Existence and Stability of Fast Pulses for the Discrete FitzHugh-Nagumo System

MIZZOU™

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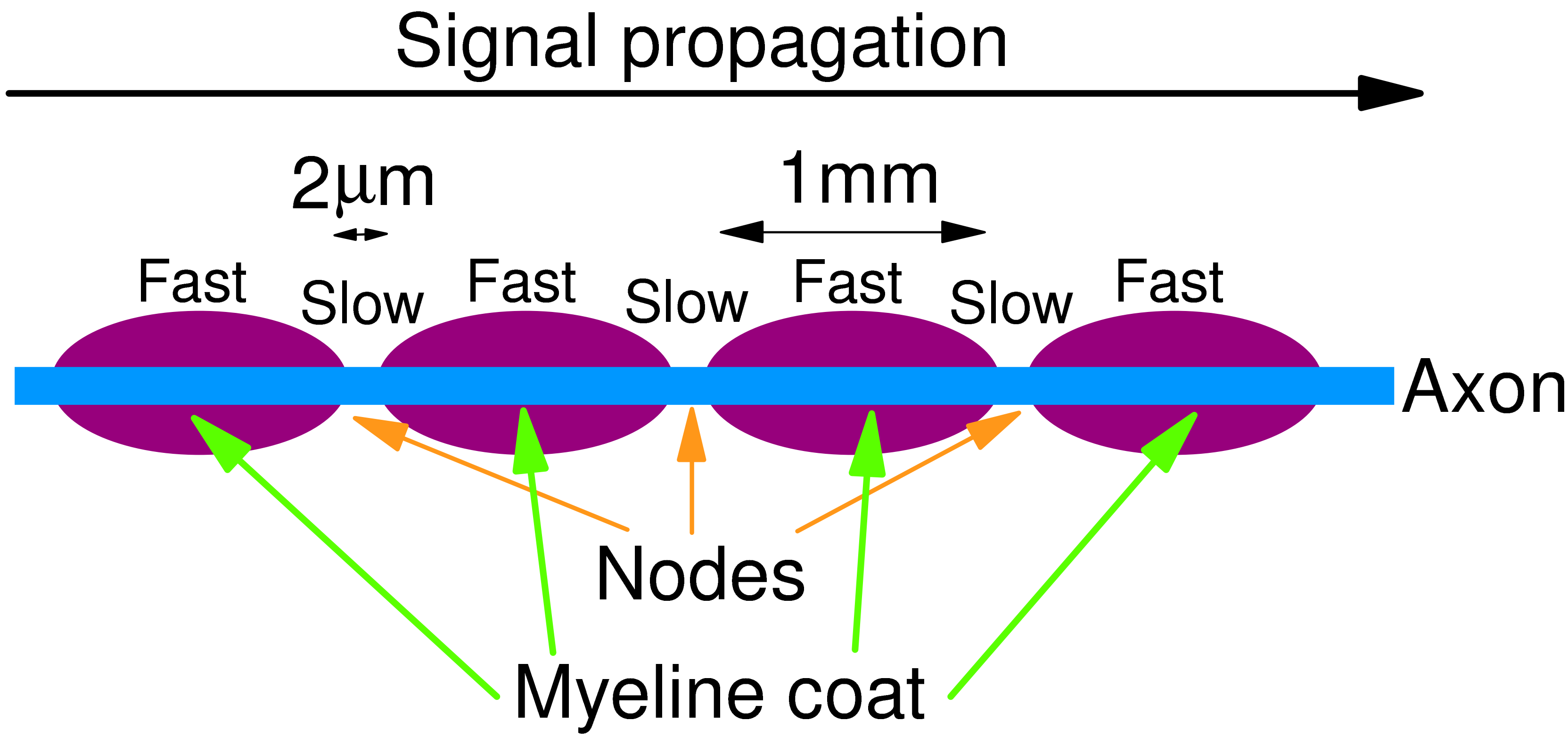
Signal Propagation through Nerves

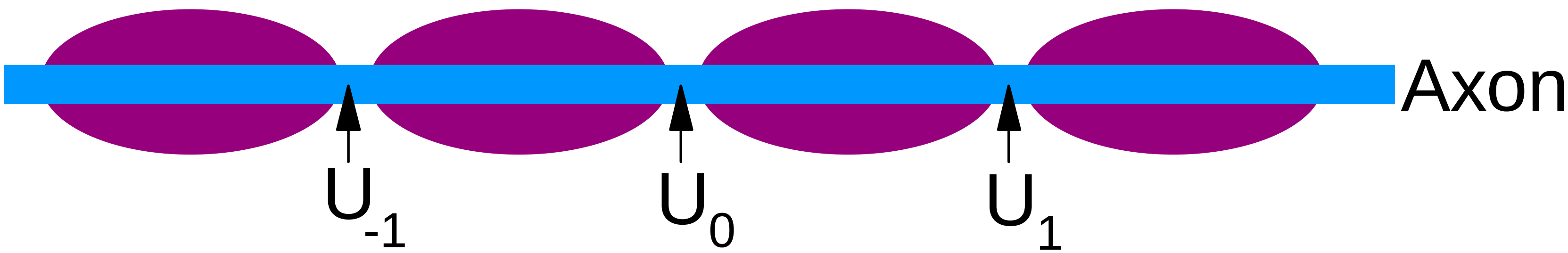
Nerve fibres carry signals over large distances (meter range).



- Fiber has myeline coating with periodic gaps called *nodes of Ranvier* .
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range)
- Slow propagation in gaps, but signal chemically reinforced.

Signal Propagation: The Model

One is interested in the potential U_j at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].

Ionic current has sodium and potassium component.

Electro-chemical analysis leads to the two component LDE [Keener and Sneyd, 1998]

$$\begin{aligned}\dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)],\end{aligned}$$

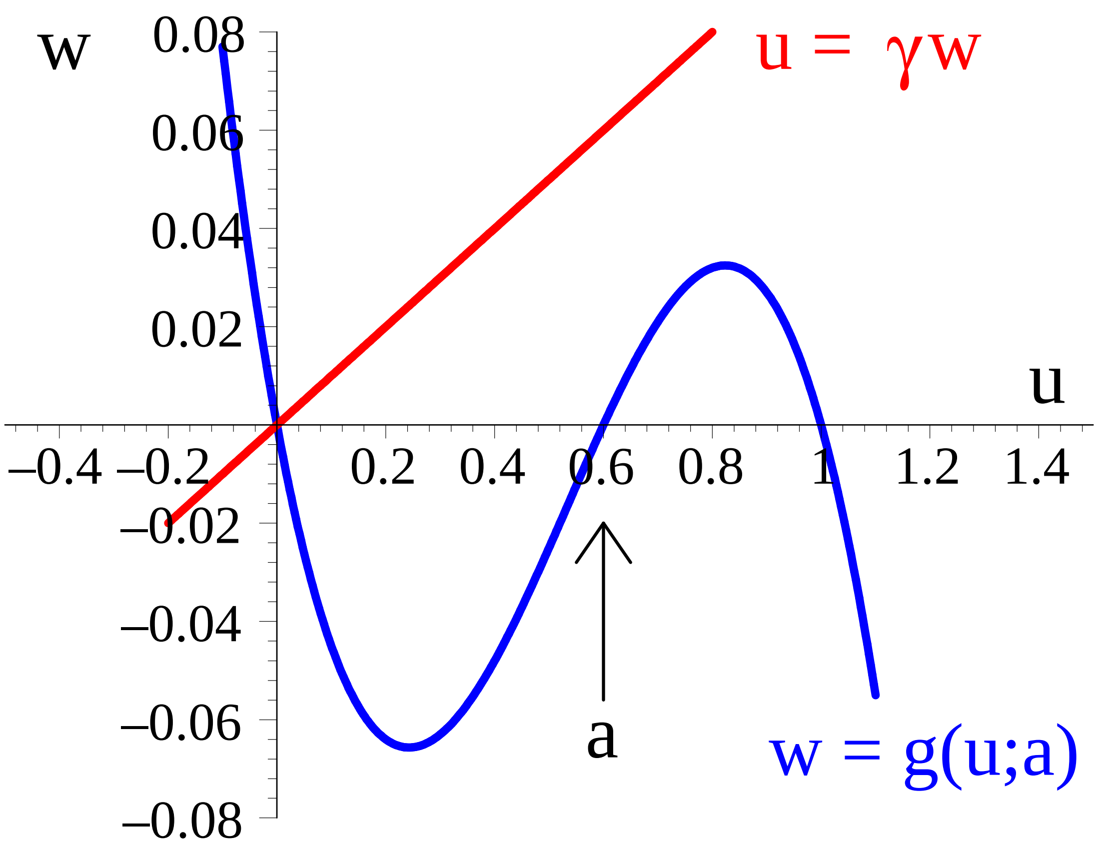
posed on a 1-dimension lattice, i.e. $j \in \mathbb{Z}$.

Potassium recovery encoded in second equation. Slow recovery $\rightarrow$ small $\epsilon > 0$.

Signal Propagation: Nonlinearity

Recall the dynamics:

$$\begin{aligned} \dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t);a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)]. \end{aligned}$$



Bistable nonlinearity g given by

$$g(u;a) = u(a-u)(u-1).$$

Parameter $\gamma > 0$ small so

$$w \neq g(\gamma w; a)$$

for $w \neq 0$.

Discrete FitzHugh-Nagumo LDE

Recall dynamics:

$$\begin{array}{rcl} \dot{U}_j(t) & = & U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t);a) - W_j(t), \\ \dot{W}_j(t) & = & \epsilon[U_j(t) - \gamma W_j(t)]. \end{array}$$

Travelling wave Ansatz $(U_j, W_j)(t) = (u, w)(j + ct)$ leads to

$$\begin{array}{rcl} cu'(\xi) & = & u(\xi + 1) + u(\xi - 1) - 2u(\xi) + g(u(\xi);a) - w(\xi), \\ cw'(\xi) & = & \epsilon[u(\xi) - \gamma w(\xi)]. \end{array}$$

This is a singularly perturbed functional differential equation of mixed type (MFDE).

