§1.7 Order bounded sets

Let $(X, \leq)$ be a pov.

<u>Def</u>. $A \subseteq X$ is <u>order bounded</u> if $\exists\, a, b \in X : \forall x \in A : a \leq x \leq b$.

<u>Def</u> order interval:

$[a,b] := \{x \in X : a \leq x \leq b\}$.

<u>Note</u>. $A \subseteq X$ order bdd $\iff$ A contained in order interval.

If u order unit in X, then:

- $X = \bigcup_{n \in \mathbb{N}} [-nu, nu]$
- X Archimedean $\iff$ $\bigcap_{n \in \mathbb{N}} [-\frac{1}{n}u, \frac{1}{n}u] = \{0\}$.

<u>Chapter 2</u> Dual spaces and operators.

Def Let X, Y be povs.

$L(X,Y) = \{T : X \to Y : T \text{ is linear}\}$.

$T \in L(X,Y)$ is called <u>positive</u>

if $\forall x \in X$:

$x \geq 0 \implies Tx \geq 0$.

(Equiv:

$x \leq y \implies Tx \leq Ty$)

Ex $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ with standard order

then A positive $\iff \forall i,j \quad a_{ij} \geq 0$.

Ex $(X, =)$.

Every $T : X \to Y$ is positive.

Thm. $K := \{T \in L(X,Y) : T \text{ is positive}\}$.

Then (1) K is a <u>wedge</u> in $L(X,Y)$

(2) If X is directed, then K is a <u>cone</u> in $L(X,Y)$.

Pf • $T \in K$, $\alpha \geq 0 \implies \alpha T \in K$:

$\forall x \in X$, $x \geq 0$: $Tx \geq 0$, so $\alpha Tx \geq 0$ so $\alpha T \in K$.

• $S, T \in K \implies S + T \in K$.

$\forall x \in X$, $x \geq 0$. $Sx \geq 0$, $Tx \geq 0$, so $(S+T)x = Sx + Tx \geq 0$

so $S+T$ is positive.

(2): $(-K) \cap K = \{0\}$.

Let $T \in (-K) \cap K$. Then $T, -T$ positive.

If $x \in X$, $x \geq 0$: $Tx \geq 0$, $-Tx \geq 0$

$Tx \leq 0$

so $Tx = 0$.

If X is directed, X is spanned by

$X^+ = \{x \in X : x \geq 0\}$.

So $T = 0$. $\square$

Def $T \in L(X, Y)$ is called order bounded if for every order bounded $A \subseteq X$ the set $T(A)$ is order bounded in Y.

Equiv: $\forall a, b \in X$:

$T([a,b])$ is contained in an order interval.

Note: T positive $\Rightarrow$ T is order bounded.

$L^{\sim}(X,Y) = \{T \in L(X,Y) : T \text{ is order bounded}\}$.

<u>Thm</u>. $L^{\sim}(X,Y)$ is a linear subspace of $L(X,Y)$.

<u>Def</u> $X^{\sim} = L^{\sim}(X,\mathbb{R})$ order dual (order bounded dual) of X

<u>Def</u> $T \in L(X,Y)$ is called <u>regular</u> if $\exists\, T_1, T_2$ positive s.t. $T = T_1 - T_2$.

$L^{r}(X,Y) = \{T \in L(X,Y) : T \text{ is regular}\}$.

<u>Thm</u>. $L^{r}(X,Y)$ is a lin. subspace of $L(X,Y)$ and $L^{r}(X,Y) \subseteq L^{\sim}(X,Y)$.

In general: $L^{r}(X,Y) \subsetneqq L^{\sim}(X,Y)$.

<u>Thm</u>. If X is directed and Y Archimedean, then $L(X,Y)$ is Archimedean.

<u>Prf</u>: Let S, T s.t. $nS \le T$ $(\forall n)$ To show $S \le 0$.

$\forall x \in X^+$: $nSx \le T$ $(\forall n)$ so $Sx \le 0$ so $S \le 0$ $\square$

<u>Cor</u> $X^{\sim}$ is Archim.

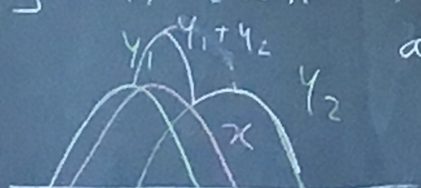
§2.? Riesz decomposition property (RDP).

