beta_y U_0] V_*(y)) = (0, -(U_0 - [\beta_y U_0] V_*(y)))$$

for all $U_0 \in E^{\mathrm{u}}_-(y)$. Thus, the space

$$E^{\mathrm{u}}_+(y) := \{U_0 \in E^{\mathrm{u}}_-(y) :\ \mathcal{J}_y U_0 = (0, -U_0)\} = \{U_0 \in E^{\mathrm{u}}_-(y) :\ \pi_1 \mathcal{J}_y U_0 = 0\}$$

is a complement of $\mathbb{R}V_*(y)$ in $E^{\mathrm{u}}_-(y)$. In particular, we have

$$\mathcal{J}_y U_0 = (0, -U_0), \qquad \mathcal{S}_y U_0 = (0, U^-), \qquad \forall U_0 \in E^{\mathrm{u}}_+(y), \tag{3.21}$$

and $U^-(x)$ is the unique bounded weak solution of $U_x = \mathcal{A}(x)U$ for $x < y$ with $U^-(y) = -U_0$.

Thus, we have $E^{\mathrm{s}}_+(y) \oplus E^{\mathrm{u}}_+(y) = Y$ for all $y \geq 0$. Using that the integral constraints involve only the values on $\mathbb{R}^-$, it is not difficult to see that the spaces $E^{\mathrm{s,u}}_+(y)$ are mapped into $E^{\mathrm{s,u}}_+(x)$ under $\Phi^{\mathrm{s,u}}_+(x, y)$. Equations (3.20) and (3.21) together with the use of weighted norms as in [27, Proof of Lemma 5.5] therefore give the statement of the lemma. ∎

The preceding lemma and its proof imply Theorem 6(i)-(ii). To show Theorem 6(iii), which asserts regularity in y, we fix $y \geq 0$, and let h be sufficiently close to 0. We know from the preceding discussion that $U(\cdot, h) = (U^+, U^-) = \mathcal{S}_{y+h} U_0$ is the unique bounded weak solution of

$$\begin{aligned} U^\pm_x &= \mathcal{A}(x)U^\pm, \qquad x \gtrless y + h \\ U^+(y+h) - U^-(y+h) &= U_0 \\ \int_{-\infty}^{-2q} \langle V_*(x), U^-(x)\rangle \,\mathrm{d}x &= 0. \end{aligned}$$

We set $\tilde{U}(x, h) = U(x + h, h)$ and see that $\tilde{U}(\cdot, h)$ is the unique weak solution of

$$\begin{aligned} \tilde{U}^\pm_x &= \mathcal{A}(x+h)\tilde{U}^\pm, \qquad x \gtrless y \\ \tilde{U}^+(y) - \tilde{U}^-(y) &= U_0 \\ \int_{-\infty}^{-2q-h} \langle V_*(x+h), \tilde{U}^-(x)\rangle \,\mathrm{d}x &= 0. \end{aligned} \tag{3.22}$$

First, we consider the related problem

$$\begin{aligned} \check{U}^\pm_x &= \mathcal{A}(x+h)\check{U}^\pm, \qquad x \gtrless y \\ \check{U}^+(y) - \check{U}^-(y) &= U_0 \\ \int_{-\infty}^{-2q} \langle V_*(x+h), \check{U}^-(x)\rangle \,\mathrm{d}x &= 0, \end{aligned} \tag{3.23}$$

and note that it has a unique solution $\check{U}(\cdot, h)$ in X_y. Since the h-dependent parts in the above equation are bounded operators from X^ℓ into $X^\ell \times \mathbb{R}$ for $\ell = 0, 1$ that depend smoothly on h, we conclude that $\check{U}(\cdot, h)$ depends smoothly on h as a function from $h \in (-\delta, \delta)$ into X_y. The function

$$\tilde{U}(\cdot, h) := \check{U}(\cdot, h) + \beta(h) V_*(\cdot + h)$$

with

$$\beta(h) := \frac{\int_{-2q-h}^{-2q} \langle V_*(x+h), \check{U}^-(x, h)\rangle \,\mathrm{d}x}{\int_{-\infty}^{-2q} |V_*(x)|^2 \,\mathrm{d}x}$$

then satisfies (3.22) since

$$\begin{aligned} \int_{-\infty}^{-2q-h} \langle V_*(x+h), \tilde{U}^-(x)\rangle \,\mathrm{d}x &= \int_{-\infty}^{-2q-h} \langle V_*(x+h), \check{U}^-(x)\rangle \,\mathrm{d}x + \beta(h) \int_{-\infty}^{-2q-h} |V_*(x+h)|^2 \,\mathrm{d}x \\ &\overset{(3.23)}{=} -\int_{-2q-h}^{-2q} \langle V_*(x+h), \check{U}^-(x)\rangle \,\mathrm{d}x + \beta(h) \int_{-\infty}^{-2q-h} |V_*(x+h)|^2 \,\mathrm{d}x = 0 \end{aligned}$$

by definition of $\beta(h)$. Next, we observe that $\beta(h)$ is smooth in h due to the regularity properties of V_* and $\check{U}^-(\cdot,h)$. Hence, $\tilde{U}(\cdot,h)$ is differentiable in y with values in X_y, and we have

$$U_h^{\pm}(x,0) = -U_x^{\pm}(x,0) + \tilde{U}_h^{\pm}(x,0) = -U_x^{\pm}(x,0) + \check{U}_h(x,0) + \beta_h(0)V_*(x)$$

for $x > y+p$ and $x < y-q$ by Theorem 4(iii). Furthermore, $|U_x^{\pm}(x,0)|_Y \le K\mathrm{e}^{-\kappa|x-y|}|$ for $x > y+p$ and $x < y-q$, respectively, by using the equation satisfied by $U^{\pm}(x,0)$. The same estimate can be shown for $\check{U}_h(x,0)$ by writing down the equation satisfied by $\check{U}(\cdot,h)$ and solving it in exponentially weighted norms. We omit the details.

It remains to consider Theorem 6(iv). To apply the preceding construction to $\mathcal{L}-\lambda$, we need to verify that it is onto and satisfies (H3)(iii).

Lemma 3.11 *If $\mathcal{L}$ is onto and satisfies (H3), then $\mathcal{L}-\lambda$ is also onto and satisfies (H3) for all λ sufficiently close to zero. In particular, elements in the null space of $\mathcal{L}-\lambda$ cannot vanish on intervals of length $p+q$ unless they vanish everywhere.*

Proof. Throughout, we assume that λ is sufficiently close to zero. Recall that $\mathcal{L}$ is onto and Fredholm due to (H3). If $\dim \mathrm{N}(\mathcal{L}) = d$, then the Fredholm index of $\mathcal{L}$ is d. These observations imply that $\mathcal{L}-\lambda$ is also onto and Fredholm with index d. Thus, $\dim \mathrm{N}(\mathcal{L}-\lambda) = d$ for all λ near zero, and we need to show that v cannot vanish on any interval of length $p+q$ for each nontrivial $v \in \mathrm{N}(\mathcal{L}-\lambda)$. We now proceed as above and construct, for each $y \in \mathbb{R}^+$, the map $\mathcal{J}_y(\lambda) : Y \to Y \times Y$, $U_0 \mapsto (U_+, U_-)$ associated with $\mathcal{L}-\lambda$. Arguing as in the proof of Lemma 3.10, we see that $\mathcal{J}_y(\lambda)$ is injective and bounded uniformly in $y \in \mathbb{R}^+$ and λ near zero. Furthermore, the spaces $\mathrm{Rg}(\pi_j\mathcal{J}_y(\lambda))$ are closed in Y, their sum is Y, and each nonzero element $V(y)$ in their intersection $\mathrm{Rg}(\pi_1\mathcal{J}_y(\lambda)) \cap \mathrm{Rg}(\pi_2\mathcal{J}_y(\lambda))$ yields a nonzero element v in $\mathrm{N}(\mathcal{L}-\lambda$ via $[\mathcal{S}_y(\lambda)V(y)](x) = (v(x+\cdot), v(x))$. Hypothesis (H3)(iii) implies that $\mathrm{Rg}(\pi_1\mathcal{J}_y(0)) \cap \mathrm{Rg}(\pi_2\mathcal{J}_y(0))$ has dimension d and is, in fact, spanned by $V_j(y) = (v_j(y+\cdot), v_j(y))$, where $\{v_j\}$ is a basis on $\mathrm{N}(\mathcal{L}-\lambda)$. The assertions of the lemma follow if we can show that $\mathrm{Rg}(\pi_1\mathcal{J}_y(\lambda)) \cap \mathrm{Rg}(\pi_2\mathcal{J}_y(\lambda))$ has dimension d for each λ near zero and each $y \in \mathbb{R}^+$ (since the same argument then applies to $y \in \mathbb{R}^-$). Since Y is a Hilbert space, we can find closed complements Y_j of the null space of $\pi_j\mathcal{J}_y(0)$ in Y and conclude that $\pi_j\mathcal{J}_y(0) : Y_j \to \mathrm{Rg}(\pi_j\mathcal{J}_y(0))$ is invertible, since $\mathrm{Rg}(\pi_j\mathcal{J}_y(\lambda))$ is closed in Y. Consider the bounded map

$$\tilde{\mathcal{J}}_y(\lambda) : \quad Y_1 \times Y_2 \longrightarrow Y, \quad (U_1, U_2) \longmapsto \pi_1\mathcal{J}_y(\lambda) - \pi_2\mathcal{J}_y(\lambda),$$

and observe that $\tilde{\mathcal{J}}_y(\lambda)$ is onto. Next, note that $(U_1,U_2) \in \mathrm{N}(\tilde{\mathcal{J}}_y(0))$ if and only if $\pi_1\mathcal{J}_y(\lambda)U_1 = \pi_2\mathcal{J}_y(\lambda)U_2$. Since $\pi_j\mathcal{J}_y(0)|_{Y_j}$ is an isomorphism onto $\mathrm{Rg}(\pi_j\mathcal{J}_y(0))$, we see that $\mathrm{N}(\tilde{\mathcal{J}}_y(0))$ is isomorphic to $\mathrm{Rg}(\pi_1\mathcal{J}_y(0)) \cap \mathrm{Rg}(\pi_2\mathcal{J}_y(0))$ and has therefore dimension d. In particular, $\tilde{\mathcal{J}}_y(0)$ is Fredholm with index d, and therefore so is $\tilde{\mathcal{J}}_y(\lambda)$. We conclude that $\dim \mathrm{N}(\tilde{\mathcal{J}}_y(\lambda)) = d$ for all λ near zero and, arguing as above, see that $\dim[\mathrm{Rg}(\pi_1\mathcal{J}_y(\lambda)) \cap \mathrm{Rg}(\pi_2\mathcal{J}_y(\lambda))] = d$ for all such λ, and the lemma is proved. ∎

The preceding lemma allows us now to proceed exactly as before to establish analyticity in λ. Define

$$\tilde{\mathcal{T}}(\lambda) : \; X^1 \to X \times \mathbb{R},\; U \mapsto (\mathcal{T}(\lambda)U, [V_*^0, U]), \qquad \tilde{\mathcal{T}}_{\mathrm{e}}(\lambda) : \; X \to [X^1]^* \times \mathbb{R},\; U \mapsto (\mathcal{T}_{\mathrm{e}}(\lambda)U, [V_*^0, U]),$$

where $\mathcal{T}(\lambda)U = U_x - A(\cdot,\lambda)U$, and V_*^0 spans the null space of $\mathcal{T}(0)$. The preceding construction then shows that the exponential dichotomies are locally analytic in λ, and the exponential bounds for derivatives with respect to λ follow again from using exponential weights. This completes the proof of Theorem 6.

For later use, we prove a result about the convergence of the stable subspaces associated with an exponential dichotomy on $\mathbb{R}^+$. We denote by $\mathcal{A}_\pm$ and $\mathcal{L}_\pm$ the operators given by (3.16) and (3.17), respectively, with coefficient matrices $A_j^\pm$ instead of A_j.

Lemma 3.12 *Assume that Hypothesis (H3) is met, then the subspace $E^{\rm s}_+(x) := \mathrm{Rg}(P^{\rm s}_+(x))$ converges to the stable subspace $E^{\rm s}_+$ associated with the asymptotic operator $\mathcal{A}_+$ as $x \to \infty$. Similarly, the unstable subspace $E^{\rm u}_-(x) := \mathrm{Rg}(P^{\rm u}_-(x))$ converges to the unstable subspace $E^{\rm u}_-$ of $\mathcal{A}_-$ as $x \to -\infty$.*

Proof. The following cutoff procedure is inspired by [19]. Let $\chi_+ : \mathbb{R} \to \mathbb{R}$ be a smooth monotone function that satisfies

$$\chi_+(x) := \begin{cases} 1 & 0 \le x \\ \text{monotone} & -1 \le x \le 0 \\ 0 & x \le -1. \end{cases} \tag{3.24}$$

Define the operator $\mathcal{L}^L_+ : H^1 \to L^2$ via

$$[\mathcal{L}^L_+ u](x) = u_x(x) - \sum_{j=-p}^{q} \left[A^+_j + (A_j(x) - A^+_j)\chi_+(x-L)\right] u(x+j) \tag{3.25}$$

and note that $\mathcal{L}_+ - \mathcal{L}^L_+$ is a bounded operator from H^1 into itself with $\|\mathcal{L}_+ - \mathcal{L}^L_+\|_{H^1} = \mathrm{o}_L(1)$, where, by definition, $\mathrm{o}_L(1)$ goes to zero as $L \to \infty$. In particular, $\mathcal{L}^L_+$ is invertible for all sufficiently large $L \gg 1$. Using the construction of exponential dichotomies given above, it follows that the dynamical system associated with $\mathcal{L}^L_+$ has exponential dichotomies $\Phi^{\rm s,u}_L(x,y)$ on $\mathbb{R}$ and that $\mathrm{Rg}(\Phi^{\rm s}_L(x,x))$ is $\mathrm{o}_L(1)$-close to $E^{\rm s}_+$ for $x \in \mathbb{R}$. Uniqueness of weak solutions to (3.2) shows that $\mathrm{Rg}(\Phi^{\rm s}_L(x,x)) = E^{\rm s}_+(x)$ for all $x \ge L$, which completes the proof for $E^{\rm s}_+(x)$.

An analogous proof works on $\mathbb{R}^-$ using a smooth cutoff function χ_- defined by

$$\chi_-(x) := \begin{cases} 1 & x \le 0 \\ \text{monotone} & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

and the operator $\mathcal{L}^L_- : H^1 \to L^2$ defined by

$$[\mathcal{L}^L_- u](x) = u_x(x) - \sum_{j=-p}^{q} \left[A^-_j + (A_j(x) - A^-_j)\chi_-(x+L)\right] u(x+j) \tag{3.26}$$

which is $\mathrm{o}_L(1)$ close to $\mathcal{L}_-$ as $L \to \infty$. ∎

4 Resolvent kernels of semidiscrete conservation laws

Throughout this section, we assume that the hypotheses formulated in Theorem 1 are met. We remark though that (S5) is not used in this section but will become crucial in §5.

