

Let S be a set, then
 $\geq$ a binary relation is a partial order if: $\forall x, y, z \in S$

(i) $x \geq x$ (reflexive)

(ii) $x \geq y, y \geq x \Rightarrow x = y$ (anti-symmetric)

(iii) $x \geq y, y \geq z \Rightarrow x \geq z$ (transitive)

Let X be a real vector space and $\geq$
a partial order then $(X, \geq)$ is a partially ordered
vector space (pov) if

(iv) $x \geq y \Rightarrow x + z \geq y + z$

(v) $x \geq 0, \alpha \geq 0 \Rightarrow \alpha x \geq 0$

Terminology:

- $x \geq 0$ is called positive

- $X^+ = \{x \in X, x \geq 0\}$ is the positive cone of X .

- $A \subseteq X$ then $x \in X$ is an upper bound
of A if $x \geq a \forall a \in A$

Example

(i) $(X, \geq)$

(ii) $\mathbb{R}^S = \{f: S \rightarrow \mathbb{R}\}$

$f \geq g \Leftrightarrow f(s) \geq_{\mathbb{R}} g(s) \forall s \in S$

(iii) X the space of symmetric real
matrices ordered by

$A \geq B \iff A - B$ is a positive semidefinite matrix

Let $(X, \geq)$ be a pov and $Y \subseteq X$ subspace, then $\geq$ restricts to a partial on Y and

$(Y, \geq|_Y)$ is a pov.

Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be pov's then $X \times Y$ can be ordered by

$$(x_1, y_1) \geq (x_2, y_2) \iff x_1 \geq_X x_2 \wedge y_1 \geq_Y y_2$$

$\mathbb{R}^n$ is ordered this way.

Definition Let X be a vector space and $C \subseteq X$

- C is called a wedge if $\alpha C, C + C \subseteq C \quad \alpha \geq 0$
- A wedge C is called a cone $C \cap -C = \{0\}$

Theorem: X a real vector space, $C \subseteq X$ Then C is a cone if and only if there a partial order $\geq$ st $(X, \geq)$ is a pov and $X^+ = C$

Proof: Suppose $(X, \geq)$ is a pos and

let $C = X^+$. By def $\forall \alpha > 0 \quad \alpha C \subseteq C$

let $x, y \in C$ then $x \geq 0$ so
 $x+y \geq y \geq 0$ so $C+C \subseteq C$

Suppose $x \in C \cap -C$ then
 $x \geq 0$ and $-x \geq 0$
 $0 \geq x$

so $x=0$
Indeed C is a cone

Suppose C is a cone. We define

$$x \geq y \iff x-y \in C$$

$$x \geq y \wedge y \geq x$$

$$x-y \in C$$

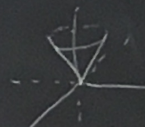
$$y-x \in C$$

$$(y-x) = -(x-y) \text{ hence } x=y$$

$$x \geq y, y \geq z \text{ then } \begin{matrix} x-y \in C \\ y-z \in C \end{matrix}$$

$$C \ni (x-y) + (y-z) = x-z \text{ so } x \geq z \quad \square$$

Let $X = \mathbb{R}^3$ and $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2, z \geq 0\}$

 is the ice cream cone.

Let $P_2(\mathbb{R})$ denote the space of real polynomials of degree up to 2, pointwise order.

Claim (X, C) is order isomorphic to $P_2(\mathbb{R})$.

We want $j: X \rightarrow P_2(\mathbb{R})$ linear bijection s.t.

$$x \geq y \Leftrightarrow j(x) \geq j(y)$$

$j: X \rightarrow P_2(\mathbb{R})$ defined by

$$(x, y, z) \mapsto p(t) = at^2 + bt + c$$

where $a = x + z$, $b = 2y$, $c = z - x$

$p \in P_2(\mathbb{R})$ is positive iff:

$$(i) \ a = b = 0, \ c \geq 0$$

$$(ii) \ b^2 - 4ac \leq 0 \text{ and } a > 0$$

Let $(x, y, z) \in C$ solving equation

$$x = \frac{a-c}{2}, \ y = \frac{b}{2}, \ z = \frac{a+c}{2} \quad \text{then}$$

$$b^2 \leq 4ac, \ a \geq 0.$$

$0 \leq z = \frac{a+c}{2}$ so $a, c \geq 0$.
 If $a=0$ then (i) holds
 If $a>0$ then (ii) holds $\left. \vphantom{\begin{matrix} \text{If } a=0 \\ \text{If } a>0 \end{matrix}} \right\} p \text{ is positive}$
 Let $p(t) = at^2 + bt + c$ positive
 (i) $y=0$ and $x=-z$ so
 $x^2 + y^2 \leq z^2$ and $0 \leq c = 2z$
 (ii) $x^2 + y^2 \leq z^2$ and, in particular,
 $|x| \leq |z|$ so $0 \leq a = x+z$, so $z \geq 0$ $\square$