Def Let X pov. X has RDP if:

$\forall\, x, y_1, y_2 \in X^+$ with $x \leq y_1 + y_2$

$\exists\, x_1, x_2 \in X^+ : \quad x = x_1 + x_2$

and $\quad x_1 \leq y_1, \; x_2 \leq y_2.$



Thm. Every Riesz space has RDP.

Prf. Take:

$x_1 = x \wedge y_1, \quad x_2 = x - x \wedge y_1.$

Then $x = x_1 + x_2$. Also:

$x_1 \geq 0,$

$x_2 = (x - x \wedge y_1) = (x - y_1)^+ \geq 0.$

$x \geq y_1: \; x - y_1 = x - y_1$

$x \leq y_1: \; x - x = 0$

$x_1 \leq x \wedge y_1 \leq y_1$

$x_2 \leq y_2:$

$$x_2 = (x - y_1)^+ \overset{x \le y_1 + y_2}{\le} (y_1 + y_2 - y_1)^+ = y_2^+ = y_2$$

$\square$

Thm. If X has RDP and $x, y_1, \dots, y_n \in X^+$ s.t. $x \le y_1 + \dots + y_n$ then $\exists\, x_1, \dots, x_n \in X^+$ s.t. $x = x_1 + \dots + x_n$ and $\forall i \;\; x_i \le y_i$.

Prf. Induction.

Thm. X has RDP $\iff \forall x_1, x_2 \in X$: $[0, x_1] + [0, x_2] = [0, x_1 + x_2]$

Thm (F. Riesz). X p.o.v. Equivalent are:

(a) X has RDP

(b) $\forall a_1, a_2, b_1, b_2 \in X$ with $a_i \le b_j \;\; \forall i, j \in \{1, 2\}$ then $\exists\, x \in X$: $a_i \le x \le b_j \;\; \forall i, j \in \{1, 2\}$

Riesz Interpolation property (RIP).

(c) $\forall a_1, \dots, a_m, b_1, \dots, b_n \in X$ with $a_i \le b_j \;\; \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ $\exists\, x \in X$ $a_i \le x \le b_j \;\; \forall i, j$

(d) $\forall x_1, \dots, x_m, y_1, \dots, y_n \in X^+$ s.t. $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j$ then $\exists\, z_{ij} \in X^+$ s.t. $x_i = \sum_{j=1}^n z_{ij}$ and $y_j = \sum_{i=1}^m z_{ij}$.

Proof

(a) $\Longrightarrow$ (d): We know: $\forall x, y_1, \dots, y_n \in X^+$ s.t. $x \le y_1 + \dots + y_n$

$\exists x_1, \dots, x_n \in X^+$: $x = x_1 + \dots + x_n$, $x_i \le y_i$, $i = 1, \dots, n$.

Induction on m.

$m = 1$: $x = y_1 + \dots + y_n$. ok.

Suppose true for m. Let

$$\sum_{i=1}^{m+1} x_i = \sum_{j=1}^{n} y_j.$$

Then $\sum_{i=1}^{m} x_i \le \sum_{j=1}^{n} y_j$ so by RDP

$\exists u_1, \dots, u_n$: $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} u_j$, $0 \le u_j \le y_j$

By hypothesis: $\exists z_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$:

$$\forall i \in \{1, \dots, m\}: \quad x_i = \sum_{j=1}^{n} z_{ij}$$

$$\forall j \in \{1, \dots, n\}: \quad u_j = \sum_{i=1}^{m} z_{ij}.$$

Take:

$$z_{m+1,j} = y_j - u_j, \quad j = 1, \dots, n.$$

Then

$$\sum_{i=1}^{m+1} z_{ij} = u_j + y_j - u_j = y_j$$

$$\sum_{j=1}^{n} z_{m+1,j} = \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} u_j = \underbrace{\sum_{1}^{m+1} x_j - \sum_{1}^{m} x_j}_{= x_{m+1}}$$

(d) $\Rightarrow$ (b):

$$\underbrace{(b_1 - \underbrace{a_1)}_{\geq 0}}_{= z_{11} + z_{12}} + \underbrace{(b_2 - \underbrace{a_2)}_{\geq 0}}_{z_{21} + z_{22}} = \underbrace{(b_1 - \underbrace{a_2)}_{\geq 0}}_{z_{11} + z_{21}} + \underbrace{(b_2 - \underbrace{a_1)}_{\geq 0}}_{z_{12} + z_{22}}$$

Take $x := a_1 + z_{12}$.

