Travelling around Obstacles in Planar Anistropic Spatial Systems

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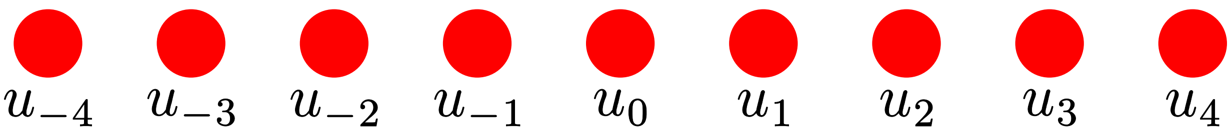
(Joint work with E. van Vleck and A. Hoffman)

Lattice Differential Equations

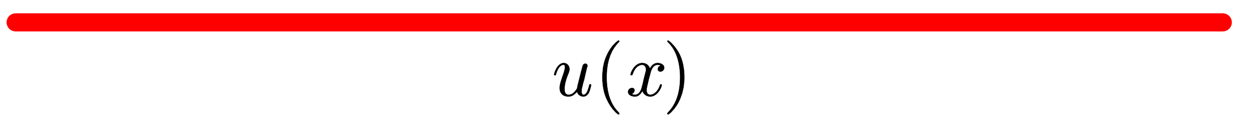
Lattice differential equations (LDEs) are ODEs indexed on a spatial lattice, e.g.

$$\dot{u}_j(t) = \alpha\big(u_{j-1}(t) + u_{j+1}(t) - 2u_j(t)\big) + f\big(u_j(t)\big), \qquad j \in \mathbb{Z}.$$



Picking $\alpha = h^{-2} \gg 1$, LDE can be seen as discretization with distance h of PDE

$$\partial_t u(t,x) = \partial_{xx} u(t,x) + f\big(u(t,x)\big), \qquad x \in \mathbb{R}.$$



- Discrete Laplacian: $u_{j-1} + u_{j+1} - 2u_j$
- Many physical models have a discrete spatial structure $\rightarrow$ LDEs.
- Main theme: qualitative differences between PDEs and LDEs.

Lattice Differential Equations

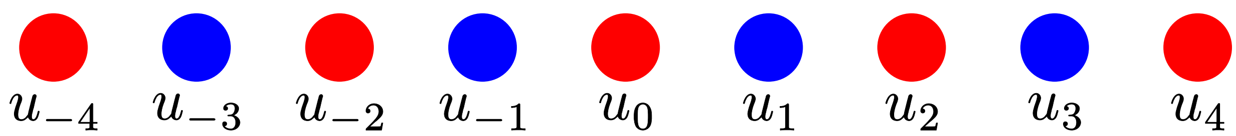
Recall LDE

$$\dot{u}_j(t) = \alpha\big(u_{j-1}(t) + u_{j+1}(t) - 2u_j(t)\big) + f\big(u_j(t)\big), \qquad j \in \mathbb{Z}.$$

- $\alpha \gg 1$ - semi-discretization of PDE. Useful discretizations should not introduce new behaviour.

- $\alpha \sim 1$ - spatial gaps as energy barriers.

- $\alpha < 0$ - anti-diffusion.

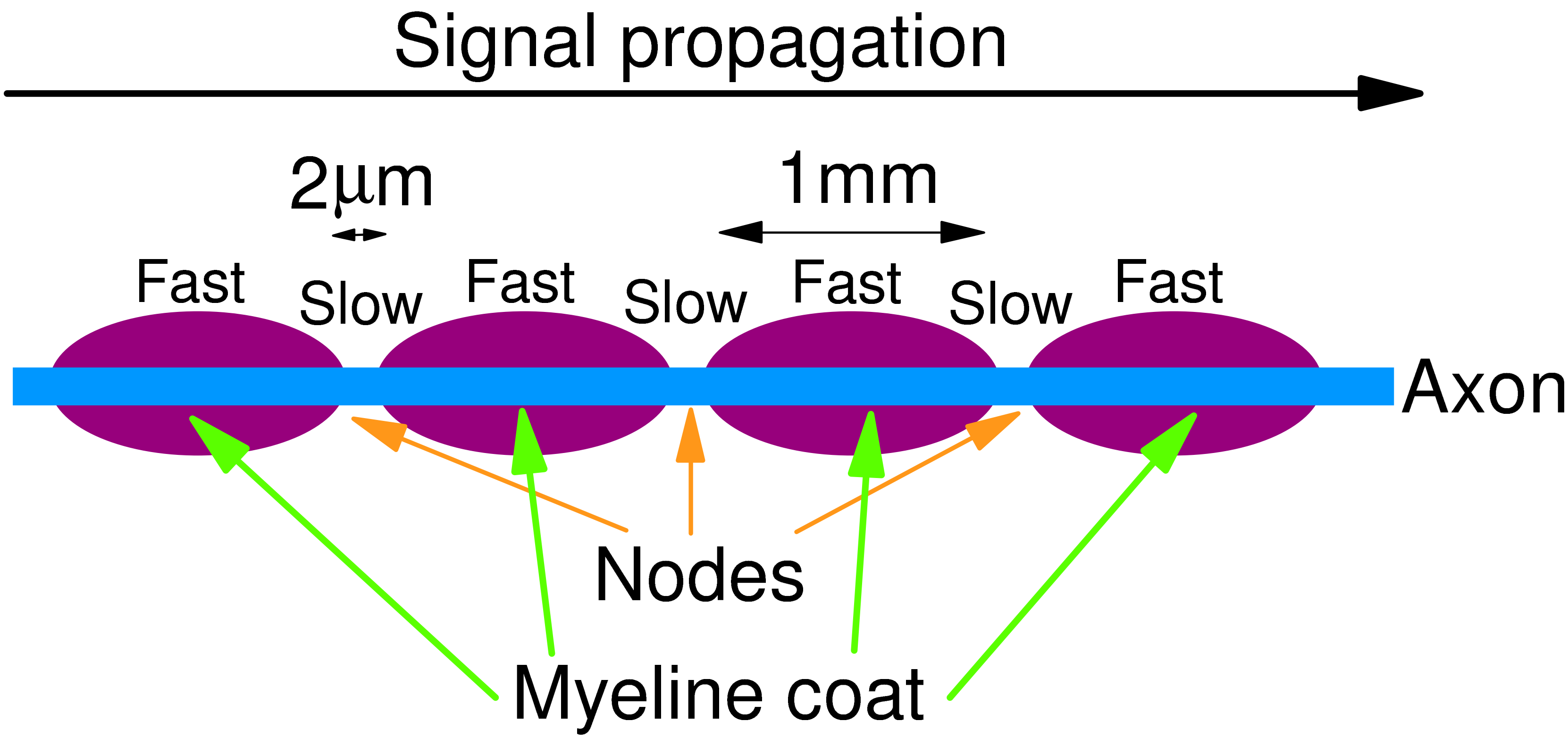


Can be restated as periodic system with positive diffusion. [Van Vleck, Vainchtein]

No clear PDE analogue.

Signal Propagation through Nerves

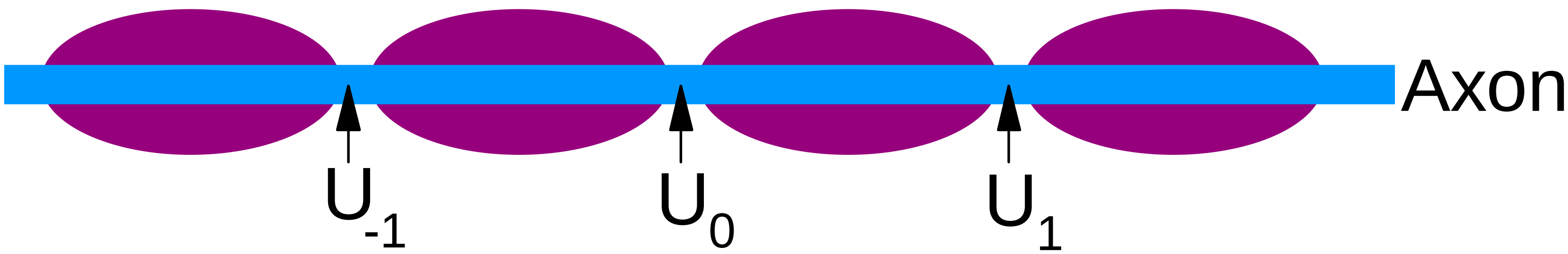
Nerve fibres carry signals over large distances (meter range).



- Fiber has myeline coating with periodic gaps called *nodes of Ranvier* .
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range)
- Slow propagation in gaps, but signal chemically reinforced.

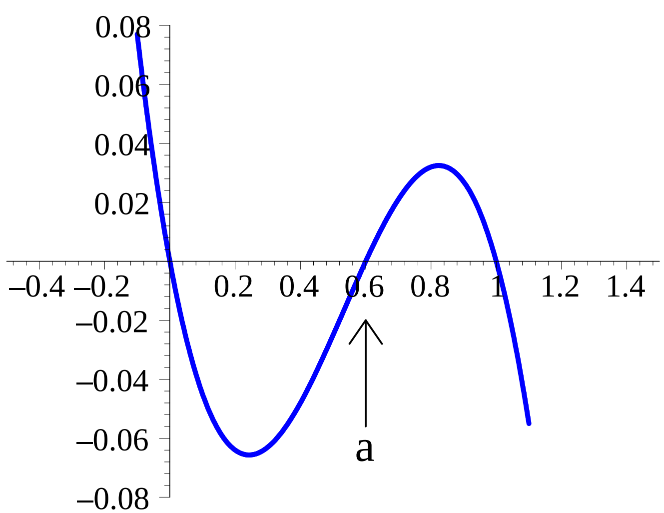
Signal Propagation: The Model

One is interested in the potential U_j at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].
Ignoring recovery, one arrives at the LDE [Keener and Sneyd, 1998]

$$\frac{d}{dt}U_j(t) \quad = \quad U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t);a), \qquad j \in \mathbb{Z}.$$



Bistable nonlinearity g given by

$$g(u;a) = u(a-u)(u-1).$$

Signal Propagation: PDE

In continuum limit: Nagumo LDE becomes Nagumo PDE

$$\partial_t u = \partial_{xx} u + u(a-u)(u-1).$$

Starting step [Fife, McLeod]: travelling waves.

Travelling wave $u(x,t) = \phi(x+ct)$ satisfies:

$$c\phi'(\xi) \quad = \quad \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

Interested in front solutions connecting 0 to 1, i.e.

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \phi(\xi) = 1.$$

Signal Propagation: PDE

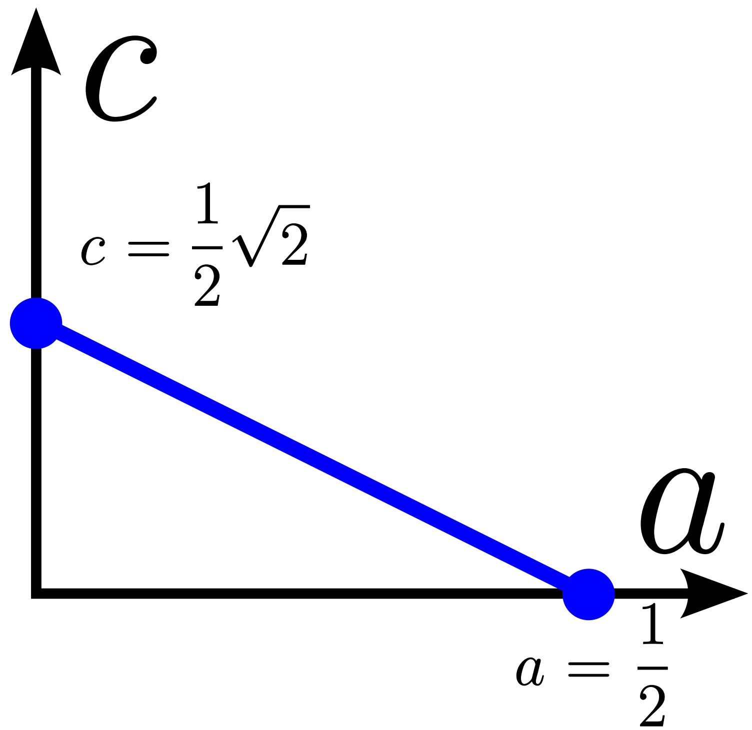
Recall travelling wave ODE

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

$$\lim_{\xi \to -\infty} \phi(\xi) = 0,$$
$$\lim_{\xi \to +\infty} \phi(\xi) = 1.$$

Explicit solutions available:

$$\phi(\xi) = \tfrac{1}{2} + \tfrac{1}{2}\tanh\left(\tfrac{1}{4}\sqrt{2}\,\xi\right),$$
$$c(a) = \tfrac{1}{\sqrt{2}}(1 - 2a).$$



Signal Propagation: LDE

Recall the Nagumo LDE

$$\frac{d}{dt}U_j(t) \quad = \quad [U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + g(U_j(t); a), \qquad j \in \mathbb{Z}.$$

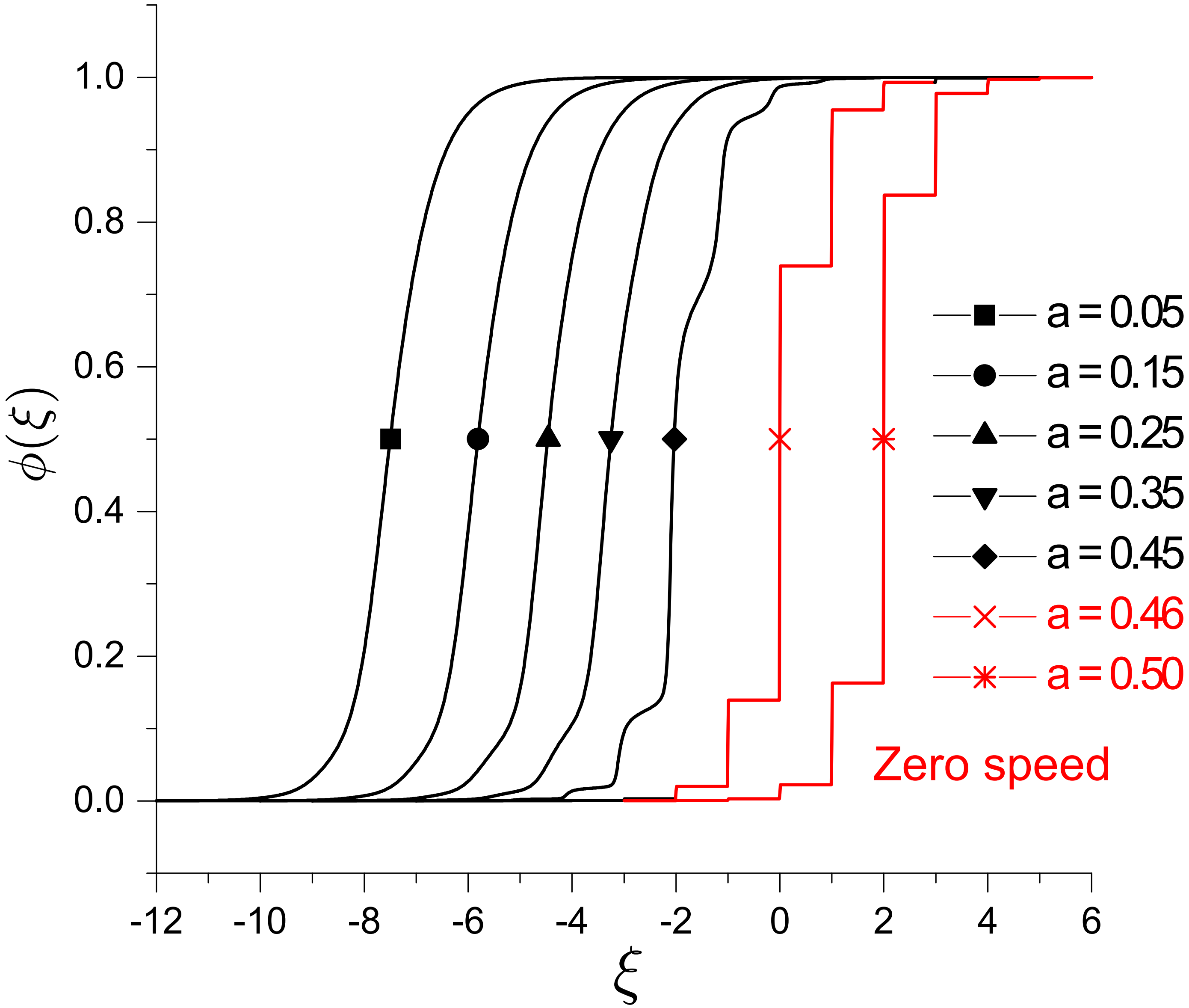
Travelling wave profile $U_j(t) = \phi(j + ct)$ must satisfy:

$$\begin{aligned} c\phi'(\xi) &= [\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] + g\big(\phi(\xi); a\big) \\ \lim_{\xi \to -\infty} \phi(\xi) &= 0, \\ \lim_{\xi \to +\infty} \phi(\xi) &= 1. \end{aligned}$$

- Notice that wave speed c enters in singular fashion.
- When $c \neq 0$, this is a functional differential equation of mixed type (MFDE).
- When $c = 0$, this is a difference equation.