Interested in pulses: $\lim_{\xi \to \pm\infty}(u, w)(\xi) = (0, 0)$.

Previous work by [Tonnelier], [Elmer and Van Vleck]; [Carpio et al]; lot of insight; rigorous results for special cases.

Signal Propagation: FitzHugh-Nagumo LDE

Reduction 1: Choose $\epsilon = 0$, which gives:

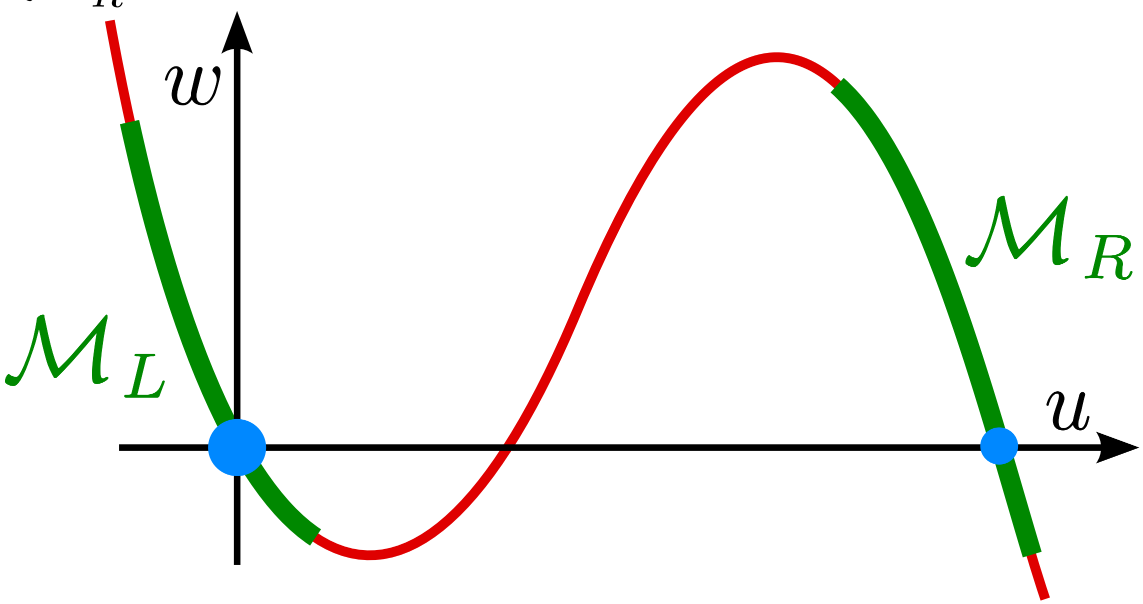
$$\begin{array}{rcl} cu'(\xi) & = & u(\xi+1) + u(\xi-1) - 2u(\xi) + g(u(\xi);a) - w(\xi), \\ cw'(\xi) & = & 0, \end{array}$$

admitting an equilibria-manifold $\mathcal{M} = (u, g(u;a))$.

Fast dynamics: u varies; w fixed.

Slow dynamics: u slaved to w by $g(u;a) = w$; movement only along $\mathcal{M}$.

Choose $\mathcal{M}_L$ and $\mathcal{M}_R$ as:



Signal Propagation: FitzHugh-Nagumo LDE

Reduction 2: Choose $\epsilon = 0$ and $W = 0$, which gives Nagumo LDE

$$\dot{U}_j(t) = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a).$$

Want: travelling fronts $U_j(t) = q_f(j + ct)$, which must solve MFDE

$$\begin{array}{rcl} cq_f'(\xi) & = & q_f(\xi + 1) + q_f(\xi - 1) - 2q_f(\xi) + g(q_f(\xi); a), \\ \lim_{\xi \to -\infty} q_f(\xi) & = & 0, \qquad \lim_{\xi \to \infty} q_f(\xi) = 1. \end{array}$$

Compare to Nagumo PDE

$$\partial_t u = \partial_{xx} u + g(u, a),$$

with traveling front ODE:

$$\begin{array}{rcl} cq_f'(\xi) & = & q_f''(\xi) + g\big(q_f(\xi); a\big) \\ \lim_{\xi \to -\infty} q_f(\xi) & = & 0, \qquad \lim_{\xi \to \infty} q_f(\xi) = 1. \end{array}$$

Signal Propagation: Comparison

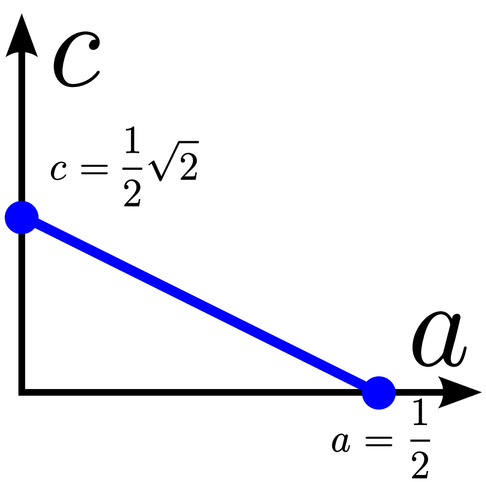
PDE

$$\partial_t u = \partial_{xx} u + g(u, a)$$

Travelling front $u = q_f(x + ct)$ satisfies:

$$c q_f'(\xi) = q_f''(\xi) + g\big(q_f(\xi); a\big)$$

Travelling fronts connecting 0 to 1:



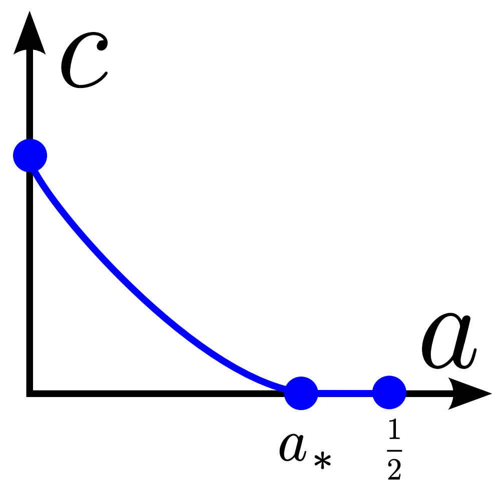
LDE

$$\dot{U}_j = U_{j+1} + U_{j-1} - 2U_j + g(U_j; a)$$

Travelling front $U_j = q_f(j + ct)$ satisfies:

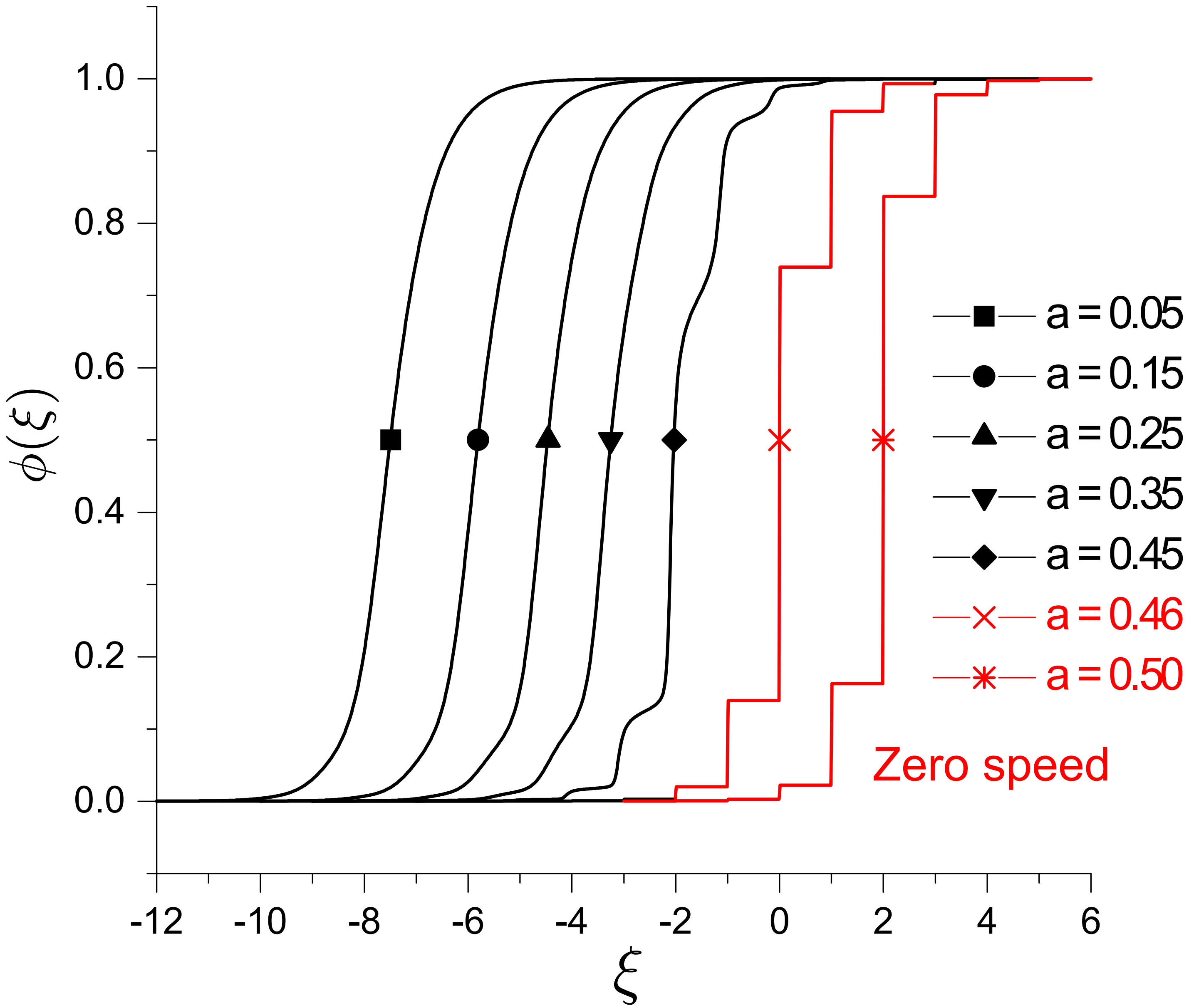
$$\begin{aligned} c q_f'(\xi) &= q_f(\xi + 1) + q_f(\xi - 1) - 2q_f(\xi) \\ &\quad + g\big(q_f(\xi); a\big) \end{aligned}$$