Thm (Riesz-Kantorovich)
Let X be pov with RDP and Y a Dedekind complete Riesz space.
Then $L^\sim(X,Y)$ is a Riesz space, even Dedekind complete.

Thm (Kantorovich) Let X be a directed pov, Y pov, and $T: X^+ \to Y^+$ st.

$T(x_1 + x_2) = Tx_1 + Tx_2$ all $x_1, x_2 \in X^+$

and $T(\lambda x) = \lambda T x$ all $x \in X^+, \lambda \geq 0$.

Then T extends uniquely to a positive linear map $X \to Y$.

Pf. ($\forall x \in X$: $x = x_1 - x_2$, $x_1, x_2 \in X^+$
$Tx := Tx_1 - Tx_2$.)

Let $x_1, x_2, x_3, x_4 \in X^+$ st.
$x_1 - x_2 = x_3 - x_4$.
Then $x_2 + x_4 = x_1 + x_3$
So $Tx_2 + Tx_4 = T(x_2 + x_4)$
$= T(x_1 + x_3) = Tx_1 + Tx_3$
So $Tx_1 - Tx_2 = Tx_3 - Tx_4$.
Hence for $x \in X$, $x_1, x_2 \in X^+$ with $x = x_1 - x_2$:
$Tx = Tx_1 - Tx_2$ is well defined.

• $T(v_1 + v_2) = Tv_1 + Tv_2$:

$v_1 = x_1 - x_2$, $x_1, \dots, x_4 \in X^+$

$v_2 = x_3 - x_4$

$v_1 + v_2 = (x_1 + x_3) - (x_2 + x_4)$

So $T(v_1 + v_2) = T(x_1 + x_3) - T(x_2 + x_2)$

To show: $= Tx_1 - Tx_2 + Tx_3 - Tx_4$.

$T(x_1 + x_3) + Tx_2 + Tx_4$

$= Tx_1 + Tx_3 + Tx_2 + Tx_4$

$= Tx_1 + Tx_3 + T(x_2 + x_4)$

□

<u>Thm</u> Let X pov with RDP, Y Ded compl Riesz space, $T \in \mathcal{L}^\sim(X, Y)$.
For $x \in X^+$:

$$Sx := \sup\{ Tu : 0 \le u \le x \}$$

exists in Y^+ and $S: X^+ \to Y^+$ is additive and positively homogeneous and its extension to X is the positive part of T (i.e. $S = T \vee 0$)

<u>Proof</u> Fix $x \in X^+$. T is order bounded so $T([0, x])$ is order bounded, hence bounded above hence the sup in Y exists.
For $u = 0$ $Tu = 0$. So $Sx \ge 0$.

For $\lambda \geq 0$: $0 \leq u \leq x \iff 0 \leq \lambda u \leq \lambda x$

and $T(\lambda u) = \lambda T u$

So $S(\lambda x) = \lambda S x$.

Next let $x_1, x_2 \in X^+$. To show: $S(x_1 + x_2) = S(x_1) + S(x_2)$.

Let $u_1 \in X^+$ $u_1 \leq x_1$

$u_2 \in X^+$ $u_2 \leq x_2$

Then $0 \leq u_1 + u_2 \leq x_1 + x_2$

and $T(u_1 + u_2) = T(u_1) + T(u_2)$.

Hence $S(x_1 + x_2) \geq T(u_1 + u_2) = T(u_1) + T(u_2)$.

Sup over u_1: $S(x_1 + x_2) \geq S(x_1) + T(u_2)$

Sup over u_2: $S(x_1 + x_2) \geq S(x_1) + S(x_2)$.

Now let $0 \leq u \leq x_1 + x_2$.

By RDP: $\exists\, u_1, u_2 \in X^+$ s.t.

$u = u_1 + u_2$, $u_1 \leq x_1$

$u_2 \leq x_2$.

and $Tu = Tu_1 + Tu_2$.

$\leq S(x_1) + S(x_2)$

So $S(x_1 + x_2) \leq S(x_1) + S(x_2)$.