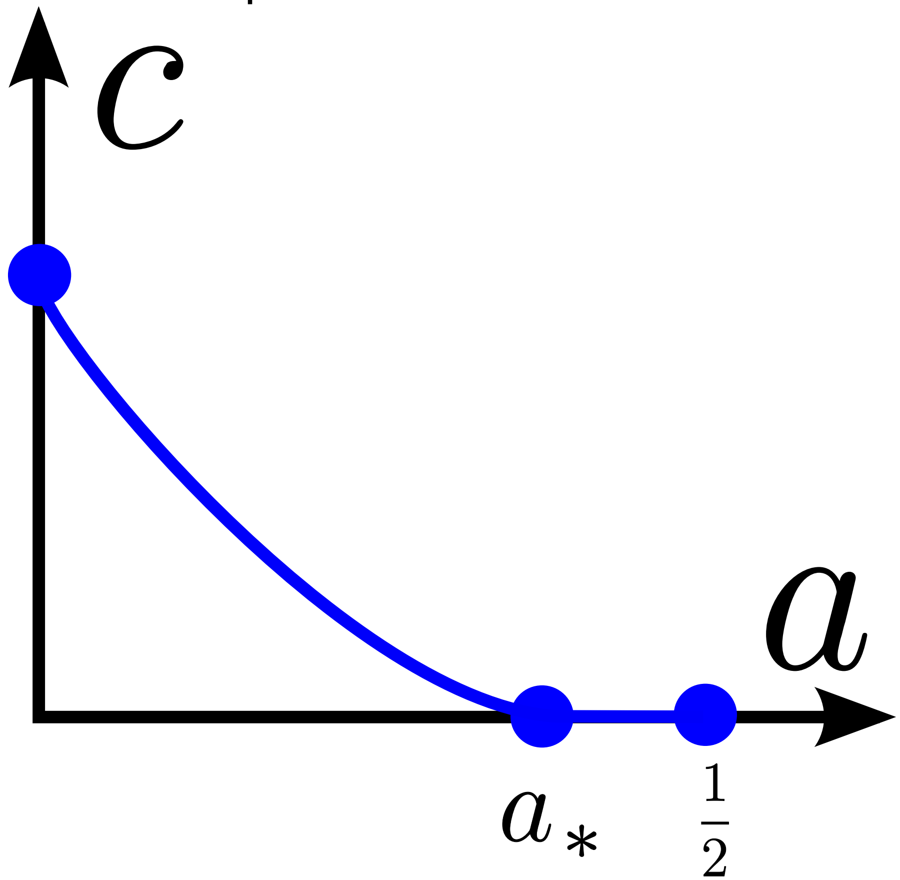
Discrete Nagumo LDE - Propagation failure

Travelling waves for the discrete Nagumo LDE connecting $0 \to 1$.



Propagation

Typical wave speed c versus a plot for discrete reaction-diffusion systems:



In principle, can have $a_* = \frac{1}{2}$ or $a_* < \frac{1}{2}$.

In case $a_* < \frac{1}{2}$, then we say that LDE suffers from propagation failure.

Propagation failure common for LDEs and widely studied; pioneed by [Keener].

Signal Propagation: Comparison

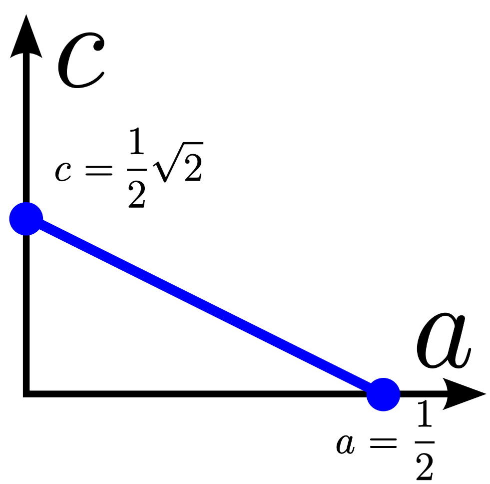
PDE

$$\partial_t u = \partial_{xx} u + g(u, a)$$

Travelling wave $u = \phi(x + ct)$ satisfies:

$$c\phi'(\xi) = \phi''(\xi) + g\big(\phi(\xi); a\big)$$

Travelling waves connecting 0 to 1:



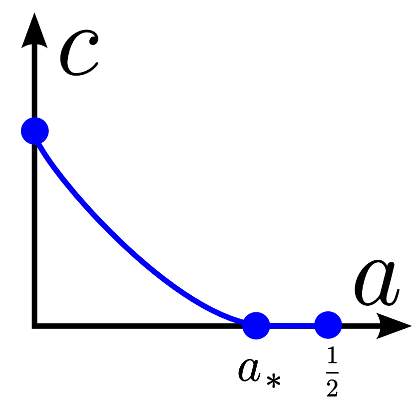
LDE

$$\frac{d}{dt} U_j = U_{j+1} + U_{j-1} - 2U_j + g(U_j; a)$$

Travelling wave $U_j = \phi(j + ct)$ satisfies:

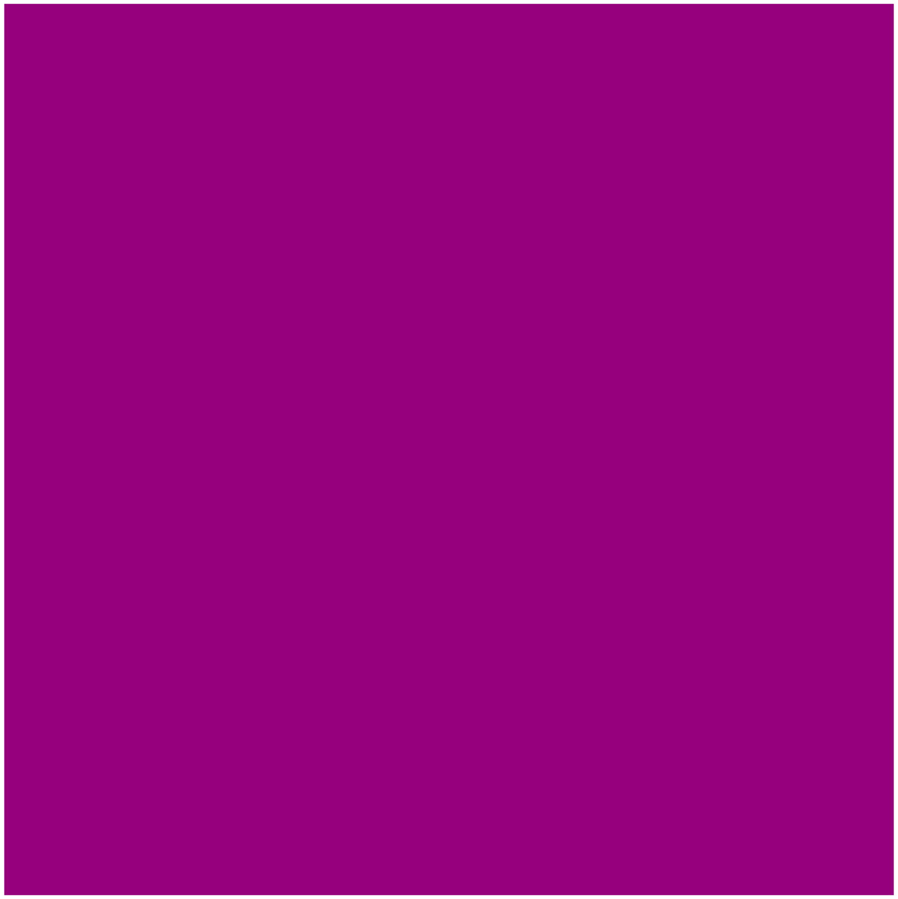
$$c\phi'(\xi) = \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) + g\big(\phi(\xi); a\big)$$

Travelling waves connecting 0 to 1:

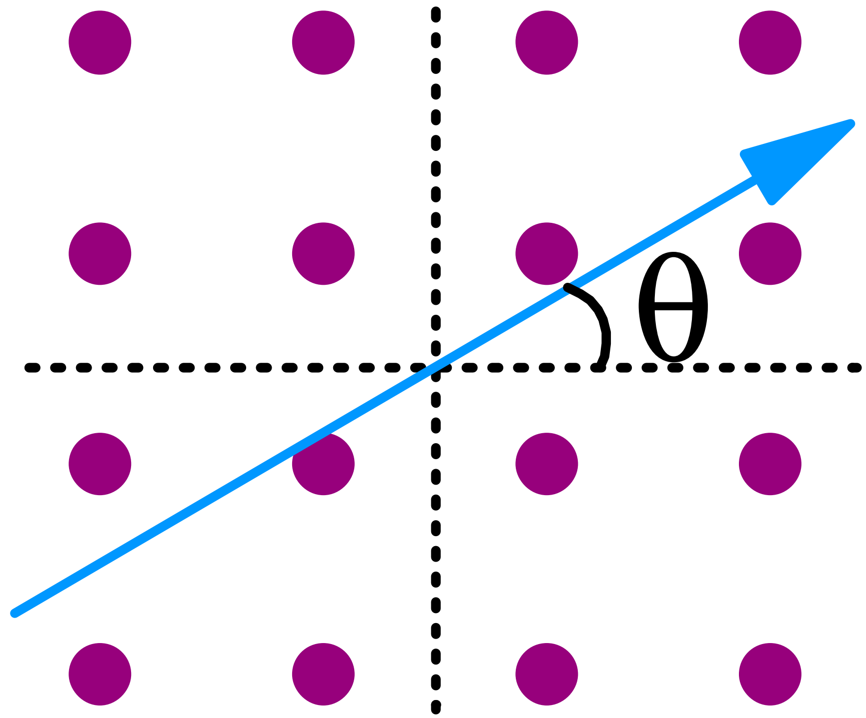


Propagation failure if $a_* < \frac{1}{2}$.

Lattice equations



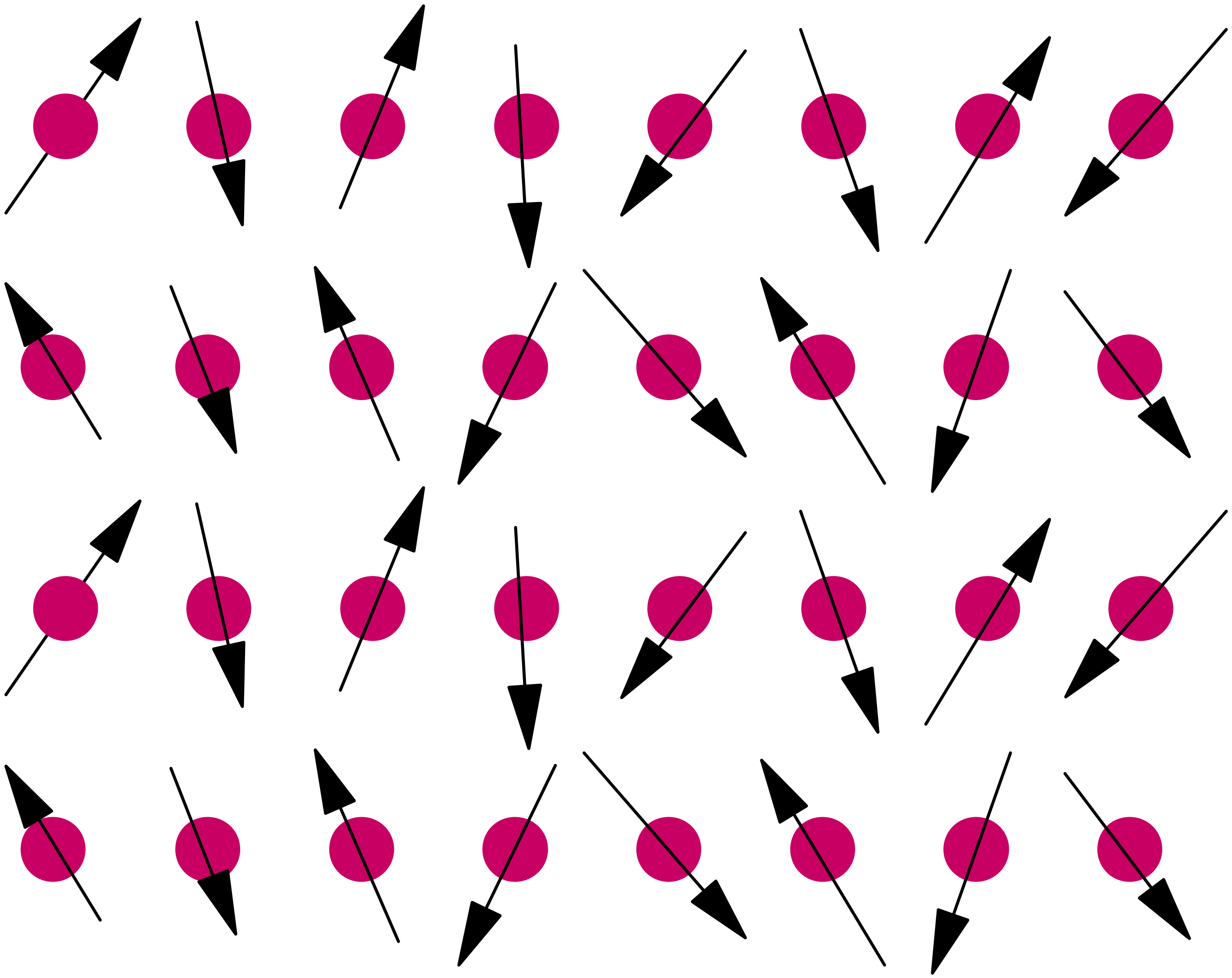
Continuous media (PDE)



Discrete media (LDE)

- In 2d even more differences between PDE and LDE appear.
- Lattice looks different from different directions!

Ising Models



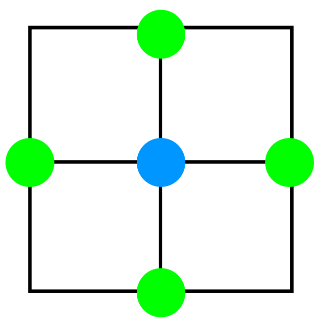
- Each lattice site occupied by block of particles that each have 2 possible states.
- Non-local interactions between lattice sites.

Lattice equations: Geometry

Dynamics for fractional occupancy $u_{i,j}$ of first state satisfies [Bates, 1999]

$$\dot{u}_{i,j}(t) \;=\; [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

- Nonlinearity g governs local fluctuations.
- The operator Δ^+ mixes the lattice sites. Typical choice:



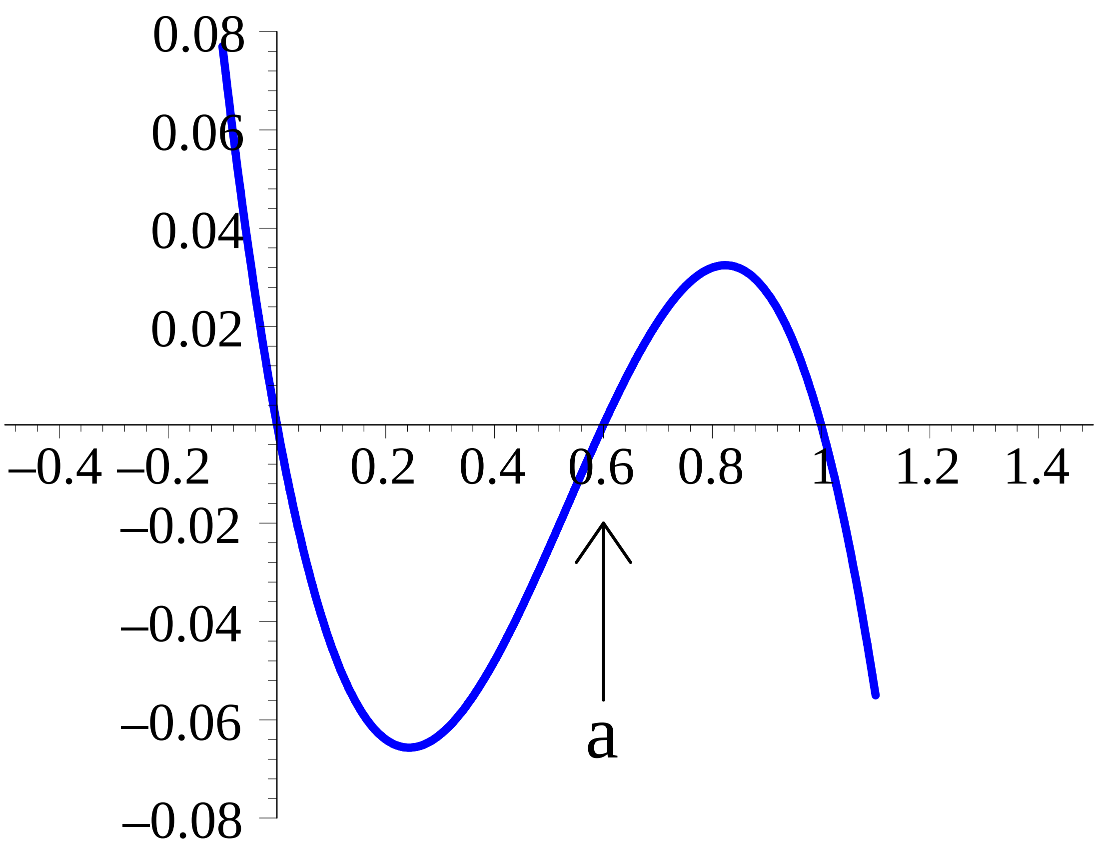
$$[\Delta^+ u]_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}.$$

Δ^+ can be seen as discrete version of Laplacian.

2d LDE: Nonlinearity

Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$



Bistable nonlinearity g given by

$$g(u; a) = u(a - u)(u - 1).$$

Two **stable** equilibria $u = 0$ and $u = 1$.

One **unstable** equilibrium $u = a$.