Travelling waves connecting 0 to 1:



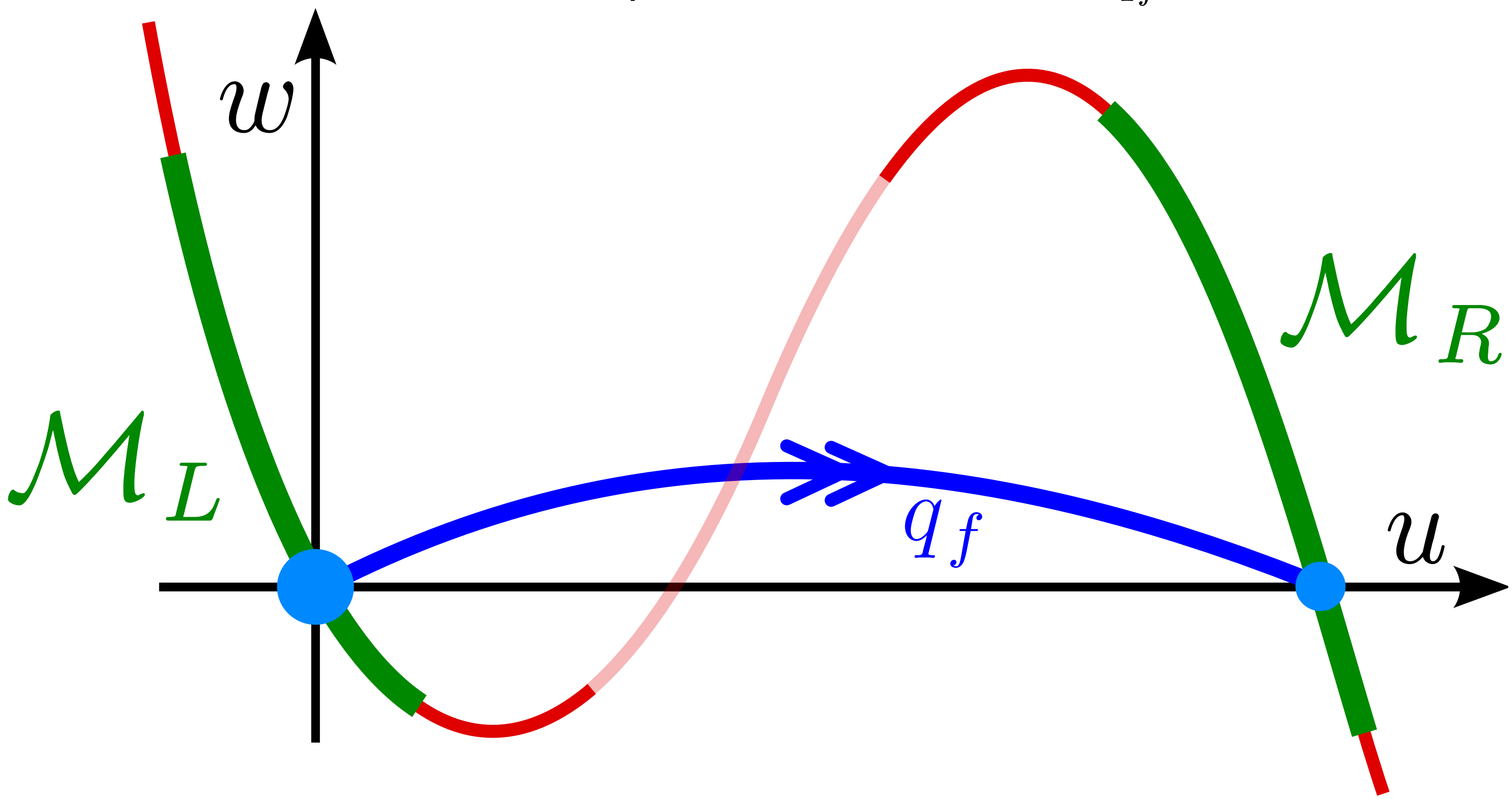
Discrete FitzHugh-Nagumo LDE - Propagation failure

Travelling fronts for the discrete Nagumo equation connecting $0 \to 1$.



Signal Propagation: FitzHugh-Nagumo LDE

Fix $0 < a < a_*$; there exists wave speed $c_* > 0$ and front q_f:

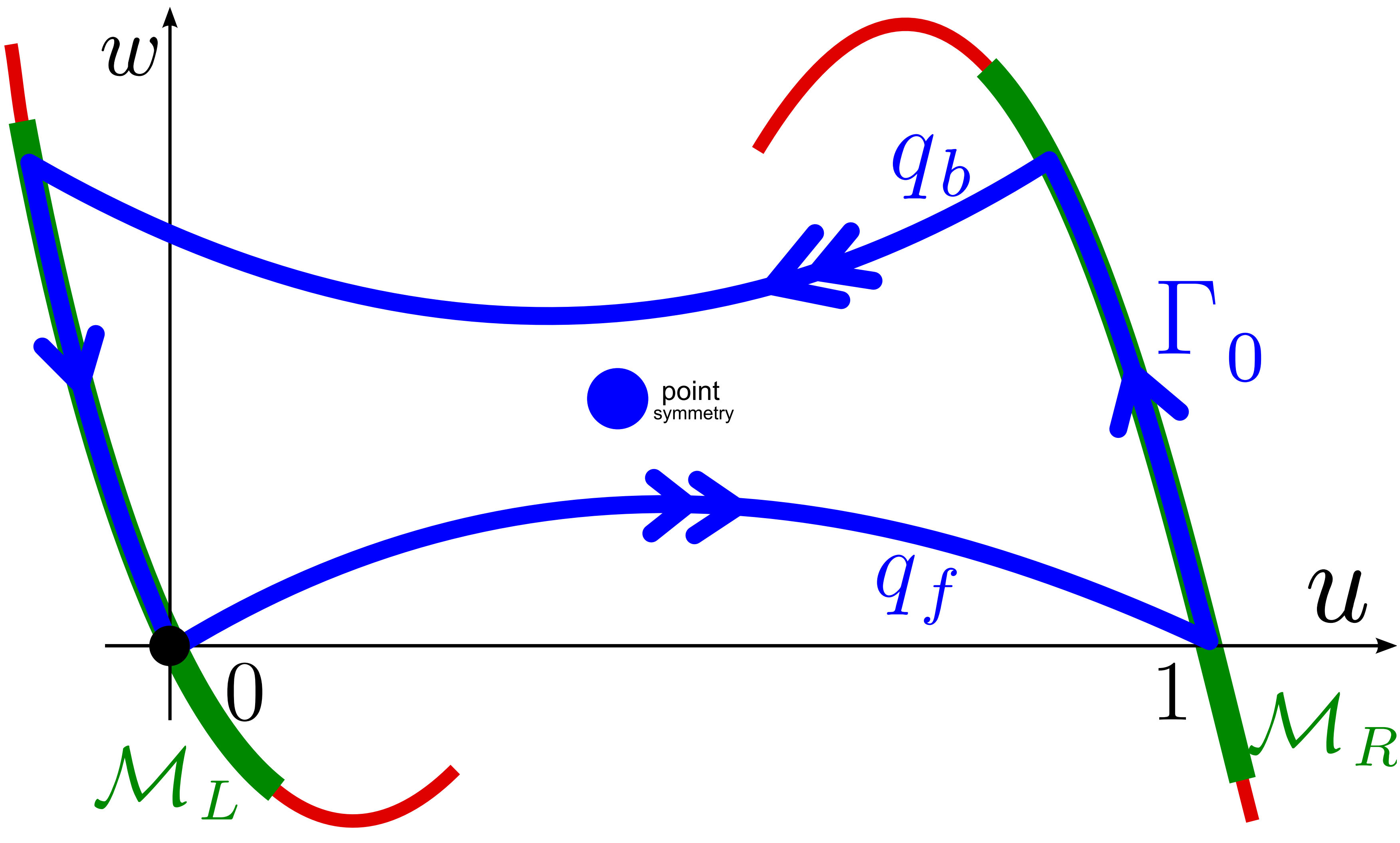


We now need to go back from $\mathcal{M}_R$ to $\mathcal{M}_L$.