Theorem 3 shows that the Green's function $\mathcal{G}(j,i,t,s)$ of the linearization of the system of semidiscrete conservation laws around the Lax shock can be calculated via

$$\mathcal{G}(j,i,t,s) = \frac{-1}{2\pi\mathrm{i}\sigma} \int_{\gamma-\mathrm{i}\pi\sigma}^{\gamma+\mathrm{i}\pi\sigma} \mathrm{e}^{\lambda(t-s)} G\left(j - \frac{\sigma t}{h}, i - \frac{\sigma s}{h}, \lambda\right) \mathrm{d}\lambda \qquad (\gamma \gg 1 \text{ fixed}) \tag{4.1}$$

from the resolvent kernel of the operator

$$\mathcal{L}: \quad L^2(\mathbb{R},\mathbb{C}^N) \longrightarrow L^2(\mathbb{R},\mathbb{C}^N), \quad u \longmapsto \sigma u_x - \sum_{j=-p}^{q} A_j(\cdot)u(\cdot + j).$$

This operator has domain $H^1(\mathbb{R},\mathbb{C}^N)$, and its coefficients are given by

$$A_j(x) = \partial_j f\left(u_*(x-p+1),\dots,u_*(x+q)\right) - \partial_{j+1} f\left(u_*(x-p),\dots,u_*(x+q-1)\right). \tag{4.2}$$

Recall that the resolvent kernel $G(x, y, \lambda)$ of $\mathcal{L}$ is defined as the solution, in the sense of distributions, of

$$[\mathcal{L} - \lambda]G(\cdot, y, \lambda) = \delta(\cdot - y).$$

Theorem 5 implies that $G(x, y, \lambda)$ is well defined and analytic in λ for all $\lambda \in \{\operatorname{Re}\lambda \geq 0\} \setminus 2\pi\mathrm{i}\sigma\mathbb{Z}$ since $\mathcal{L} - \lambda$ is invertible for these values of λ by Hypothesis (S1). To get useful pointwise bounds on the Green's function $\mathcal{G}(j, i, t, s)$ of the lattice system, it will be necessary to move the contour of integration in (4.1) into $\operatorname{Re}\lambda < 0$: this requires that we extend the resolvent kernel $G(x, y, \lambda)$ to a neighborhood of $\lambda = 0$, and the issue is that we do not know in what sense $G(x, y, 0)$ exists since $\lambda = 0$ lies in the spectrum of $\mathcal{L}$ by (S2). Thus, much of this section will be concerned with constructing $G(x, y, \lambda)$ for λ near zero.

4.1 The asymptotic operators and their adjoints

The results in §3 indicate that asymptotic hyperbolicity of $\mathcal{L} - \lambda$ is the key property that would allow us to use exponential dichotomies to construct $G(x, y, \lambda)$. Thus, we first check for which values of λ near zero the associated asymptotic operators $\mathcal{L}_\pm - \lambda$ with

$$\mathcal{L}_\pm : \quad u \longmapsto \sigma u_x - \sum_{j=-p}^{q} \underbrace{(\partial_j - \partial_{j+1})f(u_\pm, \dots, u_\pm)}_{=: A_j^\pm} u(\cdot + j)$$

are hyperbolic. The characteristic equations associated with $\mathcal{L}_\pm - \lambda$ are given by

$$\det \Delta_\pm(\nu, \lambda) := \det \left[\sigma\nu - \lambda - \sum_{j=-p}^{q} (\partial_j - \partial_{j+1}) f(u_\pm, \dots, u_\pm) \mathrm{e}^{\nu j} \right] = 0. \tag{4.3}$$

In Hypothesis (S1), we assumed that $\mathcal{L} - \lambda$ is invertible for all $\lambda \in \{\operatorname{Re}\lambda \geq 0\} \setminus 2\pi\mathrm{i}\sigma\mathbb{Z}$, and we conclude that $\det \Delta(\nu, \lambda) \neq 0$ for each $\nu \in \mathrm{i}\mathbb{R}$ and each λ in the aforementioned set. Furthermore, (S4) implies that $\det \Delta(\nu, 0) \neq 0$ does not have any solutions $\nu \in \mathrm{i}\mathbb{R} \setminus \{0\}$. Lemma 3.1 then shows that (4.3) can have roots ν near the imaginary axis for λ near the imaginary axis with $|\operatorname{Im}\lambda| \leq \pi|\sigma|$ only when both ν and λ are near zero. Thus, it suffices to understand the solutions of (4.3) for both ν and λ near zero.

Lemma 4.1 *Assume that (H1) is met, then the equation $\det \Delta_\pm(\nu, \lambda) = 0$ has precisely N solutions $\nu_n^\pm(\lambda)$ near the origin for each λ near zero. Furthermore, these solutions are analytic in λ and have the expansion*

$$\nu_n^\pm(\lambda) = \frac{\lambda}{\sigma - a_n^\pm} + \frac{\langle l_n^\pm, B^\pm r_n^\pm \rangle}{|l_n^\pm|^2} \frac{\lambda^2}{(a_n^\pm - \sigma)^3} + \mathrm{O}(\lambda^3), \qquad n = 1, \dots, N, \tag{4.4}$$

where $B^\pm = \sum_{j=-p}^{q} (1 - 2j)\partial_j f(u_\pm, \dots, u_\pm)$ is defined in (1.8). Finally, there are analytic functions $r_n^\pm(\lambda) \in \mathbb{R}^N$ with $r_n^\pm(0) = r_n^\pm$ so that $\Delta_\pm(\nu_n^\pm(\lambda), \lambda) r_n^\pm(\lambda) \equiv 0$.

Proof. Since we set $\partial_{-p} f = \partial_{q+1} f = 0$, we see that

$$\sum_{j=-p}^{q} (\partial_j - \partial_{j+1}) f(u_\pm, \dots, u_\pm) = 0. \tag{4.5}$$

Thus, Rouché's theorem implies that (4.3) has precisely N solutions ν near zero, counted with multiplicity, for each λ near zero. Next, a calculation shows that $\Delta_\pm(\nu, \lambda)$ has the expansion

$$\begin{aligned} \Delta_\pm(\nu, \lambda) &= \sigma\nu - \lambda - \sum_{j=-p}^{q} \partial_j f(u_\pm, \dots, u_\pm)\nu + \sum_{j=-p}^{q} (1 - 2j)\partial_j f(u_\pm, \dots, u_\pm) \frac{\nu^2}{2} + \mathrm{O}(\nu^3) \\ &= \sigma\nu - \lambda - \bar{f}_u(u_\pm)\nu + B^\pm \nu^2 + \mathrm{O}(\nu^3). \end{aligned}$$

A straightforward Lyapunov–Schmidt reduction applied to $\Delta_\pm(\nu,\lambda)r = 0$ now yields the statements made in the lemma upon recalling that $a_n^\pm \neq \sigma$ are the simple, real, distinct eigenvalues of $\bar{f}_u(u_\pm)$ with left and right eigenvectors $l_n^\pm$ and $r_n^\pm$, respectively. ∎

Alternatively, solving (4.3) for λ as a function of $\nu = \mathrm{i}\gamma$ with $\gamma \in \mathbb{R}$ yields the solution branches

$$\lambda_n^\pm(\gamma) = -(a_n^\pm - \sigma)\mathrm{i}\gamma - \frac{\langle l_n^\pm, B^\pm r_n^\pm\rangle}{|l_n^\pm|^2}\gamma^2 + \mathrm{O}(\gamma^3), \qquad n = 1,\ldots,N,$$

and (S1) implies that these curves, which lie in the essential spectrum of $\mathcal{L}$, are contained in the open left half-plane. (S5) implies further that these curves have a quadratic tangency to the imaginary axis at $\lambda = 0$.

Remark 4.2 *In the context of the weak semidiscrete shocks that we considered in Theorem 2, Lemma 4.1 implies that $\Delta_+(\nu,0)$ has a root of order N at $\nu = 0$. Furthermore, Lemma 3.1 and assumption (S4) for $u = u_0$ imply that $\Delta_+(\nu,0)$ cannot have any nonzero purely imaginary roots ν, which completes the proof of Theorem 2.*

Lemma 4.1 implies in particular that we cannot immediately apply the theory from §3 as the asymptotic operators are not hyperbolic at $\lambda = 0$. We shall see that we can nevertheless extend the resolvent kernel meromorphically across $\lambda = 0$.

First, we consider the dynamical-systems formulation

$$U_x = \mathcal{A}_\pm(\lambda)U, \qquad \mathcal{A}_\pm(\lambda)\begin{pmatrix}\phi\\ \alpha\end{pmatrix} = \begin{pmatrix}\phi_z \\ \frac{1}{\sigma}(A_0^\pm + \lambda)\alpha + \frac{1}{\sigma}\sum_{j=-p,\, j\neq 0}^{q} A_j^\pm \phi(j)\end{pmatrix} \tag{4.6}$$

on Y that is associated with $\mathcal{L}_\pm - \lambda$. The spectrum of $\mathcal{A}_\pm(\lambda)$ on Y consists entirely of eigenvalues (which we refer to as spatial eigenvalues), and these are in 1:1 correspondence[6] with roots ν of $\det\Delta_\pm(\nu,\lambda)$; see §3.3. The preceding discussion shows that the operators $\mathcal{A}_\pm(\lambda)$ are hyperbolic in $\operatorname{Re}\lambda > -\delta$ for some small $\delta > 0$ except when λ is close to zero, when they each have N eigenvalues $\nu_n^\pm(\lambda)$ near zero. As these roots will be crucial in the forthcoming analysis, we state the following remark that will allow us to isolate them from the remaining eigenvalues.

Remark 4.3 *If (H1), (S1), and (S4) are met, then Lemmas 3.1 and 4.1 imply that there are constants $\epsilon, \bar\eta > 0$ such that, for each $\lambda \in B_\epsilon(0)$, each solution ν of $\det\Delta_\pm(\nu,\lambda) = 0$ has distance larger than $4\bar\eta$ from the imaginary axis except for the N weak spatial eigenvalues $\nu_n^\pm(\lambda)$ given in (4.4) which have distance smaller than $\frac{1}{2}\bar\eta$ from the imaginary axis. In particular, $\det\Delta_\pm(\nu,\lambda) = 0$ has no solutions ν with $\bar\eta \le |\operatorname{Re}\nu| \le 3\bar\eta$ for $\lambda \in B_\epsilon(0)$. From now on, we fix the constants ϵ and $\bar\eta$, and we mean $\lambda \in B_\epsilon(0)$ when we say that λ is near zero.*

The eigenvectors of $\mathcal{A}_\pm(\lambda)$ belonging to the eigenvalues $\nu_n^\pm(\lambda)$ are given by

$$\mathcal{V}_n^\pm(\lambda) := r_n^\pm(\lambda)\begin{pmatrix}\mathrm{e}^{\nu_n^\pm(\lambda)z}\\ 1\end{pmatrix} \in Y, \qquad n = 1,\ldots,N, \tag{4.7}$$

and we note that they depend analytically on λ in a neighborhood of zero. We set

$$E_\pm^{\mathrm{c}}(\lambda) = \operatorname{span}\{\mathcal{V}_n^\pm(\lambda) :\ n = 1,\ldots,N\}. \tag{4.8}$$

Next, we use Remark 4.3, the weighted operators $\mathcal{L}_+^{\pm 2\bar\eta} - \lambda$ and $\mathcal{A}_+^{\pm 2\bar\eta}(\lambda)$ from (3.7), and the discussion at the end of §3.1 to obtain two sets of exponential dichotomies of $U_x = \mathcal{A}_+(\lambda)U$ on $\mathbb{R}$, one with rates $\kappa^{\mathrm{s}} = \bar\eta < 3\bar\eta = \kappa^{\mathrm{u}}$ and the other one with rates $\kappa^{\mathrm{s}} = -3\bar\eta < -\bar\eta = \kappa^{\mathrm{u}}$. We denote the associated subspaces belonging to these two

[6] We shall always count eigenvalues and roots with their algebraic multiplicity and order, respectively.

sets of dichotomies by $E_+^{\mathrm{cs,uu}}(\lambda)$ and $E_+^{\mathrm{ss,cu}}(\lambda)$, respectively[7]. In particular, these subspaces depend analytically on λ, and we have

$$E_+^{\mathrm{cs}}(\lambda) \oplus E_+^{\mathrm{uu}}(\lambda) = Y, \qquad E_+^{\mathrm{ss}}(\lambda) \oplus E_+^{\mathrm{cu}}(\lambda) = Y. \tag{4.9}$$

Furthermore, [12, Lemma 4.3 and Proposition 5.1] shows that

$$E_+^{\mathrm{cs}}(\lambda) = E_+^{\mathrm{ss}}(\lambda) \oplus E_+^{\mathrm{c}}(\lambda), \qquad E_+^{\mathrm{cu}}(\lambda) = E_+^{\mathrm{uu}}(\lambda) \oplus E_+^{\mathrm{c}}(\lambda) \tag{4.10}$$

for all $\lambda \in B_\epsilon(0)$. Analogous subspaces exist for $\mathcal{A}_-(\lambda)$.

Next, consider λ near zero with $\mathrm{Re}\,\lambda > 0$. In this region, the resolvent kernel $G(x,y,\lambda)$ of $\mathcal{L}-\lambda$ involves the stable dichotomy for $x > y$ and the unstable dichotomy for $x < y$; see (3.4). Lemma 3.12 shows that the associated stable subspace for $x > y$ converges to the stable subspace of $\mathcal{A}_+(\lambda)$ as $x \to \infty$, and the unstable subspace for $x < y$ converges similarly to the unstable subspace of $\mathcal{A}_-(\lambda)$ as $x \to -\infty$. Thus, to extend the dichotomies into $\{\mathrm{Re}\,\lambda < 0\}$ near $\lambda = 0$, we need to first find out how the stable and unstable subspaces of $\mathcal{A}_\pm(\lambda)$ behave near $\lambda = 0$, where hyperbolicity is lost due to the small spatial eigenvalues $\nu_n^\pm(\lambda)$. In particular, (4.9)-(4.10) show that it suffices to investigate this issue inside the center spaces $E_\pm^{\mathrm{c}}(\lambda)$ defined in (4.8). Using the information

$$\underbrace{a_1^- < \ldots < a_{k-1}^-}_{=:a_{n,\mathrm{out}}^-} < \sigma < \underbrace{a_k^- < \ldots < a_N^-}_{=:a_{n,\mathrm{in}}^-}, \qquad \underbrace{a_1^+ < \ldots < a_k^+}_{=:a_{n,\mathrm{in}}^+} < \sigma < \underbrace{a_{k+1}^+ < \ldots < a_N^+}_{=:a_{n,\mathrm{out}}^+}$$

given in (1.5) on the characteristic speeds $a_n^\pm$ of the Lax k-shock, we see that the spatial eigenvalues $\nu_n^\pm(\lambda)$ satisfy

$$\underbrace{\nu_1^-,\ldots,\nu_{k-1}^-}_{=:\nu_{n,\mathrm{out}}^-(\lambda)} > 0, \qquad \underbrace{\nu_k^-,\ldots,\nu_N^-}_{=:\nu_{n,\mathrm{in}}^-(\lambda)} < 0, \qquad \underbrace{\nu_1^+,\ldots,\nu_k^+}_{=:\nu_{n,\mathrm{in}}^+(\lambda)} > 0, \qquad \underbrace{\nu_{k+1}^+,\ldots,\nu_N^+}_{=:\nu_{n,\mathrm{out}}^+(\lambda)} < 0 \tag{4.11}$$

for $\lambda > 0$. Thus, for $\mathrm{Re}\,\lambda > 0$, the outgoing characteristics contribute to the unstable subspace of $\mathcal{A}_-(\lambda)$ and the stable subspace of $\mathcal{A}_+(\lambda)$, while the incoming characteristics contribute to the stable subspace of $\mathcal{A}_-(\lambda)$ and the unstable subspace of $\mathcal{A}_+(\lambda)$: we therefore define the finite-dimensional subspaces

$$\mathcal{R}_{\mathrm{out}}^\pm(\lambda) = \mathrm{span}\{\mathcal{V}_n^\pm(\lambda):\ a_n^\pm \gtrless 0\}, \quad \mathcal{R}_{\mathrm{in}}^\pm(\lambda) = \mathrm{span}\{\mathcal{V}_n^\pm(\lambda):\ a_n^\pm \lessgtr 0\}, \quad E_\pm^{\mathrm{c}}(\lambda) = \mathcal{R}_{\mathrm{out}}^\pm(\lambda) \oplus \mathcal{R}_{\mathrm{in}}^\pm(\lambda)$$

composed of outgoing and incoming modes, which are analytic in λ near zero. Note that $\dim \mathcal{R}_{\mathrm{out}}^+$ and $\dim \mathcal{R}_{\mathrm{out}}^-$ are $N-k$ and $k-1$, respectively. Using these definitions, it follows from the above discussion that the decompositions

$$\underbrace{E_-^{\mathrm{uu}}(\lambda) \oplus \mathcal{R}_{\mathrm{out}}^-(\lambda)}_{=:F_-^{\mathrm{eu}}(\lambda)} \oplus \underbrace{E_-^{\mathrm{ss}}(\lambda) \oplus \mathcal{R}_{\mathrm{in}}^-(\lambda)}_{=:F_-^{\mathrm{es}}(\lambda)} = Y, \qquad \underbrace{E_+^{\mathrm{ss}}(\lambda) \oplus \mathcal{R}_{\mathrm{out}}^+(\lambda)}_{=:F_+^{\mathrm{es}}(\lambda)} \oplus \underbrace{E_+^{\mathrm{uu}}(\lambda) \oplus \mathcal{R}_{\mathrm{in}}^+(\lambda)}_{=:F_+^{\mathrm{eu}}(\lambda)} = Y \tag{4.12}$$

and the bounded projections on Y associated with the decompositions $F_\pm^{\mathrm{es}}(\lambda) \oplus F_\pm^{\mathrm{eu}}(\lambda) = Y$ exist and are analytic in $\lambda \in B_\epsilon(0)$. In particular, the stable and unstable subspaces $E_\pm^{\mathrm{s,u}}(\lambda)$ of $\mathcal{A}_\pm(\lambda)$ for $\mathrm{Re}\,\lambda > 0$ can be extended analytically as the spaces $F_\pm^{\mathrm{es,eu}}(\lambda)$ to the ball $B_\epsilon(0)$. We emphasize that, for $\lambda < 0$, the subspaces $F_\pm^{\mathrm{es}}(\lambda)$ contain eigenvectors of $\mathcal{A}_\pm(\lambda)$ that belong to weakly unstable spatial eigenvalues; similarly, the spaces $F_\pm^{\mathrm{eu}}(\lambda)$ contain weakly stable spatial eigenvectors when $\lambda < 0$.