Lattice equations: Travelling Waves

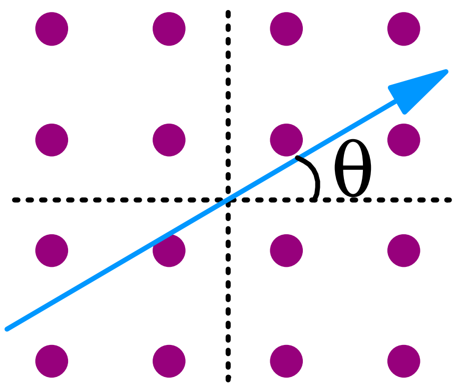
Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

The nonlinearity g 'pulls' u towards either $u = 0$ or $u = 1$ [competition].

The discrete diffusion 'smooths' out any wrinkles in u.

Travelling waves: compromise between these two forces.



Travelling waves with **profile** Φ and **speed** c connecting $u = 0$ to $u = 1$ in direction

$$\vec{k} = (\cos\theta, \sin\theta).$$

$$u_{i,j}(t) = \Phi((\cos\theta, \sin\theta) \cdot (i, j) + ct), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1.$$

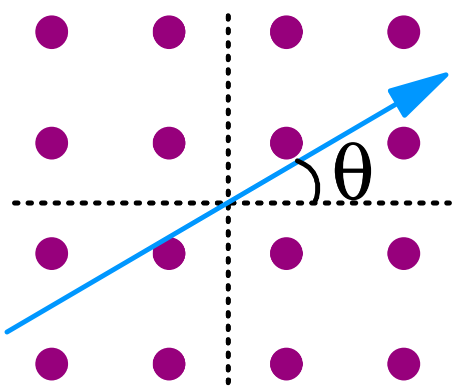
Lattice equations: Travelling Waves

Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

- Travelling waves connecting $u \equiv 0$ to $u \equiv 1$ must satisfy

$$\begin{aligned} c\Phi'(\xi) \quad = \quad & \Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi) \\ & + g(\Phi(\xi); a) \end{aligned}$$



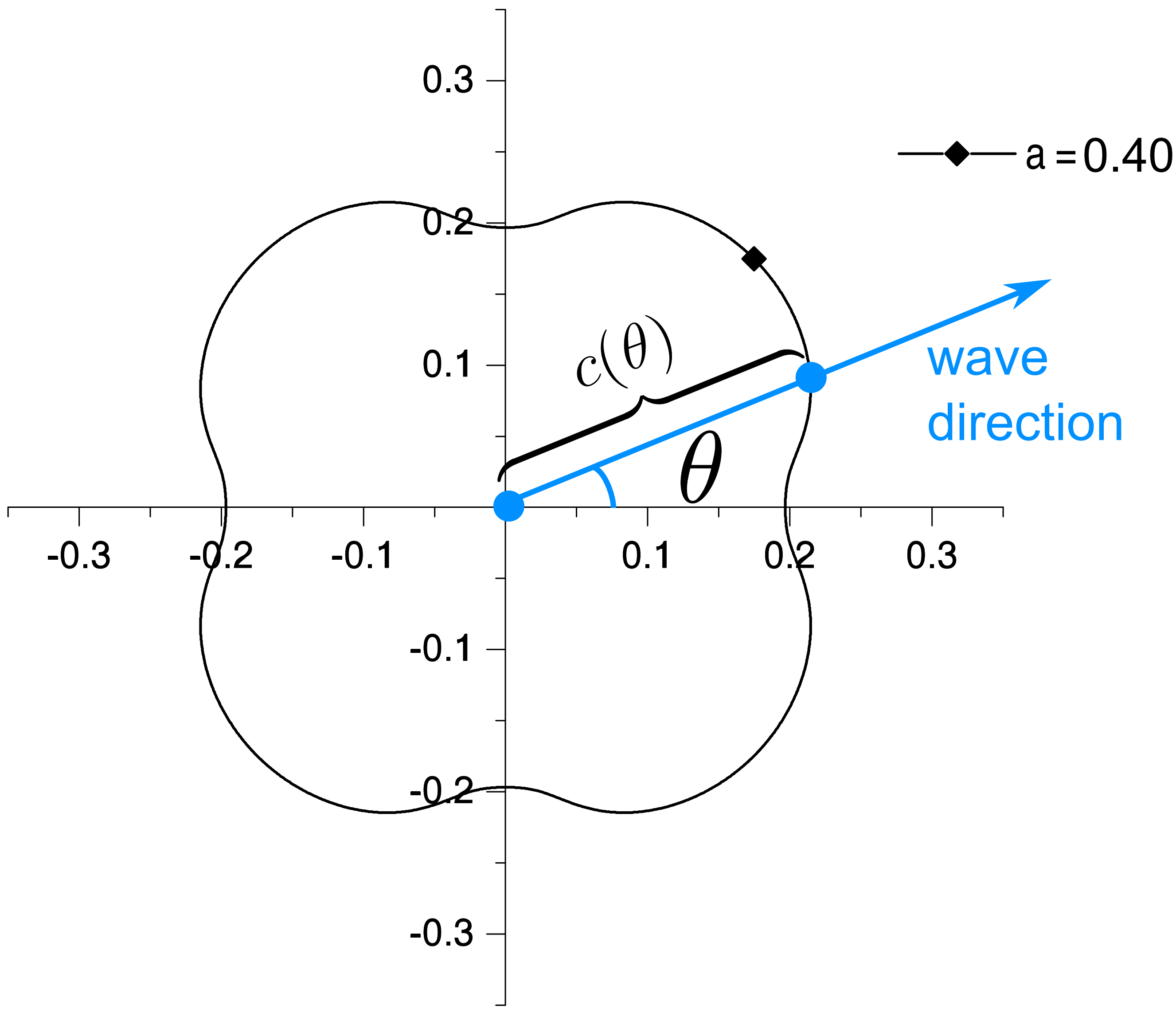
This is a mixed type functional differential equation (MFDE).

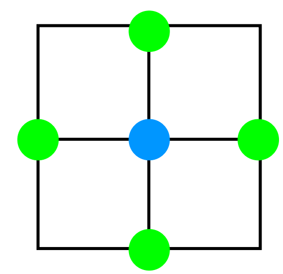
Direction θ explicitly appears in wave equation.

[Mallet-Paret]: waves exist for all directions.

Lattice equations: Spatial anisotropy

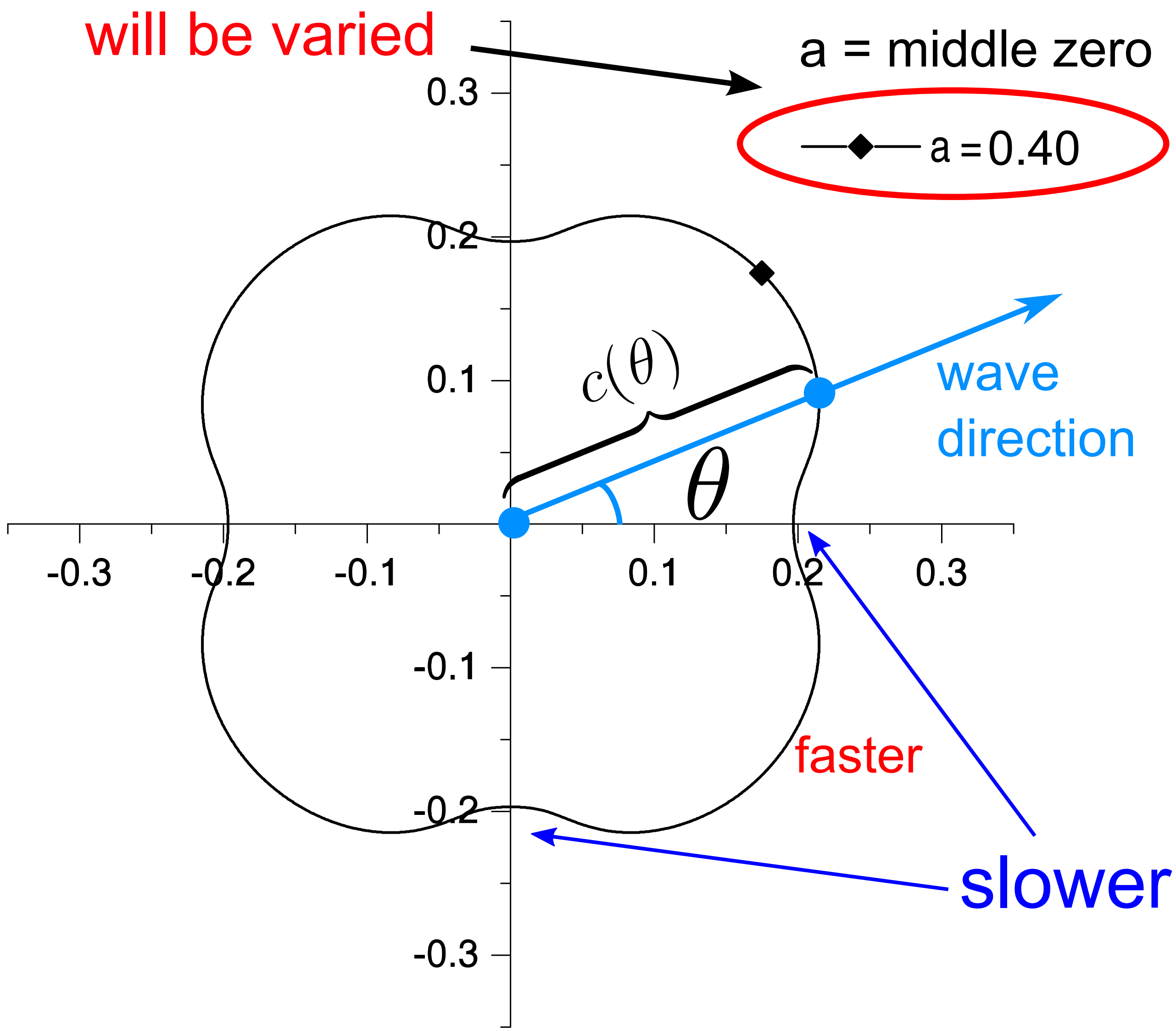
Wavespeed c depends on the angle of propagation θ.

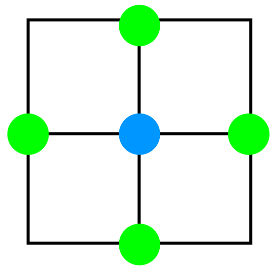




Lattice equations: Spatial anisotropy

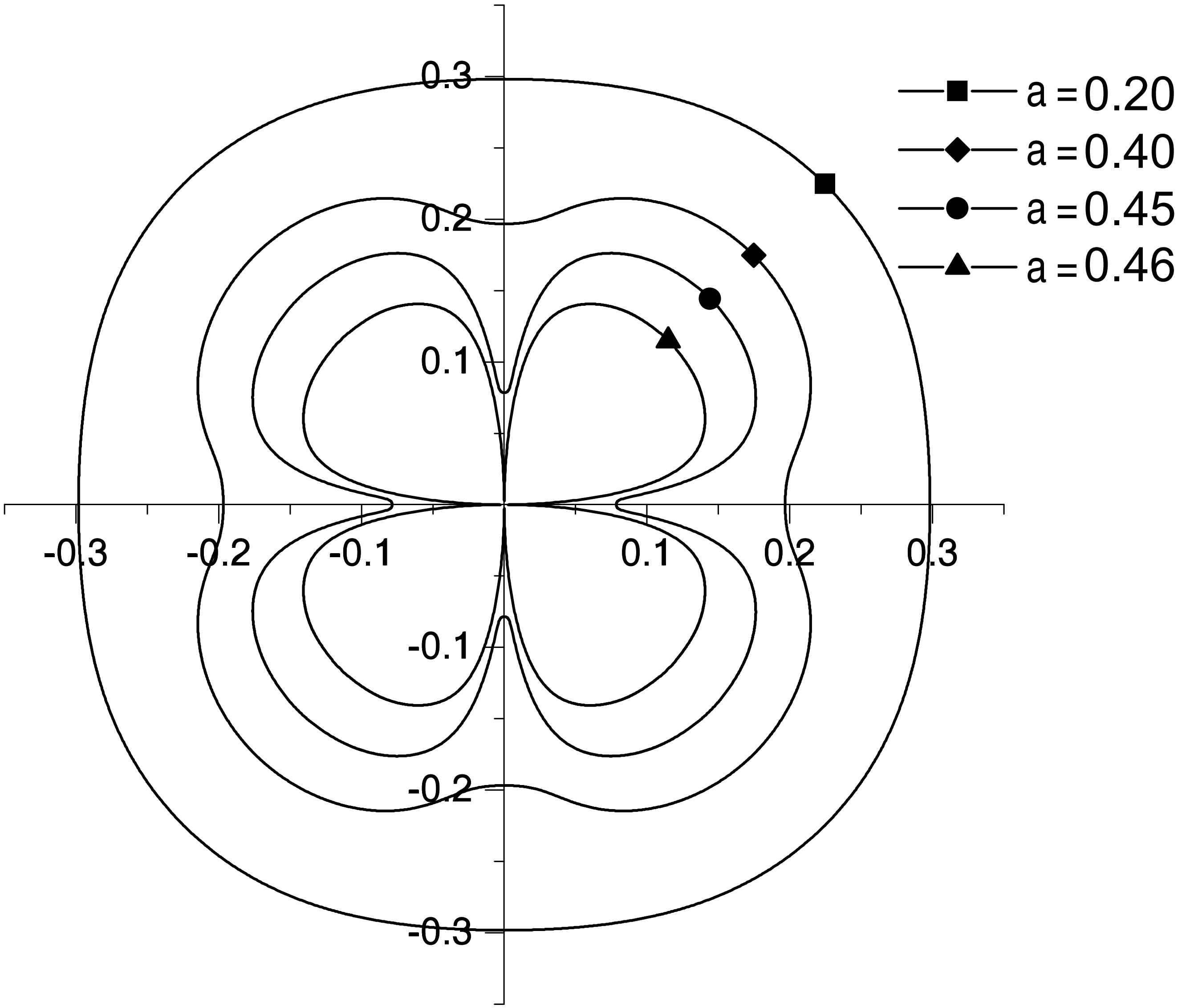
Wavespeed c depends on the angle of propagation θ.

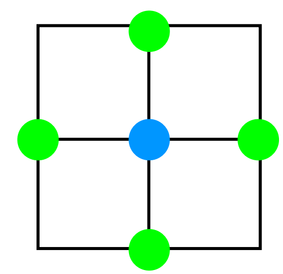




Lattice equations: Spatial anisotropy - II

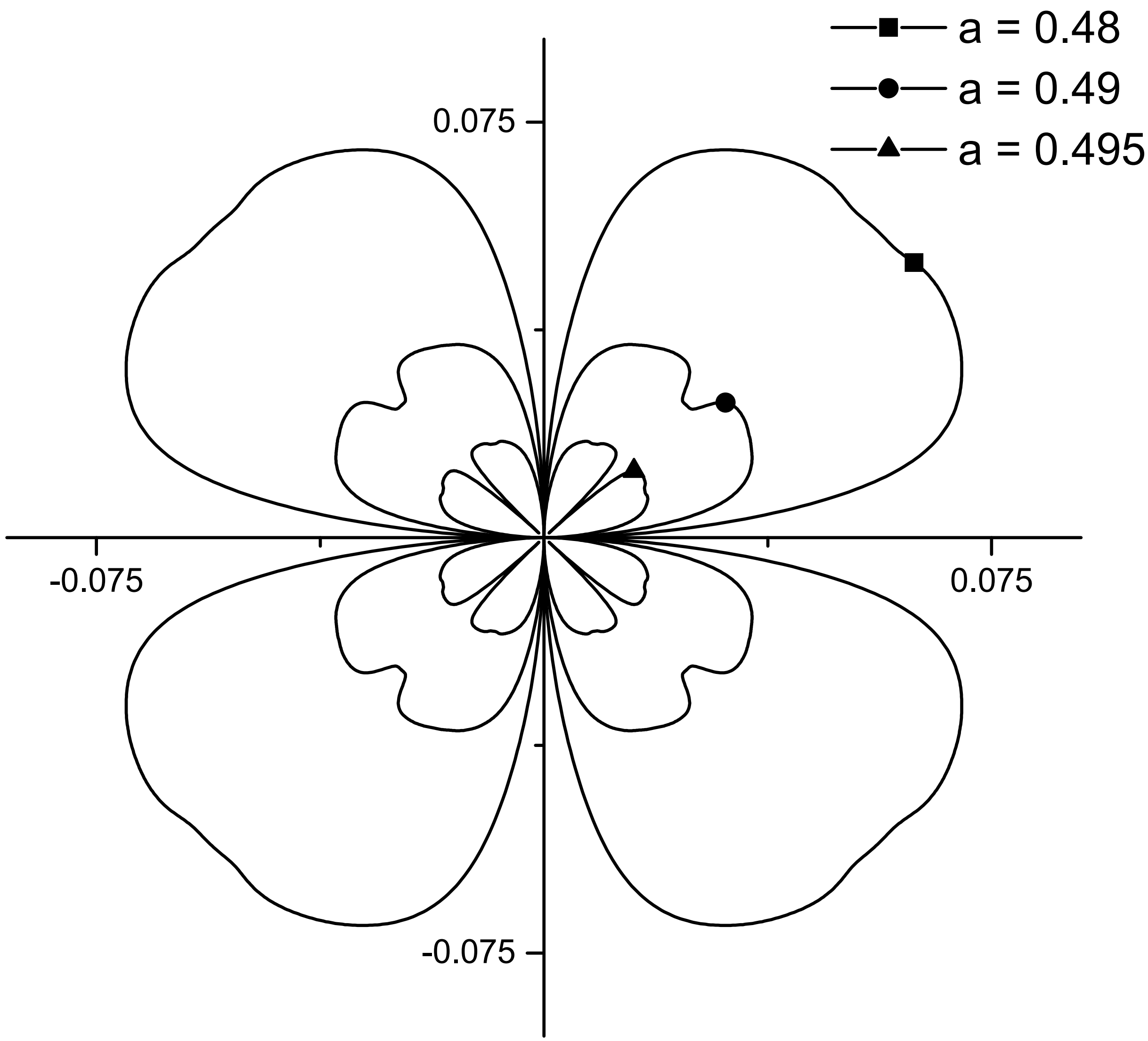
Wavespeed c depends on the angle of propagation θ.