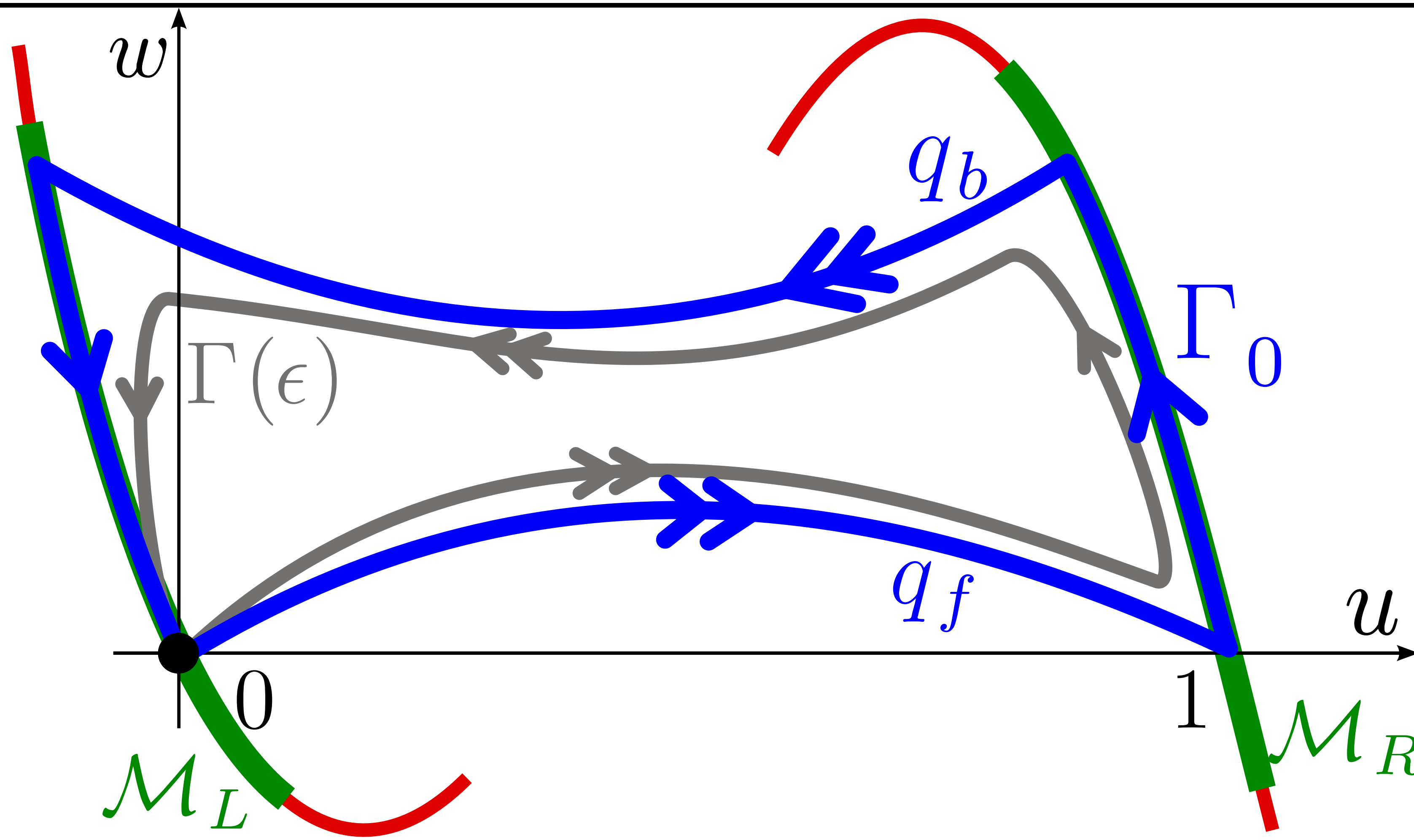
Cubic is symmetric around inflection point $\longrightarrow$ mirror q_f to find back q_b.

Signal Propagation: FitzHugh-Nagumo LDE

Connecting the pieces we find a singular homoclinic orbit Γ_0.



Discrete FitzHugh-Nagumo LDE - Main Result



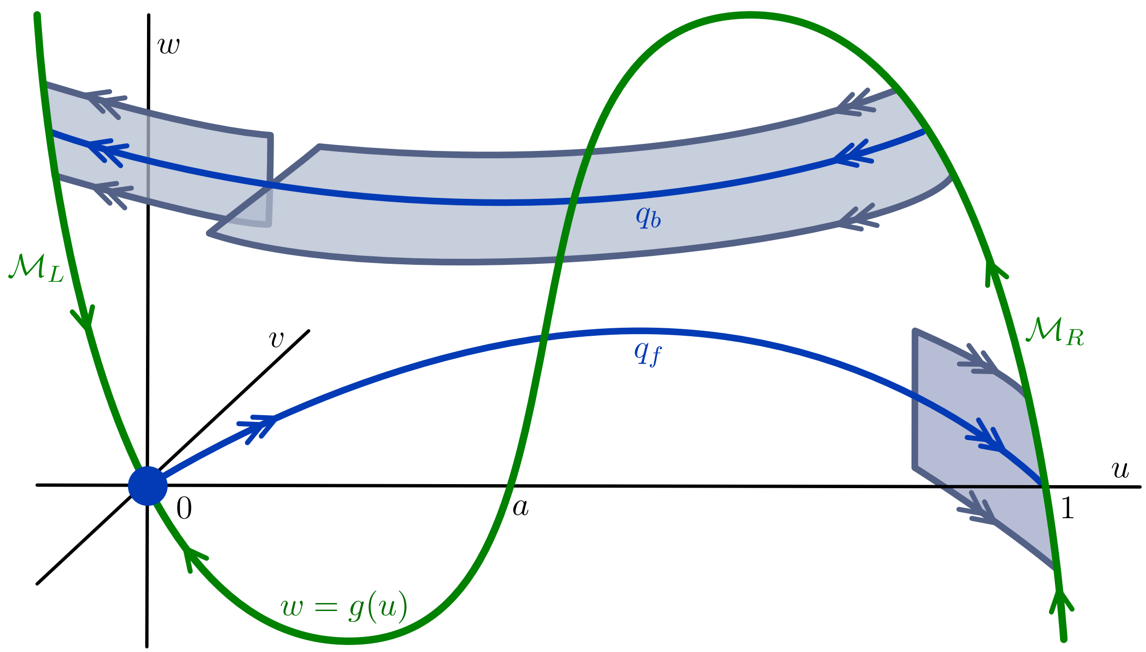
Main Result [H., Sandstede]: Choose $0 < a < a_*$ to ensure that the discrete Nagumo equation supports front with $c > 0$. For sufficiently small $\epsilon > 0$, there is a [locally unique] stable travelling pulse solution $\Gamma(\epsilon)$ to the discrete FitzHugh–Nagumo LDE that bifurcates off Γ_0 and winds around Γ_0 once.

Signal Propagation: FitzHugh-Nagumo PDE

- Result generalizes classic existence + stability theorem for FitzHugh-Nagumo PDE [Carpenter], [Hastings], [Yanagida] ('70s and '80s)

$$
\begin{aligned}
U_t &= U_{xx} + g(U; a) - W, \\
W_t &= \epsilon[U - \gamma W].
\end{aligned}
$$

- 'Modern' existence proof [Jones et al] uses Exchange Lemma to show transverse intersection of manifolds $\mathcal{W}^u(0)$ and $\mathcal{W}^s(\mathcal{M}_L)$.



The program

Main goal: lift geometric singular perturbation theory to MFDEs.

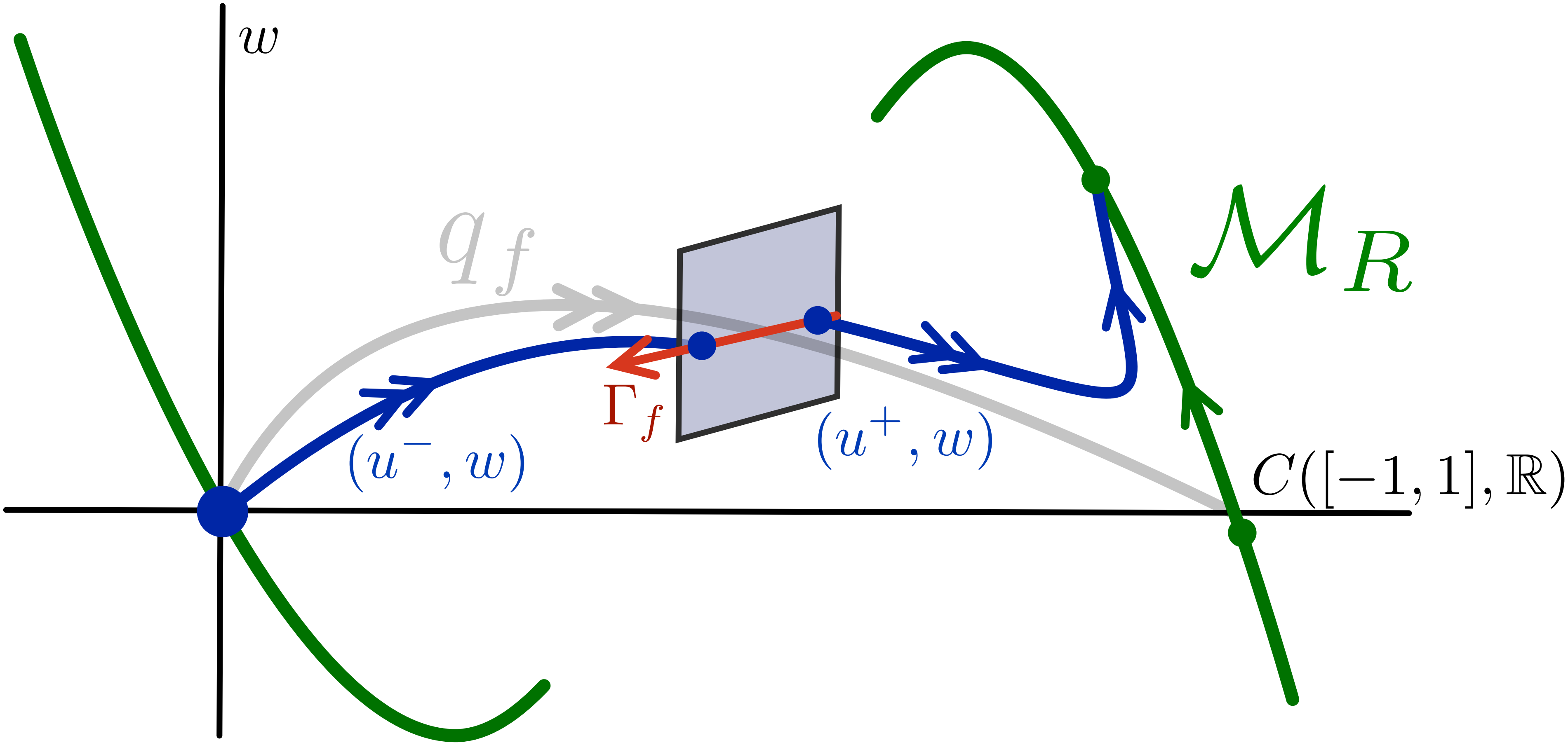
- Ill-posedness: care must be taken to define unstable / stable manifolds.
- Track intersections of ∞-dim stable / unstable manifolds.
- Exchange Lemma: Fenichel coordinates unavailable in infinite dimensions.
- Evans function: Not available for MFDEs.