Next, we collect a few useful properties of the adjoint operator associated with $\mathcal{L}$, which is given by

$$\mathcal{L}^*: \quad L^2(\mathbb{R},\mathbb{C}^N) \longrightarrow L^2(\mathbb{R},\mathbb{C}^N), \quad \hat{u} \longmapsto -\sigma\hat{u}_x - \sum_{j=-p}^{q} A_j(\cdot - j)^*\hat{u}(\cdot - j).$$

The asymptotic operators belonging to $\mathcal{L}^*$ are

$$\mathcal{L}_\pm^*: \quad \hat{u} \longmapsto -\sigma\hat{u}_x - \sum_{j=-q}^{q} (\partial_j - \partial_{j+1}) f(u_\pm,\ldots,u_\pm)^*\hat{u}(\cdot - j).$$

[7]Note that it is not clear whether these subspaces are spectral subspaces of $\mathcal{A}_+$; see the discussion preceding Lemma 3.8.

Note that the associated characteristic equations are given by $\det\Delta_\pm(-\nu,\lambda)^*$, and we obtain from Lemma 4.1 analytic nonzero solutions $l_n^\pm(\lambda)$ of $\Delta_\pm(-\nu_n^\pm(\lambda),\lambda)^*l=0$. In particular, the operators $\hat{\mathcal{A}}_\pm(\lambda)$ associated with the spatial-dynamics formulation

$$\hat{U}_x=\hat{\mathcal{A}}_\pm(\lambda)\hat{U}$$

of $(\mathcal{L}_\pm^*-\lambda)\hat{u}=0$ have each precisely N discrete eigenvalues near zero, and these are given by $-\nu_n^\pm(\lambda)$, and the associated eigenfunctions

$$\hat{\mathcal{V}}_n^\pm(\lambda):=l_n^\pm(\lambda)\begin{pmatrix}\mathrm{e}^{-\nu_n^\pm(\lambda)z}\\ 1\end{pmatrix}\in\hat{Y},\qquad n=1,\dots,N \tag{4.13}$$

are analytic in λ.

Lemma 4.4 *Assume that Hypothesis (H1) is met, then the Hale inner products of $\hat{\mathcal{V}}_n^\pm(0)$ and $\mathcal{V}_m^\pm(0)$ with respect to the asymptotic operator $\mathcal{L}_\pm$ are given by*

$$\langle\hat{\mathcal{V}}_n^\pm(0),\mathcal{V}_m^\pm(0)\rangle_\pm=\langle l_n^\pm,r_m^\pm\rangle-\sum_{j=-p}^{q}\frac{1}{\sigma}\int_0^j\langle l_n^\pm,(\partial_j-\partial_{j+1})f(u_\pm,\dots,u_\pm)r_m^\pm\rangle\,\mathrm{d}z=\left(1-\frac{a_n^\pm}{\sigma}\right)\delta_{mn}$$

for $n,m\in\{1,\dots,N\}$.

Proof. Using the relations $\langle l_n^\pm,r_m^\pm\rangle=\delta_{mn}$, we obtain:

$$\begin{aligned}
\langle\hat{\mathcal{V}}_n^\pm(0),\mathcal{V}_m^\pm(0)\rangle_\pm &= \langle l_n^\pm,r_m^\pm\rangle-\sum_{j=-p}^{q}\frac{1}{\sigma}\int_0^j\langle l_n^\pm,(\partial_j-\partial_{j+1})f(u_\pm,\dots,u_\pm)r_m^\pm\rangle\,\mathrm{d}z\\
&= \delta_{mn}-\sum_{j=-p}^{q}\frac{j}{\sigma}\langle l_n^\pm,(\partial_j-\partial_{j+1})f(u_\pm,\dots,u_\pm)r_m^\pm\rangle\\
&\overset{(4.5)}{=} \delta_{mn}-\left\langle l_n^\pm,\sum_{j=-p+1}^{q}\partial_jf(u_\pm,\dots,u_\pm)r_m^\pm\right\rangle\\
&= \delta_{mn}-\langle l_n^\pm,\bar{f}_u(u_\pm,\dots,u_\pm)r_m^\pm\rangle \;=\; \left(1-\frac{a_n^\pm}{\sigma}\right)\delta_{mn},
\end{aligned}$$

which proves the lemma. ∎

4.2 Exponential dichotomies in weighted spaces

Consider now the dynamical system

$$U_x=\mathcal{A}(x,\lambda)U,\qquad \mathcal{A}(x,\lambda)\begin{pmatrix}\phi\\ \alpha\end{pmatrix}=\begin{pmatrix}\phi_z\\ \frac{1}{\sigma}(A_0(x)+\lambda)\alpha+\frac{1}{\sigma}\sum_{j=-p,\,j\neq0}^{q}A_j(x)\phi(j)\end{pmatrix} \tag{4.14}$$

with $A_j(x)$ as in (4.2) that belongs to $\mathcal{L}-\lambda$. Similarly, we introduce the dynamical-systems formulation

$$\hat{U}_x=\hat{\mathcal{A}}(x,\lambda)\hat{U},\qquad \hat{\mathcal{A}}(x,\lambda)\begin{pmatrix}\psi\\ \beta\end{pmatrix}=\begin{pmatrix}\psi_z\\ -\frac{1}{\sigma}(A_0(x)^*+\lambda)\beta-\frac{1}{\sigma}\sum_{j=-p,\,j\neq0}^{q}A_j(x-j)^*\psi(-j)\end{pmatrix} \tag{4.15}$$

with $\hat{U}\in\hat{Y}$ that is induced by the adjoint operator $\mathcal{L}^*-\lambda$.

Lemma 4.5 *Assume that Hypothesis (H1) is met, then the constant functions $\Psi_n:=(l_n^\pm,l_n^\pm)\in\hat{Y}$ satisfy (4.15) with $\lambda=0$ for $n=1,\dots,N$.*

Proof. It suffices to prove that $\mathcal{L}^* l_n^{\pm} = 0$ for each n. Since $\partial_{-p} f = \partial_{q+1} f = 0$, we have

$$\begin{aligned}
\mathcal{L}^* l &= -\sum_{j=-p}^{q} \left[\partial_j f(u_*(x-j-p+1),\ldots,u_*(x-j+q))^* - \partial_{j+1} f(u_*(x-j-p),\ldots,u_*(x-j+q-1))^*\right] l \\
&= -\sum_{j=-p}^{q} \left[\partial_j f(u_*(x-j-p+1),\ldots,u_*(x-j+q))^* \right. \\
&\qquad \left. -\partial_j f(u_*(x-j+1-p),\ldots,u_*(x-j+1+q-1))^*\right] l \\
&= 0,
\end{aligned}$$

and the assertion is proved. ∎

We can now prove Lemma 1.1.

Proof of Lemma 1.1. We show that a Lax k-shock that satisfies Hypotheses (H1), (H2), and (S4) converges exponentially to its asymptotic states $u_\pm$ and that its speed σ is given by the Rankine–Hugoniot condition. Note that we already know that $u_*'(x)$ converges to zero as $|x| \to \infty$.

Let $\mathcal{L}_+^L$ be as in (3.25), and define $\mathcal{L}_+^{L,\eta} := \mathrm{e}^{-\eta x} \mathcal{L}_+^L \mathrm{e}^{\eta x}$ as in (3.7). Choose $\bar\eta$ as in Remark 4.3, then the discussion at the end of §3.1 implies that the operators $\mathcal{L}_+^{L,\pm 2\bar\eta}$ are invertible. Consequently, the dynamical system $U_x = \mathcal{A}(x,0)U$ we introduced in (4.14) has exponential dichotomies $\Phi^{\mathrm{cs},\mathrm{uu}}(x,y)$ with rates $\kappa^{\mathrm{s}} = \bar\eta < 3\bar\eta = \kappa^{\mathrm{u}}$ and $\Phi^{\mathrm{cs},\mathrm{uu}}(x,y)$ with rates $\kappa^{\mathrm{s}} = -3\bar\eta < -\bar\eta = \kappa^{\mathrm{u}}$ that are defined for $x, y \geq L$. Furthermore, using the relation between the weighted and unweighted systems, we conclude from Lemma 3.12 that $\mathrm{Rg}(\Phi^j(x,x)) =: E^j(x) \to E_+^j$ as $x \to \infty$ for $j = \mathrm{cs}, \mathrm{ss}$, where E_+^j are the subspaces in (4.8)-(4.10) evaluated at $\lambda = 0$. Using the properties of the spaces E_+^j given in (4.8)-(4.10), we see that $E_+^{\mathrm{ss}}(x) \to E_+^{\mathrm{ss}}$ and $E_+^{\mathrm{cs}}(x) \to E_+^{\mathrm{ss}} \oplus E_+^{\mathrm{c}}$ as $x \to \infty$.

Next, consider the Lax shock $u_*(x)$ and define $U_*'(x) := (u_*'(x+\cdot), u_*'(x)) \in Y$ which is then an H^1-solution of $U_x = \mathcal{A}(x,0)U$ on $\mathbb{R}$. Theorem 6(ii) applied to the equation that correspond to $\mathcal{L}_+^{L,2\bar\eta}$ implies that $U_*'(x) \in E_+^{\mathrm{cs}}(x)$ for all $x \geq L$. Next, Lemmas 4.5 and 3.3(i) show that the constant solutions Ψ_n have nonzero Hale inner product with elements in E_+^{c} and that $\langle \Psi_n, V^{\mathrm{ss}} \rangle_{(x)} = 0$ for all $V^{\mathrm{ss}} \in E_+^{\mathrm{s}}(x)$ with $x \geq L$ and each $n = 1,\ldots,N$. Consider now the bounded map

$$\Psi_x : \quad Y \longrightarrow \mathbb{R}^N, \quad U_0 \longmapsto \left(\langle \Psi_n, U_0 \rangle_{(x)}\right)_{n=1,\ldots,N}$$

for $x \geq L$. We claim that $\mathrm{N}(\Psi_x|_{E_+^{\mathrm{cs}}(x)}) = E_+^{\mathrm{ss}}(x)$ for $x \geq L$ sufficiently large. Indeed, we already showed that $E_+^{\mathrm{ss}}(x) \subset \mathrm{N}(\Psi_x|_{E_+^{\mathrm{cs}}(x)})$. Furthermore, $\mathrm{Rg}(\Psi_x|_{E_+^{\mathrm{cs}}(x)}) = \mathbb{R}^N$ for sufficiently large x due to continuity of the Hale inner product and since $E_+^{\mathrm{cs}}(x)$ converges to $E_+^{\mathrm{ss}} \oplus E_+^{\mathrm{c}}$. Since $E_+^{\mathrm{ss}}(x)$ has codimension N in $E_+^{\mathrm{cs}}(x)$, the claim follows. Since $U_*'(x)$ converges to zero as $x \to \infty$, we conclude from Lemma 3.3(i) that $\Psi_x U_*'(x) = 0$. Hence, $U_*'(x) \in E_+^{\mathrm{ss}}(x)$ for all sufficiently large x, which proves that $|U_*'(x)| \leq K\mathrm{e}^{-3\bar\eta x}$ for $x \geq L$ for some K that does not depend on x. The same arguments apply to $x \leq -L$ upon using $\mathcal{L}_-^L$ from (3.26). Thus, there is a constant K such that $|U_*'(x)| \leq K\mathrm{e}^{-3\bar\eta|x|}$ for $x \in \mathbb{R}$ as claimed.

It remains to establish the Rankine–Hugoniot condition for σ. Note that the shock profile $u_*(x)$ is a smooth solution of the nonlinear functional differential equation

$$\sigma u_*'(x) = f(u_*(x-p+1),\ldots,u_*(x+q)) - f(u_*(x-p),\ldots,u_*(x+q-1)),$$

and integrating this equation in x over the interval $[-L, L]$ we get

$$\begin{aligned}
\sigma(u_*(L) - u_*(-L)) &= \int_{-L}^{L} [f(u_*(x-p+1),\ldots,u_*(x+q)) - f(u_*(x-p),\ldots,u_*(x+q-1))]\,\mathrm{d}x \\
&= \int_{-L+1}^{L+1} f(u_*(x-p),\ldots,u_*(x+q-1))\,\mathrm{d}x - \int_{-L}^{L} f(u_*(x-p),\ldots,u_*(x+q-1))\,\mathrm{d}x \\
&= \underbrace{\int_{L}^{L+1} f(u_*(x-p),\ldots,u_*(x+q-1))\,\mathrm{d}x}_{\to \bar f(u_+) \text{ as } L\to\infty} - \underbrace{\int_{-L}^{-L+1} f(u_*(x-p),\ldots,u_*(x+q-1))\,\mathrm{d}x}_{\to \bar f(u_-) \text{ as } L\to\infty},
\end{aligned}$$

which shows that $\sigma[u_*] = [\bar{f}(u_*)]$ as claimed. ∎

To obtain differentiability of exponential dichotomies in y and analyticity in λ, we would like to appeal to Theorem 6 which assumes that the underlying operators are onto. Thus, we now prove Fredholm properties of the weighted operator $\mathcal{L}^\eta = \mathrm{e}^{-\eta x}\mathcal{L}\mathrm{e}^{\eta x}$ that we introduced in (3.7).

Lemma 4.6 *Assume that Hypotheses (H1)-(H2) and (S1)-(S4) are met, and pick $\bar{\eta} > 0$ as in Remark 4.3, then $\mathcal{L}^{\pm 2\bar{\eta}}$ is Fredholm with index one, and $\mathrm{N}(\mathcal{L}^{\pm 2\bar{\eta}}) = \mathrm{span}\{u_*'\}$. In particular, $\mathcal{L}^{\pm 2\bar{\eta}}$ is onto.*

We remark that Lemmas 3.11 and 4.6 imply that $\mathcal{L}^{\pm 2\bar{\eta}} - \lambda$ is then also surjective and Fredholm with index one for all λ near zero.