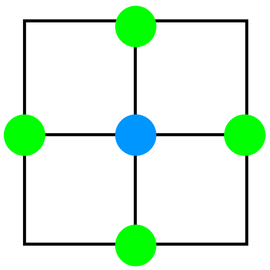




Lattice equations: Spatial anisotropy - III

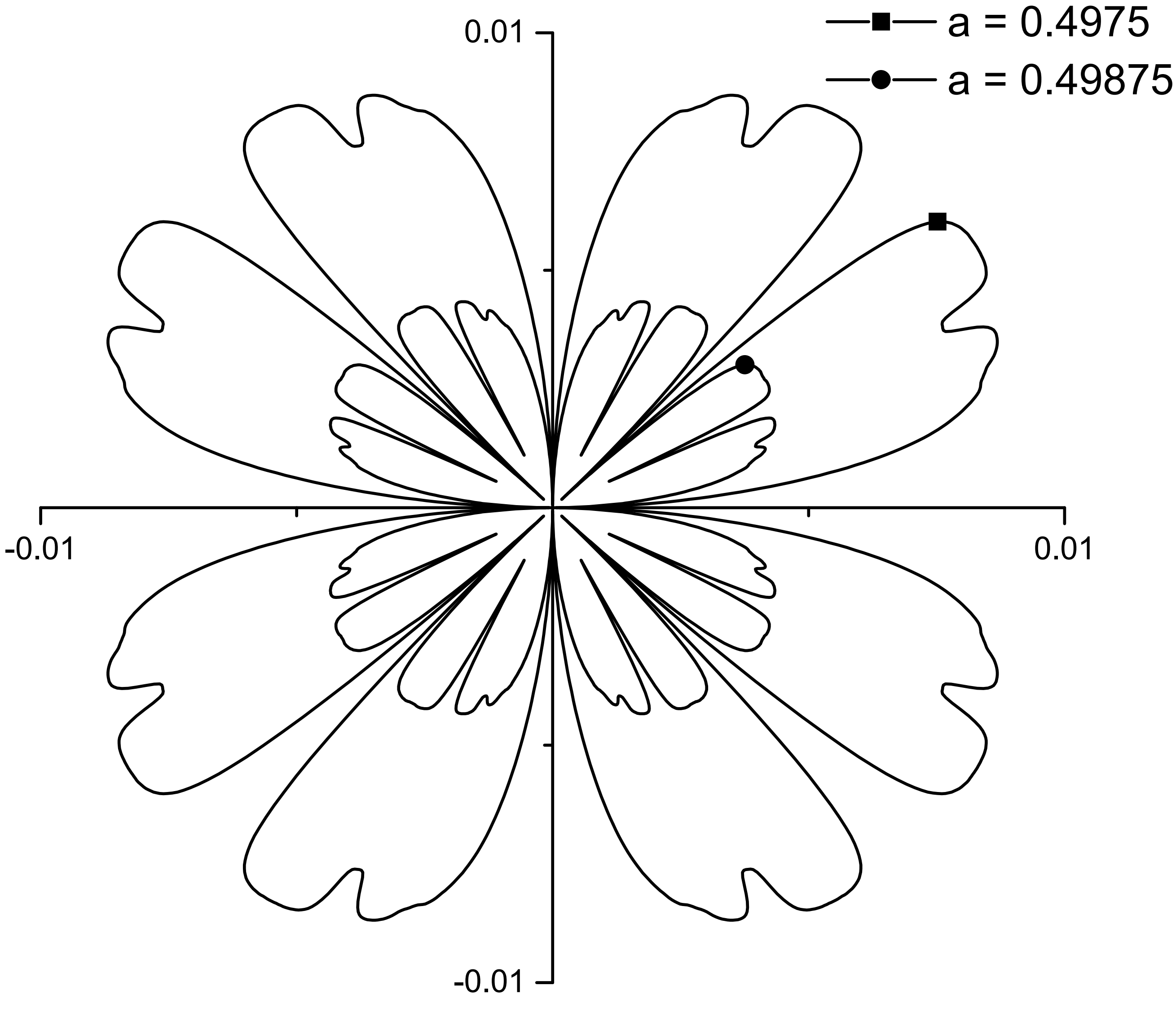
Behaviour as $a \to 0.5$ is interesting.

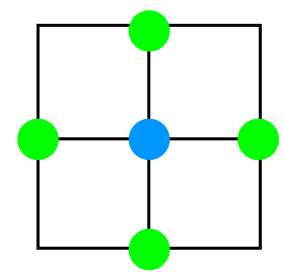




Lattice equations: Spatial anisotropy - IV

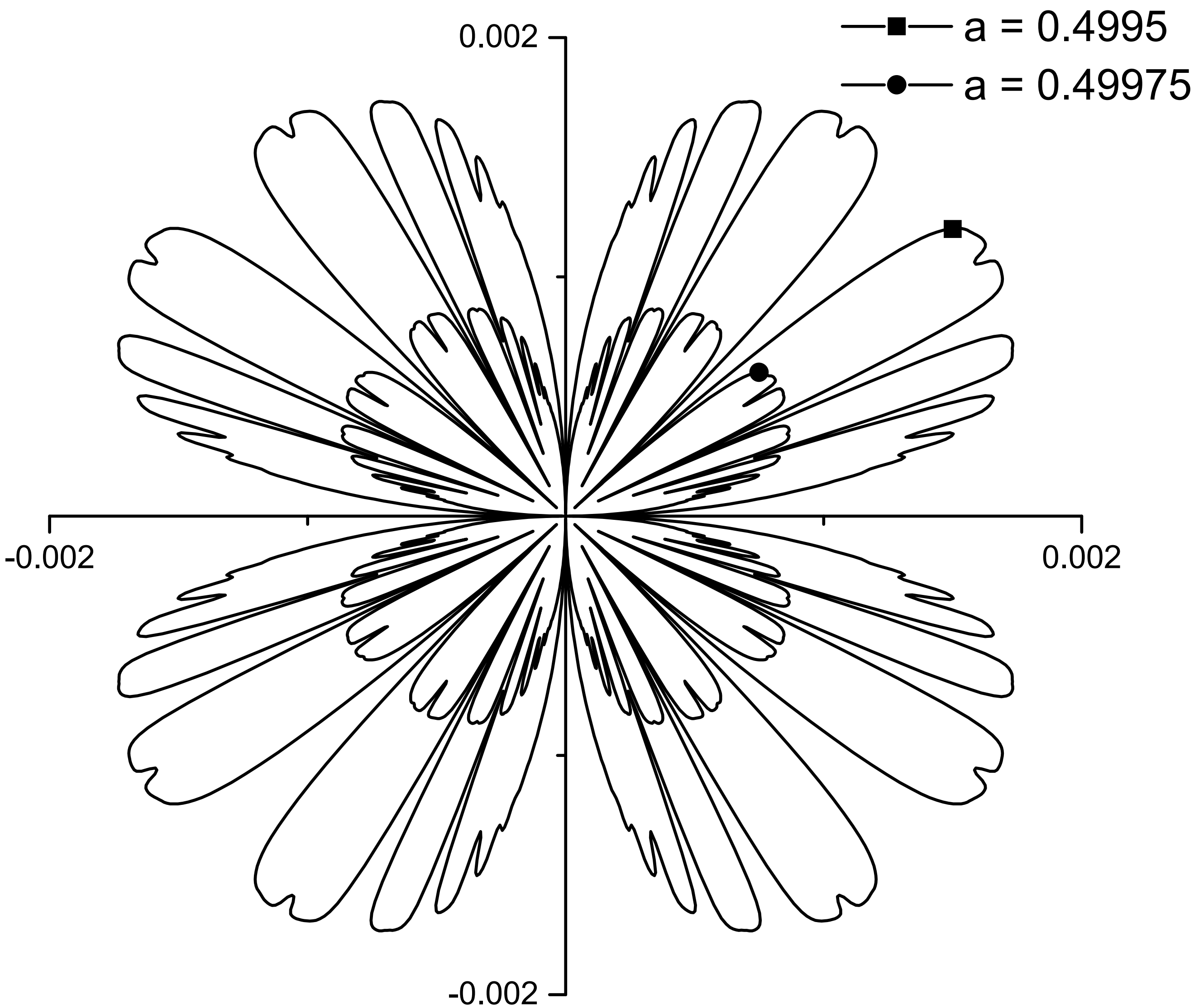
Behaviour as $a \to 0.5$ is interesting.

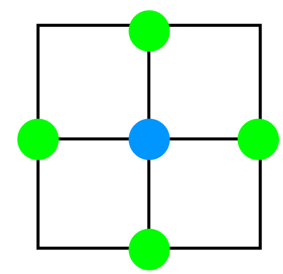




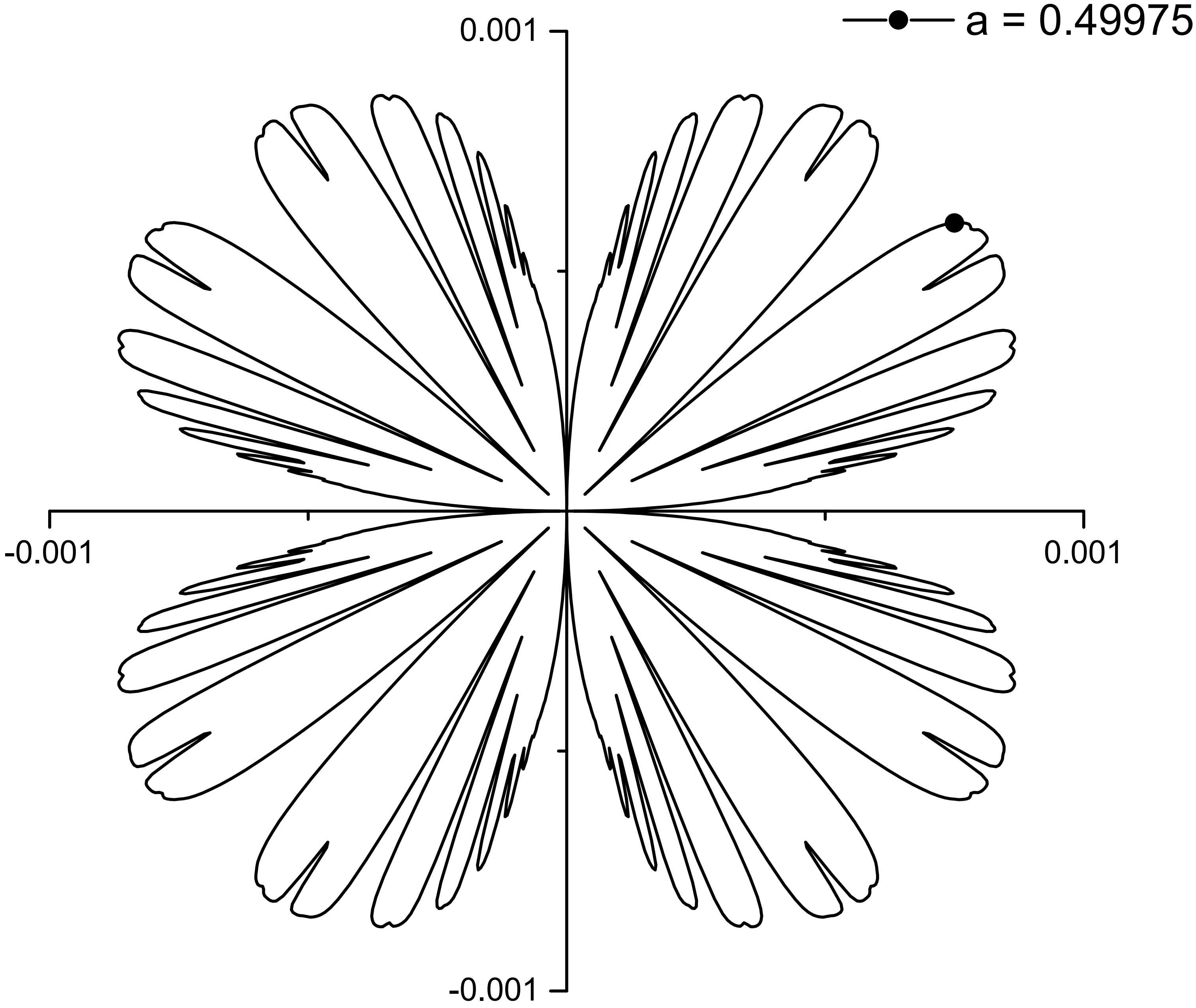
Lattice equations: Spatial anisotropy - V

Behaviour as $a \to 0.5$ is interesting.





Lattice equations: Spatial anisotropy - VII



Lattice equations: Spatial anisotropy - VI

Conjecture: Pinning is stronger in rational directions than irrational directions.

Conjecture: The more 'aligned' with lattice, the stronger the pinning is.

Partial results: [Cahn, Van Vleck, Mallet Paret, Hoffman, H.]

In this talk: we fix (a, θ) and **assume** that $c \neq 0$.

Goal: understand **stability** of the travelling wave.

Direction dependence?

PDE

Consider 2d PDE

$$u_t = u_{xx} + u_{yy} + g(u)$$

with travelling wave solution

$$u(x, y, t) = \Phi(x + ct).$$

For simplicity here: assume $c = 0$.

Wave profile satisfies:

$$0 = \Phi''(x) + g\big(\Phi(x)\big)$$

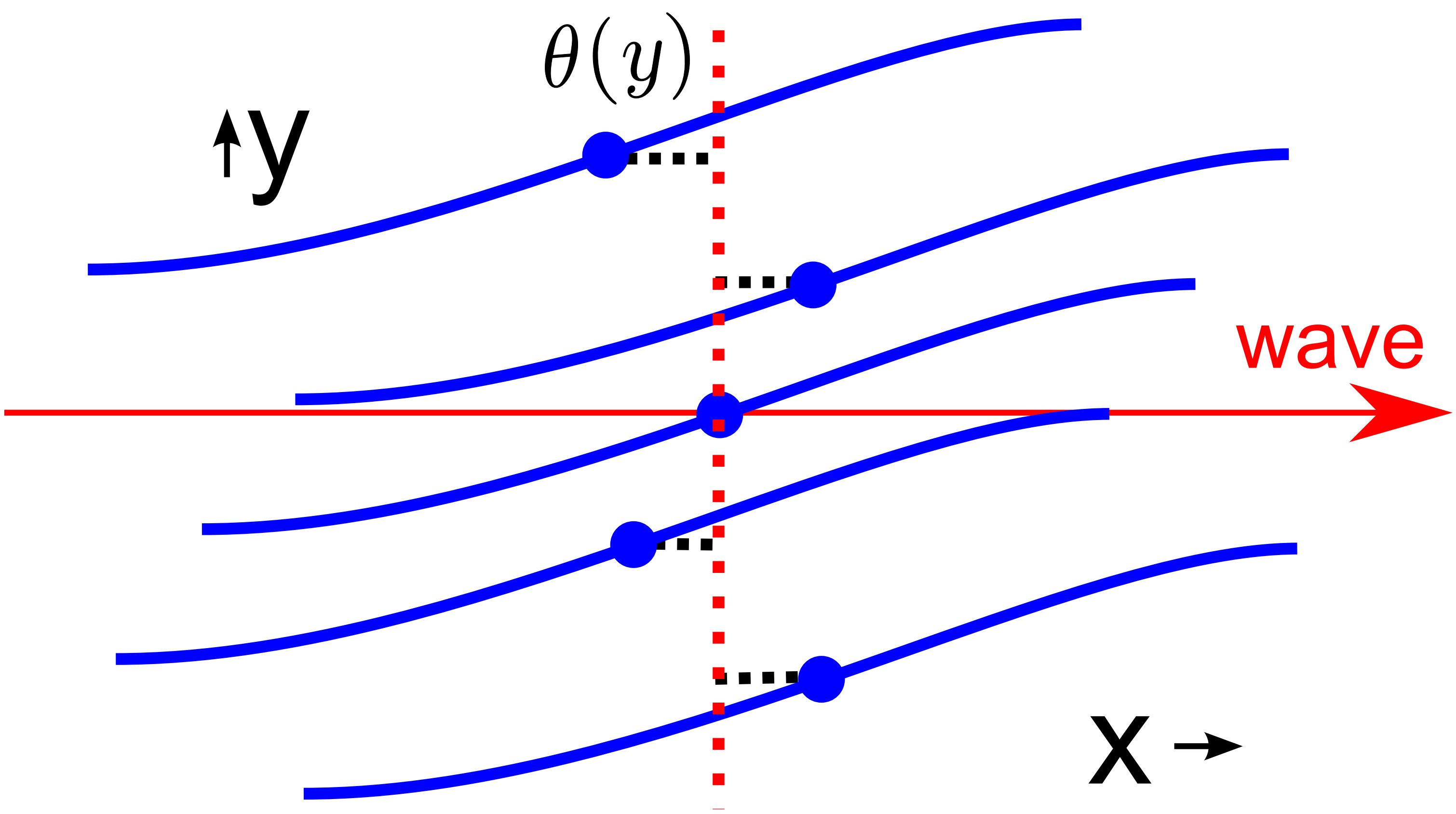
and we have stationary PDE solution:

$$u(x, y, t) = \Phi(x).$$

PDE

[Kapitula]: Study perturbations using Ansatz

$$u(t, x, y) = \Phi(x + \theta(t, y)) + v(t, x, y).$$



PDE

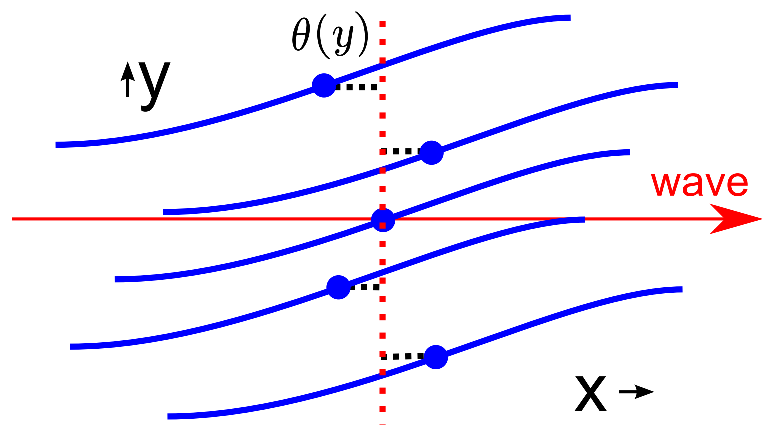
[Kapitula]: Study perturbations using Ansatz

$$u(t, x, y) = \Phi(x + \theta(t, y)) + v(t, x, y).$$

In order to separate out θ and v evolutions; need normalization:

$$\int_{-\infty}^{\infty} \Phi'(x) v(t, x, y)\, dx = 0, \qquad \text{for all } y \in \mathbb{R} \text{ and } t \geq 0.$$

Interpretation: v is orthogonal to perturbations caused by shift of profile.