Main ingredients:

- Suitable finite dimensional subspaces of $C([-1,1],\mathbb{R})$.
- Analytical underpinning for geometrical constructions.
- Direct construction of potential eigenfunctions.

Existence: Step 1 - Breaking the front

Varying ϵ and c breaks orbit q_f into **quasi-front** solution: two parts (u^-, w) and (u^+, w).

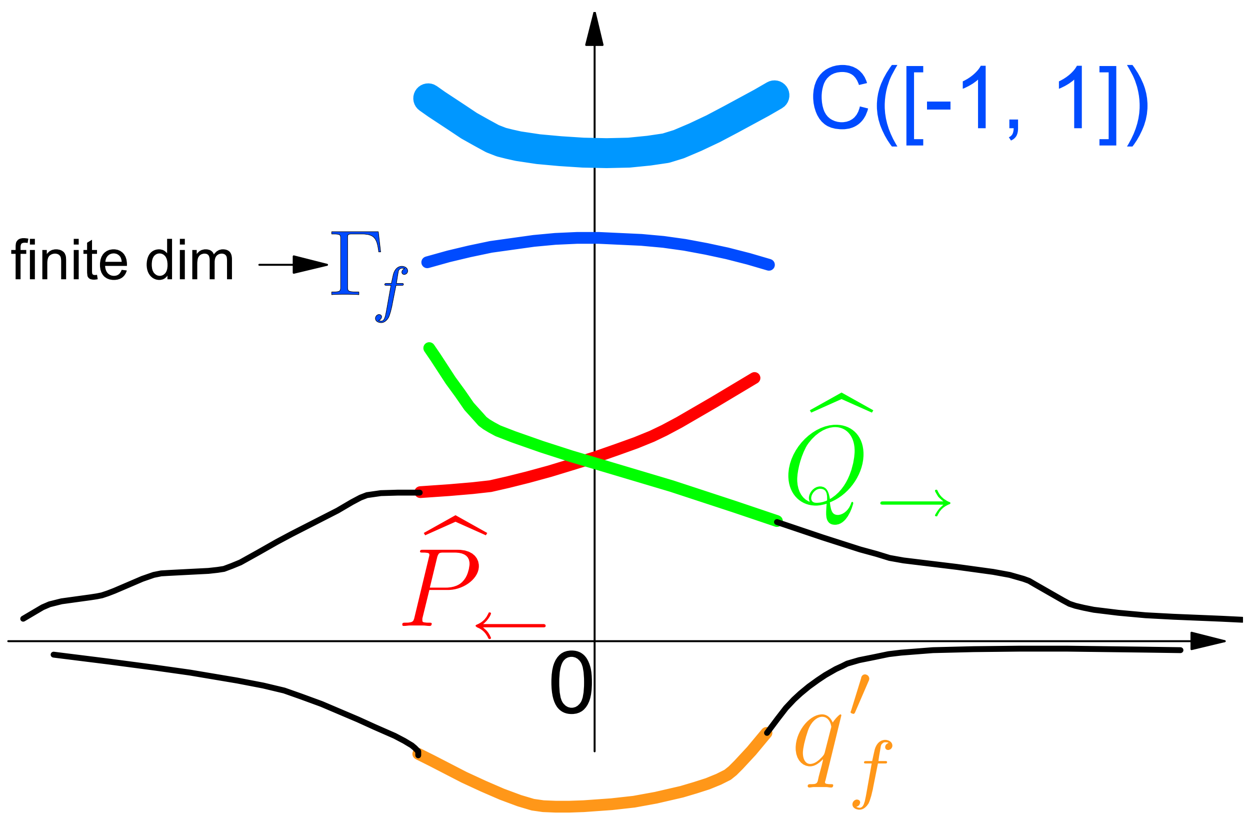


Want to contain jump in some finite-dimensional $\Gamma_f \subset C([-1,1],\mathbb{R})$.

Existence: Step 1 - Breaking the front

Construction based upon exponential dichotomies on $\mathbb{R}$ for linearization

$$cv'(\xi) = v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\big(q_f(\xi)\big)v(\xi).$$

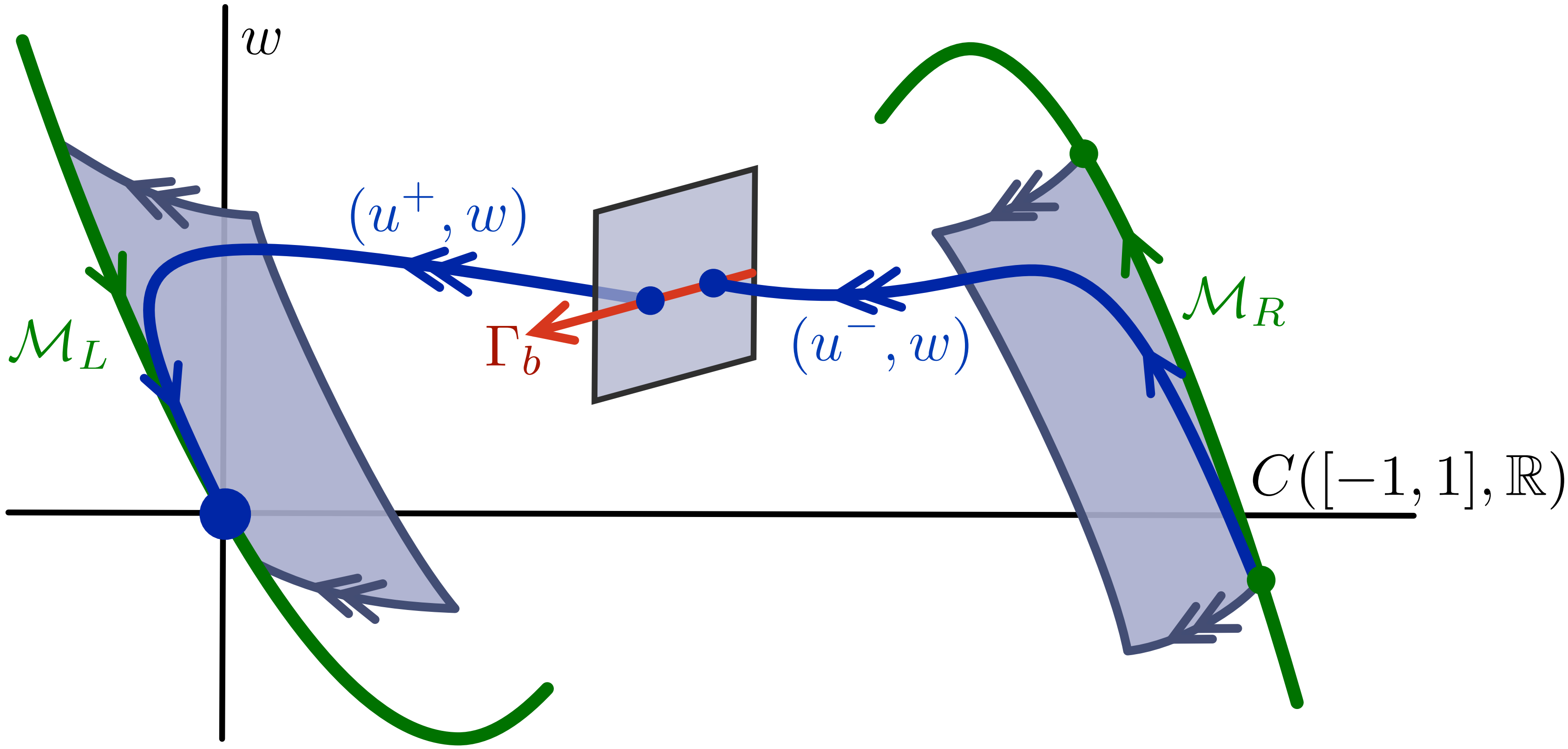


Thm. [Mallet-Paret and Verduyn Lunel, 2001]:

$$C([-1,1],\mathbb{R}) = \widehat{P}_{\leftarrow} \oplus \widehat{Q}_{\rightarrow} \oplus \{q'_f\} \oplus \Gamma_f.$$