Proof. We focus on $\mathcal{L}^{2\bar{\eta}}$ as the arguments for $\mathcal{L}^{-2\bar{\eta}}$ are analogous.

First, we fix $\lambda \in (0, \epsilon)$ and recall that the operator $\mathcal{L} - \lambda$ is invertible and therefore Fredholm with index zero. Recall also that the dispersion relations $\det \Delta_\pm(\nu, \lambda) = 0$ have precisely N roots $\nu_n^\pm(\lambda)$ with distance less than $3\bar{\eta}$ of the imaginary axis and that these roots actually lie in a ball of radius less than $\bar{\eta}$ centered at the origin. In fact, we know from (4.11) that k of the N small roots at $x = \infty$ and $k - 1$ of the small roots at $x = -\infty$ have positive real part for $\lambda > 0$. Applying [18, Theorems B and C], we see that the Fredholm index of $\mathcal{L}^{2\bar{\eta}} - \lambda$ is equal to the difference of the small roots with real part in the interval $(0, 2\bar{\eta})$ at $x = \infty$ and $x = -\infty$ so that this index is equal to $k - (k - 1) = 1$. Next, we can change λ monotonically back to zero without changing the Fredholm index of $\mathcal{L}^{2\bar{\eta}} - \lambda$ by [18, Theorems B and C] since none of the roots of the dispersion relations $\det \Delta_\pm(\nu, \lambda)$ crosses $\mathrm{Re}\, \nu = 2\bar{\eta}$. Thus, $\mathcal{L}^{2\bar{\eta}}$ is Fredholm with index one, and the small roots of $\det \Delta_\pm(\nu, 0)$ are back at the origin.

It remains to show that $\mathrm{N}(\mathcal{L}^{2\bar{\eta}}) = \mathrm{span}\{u_*'\}$. The arguments in the proof of Lemma 1.1 show that $u_*' \in \mathrm{N}(\mathcal{L}^{2\bar{\eta}})$. Assume that v is another linearly independent element in the null space of $\mathcal{L}^{2\bar{\eta}}$, then $|v(x)| \leq K\mathrm{e}^{\bar{\eta}x}$ for $x \leq 0$ by definition of $\mathcal{L}^{2\bar{\eta}}$. In particular, setting $V(x) = (v(x + \cdot), v(x))$ and recalling the constant solutions Ψ_n from Lemma 4.5, we have $\langle \Psi_n, V(x) \rangle_{(x)} = 0$ for all x, and, arguing as in the proof of Lemma 1.1 we conclude that $V(x) \in E_+^{\mathrm{ss}}(x)$ for all sufficiently large $x \gg 1$. But then we have $|v(x)| \leq K\mathrm{e}^{-\bar{\eta}|x|}$ for $x \in \mathbb{R}$ and therefore $v \in \mathrm{N}(\mathcal{L})$, which contradicts (S2). ∎

To construct analytic extensions of the exponential dichotomies from $\{\mathrm{Re}\, \lambda > 0\}$ into a neighborhood of $\lambda = 0$, we need to isolate the small spatial eigenvalues from the remainder of the spatial spectrum: this needs to be done not just for the asymptotic dynamical system (4.6) but for the full problem associated with the operator $\mathcal{L} - \lambda$. The following lemma shows that we can isolate the dynamics associated with the strongly decaying and growing directions. Recall the definitions $\iota_2 : \mathbb{C}^N \to Y$, $\alpha \mapsto (0, \alpha)$ and $\pi_2 : Y \to \mathbb{C}^N$, $(\phi, \alpha) \mapsto \alpha$ from (3.3).

Lemma 4.7 *Assume that Hypotheses (H1)-(H2) and (S4) are met, and pick $\bar{\eta} > 0$ as in Remark 4.3. For λ near zero, equation (4.14) then has exponential dichotomies $\Phi_+^{\mathrm{ss,cu}}(x, y, \lambda)$ with rates $\kappa^{\mathrm{s}} = -3\bar{\eta} < \kappa^{\mathrm{u}} = -\bar{\eta}$ and $\Phi_+^{\mathrm{cs,uu}}(x, y, \lambda)$ with rates $\kappa^{\mathrm{s}} = \bar{\eta} < \kappa^{\mathrm{u}} = 3\bar{\eta}$ on $\mathbb{R}^+$ that are analytic in λ. Furthermore, the operators $\Phi_+^{\mathrm{cs,ss}}(x, y, \lambda)$ and $\Phi_+^{\mathrm{cu,uu}}(x, y, \lambda)$ are differentiable in y for $x > y + p$ and $0 \leq x < y - q$, while the operators $\pi_2\Phi_+^{\mathrm{ss,uu}}(x, y, \lambda)\iota_2$ are differentiable pointwise in y for $x \gtrless y$: the y-derivatives of all these operators satisfy exponential bounds with the same rates as the original operators though the constants K may be larger. Finally, setting $E_+^j(x, \lambda) := \mathrm{Rg}(\Phi_+^j(x, x, \lambda))$ for $j = \mathrm{ss}, \mathrm{cs}$, we have*

$$E_+^{\mathrm{ss}}(x, \lambda) \to E_+^{\mathrm{ss}}(\lambda), \quad E_+^{\mathrm{cs}}(x, \lambda) \to E_+^{\mathrm{ss}}(\lambda) \oplus E_+^{\mathrm{c}}(\lambda)$$

as $x \to \infty$. Analogous statements hold on $\mathbb{R}^-$.

Proof. Lemma 4.6 shows that the operators $\mathcal{L}^{\pm 2\bar{\eta}}$ satisfy the assumptions of Theorem 6, and the assertions follow now from this theorem and the discussion at the end of §3.1 where we related dichotomies of the weighted and unweighted systems. ∎

4.3 Analytic extension of the exponential dichotomies on $\mathbb{R}^\pm$

We now extend the exponential dichotomies of the system

$$U_x = \mathcal{A}(x,\lambda)U, \qquad \mathcal{A}(x,\lambda)\begin{pmatrix}\phi\\ \alpha\end{pmatrix} = \begin{pmatrix}\phi_z\\ \frac{1}{\sigma}(A_0(x)+\lambda)\alpha + \frac{1}{\sigma}\sum_{j=-p,\, j\neq 0}^{q} A_j(x)\phi(j)\end{pmatrix} \tag{4.16}$$

on $\mathbb{R}^\pm$ analytically from $\operatorname{Re}\lambda > 0$ to a neighborhood of $\lambda = 0$.

Lemma 4.8 *Assume that Hypotheses (H1)-(H2) and (S1)-(S4) are met. There are then a positive constant ϵ and a family $\Phi_+^{\mathrm{s,u}}(x,y,\lambda)$ of bounded evolution operators of (4.16) that are defined for $x \gtrless y$ with $x,y\in\mathbb{R}^+$, are analytic in $\lambda\in B_\epsilon(0)$, give exponential dichotomies of (4.16) for $\operatorname{Re}\lambda>0$, and satisfy the estimates*

$$\begin{aligned}
\|\Phi_+^{\mathrm{s}}(x,y,\lambda)\|_{L(Y)} &= \sum_{\nu_{\mathrm{out}}^+} \mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)(x-y)}\right), \qquad x\geq y\geq 0 \\
\|\Phi_+^{\mathrm{u}}(x,y,\lambda)\|_{L(Y)} &= \sum_{\nu_{\mathrm{in}}^+} \mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{in}}^+(\lambda)(x-y)}\right), \qquad y\geq x\geq 0.
\end{aligned} \tag{4.17}$$

Furthermore, $\Phi_+^{\mathrm{s,u}}(x,y,\lambda)$ are differentiable in y for $x>y+p$ and $0\leq x<y-q$, respectively, with

$$\begin{aligned}
\|\partial_y\Phi_+^{\mathrm{s}}(x,y,\lambda)\iota_2\|_{L(\mathbb{R}^N,Y)} &= \mathrm{O}(\mathrm{e}^{-\bar\eta|x-y|}) + |\lambda|\sum_{\nu_{\mathrm{out}}^+} \mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)(x-y)}\right), \qquad x>y+p \\
\|\partial_y\Phi_+^{\mathrm{u}}(x,y,\lambda)\iota_2\|_{L(\mathbb{R}^N,Y)} &= \mathrm{O}(\mathrm{e}^{-\bar\eta|x-y|}) + |\lambda|\sum_{\nu_{\mathrm{in}}^+} \mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{in}}^+(\lambda)(x-y)}\right), \qquad y-q>x\geq 0,
\end{aligned} \tag{4.18}$$

while the operators $\pi_2\Phi_+^{\mathrm{ss,uu}}(x,y,\lambda)\iota_2$ that appear in the explicit expression (4.29) of the extended dichotomies are differentiable pointwise in y for $x\gtrless y$ and satisfy pointwise estimates in $\mathbb{C}^{N\times N}$ analogous to (4.18) for $x\gtrless y$. Finally, the stable subspace $\operatorname{Rg}(\Phi_+^{\mathrm{s}}(x,x,\lambda))$ converges to the extended stable subspace $E_+^{\mathrm{ss}}(\lambda)\oplus\mathcal{R}_{\mathrm{out}}^+(\lambda)$ defined in (4.12) as $x\to\infty$. The same statement with symmetric bounds holds for dichotomies on $\mathbb{R}^-$.

Proof. First, we construct H^1-solutions $U_n^+(x,\lambda)$ of (4.16) on $\mathbb{R}^+$ so that $U_n^+(x,\lambda)/|U_n^+(x,\lambda)|_Y$ converges with uniform rate $2\bar\eta$ to the element $\mathcal{V}_n^+(\lambda)\in\mathcal{R}_{\mathrm{out}}^+(\lambda)\oplus\mathcal{R}_{\mathrm{in}}^+(\lambda)$ as $x\to\infty$, where $n=1,\dots,N$. To construct $U_n^+(x,\lambda)$, we consider the system

$$(\mathcal{L}-\lambda)(r_n^+(\lambda)\mathrm{e}^{\nu_n^+(\lambda)x}+v_n)=0 \tag{4.19}$$

where the relation between $\mathcal{V}_n^+(\lambda)$ and $r_n^+(\lambda)$ is contained in (4.7). Since $(\mathcal{L}_+-\lambda)r_n^+(\lambda)\mathrm{e}^{\nu_n^+(\lambda)x}=0$ and the coefficients in $\mathcal{L}$ converge exponentially with rate $-3\bar\eta$ to those of $\mathcal{L}_+$, we see that (4.19) is of the form

$$(\mathcal{L}-\lambda)v_n = h_n(x,\lambda), \tag{4.20}$$

where $h_n(x,\lambda)$ is analytic in λ and there is a constant K so that $|h_n(x,\lambda)|\leq K\mathrm{e}^{-3\bar\eta x}$ for $x\geq 0$. Using the cutoff function χ_+ from (3.24), we define $h_n^{\mathrm{e}}(x,\lambda):=\chi_+(x+p+q)h_n(x,\lambda)$, which is analytic in λ and satisfies $|h_n^{\mathrm{e}}(x,\lambda)|\leq K\mathrm{e}^{-3\bar\eta|x|}$ for $x\in\mathbb{R}$. Following the discussion centered around (3.9)-(3.10) in §3.1, we now consider

$$(\mathcal{L}^{-2\bar\eta}-\lambda)w_n = \mathrm{e}^{2\bar\eta x}h_n^{\mathrm{e}}(x,\lambda). \tag{4.21}$$

By construction, the right-hand side is in $L^2(\mathbb{R})$, and the discussion after Lemma 4.6 shows that $\mathcal{L}^{-2\bar\eta}-\lambda$ is onto and Fredholm with index one. Thus, adding an appropriate normalization condition, equation (4.21) has a solution $w_n(\cdot,\lambda)\in H^1$ that depends analytically on λ. We conclude that $v_n(x,\lambda)=\mathrm{e}^{-2\bar\eta x}w_n(x,\lambda)$ is an H^1-solution of (4.20) that is analytic in λ and satisfies $|v_n(x)|\leq K\mathrm{e}^{-2\bar\eta x}$ for $x\geq 0$. Defining $V_n^+(x,\lambda)=(v_n^+(x+\cdot,\lambda),v_n^+(x,\lambda))\in Y^1$ for $x\geq 0$ and recalling that $\mathrm{e}^{\nu_n^+(\lambda)x}\mathcal{V}_n^+(\lambda)$ is the solution of the asymptotic system (4.6) that corresponds to the solution $r_n^+(\lambda)\mathrm{e}^{\nu_n^+(\lambda)x}$ of $(\mathcal{L}_+-\lambda)u=0$, we see that

$$U_n^+(x,\lambda)=\mathrm{e}^{\nu_n^+(\lambda)x}\mathcal{V}_n^+(\lambda)+V_n^+(x,\lambda), \qquad n=1,\dots,N$$

are H^1-solutions of (4.16) that depend analytically on λ with $|V_n^+(x,\lambda)|_Y \leq C\mathrm{e}^{-2\bar{\eta}x}$ as $x \to \infty$. For $x \geq 0$, we set

$$E_+^{\mathrm{c}}(x,\lambda) := \operatorname{span}\{U_n^+(x,\lambda)\}_{n=1,\dots,N}.$$

Next, we construct solutions $\Psi_n^+(x,\lambda)$ of the adjoint system (4.15) that converge exponentially with rate $-2\bar{\eta}$ to the adjoint eigenfunctions $\hat{\mathcal{V}}_n^{\pm}(\lambda)$ given in (4.13) as $x \to \infty$. Since the adjoint eigenfunctions $\hat{\mathcal{V}}_n^{\pm}(0)$ are solutions of the full equation $\hat{U} = \hat{A}(x,0)\hat{U}$ due to Lemma 4.5, we seek corrections to $\mathrm{e}^{-\nu_n^+(\lambda)x}\hat{\mathcal{V}}_n^{\pm}(\lambda)$ in the form $\lambda\hat{V}_n$ and therefore consider the system

$$(\mathcal{L}^* - \lambda)(l_n^+(\lambda)\mathrm{e}^{-\nu_n^+(\lambda)x} + \lambda\hat{v}_n) = 0. \tag{4.22}$$

Indeed, Lemma 4.5 implies that (4.22) is of the form

$$(\mathcal{L}^* - \lambda)\hat{v}_n = \hat{h}_n(x,\lambda), \tag{4.23}$$

where $\hat{h}_n(x,\lambda)$ is analytic in λ and $|\hat{h}_n(x,\lambda)| \leq K\mathrm{e}^{-3\bar{\eta}x}$ for $x \geq 0$ for some constant K. Set $\hat{h}_n^{\mathrm{e}}(x,\lambda) := \chi_+(x+p+q)\hat{h}_n(x,\lambda)$, which is analytic in λ and satisfies $|\hat{h}_n^{\mathrm{e}}(x,\lambda)| \leq K\mathrm{e}^{-3\bar{\eta}|x|}$ for $x \in \mathbb{R}$. As before, we now wish to solve

$$(\mathcal{L}^{*,-2\bar{\eta}} - \lambda)\hat{w}_n = \hat{h}_n^{\mathrm{e}}(x,\lambda) \tag{4.24}$$

for $\hat{w} \in H^1$. Note that the right-hand side lies in L^2. Inspecting the coefficients, it is easy to check that the operator $\mathcal{L}^{*,-2\bar{\eta}}$ coincides with $[\mathcal{L}^{2\bar{\eta}}]^*$. Recall that $\mathcal{L}^{2\bar{\eta}} - \lambda$ is surjective and Fredholm of index one, and [18, Theorem A] implies that $\mathcal{L}^{*,-2\bar{\eta}} - \lambda$ is injective and that its range has codimension one. To determine its range, note that we can choose an H^1-function $w_*(\cdot,\lambda)$ that depends analytically on λ and spans the one-dimensional null space of $\mathcal{L}^{2\bar{\eta}} - \lambda$. We also know from Lemma 3.11 that w_* cannot vanish on any interval of length $p+q$. It follows again from [18, Theorem A] that an element $\bar{h} \in L^2$ is in the range of $\mathcal{L}^{*,-2\bar{\eta}} - \lambda$ if, and only if, its scalar product with w_* vanishes. Using that w_* does not vanish everywhere on $\mathbb{R}^-$, we can now modify the right-hand side $\hat{h}_n^{\mathrm{e}}(x,\lambda)$ for $x \in \mathbb{R}^-$ so that the modified function is still analytic in λ and lies in the range of $\mathcal{L}^{*,-2\bar{\eta}} - \lambda$. Thus, (4.24) has an H^1-solution $\hat{w}_n(\cdot,\lambda)$ that is analytic in λ, which then corresponds to a solution $\hat{v}_n(x,\lambda) = \mathrm{e}^{-2\bar{\eta}x}\hat{w}_n(x,\lambda)$ is an H^1-solution of (4.23) that is analytic in λ and satisfies $|\hat{v}_n(x)| \leq K\mathrm{e}^{-2\bar{\eta}x}$ for $x \geq 0$. Setting $\hat{V}_n^+(x,\lambda) = (\hat{v}_n^+(x+\cdot,\lambda), \hat{v}_n^+(x,\lambda)) \in Y^1$ for $x \geq 0$, we obtain that

$$\Psi_n^+(x,\lambda) = \mathrm{e}^{-\nu_n^+(\lambda)x}\hat{\mathcal{V}}_n^+(\lambda) + \lambda\hat{V}_n^+(x,\lambda), \qquad n = 1,\dots,N \tag{4.25}$$

are solutions of (4.15) on $\mathbb{R}^+$, lie in $\hat{Y}^1$, are C^1 in $x \geq 0$ and analytic in λ near zero, and satisfy

$$\hat{V}_n^+(x,\lambda) = \mathrm{O}(\mathrm{e}^{-2\bar{\eta}|x|}), \qquad x \geq 0.$$