$\mathcal{Y}$ be the space of symmetric
 2×2 matrices ordered by the cone
 of positive semidefinite matrices
 Define $i: P_2(\mathbb{R}) \rightarrow \mathcal{Y}$ by
 $p(t) = at^2 + bt + c \mapsto \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$,
 then i is an order isomorphism

Definition Let $(X, \geq)$ be a pos.

- X is called directed if for all $x, y \in X$ there exists a $z \in X$ $z \geq x, y$

- $X^+ = C$ is a generating cone if $X = C - C$

Theorem Let $(X, \geq)$ a pos, $C = X^+$, TFAE

(i) X is directed

(ii) C is a generating cone

(iii) $X = \text{span}(C)$

Proof:

(ii) $\Rightarrow$ (iii) any $x \in X$ equals $x = c_1 - c_2$

(i) $\Rightarrow$ (ii) let $x \in X$ then $\exists y \in X, c_1, c_2 \in C$ s.t. $y \geq x, 0$ Now $x = y - (y - x)$

(iii) $\Rightarrow$ (i) let $x, y \in X$ and write $\overset{\text{D}}{C} \quad \overset{\text{D}}{C}$

$$x = \sum_{n=1}^m \lambda_n c_n, y = \sum_{n=m+1}^k \lambda_n c_n$$

define $A = \{n \in \mathbb{N} : 1 \leq n \leq k \text{ s.t. } \lambda_n \geq 0\}$

$$z = \sum_{n \in A} \lambda_n c_n \in X, z \geq x, y$$

$(X, \geq)$ is not directed. $\square$



Definition. Let $(X, \geq)$ be a pov. An element $u \in X$ is called an order unit if

$$\bullet \forall x \in X \exists n \in \mathbb{N} \text{ st } -nu \leq x \leq nu$$

Remark: Pov with order unit is directed.

Examples: $X \subseteq B(S)$ bounded functions and $1_S \in X$ then 1_S is an order unit.

$X = C[0,1]$ any f with $f(s) \geq 0$ is an order unit.

Example: Lexicographical order (on $\mathbb{R}^2$)

$$(x_1, y_1) \geq (x_2, y_2) \Leftrightarrow x_1 \geq x_2 \vee (x_1 = x_2 \wedge y_1 \geq y_2)$$

$$n(0,1) \leq (1,0) \quad \forall n \in \mathbb{N}$$

Definition: A pov $(X, \geq)$ is called Archimedean if $(nx \leq y \quad \forall n \in \mathbb{N}) \Rightarrow x \leq 0$

Theorem: Suppose $(X, \geq)$ is Archimedean then any linear $Y \leq X$ is also Archimedean.

Theorem: A pov $(X, \geq)$ is Archimedean if and only if for every $\mathbb{Q}$ -dim subspace $Y \subseteq X$ then $Y \cap C$ is closed in the topology of Y .

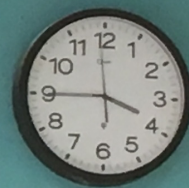
Let $(X, \geq)$ be a pov with unit $u \in X$.
We define $\|x\|_u = \inf \{ \lambda \in \mathbb{R}^+ : -\lambda u \leq x \leq \lambda u \}$
is semi-norm.

Theorem: If X is Archimedean then $\|\cdot\|_u$ is a norm.