Existence: Step 2 - Breaking the back

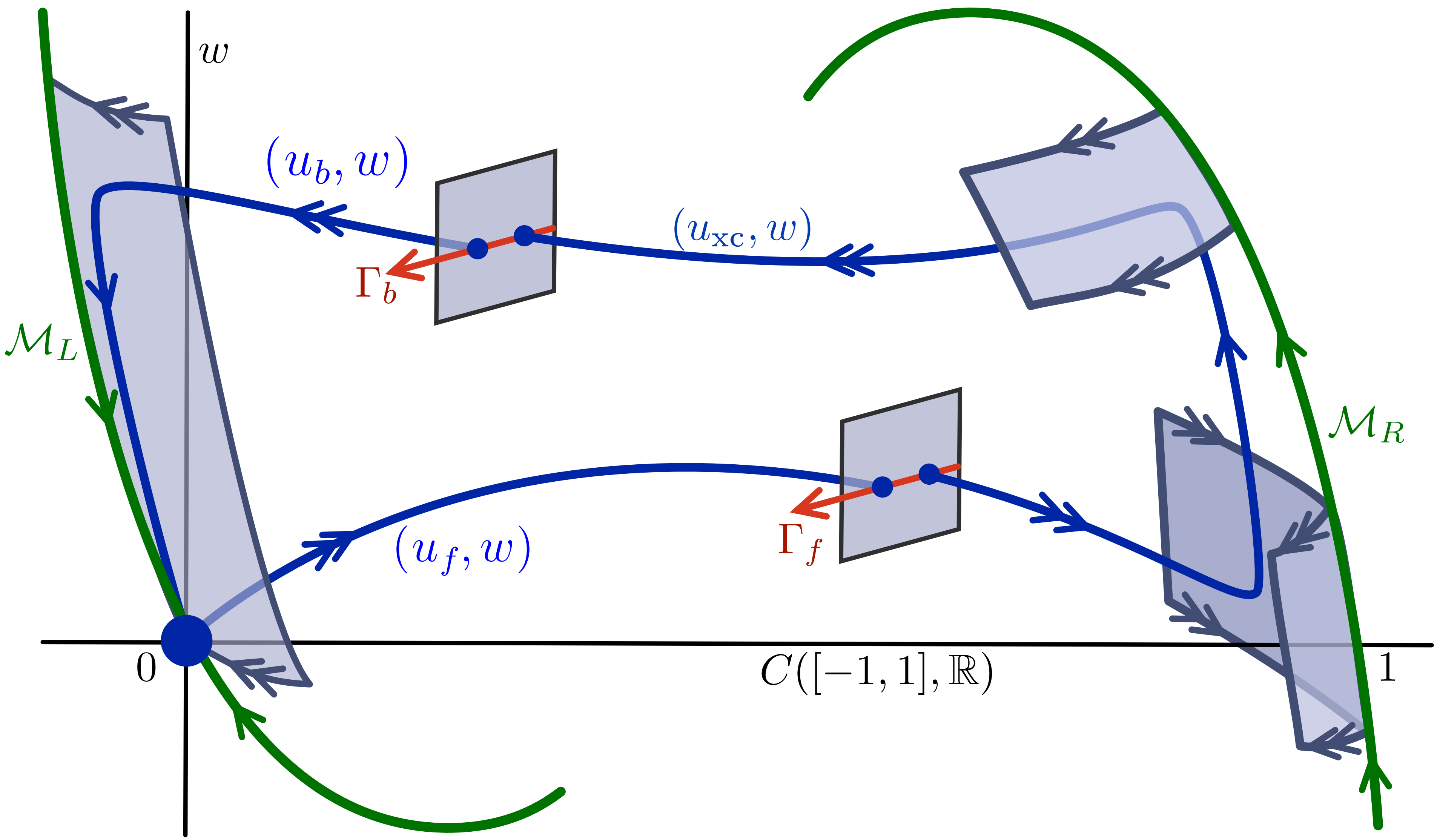
Similarly, can construct **quasi-back** solutions.



Extra degree of freedom $w_0 \approx w_*$, lifts back up and down.

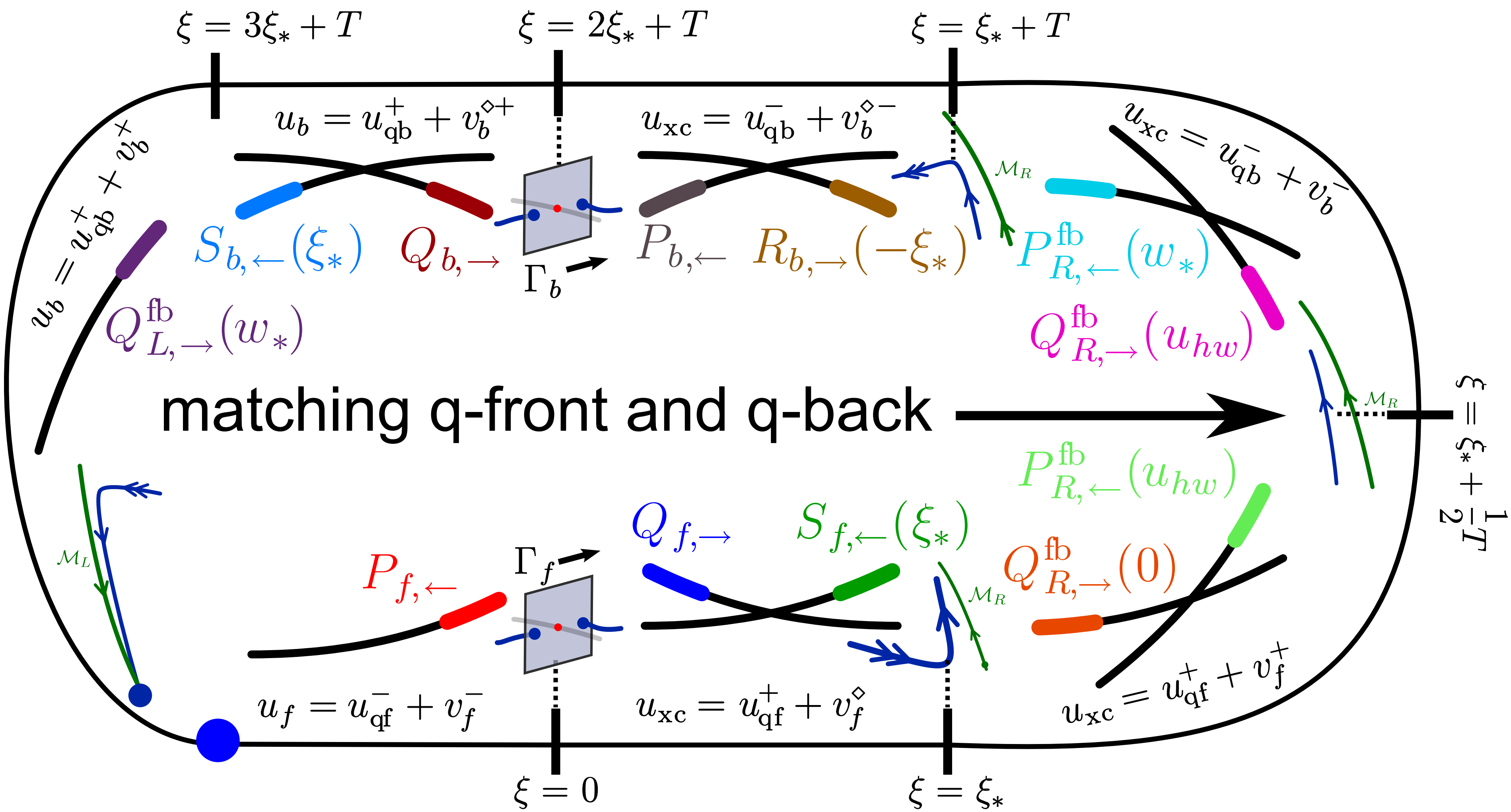
Existence: Step 3 - Exchange Lemma

Half-way along $\mathcal{M}_R$, quasi-front and quasi-back miss each other by $O(e^{-1/\epsilon})$.
Slight perturbation yields quasi-solutions:



Existence: Step 3 - Exchange Lemma

Construction uses seven distinct intervals.



Existence: Step 4 - Bifurcation equations

The jumps in Γ_f and Γ_b can be split into two parts:

- Construction of quasi-fronts and quasi-backs
  - Contribution of $O(\epsilon + |c - c_*| + |w_0 - w_*|)$.
- Modification due to Exchange Lemma
  - Contribution + derivatives are $O(e^{-1/\epsilon})$.

System to solve is hence to leading order

$$\begin{aligned} M_1(c - c_*) &= M_2\epsilon \\ M_3(c - c_*) &= M_4(w_0 - w_*) + M_5\epsilon \end{aligned}$$

The sign of M_1-M_5 can be read off from Melnikov integrals.

Three unknowns; two equations $\longrightarrow$ curve of solutions $\big(\epsilon, c(\epsilon)\big)$.

Stability

We have hence constructed travelling wave solutions

$$(U_j, W_j)(t) = \big(\bar{u}(\epsilon), \bar{w}(\epsilon)\big)\big(j + c(\epsilon)t\big).$$

Waves are shift-periodic with respect to the lattice

$$(U_j, W_j)\Big(t + 1/c(\epsilon)\Big) = (U_{j+1}, W_{j+1})(t).$$

Possible to use shift-periodic Floquet theory to study stability [Chow, Mallet-Paret, Shen].

However, we 'pretend' that $j \in \mathbb{Z}$ is continuous and study the eigenvalue MFDE

$$\begin{array}{rcl} c(\epsilon)u'(\xi) & = & u(\xi-1) + u(\xi+1) - 2u(\xi) + g'\big(\bar{u}(\epsilon)(\xi)\big)u(\xi) - w(\xi) - \lambda u(\xi), \\ c(\epsilon)w'(\xi) & = & \epsilon\big(u(\xi) - \gamma w(\xi)\big) - \lambda w(\xi), \end{array}$$

in comoving frame $\xi = j + ct$. Write as

$$\mathcal{L}(\epsilon)(u, w) = \lambda(u, w).$$

Stability - Relation between points of view

Rewrite LDE as $(\dot{U},\dot{W})(t) = \mathcal{F}\Big(U(t), W(t)\Big)$ posed on ℓ^∞.