In particular, we have

$$\partial_x\Psi_n^+(x,\lambda) = \lambda\mathrm{O}(\mathrm{e}^{-\nu_n^+(\lambda)x}), \qquad n = 1,\dots,N. \tag{4.26}$$

Theorem 6(i) and Lemmas 3.3(iii) and 4.4 show furthermore that

$$\langle\Psi_n^+(x,\lambda), U_m^+(x,\lambda)\rangle_{(x)} = \delta_{mn}, \qquad \Psi_n^+(x,\lambda) \in E_+^{\mathrm{ss}}(x,\lambda)^\perp, \qquad \Psi_n^+(x,0) \in E_+^{\mathrm{uu}}(x,0)^\perp \tag{4.27}$$

for all $x \geq 0$, where the annihilators are computed with respect to the Hale inner product.

We can now construct the analytic extensions of the exponential dichotomies. First, we claim that

$$E_+^{\mathrm{cs}}(y,\lambda) = E_+^{\mathrm{ss}}(y,\lambda) \oplus E_+^{\mathrm{c}}(y,\lambda) \tag{4.28}$$

for all $y \geq 0$. Using the convergence properties of the spaces $E_+^j(x,\lambda)$ from Lemma 4.7 and the construction of the solutions U_n^+, we know that (4.28) is true for all sufficiently large y. Using the same results, we also see easily that the right-hand side in (4.28) is always contained in $E_+^{\mathrm{cs}}(y,\lambda)$. Assume therefore that $V(y) \in E_+^{\mathrm{cs}}(y,\lambda)$ lies in a complement of $E_+^{\mathrm{ss}}(y,\lambda) \oplus E_+^{\mathrm{c}}(y,\lambda)$, and we can then arrange that $\langle\Psi_n^+(y,\lambda), V(y)\rangle_{(y)} = 0$ for all n. For

x sufficiently large, $V(x)=\Phi_+^{\rm cs}(x,y,\lambda)V(y)$ therefore satisfies $\langle\Psi_n^+(x,\lambda),V(x)\rangle_{(x)}=0$ for all n, and using that (4.28) is true for all such x we conclude that $V(x)\in E_+^{\rm ss}(x,\lambda)$ for large x. But the uniqueness property stated in Theorem 6(ii) implies that $V(y)\in E_+^{\rm ss}(y,\lambda)$ in contradiction to our assumption. Thus, (4.28) is true for all $x\geq 0$.

Since the last inclusion in (4.27) may not hold for all λ near zero, we need to construct a new strong unstable subspace that is perpendicular to $\Psi_n(y,\lambda)$ for all λ. Define the bounded and analytic mapping

$$h_+^{\rm uu}(\lambda):\quad E_+^{\rm uu}(0,\lambda)\longrightarrow E_+^{\rm c}(0,\lambda),\quad V_+^{\rm uu}\longmapsto -\sum_{n=1}^N U_n^+(0,\lambda)\langle\Psi_n^+(0,\lambda),V_+^{\rm uu}\rangle_{(0)}$$

and note that $h_+^{\rm uu}(0)=0$ by (4.27). We can now define the extended exponential dichotomies via

$$\begin{aligned}\Phi_+^{\rm s}(x,y,\lambda) &:= \Phi_+^{\rm ss}(x,y,\lambda)+\sum_{n=k+1}^N U_n^+(x,\lambda)\langle\Psi_n^+(y,\lambda),\cdot\rangle_{(y)}, \qquad x\geq y\geq 0 \\ \Phi_+^{\rm u}(x,y,\lambda) &:= \underbrace{\Phi_+^{\rm uu}(x,y,\lambda)+\Phi_+^{\rm cs}(x,0,\lambda)h_+^{\rm uu}(\lambda)\Phi_+^{\rm uu}(0,y,\lambda)}_{=:\check\Phi_+^{\rm uu}(x,y,\lambda)}+\sum_{n=1}^k U_n^+(x,\lambda)\langle\Psi_n^+(y,\lambda),\cdot\rangle_{(y)}, \qquad y\geq x\geq 0,\end{aligned} \tag{4.29}$$

where the operators $\Phi_+^{\rm cs,uu}(x,y,\lambda)$ are defined in Lemma 4.7. The operators in (4.29) are analytic in λ, and it is easy to check that they define an analytic extension of the original exponential dichotomies. Note that $\Psi_n(y,\lambda)\in{\rm Rg}(\check\Phi_+^{\rm uu}(y,y,\lambda))^\perp$ for all $y\geq 0$ and all λ near zero.

For $\alpha\in\mathbb{C}^N$, note that the Hale inner product

$$\langle\Psi_n^+(y,\lambda),\iota_2\alpha\rangle_{(y)}=\langle\pi_2\Psi_n^+(y,\lambda),\alpha\rangle_{\mathbb{C}^N}$$

of Ψ_n^+ and an element of the form $\iota_2\alpha$ depends on y only through Ψ_n^+. This property taken together with (4.26) and Theorem 6(iii) yields the desired estimates for the derivatives of $\Phi_+^{\rm s,u}(x,y,\lambda)\iota_2$ with respect to y. The remaining claims in Lemma 4.8 follow from Lemma 4.7 and the preceding analysis. ∎

4.4 Meromorphic extension of the exponential dichotomy on $\mathbb{R}$

In Lemma 4.8, we extended the exponential dichotomies $\Phi_\pm^{\rm s,u}(x,y,\lambda)$ analytically for λ near zero. The resulting dichotomies are defined separately for $x,y\in\mathbb{R}^+$ and $x,y\in\mathbb{R}^-$. To extend the Green's function analytically near $\lambda=0$, we need to construct an exponential dichotomy on $\mathbb{R}$. To see what is involved in this construction, we shall denote the ranges of the projections $\Phi_+^{\rm s}(0,0,\lambda)$ and $\Phi_-^{\rm u}(0,0,\lambda)$ by $E_+^{\rm es}(\lambda)$ and $E_-^{\rm eu}(\lambda)$, respectively. A standard argument, see for instance [25, Theorem 2], then shows that the exponential dichotomies $\Phi_+^{\rm s}(x,y,\lambda)$ and $\Phi_-^{\rm u}(x,y,\lambda)$ fit together at $x=y=0$ to produce an exponential dichotomy on $\mathbb{R}$ if and only if $E_+^{\rm es}(\lambda)\oplus E_-^{\rm eu}(\lambda)=Y$. As we shall prove below, these subspaces have a nontrivial intersection when $\lambda=0$, and the best we can do is to construct a dichotomy on $\mathbb{R}$ that is meromorphic in a ball centered at $\lambda=0$ with a simple pole at the origin.

It follows from the spectral stability assumption (S2) that

$$U_*'(x)=(u_*'(x+\cdot),u_*'(x))\in Y^1$$

is a bounded H^1-solution of the spatial-dynamics system $U_x=\mathcal{A}(x,0)U$. Similarly, the spectral stability assumption (S3) implies that there is a nonzero vector $\psi_*\in\mathbb{R}^N$ that is perpendicular to the outgoing eigenvectors of $\bar f_u(u_\pm)$ so that $\psi_*\perp[r_1^-,\ldots,r_{k-1}^-,r_{k+1}^+,\ldots,r_N^+]$. Note that the vector ψ_* is a linear combination of the incoming left eigenvectors $l_{n,\rm in}^\pm$. For later use, we remark that

$$M:=\langle\psi_*,[u_*]\rangle_{\mathbb{R}^N}\neq 0 \tag{4.30}$$

by the spectral stability assumption (S3). Finally, let

$$\Psi_*(x) = (\psi_*, \psi_*) \in \hat{Y}$$

be the associated solution of the adjoint equation (4.15), given by $\hat{U}_x = \hat{\mathcal{A}}(x,\lambda)\hat{U}$, then we have

$$\Psi_*(0) \perp \left[\mathcal{R}^+_{\text{out}}(0) + \mathcal{R}^-_{\text{out}}(0) + E^{\text{ss}}_+(0,0) + E^{\text{uu}}_+(0,0) + E^{\text{ss}}_-(0,0) + E^{\text{uu}}_-(0,0)\right] \tag{4.31}$$

by (4.27) and Lemmas 3.3(iii) and 4.4, where the annihilator is defined through the Hale inner product $\langle\cdot,\cdot\rangle_{(0)}$. We will also use the function $W_*(x)$ that is associated with $\Psi_*(x)$ via (3.14).

Lemma 4.9 *Assume that Hypotheses (H1)-(H2) and (S1)-(S4) are met, then there is an $\epsilon > 0$ such that the linear mapping*

$$\iota(\lambda): \qquad E^{\text{es}}_+(\lambda) \times E^{\text{eu}}_-(\lambda) \longrightarrow Y, \qquad (U^{\text{s}}, U^{\text{u}}) \longmapsto U^{\text{s}} - U^{\text{u}}$$

is Fredholm with index zero for all $\lambda \in B_\epsilon(0)$. Furthermore, we have $E^{\text{es}}_+(\lambda) \oplus E^{\text{eu}}_-(\lambda) = Y$ for all $\lambda \in B_\epsilon(0) \setminus \{0\}$ and

$$E^{\text{es}}_+(0) \cap E^{\text{eu}}_-(0) = \text{span}\{U_*(0)\}, \qquad [E^{\text{es}}_+(0) + E^{\text{eu}}_-(0)]^\perp = \text{span}\{\Psi_*(0)\},$$

where the annihilator $[\dots]^\perp$ is defined via the Hale inner product $\langle\cdot,\cdot\rangle_{(0)}$ at $x = 0$.

Proof. The spectral stability assumption (S1) and Theorem 5 imply that the system (4.16) has an exponential dichotomy on $\mathbb{R}$ for each λ with $\operatorname{Re}\lambda > 0$. Thus, $\iota(\lambda)$ is invertible, and therefore Fredholm with index zero, for $\operatorname{Re}\lambda > 0$. Next, we prove that $\iota(0)$ is Fredholm with index zero. Let $\chi(x)$ be a smooth monotone function with $\chi(x) = \operatorname{sgn}(x)$ for $|x| \geq 1$, and define

$$\begin{aligned}\tilde{\mathcal{L}}^\eta := \mathrm{e}^{-\eta\chi(x)x}\mathcal{L}\mathrm{e}^{\eta\chi(x)x}: \qquad & H^1(\mathbb{R},\mathbb{R}^N) \longrightarrow L^2(\mathbb{R},\mathbb{R}^N), \\ & u \longmapsto \sigma\left(u_x + \eta[\chi(x) + \chi'(x)x]u\right) - \sum_{j=-p}^{q} A_j(x)\mathrm{e}^{\eta[\chi(x+j)(x+j)-\chi(x)x]}u(\cdot + j).\end{aligned}$$

Since $\chi(x)+\chi'(x)x = \operatorname{sgn}(x)$ for $|x| \geq 1$ and $\chi(x+j)(x+j)-\chi(x)x = j\operatorname{sgn}(x)$ for $|x| \geq p+q+1$, the operator $\tilde{\mathcal{L}}^\eta$ is bounded, and the associated asymptotic operators are given by $\tilde{\mathcal{L}}^\eta_\pm = \mathrm{e}^{-\eta x}\mathcal{L}_\pm\mathrm{e}^{\eta x}$. The arguments presented in the proof of Lemma 4.6 imply that $\tilde{\mathcal{L}}^\eta_\pm$ are hyperbolic for $\eta = \pm 2\bar{\eta}$, while the discussion at the end of §3.1 implies that the N spatial eigenvalues $\nu^\pm_n(0)$ at zero associated with $\mathcal{L}_\pm$ become $\nu^\pm_n(0) \mp \eta$ for the operators $\tilde{\mathcal{L}}^\eta_\pm$. In particular, for $\eta > 0$, these eigenvalues satisfy $\nu^+_n(0) - \eta < 0$ and $\nu^-_n(0) + \eta > 0$. Thus, for $\eta > 0$, the stable and unstable eigenspaces associated with the spatial-dynamics formulation of $\tilde{\mathcal{L}}^\eta$ contain those of $\mathcal{L}$. Since $\tilde{\mathcal{L}}^\eta$ is asymptotically hyperbolic, it follows from [18, Theorem A] that the intersection of the former two spaces is finite-dimensional. The preceding arguments then show that the same is true for the intersection of $E^{\text{es}}_+(0)$ and $E^{\text{eu}}_-(0)$, and we conclude that the null space of $\iota(0)$ is finite-dimensional. Since $E^{\text{es}}_+(0)$ and $E^{\text{eu}}_-(0)$ are the ranges of bounded projections, they are closed, and the range of $\iota(0)$, which is given by their sum, is therefore also closed. Similar arguments for the operator $\tilde{\mathcal{L}}^\eta$ with $\eta < 0$ show that the codimension of the range of $\iota(0)$ is finite. Thus, $\iota(0)$ is Fredholm, and its index is therefore zero as $\iota(\lambda)$ depends continuously on λ. Continuity in λ then implies that $\iota(\lambda)$ is Fredholm with index zero for all $\lambda \in B_\epsilon(0)$.

The remaining arguments needed to complete the proof are very similar to those in [1, Proof of Lemma 5]: the only differences are that we use the Hale inner product instead of the scalar product in Y and that we appeal to Lemma 4.4 and (4.29) for properties of the Hale inner product and the extended exponential dichotomies at $\lambda = 0$. We therefore omit the details. ∎

Define the spaces

$$E^{\text{pt}}_0 := \text{span}\{U'_*(0)\}, \qquad E^\psi_0 := \text{span}\{W_*(0)\},$$

then Lemma 4.9 implies that there are closed subspaces E^{s}_0 and E^{u}_0 of Y such that

$$E^{\mathrm{es}}_+(0) = E^{\mathrm{s}}_0 \oplus E^{\mathrm{pt}}_0, \qquad E^{\mathrm{eu}}_-(0) = E^{\mathrm{u}}_0 \oplus E^{\mathrm{pt}}_0. \tag{4.32}$$

The following lemma can be viewed an an replacement of the more common Evans-function analysis in the meromorphic extension of resolvent kernels.