Proof: Let $x \in X$ with $\|x\|_u = 0$

Then $\forall n \in \mathbb{N} : -\frac{1}{n}u \leq x \leq \frac{1}{n}u$

$$\left. \begin{array}{l} nx \leq u \quad \text{so} \quad x \leq 0 \\ n(-x) \leq u \quad \text{so} \quad x \geq 0 \end{array} \right\} x = 0 \quad \square$$

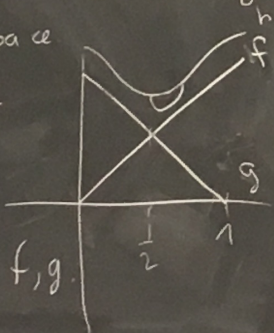


Defn $(X, \geq)$ is a Riesz space if for any $x, y \in X$ a supremum $x \vee y$ exists in X (smallest upper bound)
 $(x \vee y) \geq x, y$
and $z \geq x, y \Rightarrow z \geq (x \vee y)$

Examples: $C[0,1]$, $B(S)$ are Riesz spaces
 $f, g \in C[0,1]$ then $(f \vee g)(x) = \max(f(x), g(x))$

$C^1[0,1]$ is not a Riesz space
 $f(x) = x$, $g(x) = 1-x$ and let
 $h(x) = \max(f(x), g(x))$

Suppose h is an upper bound of f, g .



$h \geq k$ but $h \neq k$
 $\exists U \in [0,1], \varepsilon > 0$ with $h(y) - k(y) \geq \varepsilon \quad \forall y \in U$

Now we can find a $\tilde{h} \in C^1[0,1]$ with

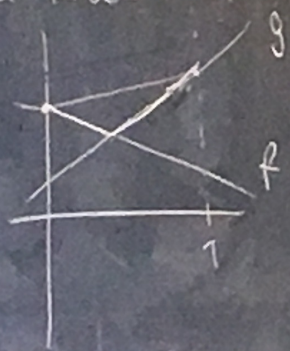
$\tilde{h}|_U = 0$ and non-zero

scale $\tilde{h}$ down to ε , and take $h - \tilde{h} \in C^1$

Namioka Space $X = \{f \in C[0,1] : 2f(\frac{1}{2}) = f(0) + f(1)\}$
has codimension 1 in $C[0,1]$ is not a Riesz space.

Definition: Let $(X, \geq)$ be a Riesz space. A linear subspace $Y \subseteq X$ is a Riesz subspace if for all $x, y \in Y$, $(x \vee y)$ taken in X is contained in Y .

Example: $\text{Aff}[0, 1] = \{x \mapsto ax + b : a, b \in \mathbb{R}\}$ is a Riesz space.



However, $\text{Aff}[0, 1]$ is not a Riesz subspace of $([0, 1])^I$.

Definition: Let $(X, \geq)$ be a Riesz space.

(i) $x^+ = x \vee 0$, positive part of x .

(ii) $x^- = (-x) \vee 0$, negative part of x .

(iii) $|x| = x^+ + x^- = x \vee (-x)$.

(iv) $x, y \in X$ are disjoint if

$$x \perp y \Leftrightarrow |x| \wedge |y| = 0$$

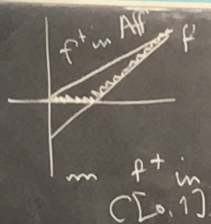
In $C[0,1]$:

$$f^+(x) = \max(f(x), 0)$$

$$|f|(x) = |f(x)|$$

$f, g \in C[0,1]$ are disjoint if

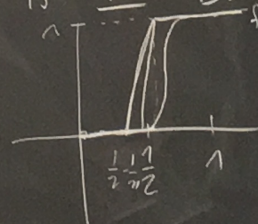
$$\forall x \in [0,1]: f(x) = 0 \vee g(x) = 0.$$



A Riesz space $(X, \geq)$ is Dedekind complete if for every subset $Y \subseteq X$ that is bounded from above $\sup Y$ exists in X .

Examples:- $\mathbb{R}^n$, $B(S)$ are Dedekind complete.

- $C[0,1]$ is not Dedekind complete.



Theorem: Every Dedekind complete Riesz space is Archimedean.

In particular, every subspace of X is Archimedean.

Proof: Let $(X, \geq)$ be a Ded complete Riesz space and $x, y \in X$ st. $nx \leq y \forall n$

$A = \{nx : n \in \mathbb{N}\}$, let $z = \sup A$.

then $2z = \sup(2A) \leq \sup(A) = z$

So $z \leq 0$

$x \leq 0$. $\square$

Theorem: Every Archimedean directed pos is a linear subspace of a Dedekind complete Riesz space.