Green's function $\mathcal{G}_{jj_0}(t,t_0,\epsilon)$ is unique solution to linearized LDE

$$\begin{array}{rcl} (\dot{U},\dot{W})(t) & = & D\mathcal{F}\Big((\bar{u}(\epsilon),\bar{w}(\epsilon))(\cdot + c(\epsilon)t)\Big)\Big(U,W\Big)(t) \\ (U_j,W_j)(t_0) & = & \delta_{jj_0}. \end{array}$$

Resolvent kernel $G_\lambda(\xi,\xi_0,\epsilon)$ is unique solution to linearized MFDE

$$(\mathcal{L}(\epsilon)-\lambda)G_\lambda(\cdot,\xi_0,\epsilon) = \delta(\xi-\xi_0).$$

Lattice does not see modulations $e^{2\pi i \xi}$. In particular,

$$G_{\lambda+2\pi i c(\epsilon)}(\xi,\xi_0,\epsilon) = e^{2\pi i(\xi_0-\xi)}G_\lambda(\xi,\xi_0,\epsilon).$$

Thm. [Benzoni-Gavage, Huot, Rousset] For $\gamma \gg 1$ and $t > 0$,

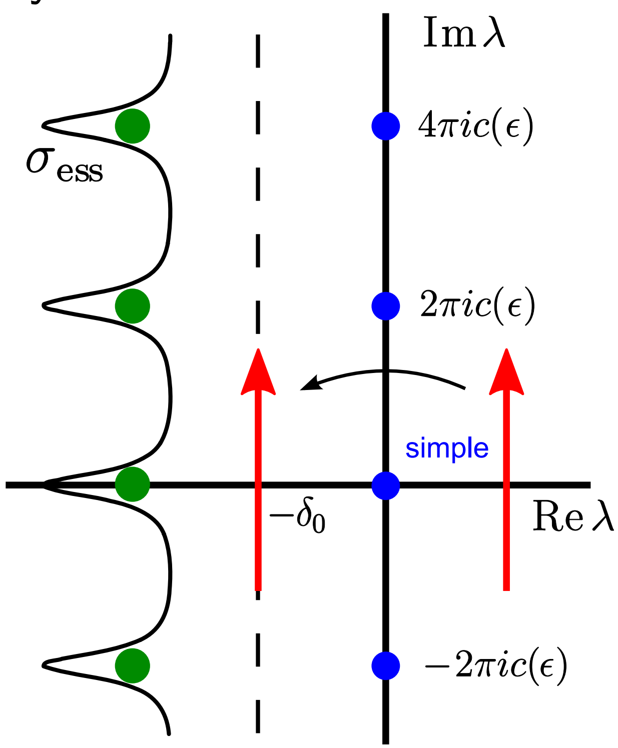
$$\mathcal{G}_{jj_0}(t,t_0,\epsilon) = \frac{-1}{2\pi i}\int_{\gamma - i\pi c(\epsilon)}^{\gamma + i\pi c(\epsilon)} e^{\lambda(t-t_0)}G_\lambda(j+ct, j_0+ct_0,\epsilon)d\lambda.$$

Stability

Goal is to shift contour of integration in

$$\mathcal{G}_{jj_0}(t, t_0, \epsilon) = \frac{-1}{2\pi i} \int_{\gamma - i\pi c(\epsilon)}^{\gamma + i\pi c(\epsilon)} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0, \epsilon) d\lambda$$

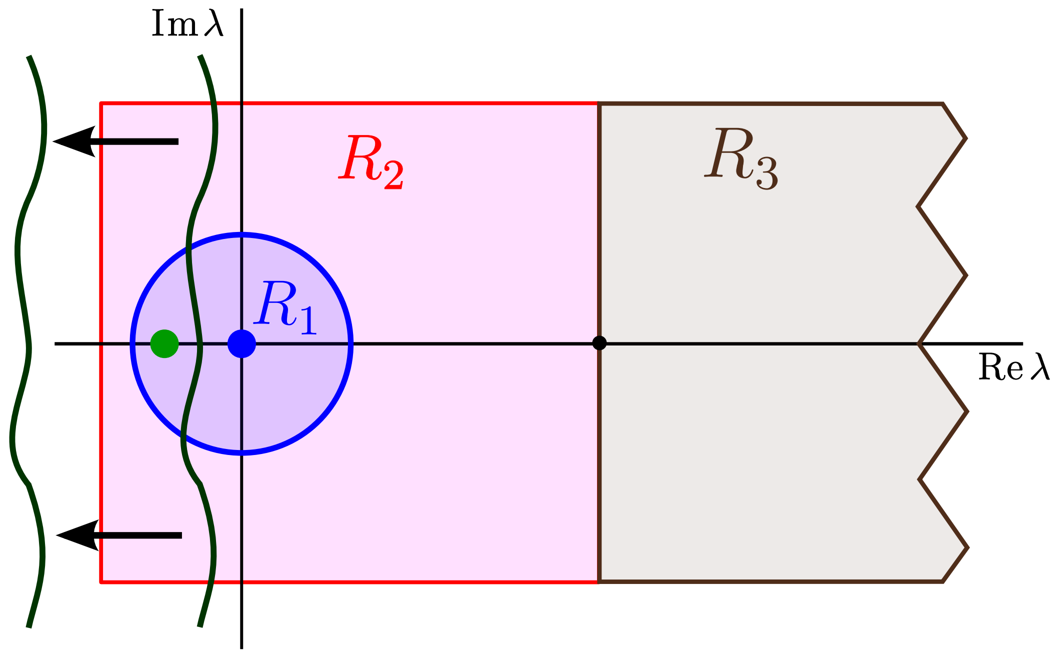
to the line $\gamma = -\delta_0$. Need to extend resolvent kernel G_λ meromorphically through imaginary axis.



Will show: Spectrum of $\mathcal{L}(\epsilon)$ admits gap.

Translational eigenvalues at $2\pi ic(\epsilon)\mathbb{Z}$ contribute simple poles to resolvent kernel $G_\lambda(\xi, \xi_0, \epsilon)$.

Stability

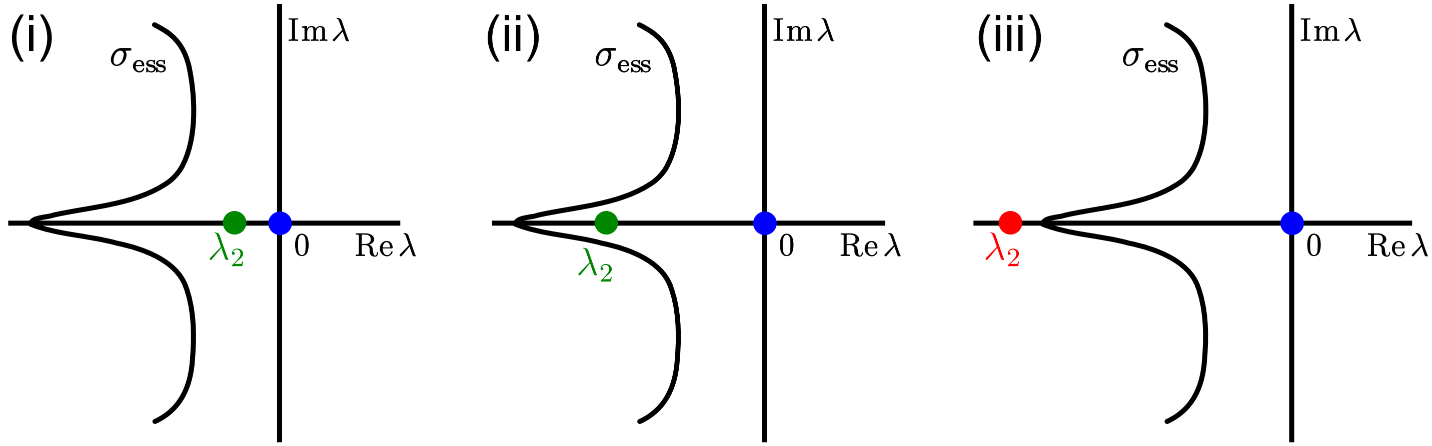


Goal is to characterize eigenvalues for $\mathcal{L}(\epsilon)$ in three regions R_1, R_2 and R_3 simultaneously for all small $\epsilon > 0$ by direct construction.

Essential spectrum is $O(\epsilon)$ to left of imaginary axis.

Push out of the way by using exponential weights, i.e., choose small $\eta > 0$ and look for solutions $\Lambda(\epsilon)(u, w) = \lambda(u, w)$ that behave as $(u, w)(\xi) = O(e^{\eta\xi})$ as $\xi \to \pm\infty$.

Stability - Resonance pole or eigenvalue



Translational eigenvalue at $\lambda = 0$.

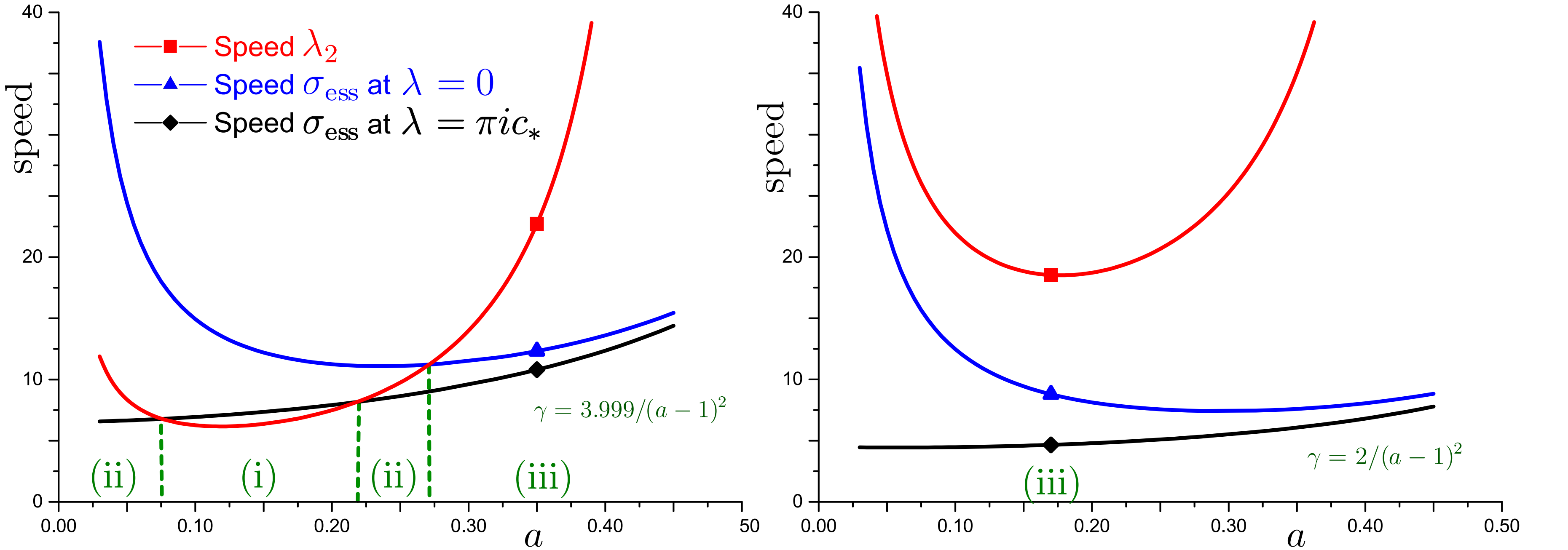
The pulse $(\bar{u}, \bar{w})(\epsilon)$ can be thought of as bound state of front q_f and back q_b.