Lemma 4.10 *Assume that Hypotheses (H1)-(H2) and (S1)-(S4) are met. For each $\lambda \in B_\epsilon(0) \setminus \{0\}$, there is a unique mapping $h_+(\lambda) : E^{\mathrm{eu}}_+(\lambda) \to E^{\mathrm{es}}_+(\lambda)$ so that $E^{\mathrm{eu}}_-(\lambda) = \operatorname{graph} h_+(\lambda)$. This mapping is of the form*

$$h_+(\lambda) = h^+_{\mathrm{a}}(\lambda) + h^+_{\mathrm{p}}(\lambda),$$

where $h^+_{\mathrm{a}}(\lambda)$ is analytic in $\lambda \in B_\epsilon(0)$ with $h^+_{\mathrm{a}}(0)|_{\mathrm{Rg}(\Phi^{\mathrm{uu}}_+(0,0,0))} = 0$, while $h^+_{\mathrm{p}}(\lambda)$ is given by

$$h^+_{\mathrm{p}}(\lambda)V = \frac{1}{M\lambda}\langle \Psi_*(0), V\rangle_{(0)} U'_*(0),$$

where $M \neq 0$ is given in (4.30). Similarly, for each $\lambda \in B_\epsilon(0) \setminus \{0\}$, there is a unique map $h_-(\lambda) : E^{\mathrm{es}}_-(\lambda) \to E^{\mathrm{eu}}_-(\lambda)$ so that $E^{\mathrm{es}}_+(\lambda) = \operatorname{graph} h_-(\lambda)$. This map has a meromorphic representation analogous to the one given above for $h_+(\lambda)$ with $h^-_{\mathrm{p}}(\lambda) = -h^+_{\mathrm{p}}(\lambda)$.

In particular, the maps $h_\pm(\lambda)$ are meromorphic on $B_\epsilon(0)$ with a simple pole at $\lambda = 0$.

Proof. Our proof is similar to [1, Proof of Lemma 6] but avoids the need for changing the extended exponential dichotomies we constructed in Lemma 4.8. We use the coordinates

$$(V^{\mathrm{s}}, V^{\mathrm{pt}}, V^{\mathrm{u}}, V^{\psi}) \in E^{\mathrm{s}}_0 \oplus E^{\mathrm{pt}}_0 \oplus E^{\mathrm{u}}_0 \oplus E^{\psi}_0$$

and indicate the range of operators by the appropriate superscript: the mapping $g^{\mathrm{u}\psi}(\lambda)$, for instance, maps into $E^{\mathrm{u}}_0 \oplus E^{\psi}_0$. Using (4.32) and analyticity of the extended dichotomies on $\mathbb{R}^\pm$, we see that

$$\begin{aligned}
E^{\mathrm{u}}_-(\lambda):& \qquad \tilde V = \tilde V^{\mathrm{u}} + \tilde V^{\mathrm{pt}} + \lambda h^{\mathrm{s}\psi}(\lambda)(\tilde V^{\mathrm{u}} + \tilde V^{\mathrm{pt}})\\
E^{\mathrm{s}}_+(\lambda):& \qquad V = V^{\mathrm{s}} + V^{\mathrm{pt}} + \lambda g^{\mathrm{u}\psi}(\lambda)(V^{\mathrm{s}} + V^{\mathrm{pt}})\\
E^{\mathrm{u}}_+(\lambda):& \qquad V = V^{\mathrm{u}} + V^{\psi} + g^{\mathrm{s,pt}}(\lambda)(V^{\mathrm{u}} + V^{\psi})
\end{aligned} \tag{4.33}$$

for unique analytic mappings $h^{\mathrm{s}\psi}(\lambda)$, $g^{\mathrm{u}\psi}(\lambda)$, and $g^{\mathrm{s,pt}}(\lambda)$ that are all analytic in λ near zero. Our goal is to write $E^{\mathrm{u}}_-(\lambda)$ as a graph over $E^{\mathrm{u}}_+(\lambda)$ with values in $E^{\mathrm{s}}_+(\lambda)$. Thus, consider

$$\tilde V^{\mathrm{u}} + \tilde V^{\mathrm{pt}} + \lambda h^{\mathrm{s}\psi}(\lambda)(\tilde V^{\mathrm{u}} + \tilde V^{\mathrm{pt}}) = \left[V^{\mathrm{u}} + V^{\psi} + g^{\mathrm{s,pt}}(\lambda)(V^{\mathrm{u}} + V^{\psi})\right] + \left[V^{\mathrm{s}} + V^{\mathrm{pt}} + \lambda g^{\mathrm{u}\psi}(\lambda)(V^{\mathrm{s}} + V^{\mathrm{pt}})\right], \tag{4.34}$$

where, for given $(\tilde V^{\mathrm{u}}, \tilde V^{\mathrm{pt}})$, we need to express $(V^{\mathrm{s}}, V^{\mathrm{pt}})$ in terms of $(V^{\mathrm{u}}, V^{\psi})$ so that (4.34) is true. Writing (4.34) in components, we obtain

$$\tilde V^{\mathrm{u}} = V^{\mathrm{u}} + \lambda g^{\mathrm{u}}(\lambda)(V^{\mathrm{s}} + V^{\mathrm{pt}}), \qquad \tilde V^{\mathrm{pt}} = V^{\mathrm{pt}} + g^{\mathrm{pt}}(\lambda)(V^{\mathrm{u}} + V^{\psi}).$$

Substituting these expressions into the stable component of (4.34), we obtain

$$\lambda h^{\mathrm{s}}(\lambda)\left[V^{\mathrm{u}} + \lambda g^{\mathrm{u}}(\lambda)(V^{\mathrm{s}} + V^{\mathrm{pt}}) + V^{\mathrm{pt}} + g^{\mathrm{pt}}(\lambda)(V^{\mathrm{u}} + V^{\psi})\right] = V^{\mathrm{s}} + g^{\mathrm{s}}(\lambda)(V^{\mathrm{u}} + V^{\psi}),$$

which we can solve for V^{s} so that

$$\begin{aligned}
V^{\mathrm{s}} \quad &= \quad \left[\mathrm{id}_{E^{\mathrm{s}}_0} - \lambda^2 h^{\mathrm{s}}(\lambda) g^{\mathrm{u}}(\lambda)\right]^{-1}\\
&\qquad\quad \times \left(\lambda h^{\mathrm{s}}(\lambda)[V^{\mathrm{u}} + \lambda g^{\mathrm{u}}(\lambda)V^{\mathrm{pt}} + V^{\mathrm{pt}} + g^{\mathrm{pt}}(\lambda)(V^{\mathrm{u}} + V^{\psi})] - g^{\mathrm{s}}(\lambda)(V^{\mathrm{u}} + V^{\psi})\right)\\
&=: \quad h^{\mathrm{s}}_1(\lambda)(V^{\mathrm{u}} + V^{\psi}) + \lambda h^{\mathrm{s}}_2(\lambda)V^{\mathrm{pt}},
\end{aligned} \tag{4.35}$$

where $h_j^{\rm s}(\lambda)$ is analytic in λ near zero. Finally, the E_0^ψ-component of equation (4.34) is given by

$$\begin{aligned}\lambda h^\psi(\lambda)\left[V^{\rm u}+\lambda g^{\rm u}(\lambda)(h_1^{\rm s}(\lambda)(V^{\rm u}+V^\psi)+\lambda h_2^{\rm s}(\lambda)V^{\rm pt}+V^{\rm pt})+V^{\rm pt}+g^{\rm pt}(\lambda)(V^{\rm u}+V^\psi)\right]\\ = \quad V^\psi+\lambda g^\psi(\lambda)(h_1^{\rm s}(\lambda)(V^{\rm u}+V^\psi)+\lambda h_2^{\rm s}(\lambda)V^{\rm pt}+V^{\rm pt}),\end{aligned}$$

which is of the form

$$\left[{\rm id}_{E_0^\psi}+\lambda h_1^\psi(\lambda)\right]V^\psi=\lambda\underbrace{\left[h^\psi(\lambda)-g^\psi(\lambda)+\lambda h_2^\psi(\lambda)\right]}_{=:M^\psi(\lambda)}V^{\rm pt}+\lambda h_3^\psi(\lambda)V^{\rm u}, \tag{4.36}$$

where all mappings are analytic in λ near zero. We claim that

$$M^\psi(0)=h^\psi(0)-g^\psi(0):\quad E_0^{\rm pt}\longrightarrow E_0^\psi$$

is invertible with inverse given by

$$V^{\rm pt}=M^{\rm pt}(0)V^\psi=\frac{1}{\langle\psi_*,[u_*]\rangle_{\mathbb{R}^N}}\langle\Psi_*(0),V^\psi\rangle_{(0)}U_*'(0)=\frac{1}{M}\langle\Psi_*(0),V^\psi\rangle_{(0)}U_*'(0), \tag{4.37}$$

where $M\neq 0$ by (4.30): we shall prove this claim below and proceed now assuming that (4.37) is true. Using (4.37), we can solve (4.36) uniquely for $V^{\rm pt}$ to get

$$V^{\rm pt}=\frac{1}{\lambda}M^\psi(\lambda)^{-1}[{\rm id}_{E_0^\psi}+\lambda h_2^\psi(\lambda)]V^\psi-M^\psi(\lambda)^{-1}h_3^\psi(\lambda)V^{\rm u}=:\frac{1}{\lambda}M^{\rm pt}(0)V^\psi+h_1^{\rm pt}(\lambda)(V^{\rm u}+V^\psi), \tag{4.38}$$

where $h_1^{\rm pt}(\lambda)$ is analytic in λ near zero. Using this expression in (4.35), we obtain

$$V^{\rm s}=h_3^{\rm s}(\lambda)(V^{\rm u}+V^\psi), \tag{4.39}$$

where $h_3^{\rm s}(\lambda)$ is analytic in λ near zero. Finally, substituting (4.38) and (4.39) into the graph representation (4.33) of $E_+^{\rm es}(\lambda)$, we see that

$$V^{\rm s}+V^{\rm pt}+\lambda g^{{\rm u}\psi}(\lambda)(V^{\rm s}+V^{\rm pt})=\frac{1}{M\lambda}\langle\Psi_*(0),V^\psi\rangle_{(0)}U_*'(0)+h_4^{\rm s,pt}(\lambda)(V^{\rm u}+V^\psi)\in E_+^{\rm es}(\lambda),$$

where $h_4^{\rm s,pt}(\lambda)$ is analytic in λ near zero. Thus, using the continuous projection $\mathcal{P}_0:Y\to Y$ onto $E_0^{\rm u}\oplus E_0^\psi$ with null space $E_0^{\rm s}\oplus E_0^{\rm pt}$, we arrive at the desired graph representation

$$h_+(\lambda)V=\frac{1}{M\lambda}\langle\Psi_*(0),V\rangle_{(0)}U_*'(0)+h_4^{\rm s,pt}(\lambda)\mathcal{P}_0V,\qquad V\in E_+^{\rm eu}(\lambda)$$

of $E_-^{\rm eu}(\lambda)$, where we used that $\Psi_*(0)$ lies in the annihilator of $E_-^{\rm eu}(0)+E_+^{\rm s}(0)$. Finally, for $\lambda=0$, we know from (4.29) and Theorem 6(i) that ${\rm Rg}(\Phi_+^{\rm uu}(0,0,0))\subset E_-^{\rm eu}(0)$, which shows that $h_{\rm a}^+(0)|_{{\rm Rg}(\Phi_+^{\rm uu}(0,0,0))}=0$.

It remains to verify that $M^\psi(0)=h^\psi(0)-g^\psi(0):E_0^{\rm pt}\to E_0^\psi$ is invertible with inverse given by (4.37). Hence, we need to find expressions for the E_0^ψ-components of the graphs of $E_+^{\rm es}(\lambda)$ and $E_-^{\rm eu}(\lambda)$ over $E_0^{\rm pt}$. We focus on $E_+^{\rm es}(\lambda)={\rm Rg}(\Phi_+^{\rm s}(0,0,\lambda))$. Using the definition (4.33) of $g^\psi(\lambda)$, we see that

$$\langle\Psi_*(0),g^\psi(0)U_*'(0)\rangle_{(0)}=\langle\Psi_*(0),\partial_\lambda\Phi_+^{\rm s}(0,0,0)U_*'(0)\rangle_{(0)}.$$

Therefore, exploiting the expression (4.29) for $\Phi_+^{\rm s}(x,y,\lambda)$, we obtain

$$\Phi_+^{\rm s}(x,0,\lambda)U_*'(0)=\Phi_+^{\rm ss}(x,0,\lambda)U_*'(0)+\sum_{n=k+1}^N U_n^+(x,\lambda)\langle\Psi_n^+(0,\lambda),U_*'(0)\rangle_{(0)}.$$

Since the Hale inner products terms in the rightmost sum vanish at $\lambda=0$, we find

$$V^{\rm s}(x):=\partial_\lambda\Phi_+^{\rm s}(x,0,0)U_*'(0)=\partial_\lambda\Phi_+^{\rm ss}(x,0,0)U_*'(0)+\sum_{n=k+1}^N U_n^+(x,0)\langle\partial_\lambda\Psi_n^+(0,0),U_*'(0)\rangle_{(0)}. \tag{4.40}$$

Setting $v^{\mathrm{s}} := \pi_2 V^{\mathrm{s}}$, we also know that v^{s} is an H^1-solution of $\mathcal{L}v = u_*'$ on $\mathbb{R}^+$. Hence, by [11, (4.8)], we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle \psi_*, v^{\mathrm{s}}(x)\rangle_{(x)} = \langle \psi_*, u_*'(x)\rangle_{\mathbb{R}^N},$$

and we conclude that

$$\langle \psi_*, v^{\mathrm{s}}(x)\rangle_{(x)} - \langle \psi_*, v^{\mathrm{s}}(0)\rangle_{(0)} = \langle \psi_*, u_*(x)\rangle_{\mathbb{R}^N} - \langle \psi_*, u_*(0)\rangle_{\mathbb{R}^N}.$$

Using (4.31), (4.40), and the fact that the term $\partial_\lambda \Phi_+^{\mathrm{ss}}(x,0,0)U_*'(0)$ in (4.40) decays to zero exponentially in x, we find

$$-\langle \psi_*, v^{\mathrm{s}}(0)\rangle_{(0)} = \langle \psi_*, u_+\rangle_{\mathbb{R}^N} - \langle \psi_*, u_*(0)\rangle_{\mathbb{R}^N}.$$

The analogous calculation on $\mathbb{R}^-$ for $h^\psi(0)$ gives

$$\langle \psi_*, v^{\mathrm{u}}(0)\rangle_{(0)} = -\langle \psi_*, u_-\rangle_{\mathbb{R}^N} + \langle \psi_*, u_*(0)\rangle_{\mathbb{R}^N}.$$

Thus,

$$\langle \Psi_*(0), [h^\psi(0) - g^\psi(0)]U_*'(0)\rangle_{(0)} = \langle \psi_*, v^{\mathrm{u}}(0) - v^{\mathrm{s}}(0)\rangle_{(0)} = \langle \psi_*, u_+ - u_-\rangle_{\mathbb{R}^N} = \langle \psi_*, [u_*]\rangle_{\mathbb{R}^N},$$

and (4.37) follows easily.