PDE

Normalization decouples v and θ evolutions at linear level.

$$\begin{aligned} v_t &= v_{xx} + v_{yy} + Dg(\Phi(x))v + \mathcal{N}_v(v,\theta) \\ \theta_t &= \theta_{yy} + \mathcal{N}_\theta(v,\theta), \end{aligned}$$

with nonlinearities [Notice: no θ^2]:

$$\mathcal{N}_* = O(v^2 + \theta_y^2 + \theta v + \theta\theta_{yy}), \qquad * = \theta, v$$

Write solution as [Duhamel]

$$\begin{aligned} \begin{pmatrix} v(t) \\ \theta(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{G}_{vv}(t) & 0 \\ 0 & \mathcal{G}_{\theta\theta}(t) \end{pmatrix} \begin{pmatrix} v(0) \\ \theta(0) \end{pmatrix} \\ &\quad + \int_{s=0}^{t} \begin{pmatrix} \mathcal{G}_{vv}(t-s) & 0 \\ 0 & \mathcal{G}_{\theta\theta}(t-s) \end{pmatrix} \begin{pmatrix} \mathcal{N}_v\big(v(s),\theta(s)\big) \\ \mathcal{N}_\theta\big(v(s),\theta(s)\big) \end{pmatrix} ds. \end{aligned}$$

PDE

Recall Duhamel expression:

$$\begin{pmatrix} v(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \mathcal{G}_{vv}(t) & 0 \\ 0 & \mathcal{G}_{\theta\theta}(t) \end{pmatrix} \begin{pmatrix} v(0) \\ \theta(0) \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} \mathcal{G}_{vv}(t-s) & 0 \\ 0 & \mathcal{G}_{\theta\theta}(t-s) \end{pmatrix} \begin{pmatrix} \mathcal{N}_v\big(v(s), \theta(s)\big) \\ \mathcal{N}_\theta\big(v(s), \theta(s)\big) \end{pmatrix} ds.$$

Here $\mathcal{G}_{vv}(t)v_0$ solution to

$$v_t(t,x,y) = v_{xx}(t,x,y) + v_{yy}(t,x,y) + Dg(\Phi(x))v(t,x,y), \qquad v(0,x,y) = v_0(x,y)$$

while $\mathcal{G}_{\theta\theta}(t)\theta_0$ solution to

$$\theta_t(t,y) = \theta_{yy}(t,y), \qquad \theta(0,y) = \theta_0(y).$$

PDE

Recall $\mathcal{G}_{vv}(t)v_0$ solution to

$$v_t(t,x,y) = v_{xx}(t,x,y) + v_{yy}(t,x,y) + Dg(\Phi(x))v(t,x,y), \qquad v(0,x,y) = v_0(x,y).$$

Fourier transform in y-direction:

$$\partial_t \widehat{v}(t,x,\omega) = \underbrace{\partial_{xx}\widehat{v}(t,x,\omega) + Dg(\Phi(x))\widehat{v}(t,x,\omega)}_{\text{Linearization around 1d wave}} \quad \underbrace{-\omega^2\widehat{v}(t,x,\omega)}_{\text{Nice rigid shift in spectrum}} \quad .$$

Normalization condition ensures $\widehat{v}_0(x,\omega)$ in exp decaying subspace for all frequencies ω.

$$\|\mathcal{G}_{vv}(t)v_0\| \sim e^{-\eta t}\,\|v_0\|\,.$$

[Norm deliberately suppressed - think L^2-summability in y-direction.]

PDE

Recall $\mathcal{G}_{\theta\theta}(t)\theta_0$ solution to

$$\theta_t(t,y) = \theta_{yy}(t,y), \qquad \theta(0,y) = \theta_0(y).$$

Heat equation; so

$$\|\mathcal{G}_{\theta\theta}(t)\theta_0\|_{L^2} \sim t^{-1/4}\,\|\theta_0\|_{L^1}\,.$$

Derivatives get more decay:

$$\begin{aligned}\|\partial_y \mathcal{G}_{\theta\theta}(t)\theta_0\|_{L^2} &\sim t^{-3/4}\,\|\theta_0\|_{L^1}\\ \|\partial_{yy} \mathcal{G}_{\theta\theta}(t)\theta_0\|_{L^2} &\sim t^{-5/4}\,\|\theta_0\|_{L^1}\end{aligned}$$

PDE

Nonlinear terms

$$\mathcal{N}_*(v,t) = O(v^2 + \theta_y^2 + \theta v + \theta\theta_{yy})$$

Slowest expected decay comes from θ_y^2 and $\theta\theta_{yy}$ terms, both giving $t^{-3/4}t^{-3/4} = t^{-3/2}$ and $t^{-1/4}t^{-5/4} = t^{-3/2}$ decay.

Recall Duhamel expression:

$$\begin{pmatrix} v(t) \\ \theta(t) \end{pmatrix} \sim \begin{pmatrix} e^{-\eta t} & 0 \\ 0 & t^{-1/4} \end{pmatrix} \begin{pmatrix} v(0) \\ \theta(0) \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} e^{-\eta(t-s)} & 0 \\ 0 & (t-s)^{-1/4} \end{pmatrix} \begin{pmatrix} s^{-3/2} \\ s^{-3/2} \end{pmatrix} ds.$$

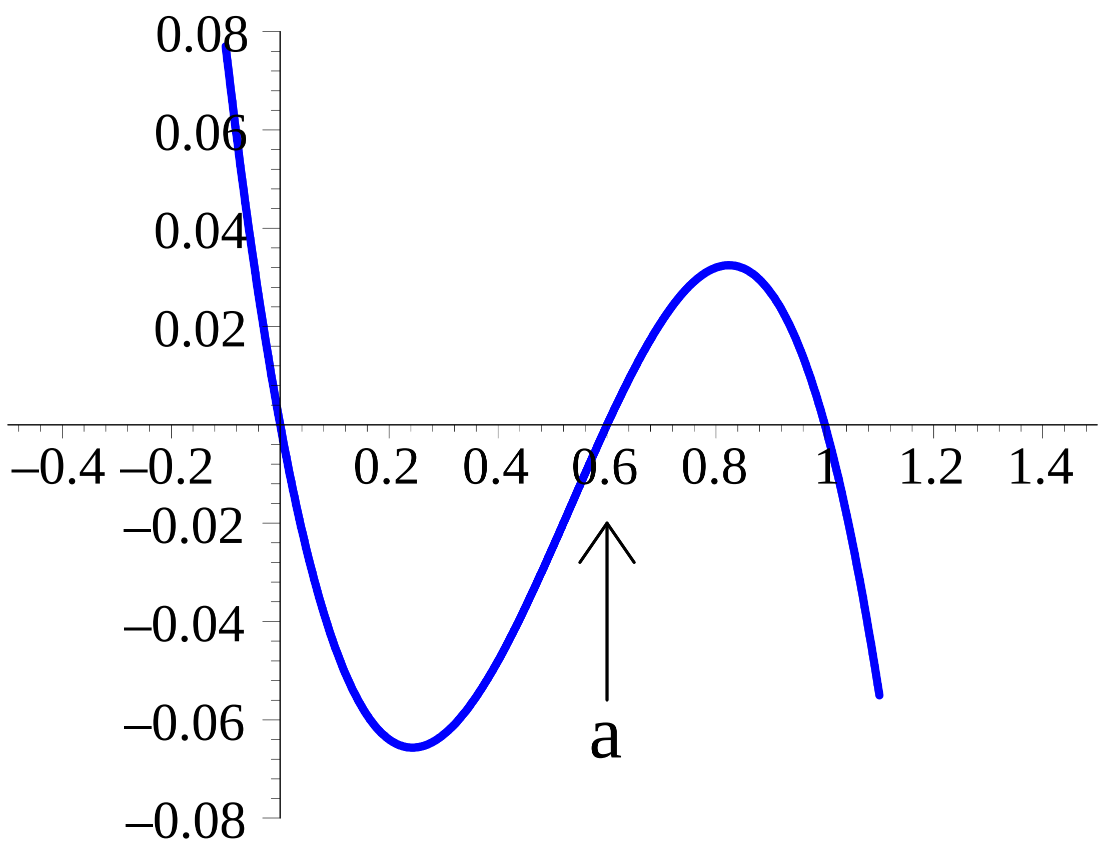
Self consistent since

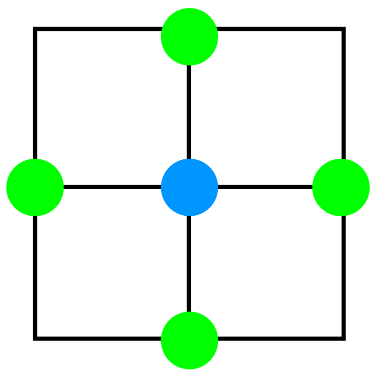
$$\int_{s=1}^{t} (t-s)^{-1/4} s^{-3/2}\, ds \sim t^{-1/4}.$$

2d Lattice Differential Equation

Back to the 2d LDE

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^{+}u(t)]_{i,j} + g(u_{i,j}(t);a).$$



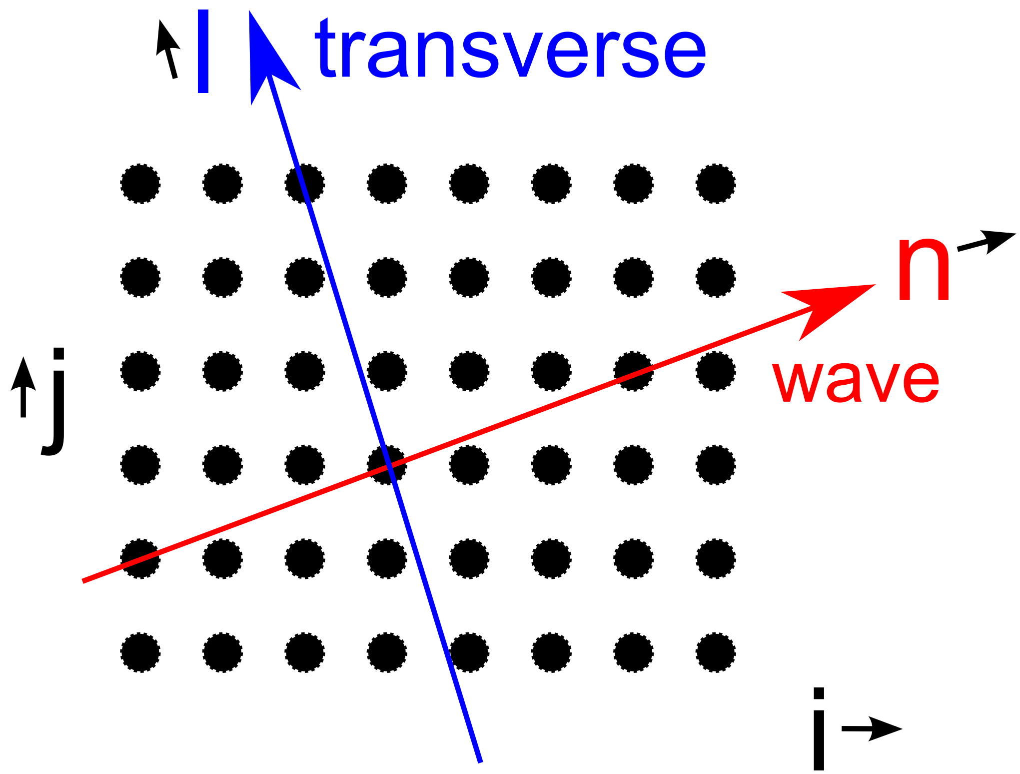


2d Lattice Differential Equation

Back to the 2d LDE (fix a from now on)

$$\dot{u}_{i,j}(t) \;=\; [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t)).$$

Assumption: we have a wave solution (c, Φ) travelling $(c \neq 0)$ in **rational** direction $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$.



New coordinates:

$$\begin{aligned} n &= i\sigma_1 + j\sigma_2 && \text{parallel} \\ l &= i\sigma_2 - j\sigma_1 && \text{transverse.} \end{aligned}$$

Old coordinates:

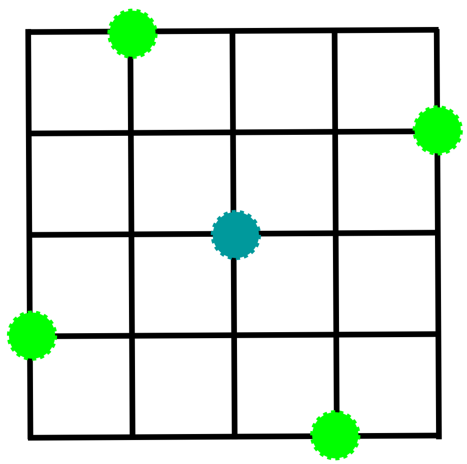
$$\begin{aligned} i &= [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_1 + l\sigma_2] \\ j &= [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_2 - l\sigma_1] \end{aligned}$$

Stability - Coordinate System

In new coordinates, LDE becomes

$$\dot{u}_{nl}(t) = [\Delta^{\times} u(t)]_{nl} + g\big(u_{nl}(t)\big).$$

The discrete operator $\Delta^{\times}$ now acts as



$$\begin{aligned}[\Delta^{\times} u]_{n,l} \quad = \quad & u_{n+\sigma_1,l+\sigma_2} + u_{n+\sigma_2,l-\sigma_1} \\ & +u_{n-\sigma_1,l-\sigma_2} + u_{n-\sigma_2,l+\sigma_1} \\ & -4u_{n,l}.\end{aligned}$$

All geometrical information encoded in $\Delta^{\times}$.

Travelling wave becomes: $u_{nl}(t) = \Phi(n + ct)$

Special cases $(\sigma_1, \sigma_2) = (1, 0)$ or $(0, 1)$ (horizontal or vertical waves): $\Delta^{\times} = \Delta^{+}$.