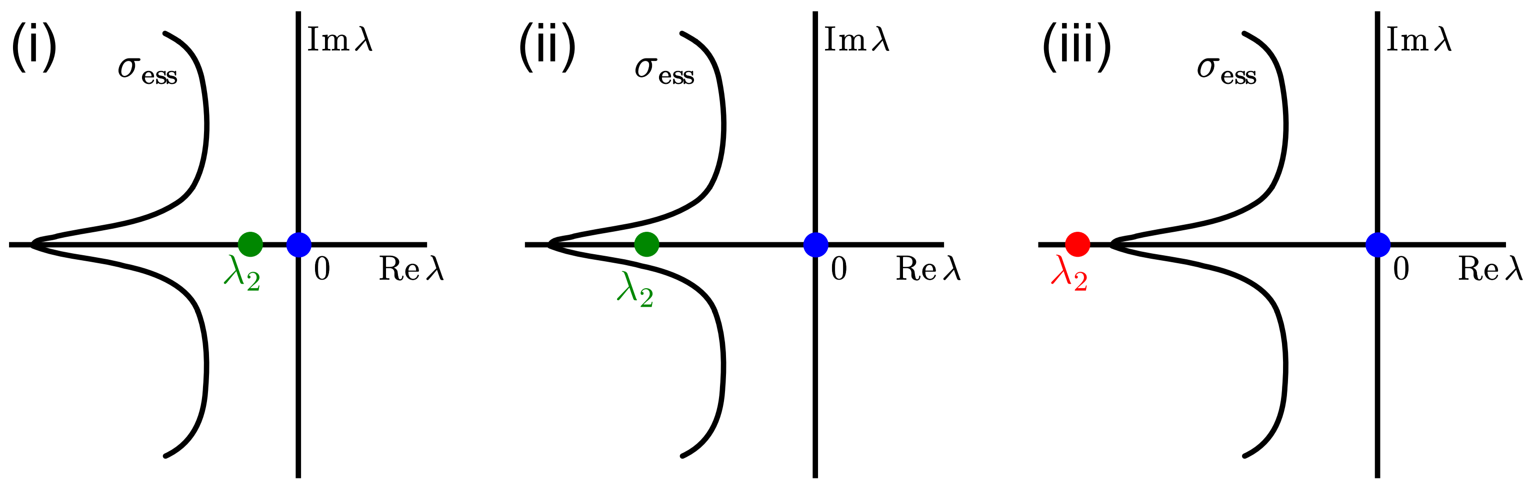
Expect second potential eigenvalue $\lambda_2 = O(\epsilon)$, with eigenfunction centered on the back q_b.

Whether λ_2 is an eigenvalue or resonance pole depends on location with respect to imaginary axis.

Our direct construction of eigenfunctions yields explicit expression for the speeds with which λ_2 and the essential spectrum move to the left.

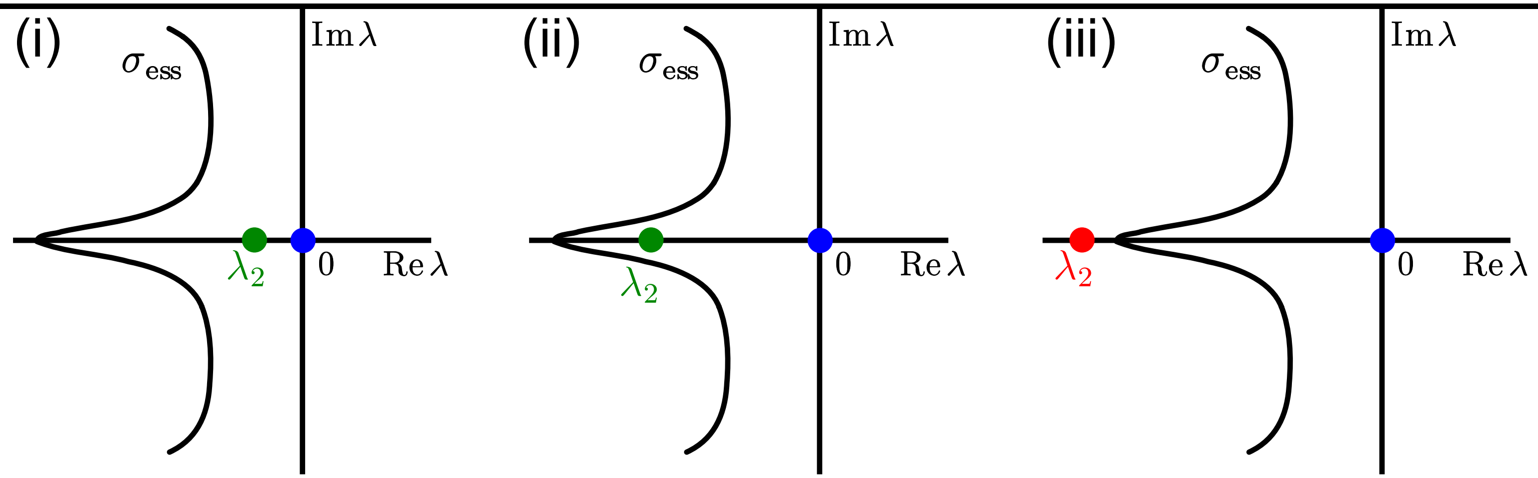
Stability - Resonance pole or eigenvalue





All three scenario's can occur.

Stability - Interpretation



Situation (i): λ_2 is an eigenvalue to the right of essential spectrum. Perturbations that change only the position of the back will decay without interacting with the front.

Other perturbations lead to a translation of the pulse profile and a movement of the back relative to the front.

Situation (ii): λ_2 is eigenvalue. Effect should still be felt for localized perturbations, affects relative position of front and back. Essential spectrum transports perturbations of background state $(u, w) = 0$ to $j = \infty$.

Situation (iii): λ_2 is resonance pole. Unclear. More detailed analysis of resolvent kernel may lead to insight.

Summary / Outlook

- Travelling pulses for discrete FHN constructed using ∞-d Exchange Lemma.
- Stability established by direct construction of potential eigenfunctions.

Number of issues to explore:

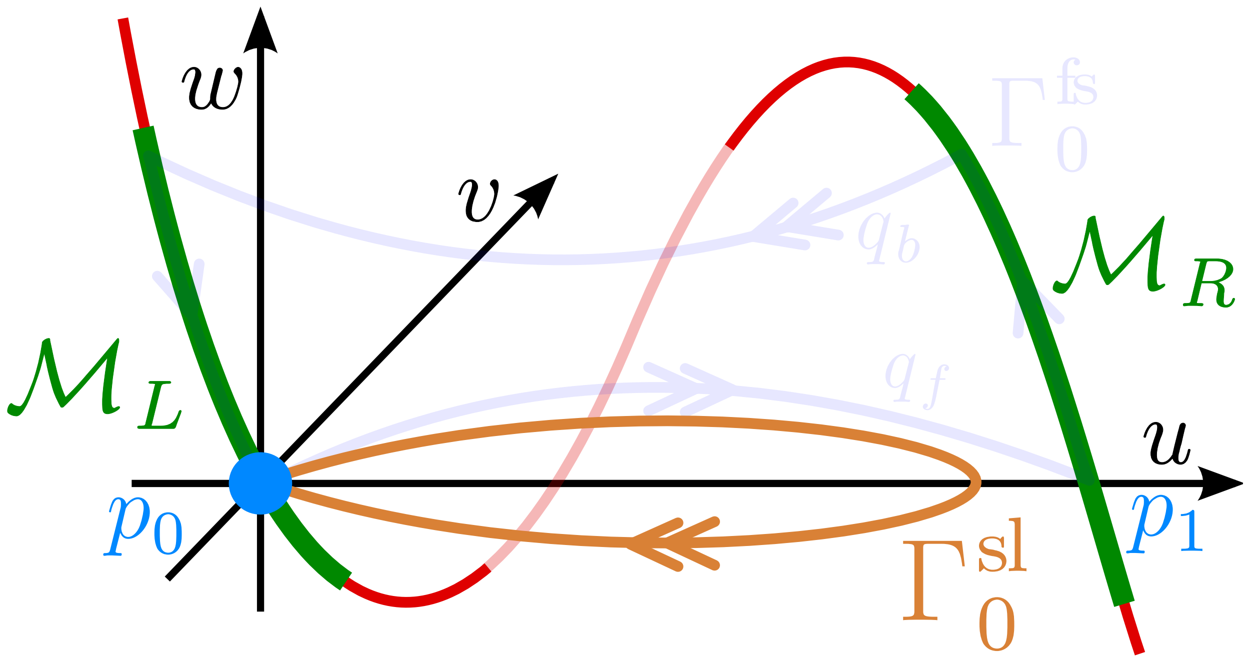
- Multi-pulses, homoclinic blow-up etc in other singularly perturbed lattice problems.
- What happens to pulses as propagation failure region is encountered?

FitzHugh-Nagumo PDE: Slow Pulses

Recall the travelling wave ODE

$$\begin{aligned} u' &= v, \\ v' &= cv - g(u;a) + w, \\ w' &= \tfrac{\epsilon}{c}(u - \gamma w). \end{aligned}$$

In the singular limit $c \longrightarrow 0$ and $\frac{\epsilon}{c} \longrightarrow 0$, one finds an additional slow-singular orbit Γ_0^{sl}.



FitzHugh-Nagumo PDE: Status

Conjecture [Yanagida]: fast and slow branches are connected.