The proof for $h_-(\lambda)$ is analogous: the only difference is that $M^\psi(0)$ is now given by $g^\psi(0) - h^\psi(0)$ which shows that $h_{\mathrm{p}}^-(\lambda) = -h_{\mathrm{p}}^+(\lambda)$. ∎

We can now proceed as in [1, §4.2] to extend the exponential dichotomy on $\mathbb{R}$ meromorphically from $\operatorname{Re}\lambda > 0$ to a neighborhood of $\lambda = 0$. We shall only state the results and refer to [1] for further details. First, using the projections $P_+^{\mathrm{s}}(x,\lambda) := \Phi_+^{\mathrm{s}}(x,x,\lambda)$ and $P_-^{\mathrm{u}}(x,\lambda) := \Phi_-^{\mathrm{u}}(x,x,\lambda)$, we see that the operators

$$\begin{aligned}
\tilde{P}_+^{\mathrm{s}}(x,\lambda) &:= P_+^{\mathrm{s}}(x,\lambda) - \Phi_+^{\mathrm{s}}(x,0,\lambda)h_+(\lambda)\Phi_+^{\mathrm{u}}(0,x,\lambda) && x \geq 0\\
\tilde{\Phi}_+^{\mathrm{s}}(x,y,\lambda) &:= \Phi_+^{\mathrm{s}}(x,y,\lambda)\tilde{P}_+^{\mathrm{s}}(y,\lambda) && x \geq y \geq 0\\
\tilde{\Phi}_+^{\mathrm{u}}(x,y,\lambda) &:= (1 - \tilde{P}_+^{\mathrm{s}}(x,\lambda))\Phi_+^{\mathrm{u}}(x,y,\lambda) && y \geq x \geq 0
\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}
\tilde{P}_-^{\mathrm{u}}(x,\lambda) &:= P_-^{\mathrm{u}}(x,\lambda) - \Phi_-^{\mathrm{u}}(x,0,\lambda)h_-(\lambda)\Phi_-^{\mathrm{s}}(0,x,\lambda) && x \leq 0\\
\tilde{\Phi}_-^{\mathrm{u}}(x,y,\lambda) &:= \Phi_-^{\mathrm{u}}(x,y,\lambda)\tilde{P}_-^{\mathrm{u}}(y,\lambda) && x \leq y \leq 0\\
\tilde{\Phi}_-^{\mathrm{s}}(x,y,\lambda) &:= (1 - \tilde{P}_-^{\mathrm{u}}(x,\lambda))\Phi_-^{\mathrm{s}}(x,y,\lambda) && y \leq x \leq 0
\end{aligned} \tag{4.42}$$

define exponential dichotomies on $\mathbb{R}^\pm$ with projections $\tilde{P}_\pm^{\mathrm{s,u}}(x,\lambda)$. By construction

$$\tilde{P}_+^{\mathrm{s}}(0,\lambda) = 1 - \tilde{P}_-^{\mathrm{u}}(0,\lambda) \qquad \forall\, \lambda \neq 0, \tag{4.43}$$

and the Laurent series of these two operators coincide at $\lambda = 0$, since the contribution of the pole at $\lambda = 0$ is, in both cases, given by the matrix $\tilde{M}_0$. Thus, the dichotomies in (4.41) and (4.42) fit together at $x = y = 0$ and give the desired meromorphic exponential dichotomy on $\mathbb{R}$ for λ near zero via

$$\Phi^{\mathrm{s}}(x,y,\lambda) := \begin{cases} \tilde{\Phi}_+^{\mathrm{s}}(x,y,\lambda) & x > y \geq 0\\ \tilde{\Phi}_+^{\mathrm{s}}(x,0,\lambda)\tilde{\Phi}_-^{\mathrm{s}}(0,y,\lambda) & x \geq 0 > y\\ \tilde{\Phi}_-^{\mathrm{s}}(x,y,\lambda) & 0 > x > y \end{cases} \tag{4.44}$$

for $x > y$, and an analogous expression for $\Phi^{\mathrm{u}}(x,y,\lambda)$ for $x < y$. This completes the meromorphic extension of the exponential dichotomies on $\mathbb{R}$ for $\lambda \in B_\epsilon(0)$.

4.5 Pointwise bounds for the resolvent kernel

Theorem 5 shows that

$$G(x,y,\lambda) := \begin{cases} \pi_2\Phi^{\mathrm{s}}(x,y,\lambda)\iota_2 & \text{for } x>y, \\ -\pi_2\Phi^{\mathrm{u}}(x,y,\lambda)\iota_2 & \text{for } x<y \end{cases}$$

is the meromorphic extension of the resolvent kernel of $\mathcal{L}-\lambda$ from $\operatorname{Re}\lambda>0$ to a neighborhood of $\lambda=0$, where

$$\iota_2: \quad \mathbb{C}^N \longrightarrow Y, \quad \alpha \longmapsto (0,\alpha), \qquad\qquad \pi_2: \quad Y \longrightarrow \mathbb{C}^N, \quad (\phi,\alpha)\longmapsto \alpha.$$

The following theorem contains the desired pointwise bounds of the resolvent kernel $G(x,y,\lambda)$ and its y-derivative $G_y(x,y,\lambda)$.

Theorem 7 *Under the assumptions of Theorem 1, there are positive constants $\epsilon,\bar\eta>0$ so that the resolvent kernel $G(x,y,\lambda)$ of $\mathcal{L}_*-\lambda$ has a meromorphic extension for $\lambda\in B_\epsilon(0)$ of the form*

$$G(x,y,\lambda) = \frac{1}{\lambda}u_*'(x)\sum_{\nu_{\mathrm{in}}^\pm}\mathrm{e}^{-\nu_{\mathrm{in}}^\pm(\lambda)y}\langle l_{\mathrm{in}}^\pm(\lambda),\cdot\rangle_{\mathbb{C}^N} + \tilde G(x,y,\lambda), \qquad y\gtrless 0, \tag{4.45}$$

where $l_{\mathrm{in}}^\pm(\lambda)$ is analytic in $\lambda\in B_\epsilon(0)$, and we have the pointwise decomposition

$$\partial_y^\ell \tilde G(x,y,\lambda) = \mathrm{O}(\mathrm{e}^{-\bar\eta|x-y|}) + |\lambda|^\ell\cdot\begin{cases} \sum_{\nu_{\mathrm{out}}^+,\nu_{\mathrm{in}}^+}\left[\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)(x-y)}\right)+\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)x}\mathrm{e}^{-\nu_{\mathrm{in}}^+(\lambda)y}\right)\right], & x>y>0 \\ \sum_{\nu_{\mathrm{in}}^-,\nu_{\mathrm{out}}^+}\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)x}\mathrm{e}^{-\nu_{\mathrm{in}}^-(\lambda)y}\right), & x>0>y \\ \sum_{\nu_{\mathrm{out}}^-,\nu_{\mathrm{in}}^-}\left[\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{in}}^-(\lambda)(x-y)}\right)+\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^-(\lambda)x}\mathrm{e}^{-\nu_{\mathrm{in}}^-(\lambda)y}\right)\right], & 0>x>y \end{cases} \tag{4.46}$$

for $\ell=0,1$, where each term of the form $\mathrm{O}(\dots)$ is analytic in $\lambda\in B_\epsilon(0)$, and analogous estimates for $x<y$.

Proof. The estimates asserted above are stronger than those proved in [1] for time-periodic viscous shocks, and we therefore give a complete proof. We focus on the case $x>y$ as the case $x<y$ is completely analogous. Recall that $h_\pm(\lambda)=h_{\mathrm{a}}^\pm(\lambda)+h_{\mathrm{p}}^\pm(\lambda)$, where $h_{\mathrm{a}}^\pm(\lambda)$ is analytic in λ, while

$$h_{\mathrm{p}}^\pm(\lambda): \quad Y\longrightarrow Y, \quad V\longmapsto \frac{\pm 1}{M\lambda}\langle\Psi_*(0),V\rangle_{(0)}U_*'(0).$$

Using (4.41)-(4.44), we find that

$$\begin{aligned} \Phi^{\mathrm{s}}(x,y,\lambda) &= \underbrace{\Phi_+^{\mathrm{s}}(x,y,\lambda)-\Phi_+^{\mathrm{s}}(x,0,\lambda)h_{\mathrm{a}}^+(\lambda)\Phi_+^{\mathrm{u}}(0,y,\lambda)}_{\text{(a)}} - \underbrace{\Phi_+^{\mathrm{s}}(x,0,\lambda)h_{\mathrm{p}}^+(\lambda)\Phi_+^{\mathrm{u}}(0,y,\lambda)}_{\text{(i)}} && x>y\ge 0 \\ \Phi^{\mathrm{s}}(x,y,\lambda) &= \underbrace{\Phi_+^{\mathrm{s}}(x,0,\lambda)\Phi_-^{\mathrm{s}}(0,y,\lambda)+\Phi_+^{\mathrm{s}}(x,0,\lambda)\Phi_-^{\mathrm{u}}(0,0,\lambda)h_{\mathrm{a}}^-(\lambda)\Phi_-^{\mathrm{s}}(0,y,\lambda)}_{\text{(b)}} && x>0>y \\ &\quad + \underbrace{\Phi_+^{\mathrm{s}}(x,0,\lambda)\Phi_-^{\mathrm{u}}(0,0,\lambda)h_{\mathrm{p}}^-(\lambda)\Phi_-^{\mathrm{s}}(0,y,\lambda)}_{\text{(ii)}} \\ \Phi^{\mathrm{s}}(x,y,\lambda) &= \underbrace{\Phi_-^{\mathrm{s}}(x,y,\lambda)+\Phi_-^{\mathrm{u}}(x,0,\lambda)h_{\mathrm{a}}^-(\lambda)\Phi_-^{\mathrm{s}}(0,y,\lambda)}_{\text{(c)}} + \underbrace{\Phi_-^{\mathrm{u}}(x,0,\lambda)h_{\mathrm{p}}^-(\lambda)\Phi_-^{\mathrm{s}}(0,y,\lambda)}_{\text{(iii)}} && 0>x>y, \end{aligned} \tag{4.47}$$

and we need to estimate $\partial_y^\ell\pi_2\Phi^{\mathrm{s}}(x,y,\lambda)\iota_2$ for $\ell=0,1$. Recall from (4.29) that

$$\Phi_+^{\mathrm{s}}(x,y,\lambda) = \Phi_+^{\mathrm{ss}}(x,y,\lambda)+\sum_{n=k+1}^N U_n^+(x,\lambda)\langle\Psi_n^+(y,\lambda),\cdot\rangle_{(y)}, \qquad x\ge y\ge 0 \tag{4.48}$$

$$\Phi_+^{\mathrm{u}}(x,y,\lambda) = \Phi_+^{\mathrm{uu}}(x,y,\lambda)+\Phi_+^{\mathrm{cs}}(x,0,\lambda)h_+^{\mathrm{uu}}(\lambda)\Phi_+^{\mathrm{uu}}(0,y,\lambda)+\sum_{n=1}^k U_n^+(x,\lambda)\langle\Psi_n^+(y,\lambda),\cdot\rangle_{(y)}, \qquad y\ge x\ge 0,$$

where each single term on the right-hand side is analytic in λ, and analogous expressions for $\Phi_-^{\mathrm{s,u}}(x,y,\lambda)$.

The terms labelled (a)-(c) are analytic in λ and satisfy the estimates (4.46) with $\ell=0$ due to (4.17). Lemma 4.8 implies that the terms $\pi_2\Phi_\pm^{\mathrm{ss,uu}}(x,y,\lambda)\iota_2$ are differentiable in y and that their y-derivatives satisfy (4.46) with $\ell=1$. Similarly, we find from (4.18), (4.48), and Lemma 4.8 that the remaining y-dependent terms $\Phi_+^{\mathrm{u}}(0,y,\lambda)$ and $\Phi_-^{\mathrm{s}}(0,y,\lambda)$ in (a)-(c) are differentiable in y with estimates that are compatible with (4.46) for $\ell=0$ provided $y>q$ and $y<-p$, respectively. Furthermore, by (4.26), each y-derivative of $\Psi_n^\pm(y,\lambda)$ gives an additional factor λ. Since $h_+^{\mathrm{uu}}(0)=h_{\mathrm{a}}^+(0)|_{\mathrm{Rg}(\Phi_+^{\mathrm{uu}}(0,0,0))}=0$ by Lemmas 4.8 and 4.10, it follows that the remaining terms either obey a uniform exponential estimate of the form $\mathrm{O}(\mathrm{e}^{-\bar\eta|x-y|})$ or else also generate an extra factor λ. It remains to discuss the differentiability of $\Phi_+^{\mathrm{u}}(0,y,\lambda)$ and $\Phi_-^{\mathrm{s}}(0,y,\lambda)$ with respect to y for $y\in[0,q+1]$ and $y\in[-p-1,0]$, respectively. For these values of y, we can simply replace the value $x_0=0$ at which we split the exponential dichotomies on $[0,\infty)$ and $(-\infty,0]$ by points $x_0=-q-1$ and $x_0=p+1$, respectively, and repeat the extension given in the preceding section to obtain differentiability with respect to y for $y\in[0,q+1]$ and $y\in[-p-1,0]$, respectively. The resulting bounds in y will look slightly different but this is irrelevant as y is in a bounded interval. This completes the discussion of the analytic terms (a)-(c).

Next, we discuss the terms (i)-(iii) in (4.47). We focus first on the term (i). Upon substituting the expression for $h_{\mathrm{p}}^+(\lambda)$ into (i) and multiplying from the left and right by π_2 and ι_2, respectively, we obtain

$$\begin{aligned}&\pi_2\Phi_+^{\mathrm{s}}(x,0,\lambda)h_{\mathrm{p}}^+(\lambda)\Phi_+^{\mathrm{u}}(0,y,\lambda)\iota_2u_0\\ &\quad=\frac{1}{M\lambda}\left\langle\Psi_*(0),\Phi_+^{\mathrm{u}}(0,y,\lambda)\iota_2u_0\right\rangle_{(0)}\pi_2\Phi_+^{\mathrm{s}}(x,0,\lambda)U_*'(0),\end{aligned}\tag{4.49}$$

where $x>y>0$ and $u_0\in\mathbb{C}^N$ is arbitrary. First, consider the term $\Phi_+^{\mathrm{s}}(x,0,\lambda)U_*'(0)$. Since $U_*'(x)$ is a solution of (4.16) for $\lambda=0$ that decays to zero exponentially, we have

$$P_+^{\mathrm{ss}}(0,0)U_*'(0)=U_*'(0)$$

from Lemma 4.7. Using this expression, we obtain

$$\begin{aligned}\Phi_+^{\mathrm{s}}(x,0,\lambda)U_*'(0)&=\Phi_+^{\mathrm{s}}(x,0,\lambda)(P_+^{\mathrm{ss}}(0,0)-P_+^{\mathrm{ss}}(0,\lambda)+P_+^{\mathrm{ss}}(0,\lambda))U_*'(0)\\ &=\Phi_+^{\mathrm{ss}}(x,0,\lambda)U_*'(0)+\lambda\mathrm{O}(\|\Phi_+^{\mathrm{s}}(x,0,\lambda)\|)\\ &=\Phi_+^{\mathrm{ss}}(x,0,0)U_*'(0)+\lambda\mathrm{O}(\|\Phi_+^{\mathrm{s}}(x,0,\lambda)\|),\end{aligned}\tag{4.50}$$

where all O(...) terms are analytic in λ. Exploiting (4.17), we arrive at

$$\pi_2\Phi_+^{\mathrm{s}}(x,0,\lambda)U_*'(0)=u_*'(x)+\lambda\sum_{\nu_{\mathrm{out}}^+}\mathrm{O}\left(\mathrm{e}^{\nu_{\mathrm{out}}^+(\lambda)x}\right).\tag{4.51}$$