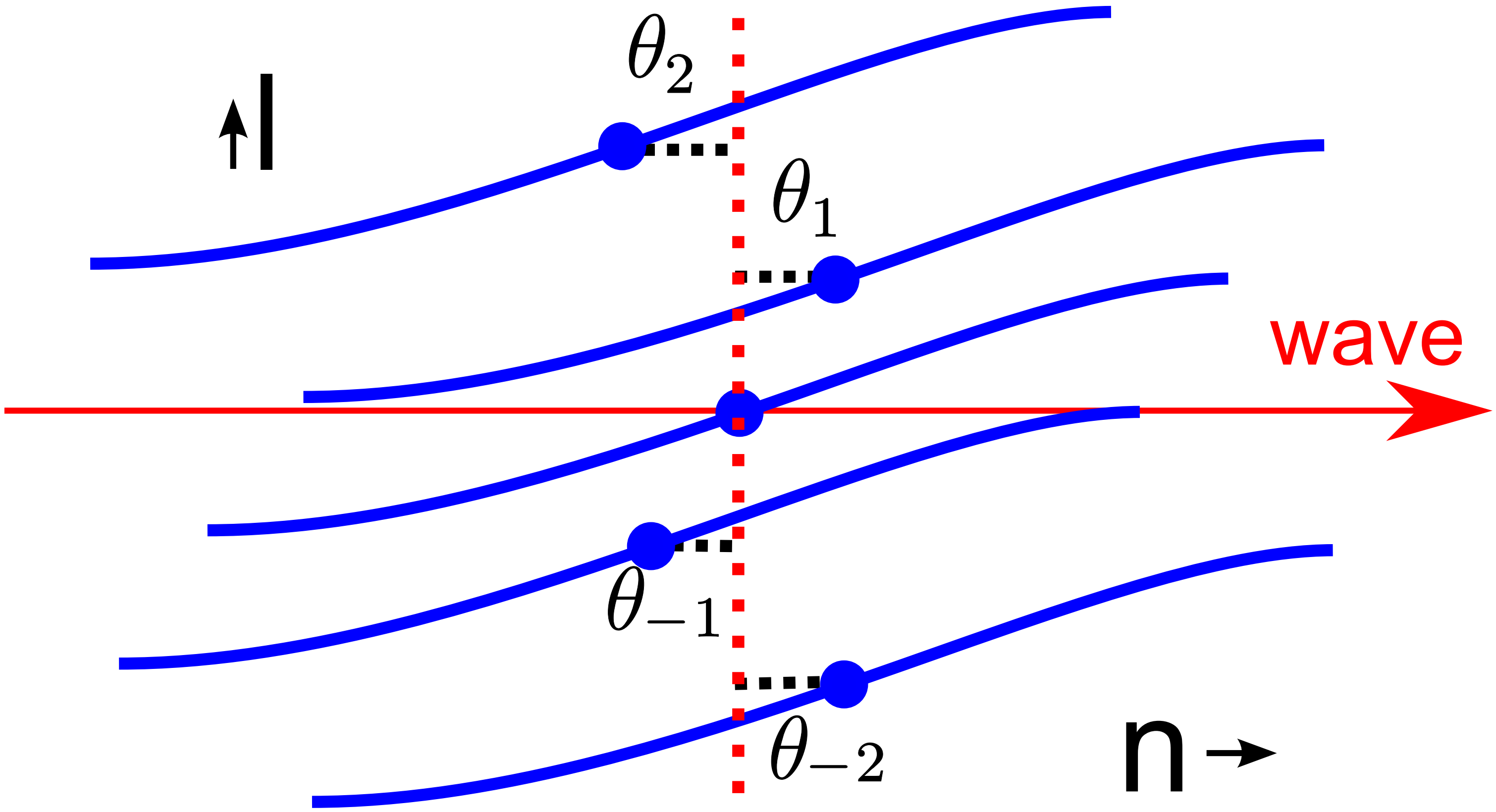
Stability - Refined Ansatz

Refined perturbation Ansatz

$$u_{nl}(t) = \Phi\big(n + ct + \theta_l(t)\big) + v_{nl}(t).$$

Here $\theta_l(t)$ measures deformation of wave profile (expect slow decay).

Remainder included in $v(t)$ (expect faster decay).



Stability - Linear System

Focus on linear LDE posed on $\mathbb{Z}^2$:

$$\dot{v}_{nl}(t) = [\Delta^{\times} v(t)]_{nl} + Dg\big(\Phi(n + ct)\big)v_{nl}(t).$$

As before: transverse coordinate l does **not** appear in coefficients.

Ideal for Fourier transform in transverse direction.

System is decoupled into

$$\frac{d}{dt}\widehat{v}_n(\omega, t) = [\widehat{\Delta}^{\times}(\omega)\widehat{v}(\omega, t)]_n + Dg(\Phi(n + ct))\widehat{v}_n(\omega, t),$$

with

$$[\widehat{\Delta}^{\times}(\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.$$

In other words, for each frequency ω we have an LDE posed on a 1d lattice (in parallel coordinate n).

Frequency dependence is horrible!

LDE - Duhamel

Duhamel formula now becomes

$$\begin{pmatrix} v(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \mathcal{G}_{vv}(t) & \mathcal{G}_{v\theta}(t) \\ \mathcal{G}_{\theta v}(t) & \mathcal{G}_{\theta\theta}(t) \end{pmatrix} \begin{pmatrix} v(0) \\ \theta(0) \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} \mathcal{G}_{vv}(t-s) & \mathcal{G}_{v\theta}(t-s) \\ \mathcal{G}_{\theta v}(t-s) & \mathcal{G}_{\theta\theta}(t-s) \end{pmatrix} \begin{pmatrix} \mathcal{N}_v\big(v(s),\theta(s)\big) \\ \mathcal{N}_\theta\big(v(s),\theta(s)\big) \end{pmatrix} ds.$$

Now with:

$$\mathcal{N}_*(v,\theta) = O(\theta v + \theta\theta^\diamond) + h.o.t.$$

where $[\theta^\diamond]_l \sim \theta_{l+1} - \theta_l$ denotes a discrete spatial derivative. Think:

$$\begin{pmatrix} \mathcal{G}_{vv}(t) & \mathcal{G}_{v\theta}(t) \\ \mathcal{G}_{\theta v}(t) & \mathcal{G}_{\theta\theta}(t) \end{pmatrix} \sim \begin{pmatrix} t^{-5/4} & t^{-3/4} \\ t^{-3/4} & t^{-1/4} \end{pmatrix}, \qquad \mathcal{N}_*(v(t),\theta(t)) \sim t^{-1}.$$

We lose everything that is nice!

$$\int_1^t (t-s)^{-1/4} s^{-1}\, ds \sim \ln(t) t^{-1/4}$$

Stability in 2d

Recall Ansatz

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t).$$

Thm. [H., Hoffman, Van Vleck, 2012] Travelling wave ($c \neq 0$) in any rational direction is nonlinearly stable under small perturbations

$$\begin{array}{lcl} \sum_{l \in \mathbb{Z}} |\theta_l(0)| & \ll & 1 \\ \sup_{n \in \mathbb{Z}} [\sum_{l \in \mathbb{Z}} |v_{nl}(0)|] & \ll & 1. \end{array}$$

Note: perturbations need to be summable in transverse direction.

We have $\theta_l(t) \to 0$ and $v_{nl}(t) \to 0$ as $t \to \infty$.

In other words, deformations of interface diffuse in transverse direction.

It does NOT lead to a shift in the wave.

Stability in 2d

Recall Ansatz

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t).$$

Algebraic decay rates depend on direction of propagation!

Horizontal waves [Norm is ℓ^∞ parallel to wave, ℓ^2 transverse to wave]

$$\theta(t) \sim t^{-1/4}, \qquad v(t) \sim t^{-3/2}.$$

Diagonal waves

$$\theta(t) \sim t^{-1/4}, \qquad v(t) \sim t^{-5/4}.$$

Other rational directions: (very slow decay - delicate nonlinear analysis needed)

$$\theta(t) \sim t^{-1/4}, \qquad v(t) \sim t^{-3/4}.$$

Sketch of Proof

The **actual** Ansatz that we use is:

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + \big(\theta_{l+1}(t) - \theta_l(t)\big)p(n + ct) + w_{nl}(t),$$

with $p : \mathbb{R} \to \mathbb{R}$ a function related to

$$[\partial_\omega \Phi_\omega]_{\omega=0},$$

where $\omega \mapsto \Phi_\omega$ is the branch of eigenfunctions

$$\mathcal{L}_\omega \Phi_\omega = \lambda_\omega \Phi_\omega; \qquad \Phi_{\omega=0} = \Phi', \qquad \lambda_{\omega=0} = 0,$$

with

$$[\Lambda_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi),$$

i.e. the linearization related to Fourier frequency ω.

Sketch of Proof

Recall **actual** Ansatz:

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + \big(\theta_{l+1}(t) - \theta_l(t)\big)p(n + ct) + w_{nl}(t).$$

Explicitly need to understand dangerous nonlinear terms

$$\theta_l(t)(\theta_{l+1}(t) - \theta_l(t)) \sim t^{-1}.$$

Key trick: $\theta_l(\theta_{l+1} - \theta_l) = \frac{1}{2}\big(\underbrace{\theta_{l+1}^2 - \theta_l^2}_{t^{-1/2}} - \underbrace{(\theta_{l+1} - \theta_l)^2}_{t^{-3/2}}\big).$

This is discrete version of conservation law trick:

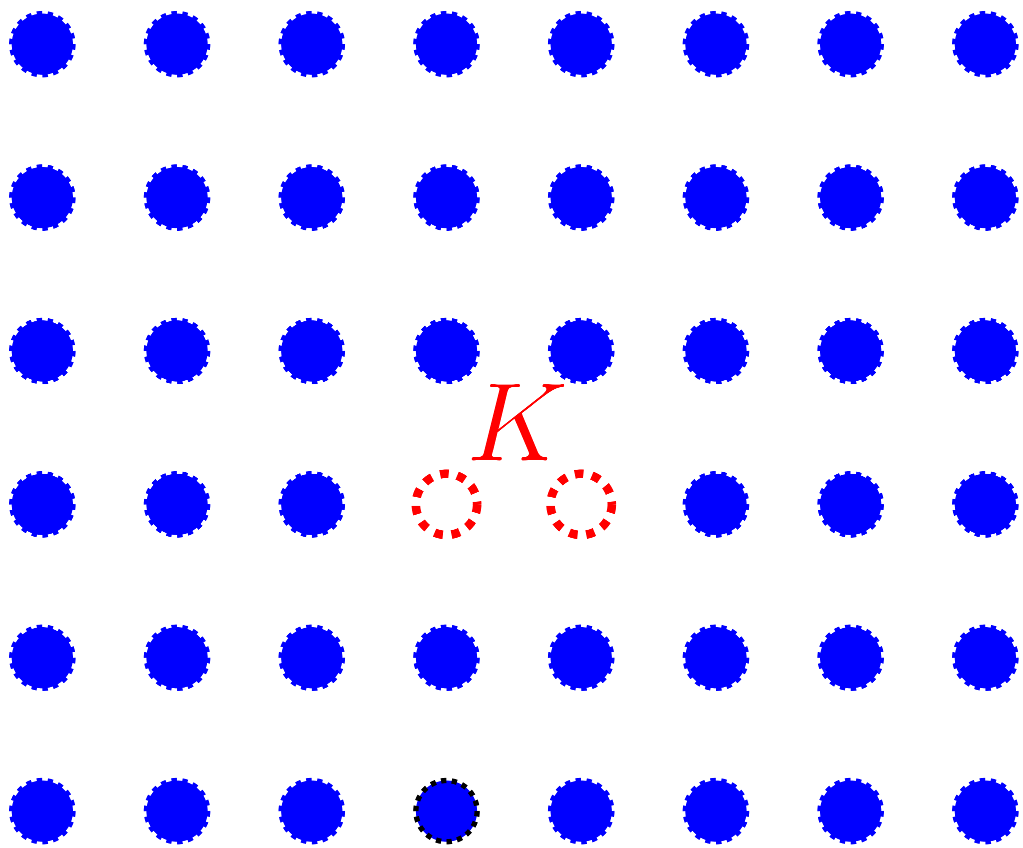
$$uu_x = \frac{1}{2}(u^2)_x,$$

$$\int_0^t (1 + t - t_0)^{-1/4}(1 + t_0)^{-1}\, dt_0 \sim \ln(1 + t)(1 + t)^{-1/4} \quad BAD$$
$$\int_0^t (1 + t - t_0)^{-3/4}(1 + t_0)^{-1/2}\, dt_0 \sim (1 + t)^{-1/4} \quad GOOD.$$

Obstacles

- Philosophy: choosing lattice directions breaks isotropy $\mathbb{R}^2$.
- Now break resulting discrete symmetry [remove set K].

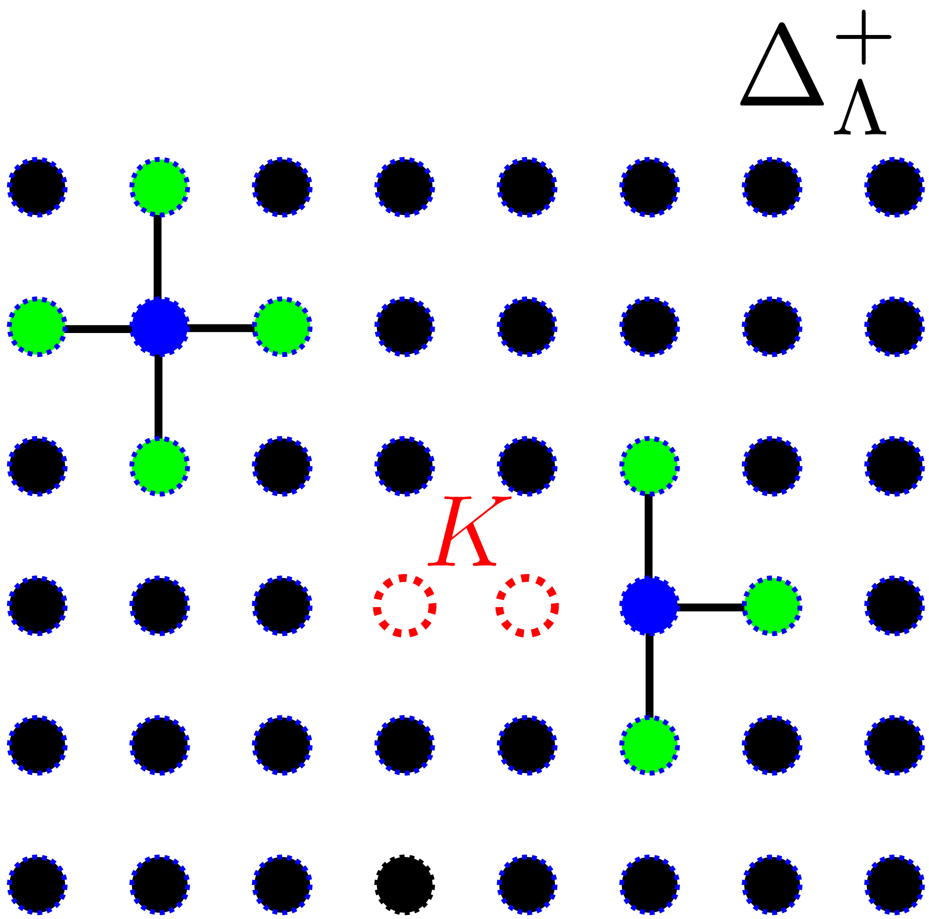
$$\Lambda = \mathbb{Z}^2 \setminus K$$



Obstacles

Punctured discrete Laplacian [think Neumann boundary conditions]

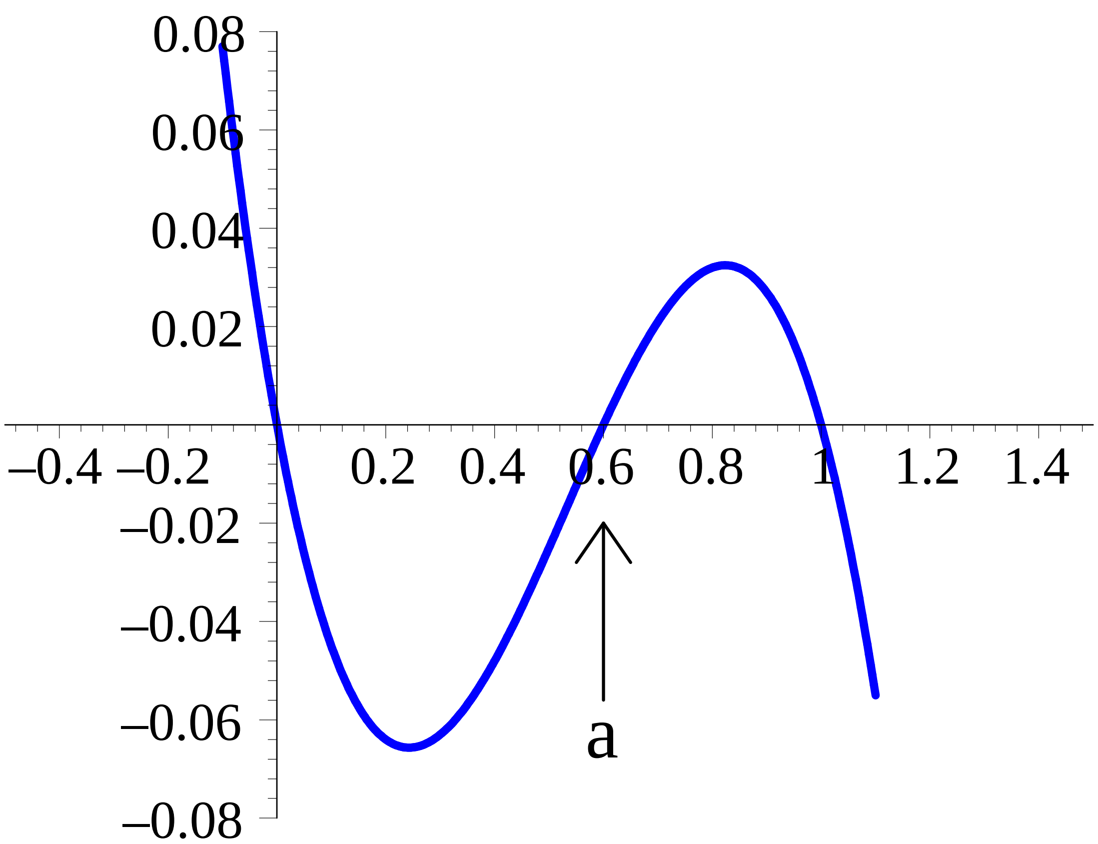
$$[\Delta_\Lambda^+ u]_{ij} = \sum_{|i-i'|+|j-j'|=1} (u_{i'j'} - u_{ij})\mathbf{1}_{(i',j')\in\Lambda}$$

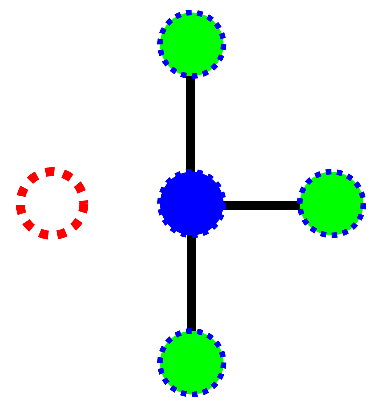


Obstacles

Now consider LDE for $(i,j) \in \Lambda$:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta_\Lambda^+ u(t)]_{i,j} + g(u_{i,j}(t);a).$$