Next, consider the term $\left\langle\Psi_*(0),\Phi_+^{\mathrm{u}}(0,y,\lambda)\iota_2u_0\right\rangle_{(0)}$ in (4.49). Substituting the expression (4.29) for $\Phi_+^{\mathrm{u}}(0,y,\lambda)$ yields

$$\begin{aligned}\left\langle\Psi_*(0),\Phi_+^{\mathrm{u}}(0,y,\lambda)\iota_2u_0\right\rangle_{(0)}&=\left\langle\Psi_*(0),\Phi_+^{\mathrm{ss}}(0,y,\lambda)\iota_2u_0\right\rangle_{(0)}+\sum_{n=k+1}^{N}\left\langle\Psi_*(0),U_n^+(0,\lambda)\right\rangle_{(0)}\left\langle\Psi_n^+(y,\lambda),\iota_2u_0\right\rangle_{(y)}\\ &\overset{(4.31)}{=}\sum_{n=k+1}^{N}\left\langle\Psi_*(0),U_n^+(0,\lambda)\right\rangle_{(0)}\left\langle\Psi_n^+(y,\lambda),\iota_2u_0\right\rangle_{(y)}+\lambda\mathrm{O}\left(\mathrm{e}^{-2\bar\eta y}\right)\\ &\overset{(4.25)}{=}\sum_{n=k+1}^{N}\left\langle\Psi_*(0),U_n^+(0,\lambda)\right\rangle_{(0)}\mathrm{e}^{-\nu_n^+(\lambda)y}\left\langle\hat{\mathcal{V}}_n^+(\lambda),\iota_2u_0\right\rangle_{(y)}+\lambda\mathrm{O}\left(\mathrm{e}^{-2\bar\eta y}\right)\\ &=\sum_{\nu_{\mathrm{in}}^+}\mathrm{e}^{-\nu_{\mathrm{in}}^+(\lambda)y}\left\langle\Psi_*(0),U_n^+(0,\lambda)\right\rangle_{(0)}\langle l_{\mathrm{in}}^+(\lambda),u_0\rangle_{\mathbb{C}^N}+\lambda\mathrm{O}\left(\mathrm{e}^{-2\bar\eta y}\right),\end{aligned}$$

where all O(...) terms are analytic in λ. Subsuming the analytic coefficients $\langle\Psi_*(0), U_n^+(0,\lambda)\rangle_{(0)}$ into $l_{\rm in}^+(\lambda)$, we obtain

$$\left\langle\Psi_*(0), \Phi_+^{\rm u}(0,y,\lambda)\iota_2 u_0\right\rangle_{(0)} = \sum_{\nu_{\rm in}^+} {\rm e}^{-\nu_{\rm in}^+(\lambda)y}\langle l_{\rm in}^+(\lambda), u_0\rangle_{\mathbb{C}^N} + \lambda {\rm O}\left({\rm e}^{-2\bar\eta y}\right). \tag{4.52}$$

Finally, substituting (4.51) and (4.52) into (4.49), we arrive at

$$\begin{aligned}\pi_2\Phi_+^{\rm s}(x,0,\lambda)h_{\rm p}^+(\lambda)\Phi_+^{\rm u}(0,y,\lambda)\iota_2 u_0 \;&=\; \frac{1}{\lambda}u_*'(x)\sum_{\nu_{\rm in}^+}{\rm e}^{-\nu_{\rm in}^+(\lambda)y}\langle l_{\rm in}^+(\lambda), u_0\rangle_{\mathbb{C}^N}\\ &+\textstyle\sum_{\nu_{\rm out}^+,\nu_{\rm in}^+}{\rm O}\left({\rm e}^{\nu_{\rm out}^+(\lambda)x}\right){\rm e}^{-\nu_{\rm in}^+(\lambda)y} + {\rm O}\left({\rm e}^{-2\bar\eta(x+y)}\right) + \lambda\sum_{\nu_{\rm out}^+}{\rm O}\left({\rm e}^{\nu_{\rm out}^+(\lambda)x}{\rm e}^{-2\bar\eta y}\right),\end{aligned}$$

where the O(...) terms are analytic in λ. This, and an analogous estimate for the y-derivative, establishes (4.45) and (4.46) for the term (i) in (4.47). Finally, the analysis of the term (iii) in (4.47) proceeds in exactly the same fashion. The only difference for term (ii) is the appearance of the extra term $\Phi_-^{\rm u}(0,0,\lambda) = P_-^{\rm u}(0,\lambda)$, which changes the computations in (4.50). Using that $P_-^{\rm u}(0,0)U_*'(0) = U_*'(0)$, we obtain

$$\begin{aligned}\Phi_+^{\rm s}(x,0,\lambda)P_-^{\rm u}(0,\lambda)U_*'(0) \;&=\; \Phi_+^{\rm s}(x,0,\lambda)(P_-^{\rm u}(0,\lambda) - P_-^{\rm u}(0,0) + P_-^{\rm u}(0,0))U_*'(0)\\ &=\; \Phi_+^{\rm s}(x,0,\lambda)U_*'(0) + \lambda{\rm O}(\|\Phi_+^{\rm s}(x,0,\lambda)\|)\end{aligned}$$

and can now proceed exactly as in (4.50). This observation completes the proof of Theorem 7. ■

5 Pointwise bounds on the Green's function and nonlinear stability

In this section, we complete the proof of Theorem 1. First, we show that the analytic extension and the pointwise bounds of the resolvent kernel $G(x,y,\lambda)$ that we established in Theorem 7 allow us to obtain pointwise bounds of the Green's function $\mathcal{G}(j,i,t,s)$ of the lattice dynamical system (2.2). The pointwise bounds for $\mathcal{G}(j,i,t,s)$ can then be used as in [4, §5] to establish nonlinear stability.

Recall from Theorem 3 and equation (2.3) in §2 that $\mathcal{G}(j,i,t,s)$ can be found from $G(x,y,\lambda)$ via the inverse Laplace transform formula

$$\mathcal{G}(j,i,t,s) = \mathcal{G}\left(j-\frac{\sigma t}{h}, i-\frac{\sigma s}{h}, t-s\right) \quad \text{where} \quad \mathcal{G}(x,y,\tau) = \frac{-1}{2\pi {\rm i}\sigma}\int_{\gamma-{\rm i}\pi\sigma}^{\gamma+{\rm i}\pi\sigma} {\rm e}^{\lambda\tau}G(x,y,\lambda)\,{\rm d}\lambda. \tag{5.1}$$

The following theorem gives the desired pointwise bounds for $\mathcal{G}(x,y,\tau)$.

Theorem 8 *Under the assumptions of Theorem 1, the temporal Green's function $\mathcal{G}(x,y,\tau)$ from (5.1) can be written as $\mathcal{G}(x,y,\tau) = \mathcal{E}(x,y,\tau) + \tilde{\mathcal{G}}(x,y,\tau)$ so that the following is true for $y \le 0$ (with analogous expressions and estimates for $y \ge 0$): we have*

$$\mathcal{E}(x,y,\tau) = u_*'(x)\sum_{a_n^->0}\left[{\rm errfn}\left(\frac{y+a_n^-\tau}{\sqrt{4(\tau+1)}}\right) - {\rm errfn}\left(\frac{y-a_n^-\tau}{\sqrt{4(\tau+1)}}\right)\right] l_n^-$$

for some appropriate constants $l_n^- \in \mathbb{R}^N$, and there are constants $\eta, K, M > 0$ with

$$|\tilde{\mathcal{G}}(x,y,\tau)| + |\partial_y\tilde{\mathcal{G}}(x,y,\tau)| \le K{\rm e}^{-\eta|x-y|}$$

for $0 \le \tau \le 1$ and

$$\begin{aligned}|\partial_y^\ell\tilde{\mathcal{G}}(x,y,\tau)| \;&\le\; K{\rm e}^{-\eta(|x-y|+\tau)} + K\tau^{-\frac{\ell}{2}}\left[\sum_{n=1}^N \tau^{-\frac12}{\rm e}^{-(x-y-a_n^-\tau)^2/M\tau}{\rm e}^{-\eta\max(x,0)}\right.\\ &+ \sum_{a_n^->0,\,a_m^-<0}\chi_{\{|a_n^-\tau|\ge|y|\}}\tau^{-\frac12}{\rm e}^{-(x-a_m^-(\tau-|y/a_n^-|))^2/M\tau}{\rm e}^{-\eta\max(x,0)}\\ &\left.+ \sum_{a_n^->0,\,a_m^+>0}\chi_{\{|a_n^-\tau|\ge|y|\}}\tau^{-\frac12}{\rm e}^{-(x-a_m^+(\tau-|y/a_n^-|))^2/M\tau}{\rm e}^{\eta\min(x,0)}\right]\end{aligned}$$

for $\ell = 0,1$ and $\tau \geq 1$, where $a_n^\pm$ are the eigenvalues of $\bar{f}_u(u_\pm)$, and the indicator function $\chi_I(x)$ of an interval $I \subset \mathbb{R}$ is one for $x \in I$ and zero otherwise.

We remark that the preceding theorem gives only pointwise bounds on the remainder term Green's function $\tilde{\mathcal{G}}(x, y, \tau)$, while, in contrast, [4, Theorem 4.8] provides a leading-order expansion of this term for one-sided schemes into an explicit sum of moving Gaussians plus a faster-decaying remainder. With a little more effort, we can derive a similar expansion for general schemes but decided to omit the derivation as it is not needed in the nonlinear stability proof.

Proof. First, consider the case $0 \leq \tau \leq 1$. Recall that $\mathcal{G}(x, y, \tau)$ is given by

$$\mathcal{G}(x, y, \tau) = \frac{-1}{2\pi \mathrm{i}\sigma} \int_{\gamma - \mathrm{i}\pi\sigma}^{\gamma + \mathrm{i}\pi\sigma} \mathrm{e}^{\lambda\tau} G(x, y, \lambda)\, \mathrm{d}\lambda \tag{5.2}$$

due to (5.1). Condition (S1) and Theorem 5 imply that $G(x, y, \lambda)$ is analytic in λ for $\mathrm{Re}\,\lambda > 0$ and satisfies the pointwise bound $|\partial_y^\ell G(x, y, \lambda)| \leq K\mathrm{e}^{-\eta|x-y|}$ with $\ell = 0, 1$ for constants K and $\eta > 0$ that are locally uniform near each λ. Using these bounds and the fact that the integral in (5.2) is over an interval of bounded length, we can evaluate the integral for some $\gamma > 0$ and obtain the estimate

$$|\partial_y^\ell \mathcal{G}(x, y, \tau)| \leq K\mathrm{e}^{-\eta|x-y|}$$

for some $\eta > 0$ with $\ell = 0, 1$. Since $\mathcal{E}(x, y, \tau)$ satisfies a similar bound for $0 \leq \tau \leq 1$, we obtain the result.

Next, consider the case $\tau \geq 1$. Cauchy's integral theorem applied to (5.2) implies that

$$\left[\mathrm{e}^{\lambda\tau}\tilde{G}(x, y, \lambda)\right]_{\lambda=\gamma-\mathrm{i}\pi\sigma}^{\lambda=\gamma+\mathrm{i}\pi\sigma} \equiv 0 \tag{5.3}$$

for each λ with $\mathrm{Re}\,\lambda \geq -\epsilon$. Equation (5.3) may be recognized as the key property [4, (4.5)] of relative periodicity with respect to the underlying lattice. Applying Cauchy's integral theorem a second time to (5.2), we thus have

$$\tilde{\mathcal{G}}(x, y, \tau) := \frac{-1}{2\pi \mathrm{i}\sigma} \oint_\Gamma \mathrm{e}^{\lambda\tau} \tilde{G}(x, y, \lambda)\, \mathrm{d}\lambda,$$

where

$$\Gamma = \left[-\frac{\epsilon}{2} - \mathrm{i}\pi\sigma, -\frac{\epsilon}{2} - \mathrm{i}r\right] \cup \left[-\frac{\epsilon}{2} - \mathrm{i}r, r - \mathrm{i}r\right] \cup [r - \mathrm{i}r, r + \mathrm{i}r] \cup \left[r + \mathrm{i}r, -\frac{\epsilon}{2} + \mathrm{i}r\right] \cup \left[-\frac{\epsilon}{2} + \mathrm{i}r, -\frac{\epsilon}{2} + \mathrm{i}\pi\sigma\right]$$

for any r with $0 < r \ll 1$. This is a representation on a contour that corresponds exactly to the low-frequency part of the contour used to begin the arguments of [28, §8] and [21, §7]. Since our bounds for the individual meromorphic pieces of $\tilde{G}(x, y, \lambda)$ as well as the initial contour Γ are the same as for the low-frequency estimates in the viscous case treated there, we obtain the same bounds for $\tilde{\mathcal{G}}(x, y, \tau)$ with $\tau \geq 1$ that were obtained in [21, 28] in the viscous case. It is here where the hypothesis (S5) enters crucially to guarantee the Gaussian nature of the estimates for $\tilde{\mathcal{G}}(x, y, \tau)$. Finally, since we have the same description of

$$E(x, y, \lambda) := \frac{1}{\lambda} u_*'(x) \sum_{\nu_{\mathrm{in}}^\pm} \mathrm{e}^{-\nu_{\mathrm{in}}^\pm(\lambda) y} \langle l_{\mathrm{in}}^\pm(\lambda), \cdot \rangle_{\mathbb{C}^N}, \qquad y \gtrless 0$$

as in the viscous case, we obtain the corresponding description of

$$\mathcal{E}(x, y, \tau) := \frac{-1}{2\pi \mathrm{i}\sigma} \oint_\Gamma \mathrm{e}^{\lambda\tau} E(x, y, \lambda)\, \mathrm{d}\lambda$$

for $\tau \geq 1$ that was obtained in [21, 28] in the viscous case. This completes the proof. ■

We remark that, compared with [4], we have somewhat simplified the proof for the pointwise estimates by working directly with the finite contour integral representation (5.2); see [1] for a similar argument in the time-periodic case. In particular, the short-time bounds for $0 \leq \tau \leq 1$ stated here are somewhat sharper than the ones formulated in [4, Theorem 4.11], which were singular as $\tau \to 0$. This is, however, a minor point as the same estimates are available by the techniques of [4].

Proof of Theorem 1. Since the nonlinear stability analysis in [4, §5] uses only the pointwise bounds of Theorem 8, we can proceed in exactly the same fashion to prove Theorem 1. We refer the reader to [4, §5] for the details. ∎

6 Discussion

Semidiscrete conservation laws arise most commonly through spatial discretizations of conservation laws posed on the line $\mathbb{R}$. In this paper, we kept the spatial step size h fixed, but it is natural to ask what happens when h goes to zero. In particular, it would be interesting to see whether the bounds on the temporal Green's function are uniform in the step size h or, more ambitiously, whether convergence, in an appropriate sense, of the temporal Green's function of the semidiscrete system to that of the limiting inviscid system can be proved. We remark that the resolvent kernel sees the spatial step size h only through a linear scaling of the eigenvalue parameter λ, so the key issue is to carefully analyse the estimation of the temporal Green's function that we outlined in §5.

Discretizing a system of conservation laws in both space and time leads to a discrete dynamical system posed on a lattice. Establishing the existence of discrete shocks in this setting is very difficult and has only been carried out in special cases. In [13, 17, 23], the existence of weak discrete shocks with vanishing or rational speeds has been shown. If the speed satisfies certain Diophantine conditions, the existence and stability of weak discrete shocks was proved in [15, 16]. In [7], Green's function bounds for stationary discrete shocks of arbitrary shock strength were obtained. Nonlinear stability of stationary weak discrete shocks was shown in [24] under assumptions that are weaker than those in [16]. One interesting problem in this context is to prove the nonlinear stability of spectrally stable discrete shocks with rational speed and arbitrary strength: such an analysis could build on the Green's function bounds obtained in [7].

The existence of discrete travelling waves with arbitrary wave speeds for dissipative lattice systems was addressed in [5] using a very different approach: The authors of [5] started with the assumption that a semidiscrete travelling wave has been found such that the spectrum of the linearization $\mathcal{L}$, given in (1.6), has a simple eigenvalue at zero, while the rest of the spectrum is contained in the open left half plane. They then proved in [5, Theorem B] that this assumption implies the existence of discrete travelling shocks with wave speeds that depend continuously on $h = \Delta x/\Delta t$ for h near zero, where Δx is the spatial step size, and Δt the temporal step size of an Euler scheme. Thus, in short, existence and strong spectral stability of a semidiscrete shock implies the existence of discrete shocks without any restriction on the wave speed. Shock profiles of semidiscrete conservation laws cannot be strongly spectrally stable. It is an open and very interesting problem to see whether the assumption of strong spectral stability could be replaced by sufficiently fast algebraic decay of solutions to the linearized semidiscrete equation.

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