Obstacles

Recall LDE

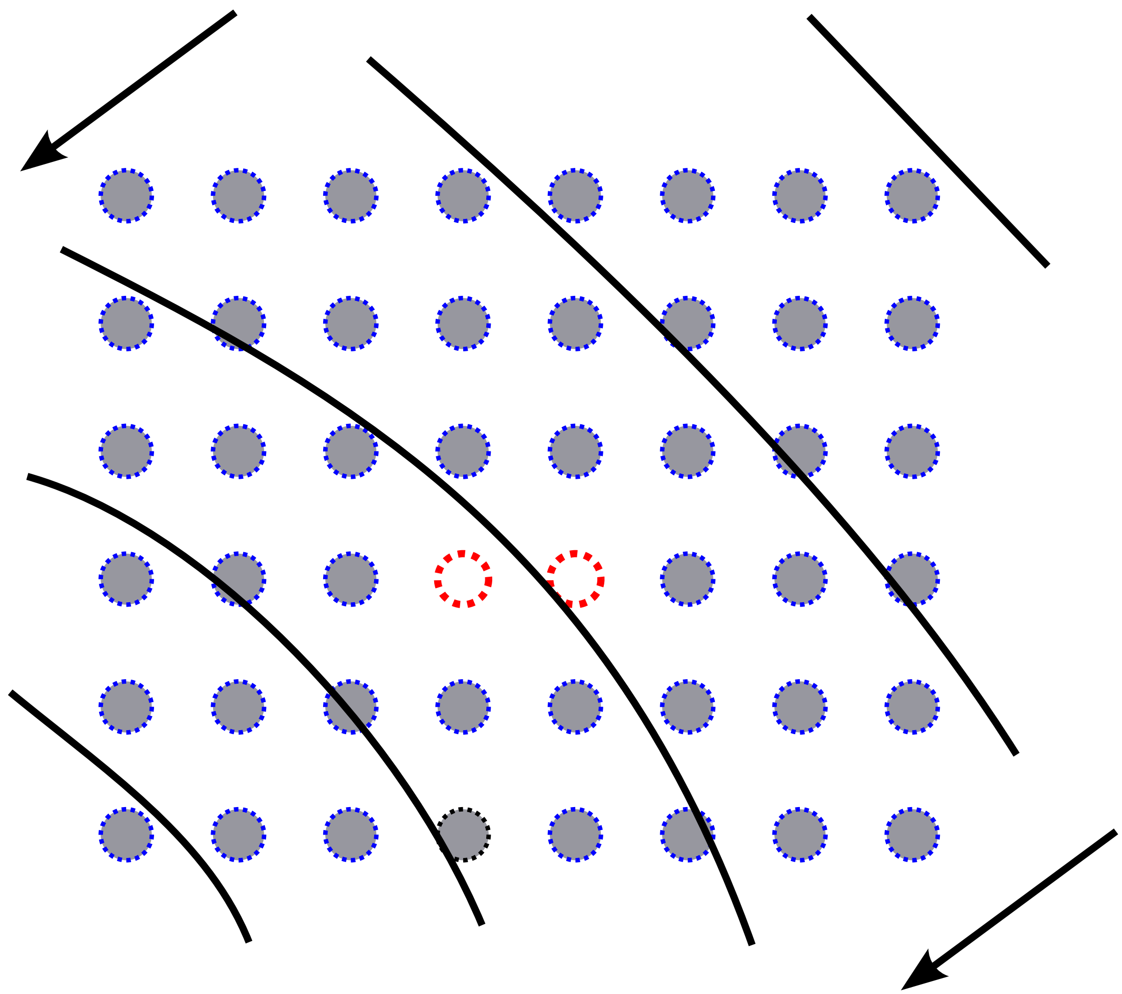
$$\dot{u}_{i,j}(t) \quad = \quad [\Delta_\Lambda^+ u(t)]_{i,j} + g(u_{i,j}(t); a), \qquad (i,j) \in \Lambda.$$

Main questions:

- How are planar fronts affected?
- Will $u = 1$ still invade the domain?
- Geometry of obstacle K?

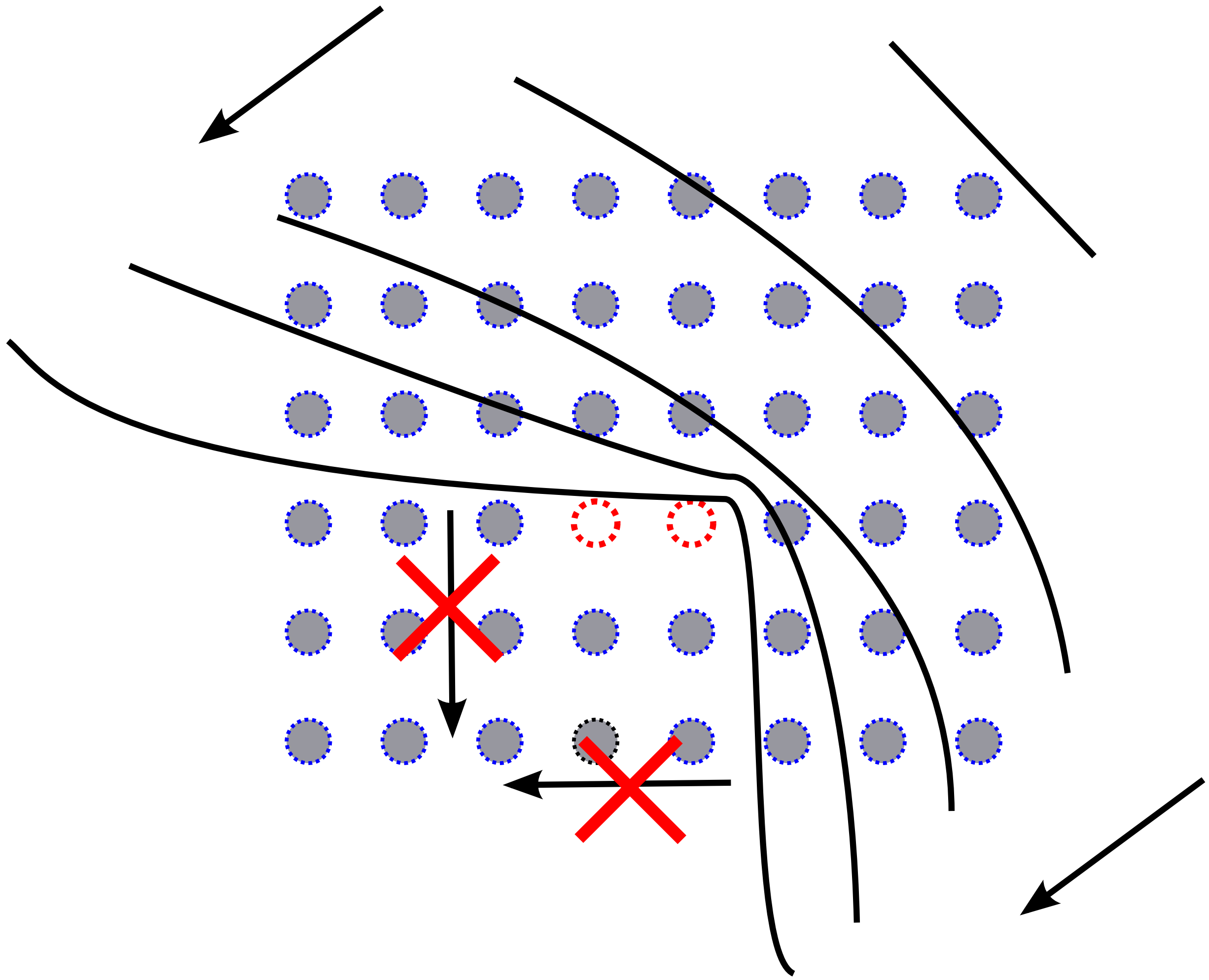
Obstacles

On the horizon, wave will propagate 'as normal'. Sufficient to pull level curves through obstacle?



Obstacles

What if propagation is blocked in vertical and horizontal directions [but not in diagonal]? Potential scenario:



Obstacles

Recall LDE

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta_{\Lambda}^{+} u(t)]_{i,j} + g(u_{i,j}(t); a), \qquad (i,j) \in \Lambda.$$

Thm. [H., Hoffman, Van Vleck, 2013]

- Suppose obstacle K is finite and 'convex' [E.g. K single point]
- Suppose $c(\theta) > 0$ for all $\theta \in [0, 2\pi]$ [All directions: no pinning]

Consider any **rational** direction $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$ and write (Φ, c) for wave in this direction.

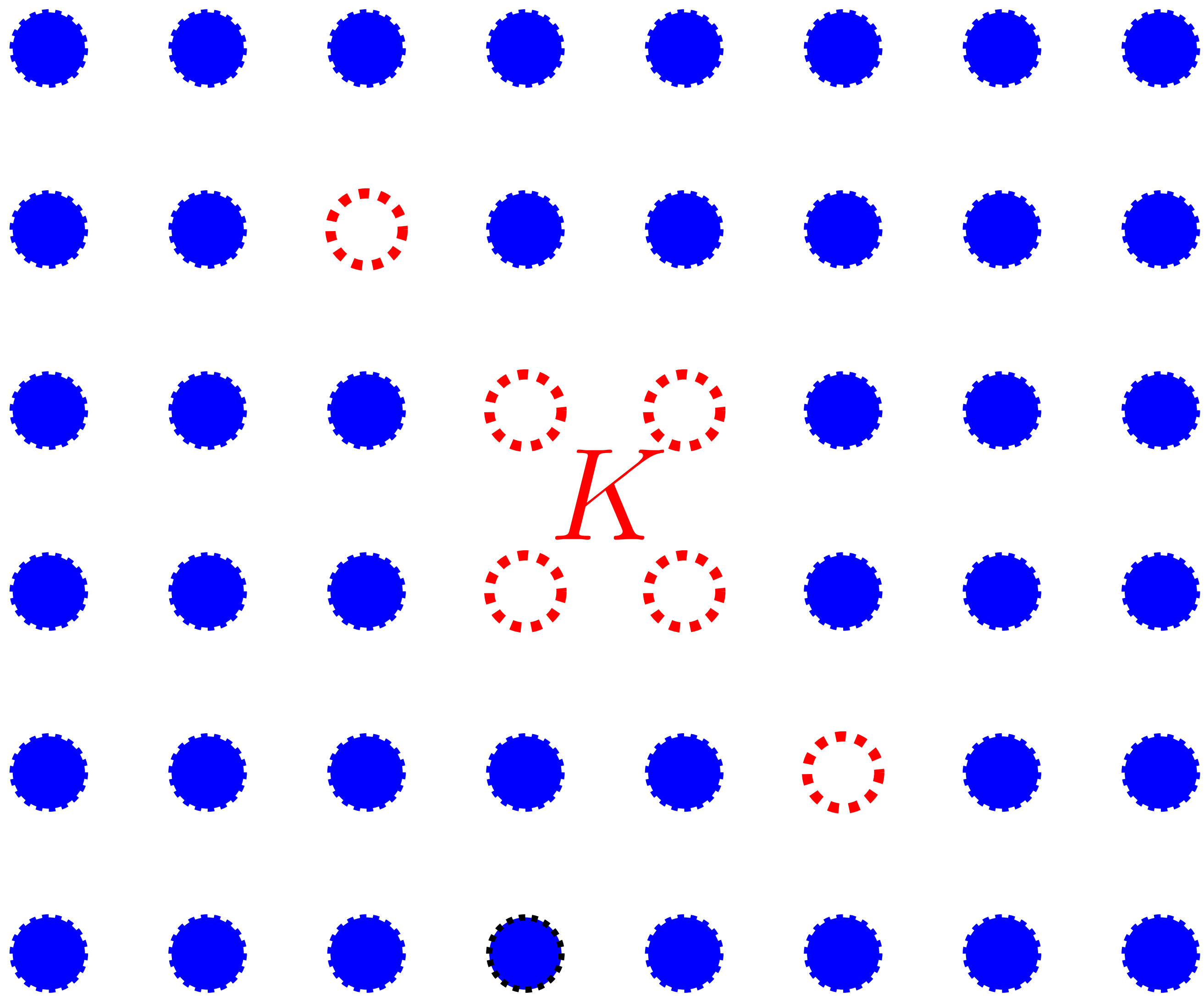
Then there is a unique entire solution u with

$$\lim_{|t| \to \infty} \sup_{(i,j) \in \Lambda} [u_{ij}(t) - \Phi(i\sigma_1 + j\sigma_2 + ct)] = 0.$$

[Distortions due to obstacle die out]

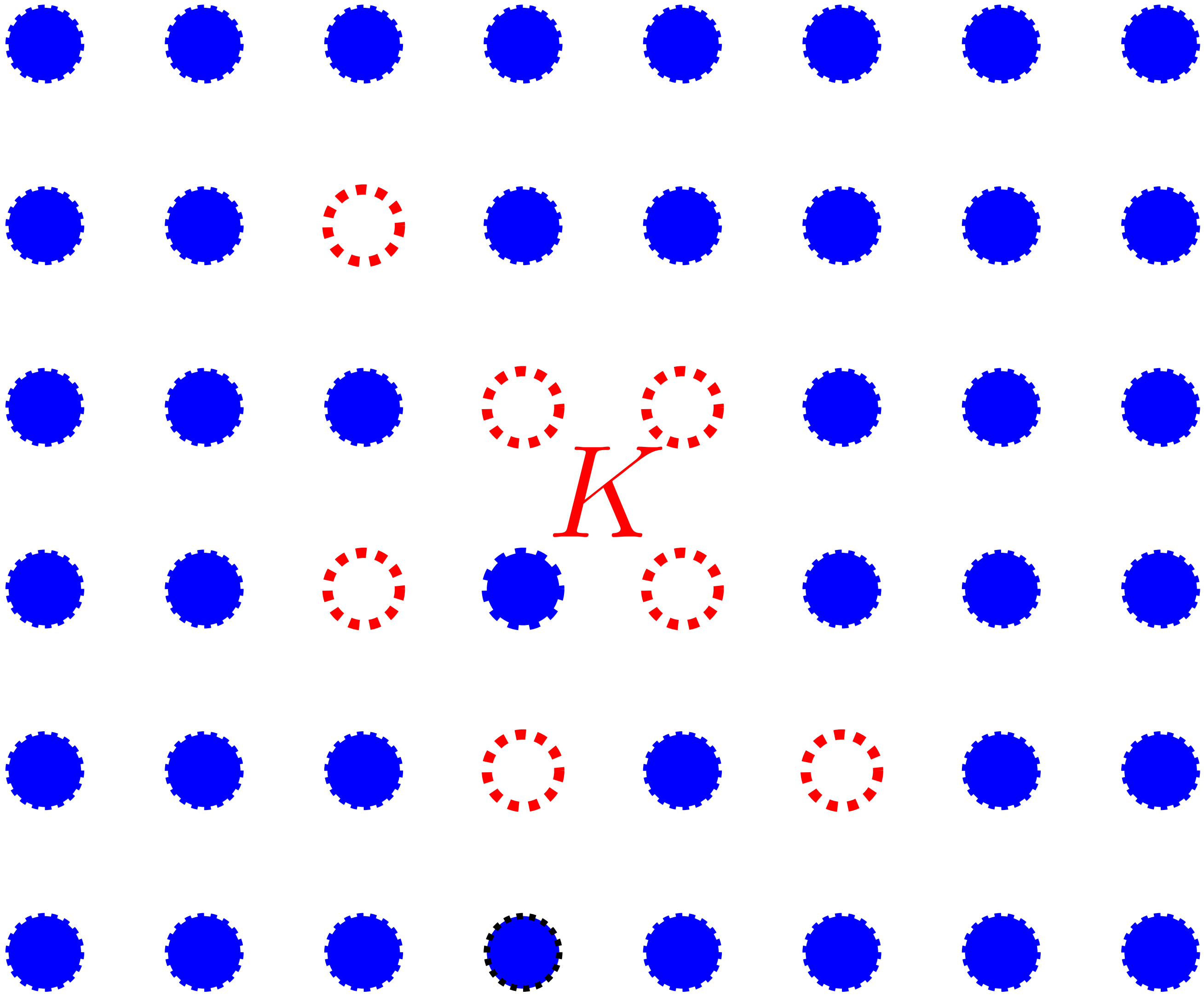
Admitted Obstacles

Covered by Thm:



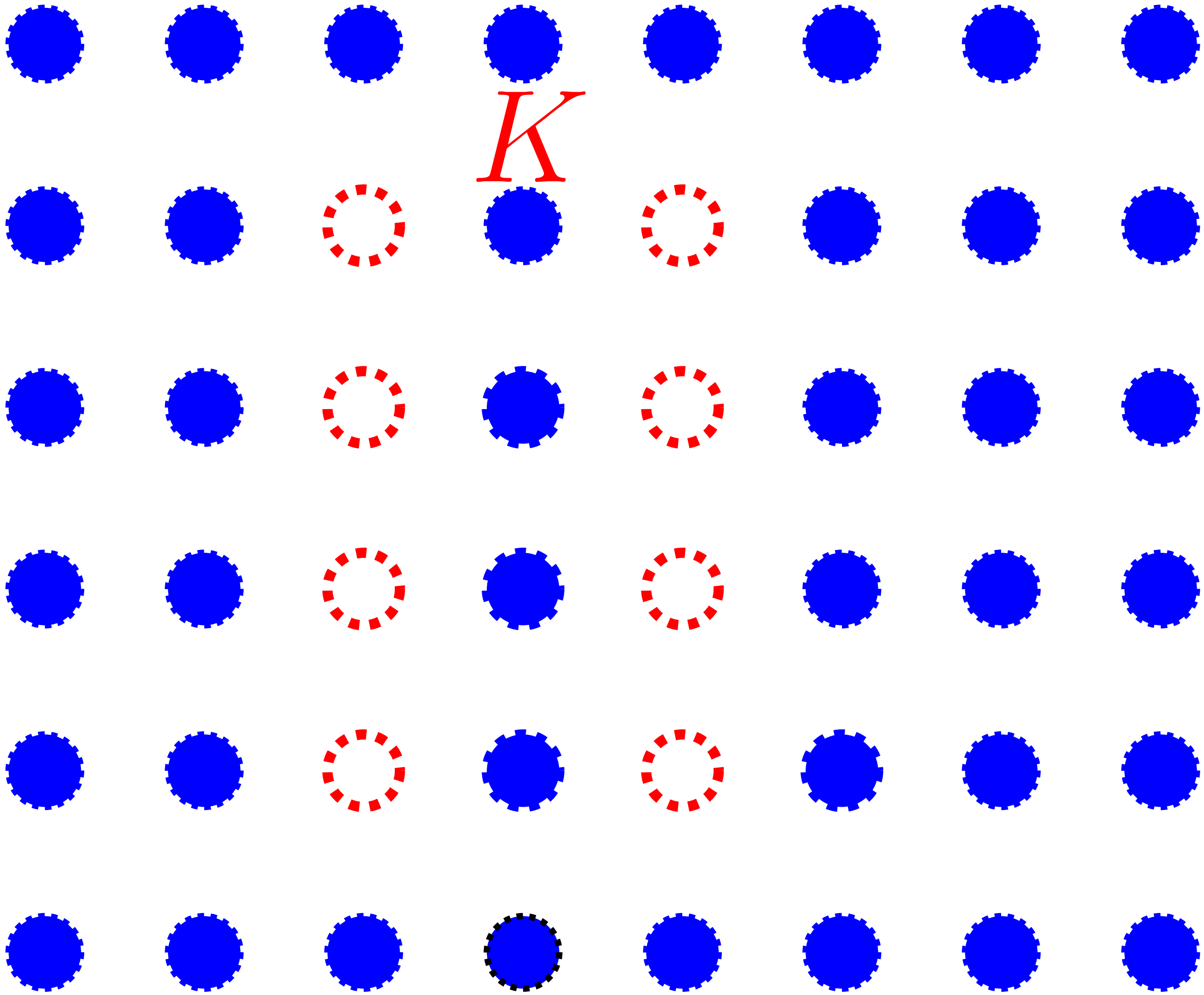
Admitted Obstacles

Not covered by Thm:



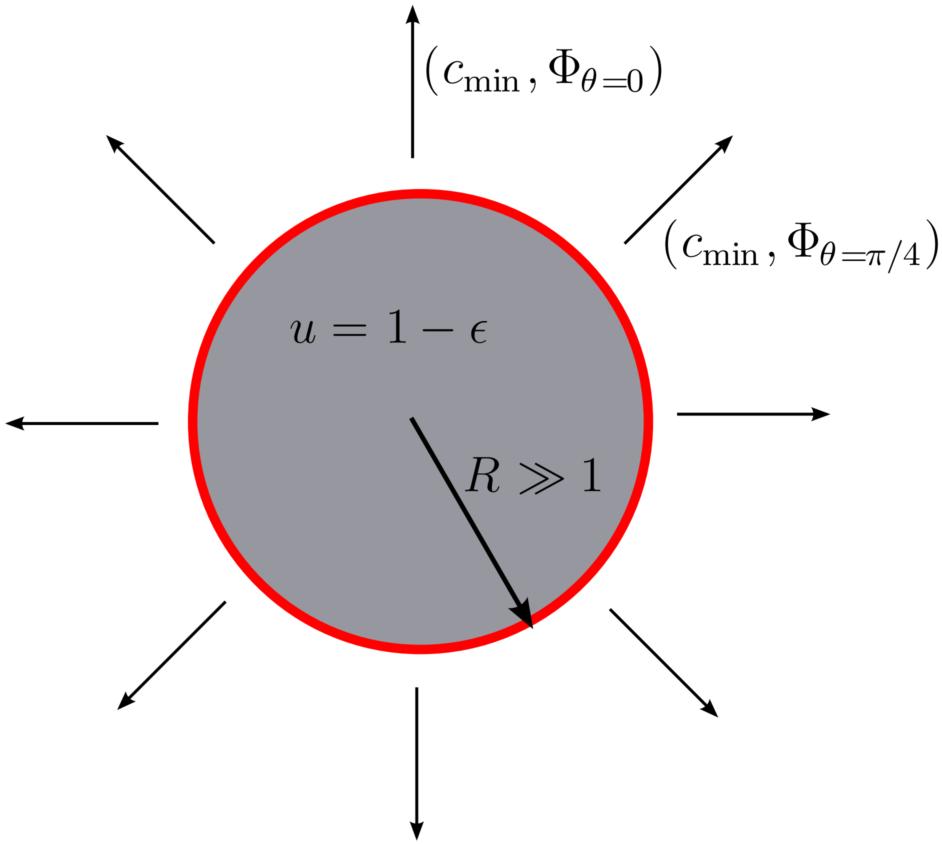
Admitted Obstacles

Not covered by Thm:



Ingredients - 1

On unobstructed lattice, large blobs where $u \sim 1$ will expand.



Prop. Suppose $c(\theta) > 0$ for all $\theta \in [0, 2\pi]$. Then for $\epsilon > 0$ there is $R \gg 1$ such that

$$1 - \epsilon < u_{ij}(0) \leq 1, \qquad i^2 + j^2 \leq R^2,$$

together with

$$0 \leq u_{ij}(0) \leq 1, \qquad (i, j) \in \mathbb{Z}^2$$

implies

$$1 - \epsilon < u_{ij}(t) \leq 1$$

whenever $i^2 + j^2 \leq (R + \frac{1}{2} c_{\min} t)^2$.

[Mechanism for waves to 'flow around' obstacle.]

Ingredients - 2

Must construct subsolutions to deal with **large** distortions post-obstacle.

PDE case: [Berestycki, Hamel, Matano (2009)]

$$u^{-}(x,y,t) = \Phi\big(x + ct - \theta(y,t) - Z(t)\big) - z(t),$$

with

$$\begin{array}{lcllll} \theta(y,t) & = & \beta t^{-\alpha} \exp[-\frac{y^2}{\gamma t}], & \beta \gg 1, & \gamma \gg 1, & 0 < \alpha \ll 1 \\ z(t) & = & \epsilon e^{-\nu t}, & 0 < \nu \ll 1 & & \\ Z(t) & = & K_Z \int_{s=0}^{t} z(s)\, ds, & K_Z \gg 1. & & \end{array}$$

To control large distortions: pick $\beta \gg 1$ as large as you need.

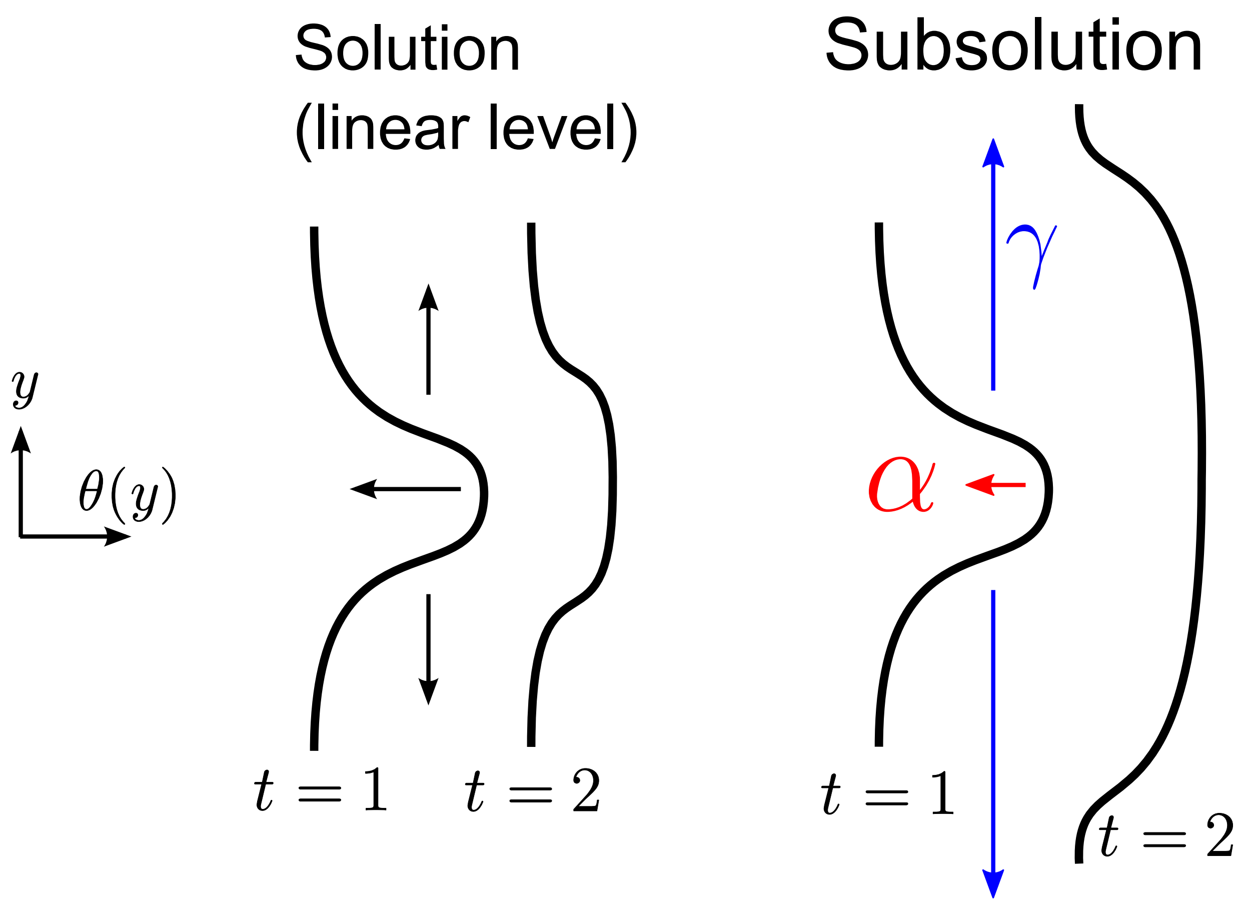
Tails $[y \to \infty]$ controlled by $z(t)$.

Main intuition: speed up the spreading out part of diffusion $[\gamma]$; slow down the decay part $[\alpha]$.

Ingredients - 2

Recall phase evolution:

$$\theta(y,t) \quad = \quad \beta t^{-\alpha} \exp[-\frac{y^2}{\gamma t}], \qquad \beta \gg 1, \qquad \gamma \gg 1, \qquad 0 < \alpha \ll 1$$



PDE vs LDE

PDE: Explicit subsolution

$$u^-(x, y, t) = \Phi(x + ct - \theta(y, t) - Z(t)\big) - z(t),$$

works because all important linear terms multiply

$$\Phi'(x + ct - \theta(y, t))$$

LDE case: if you try

$$u_{nl}^-(t) = \Phi\big(n + ct - \theta_l(t) - Z(t)\big) - z(t),$$

important linear terms will multiply one of

$$\Phi'\big(n + ct - \theta_l(t) - Z(t)\big), \qquad \Phi'\big(n + ct - \theta_l(t) - Z(t) \pm \sigma_i\big).$$

You get an n-dependent system for θ_l [BAD].

LDE : subsolution

Introduce $\overline{\sigma} = \big(\sigma_1, \sigma_2, -\sigma_1, -\sigma_2\big)$. Ansatz for LDE subsolution:

$$
\begin{aligned}
u_{nl}^{-}(t) \quad = \quad & \Phi(n + ct - \theta_l(t) - Z(t)\big) - z(t) \\
& + \textstyle\sum_{i=1}^{4} [\theta_{l+\overline{\sigma}_i}(t) - \theta_l(t)] p_i\big(n + ct - \theta_l(t) - Z(t)\big) \\
& + \textstyle\sum_{i=1}^{4} \sum_{j=1}^{4} [\theta_{l+\overline{\sigma}_i+\overline{\sigma}_j}(t) - \theta_{l+\overline{\sigma_j}}(t) - \theta_{l+\overline{\sigma_i}}(t) + \theta_l(t)] \\
& \qquad \times q_{ij}\big(n + ct - \theta_l(t) - Z(t)\big) \\
& + \textstyle\sum_{i=1}^{4} \sum_{j=1}^{4} [\theta_{l+\overline{\sigma}_i}(t) - \theta_l(t)][\theta_{l+\overline{\sigma}_j}(t) - \theta_l(t)] \\
& \qquad \times r_{ij}\big(n + ct - \theta_l(t) - Z(t)\big)
\end{aligned}
$$

[38 terms!] where the functions p_i, q_{ij} and r_{ij} are all related to the eigenvalue system

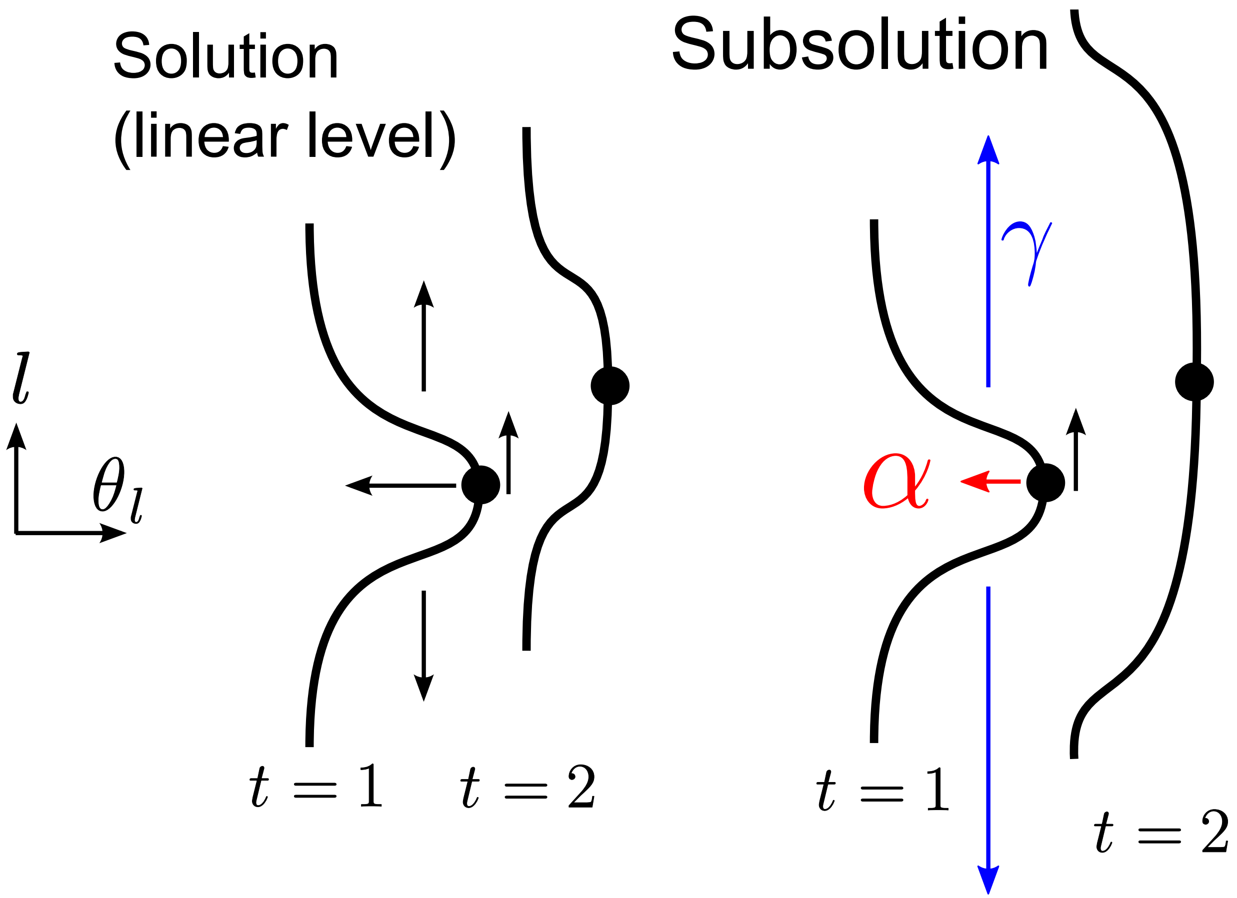
$$\mathcal{L}_\omega \Phi_\omega = \lambda_\omega \Phi_\omega.$$

Function $\theta_l(t)$ is now a convecting (modified) Gaussian.

Ingredients - 2

Actual phase evolution:

$$\theta_l(t) \quad = \quad \beta t^{-\alpha} \exp[-\frac{(l+\nu_1 t)^2}{\gamma t}], \qquad \beta \gg 1, \qquad \gamma \gg 1, \qquad 0 < \alpha \ll 1$$



Summary

- Obtained stability in 2d for rational directions
- Only spectral conditions imposed on wave.
- Works even in absence of comparison principles.
- For obstacle problems: use comparison principles.
- Waves persist if no direction is pinned and obstacle is nice.

Outlook:

- What about irrational directions ?
- What about standing waves ($c = 0$) ?
- What about pinning + obstacles ?