Hence S is additive.

By Kant. S extends to positive linear map $\bar{S}: X \to Y$.

Claim: $\bar{S} = T^+$.

<u>Prf</u>: $\bar{S} \geq 0$. For $x \in X^+$: $Sx \geq Tx$ (take $u = x$)

so $\bar{S} \geq T$. Hence $\bar{S} \geq T^+$.

Suppose R is $\geq 0, T$.

To show: $\bar{S} \leq R$.

For $x \in X^+$ and $0 \leq u \leq x$:

$Tu \leq Ru \leq Rx$

so $Sx \leq Rx$.

Hence $\bar{S} \leq R$. $\square$

<u>Thm</u> Let X be a pov with RDP and Y be a Ded. compl. Riesz space. Then $L^{\sim}(X, Y)$ is Ded. complete.

"<u>Prf</u>": Let $\mathcal{T} \subseteq L^{\sim}(X, Y)$ be bounded above, i.e. $\exists\, T_0 \in L^{\sim}(X, Y)$ s.t. $T \leq T_0 \quad \forall T \in \mathcal{T}$.

Define:

$$\hat{\mathcal{T}} = \{T_1 \vee \dots \vee T_n : T_1, \dots, T_n \in \mathcal{T}\}.$$

still bdd above by T_0.

$x \in X^+$

$Sx := \sup\{Tx : T \in \mathcal{T}\}$.

$Sx := \sup\{Tx : T \in \hat{\mathcal{T}}\}$, $x \in X^+$ is linear. $\square$

Thm (Riesz-Kantorovich formulas)

X pov, RDP, Y Ded compl. For $S, T \in L^{\sim}(X, Y)$:

for every $x \in X^+$:

$$|T| = \sup\{ Tu : -x \leq u \leq x \}$$

$$S \vee T = \sup\{ Su + T(x-u) : 0 \leq u \leq x \}$$

Open problem. Consider X, Y Riesz spaces and $T \in L^{\sim}(X, Y)$ s.t. $\sup\{T, 0\}$ exists.

Is it true that $\forall x \in X^+$:

$$T^+ x = \sup\{ Tu : 0 \leq u \leq x \} \;?$$

Cor. If X is a Riesz space then $X^{\sim}$ is a Dedekind complete Riesz space.

CORRECTION:

in all results on $L^{\sim}(X, Y)$

replace RDP by

RDP + directed

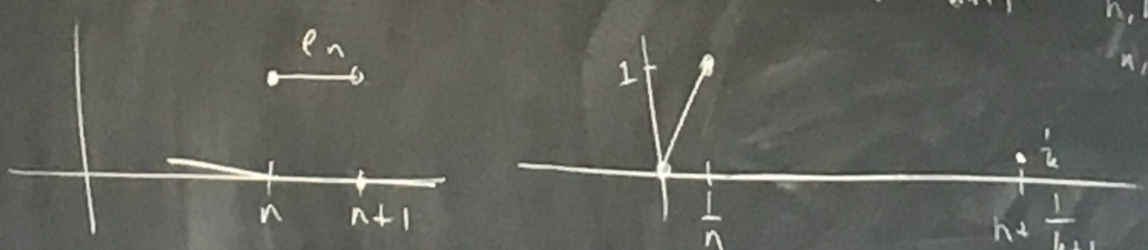
let

$$e_n := 1_{[n,n+1)}, \quad n \in \mathbb{N} = \{0,1,\dots\}.$$

$$u_{n,k}(t) = nt\, 1_{[0,\frac{1}{n}]}(t) + \frac{1}{k} 1_{\{n+\frac{1}{k+1}\}}(t), \quad t \in [0,\infty)$$

$n,k \in \mathbb{N}$

$n,k \geq 1.$



$$X = \operatorname{span}\{e_n, u_{m,k},\ n \in \mathbb{N},\ m,k \in \mathbb{N},\ m,k \geq 1\} \subseteq \{f : [0,\infty) \to \mathbb{R} \text{ bdd}\}$$

$\exists\ \lambda_{n,k},\ \mu_n$ s.t.

$$x = \sum_{n,k} \lambda_{n,k} u_{n,k} + \sum_n \mu_n e_n$$

(finite sums).

As $x = 0$ on (N, ∞)

so for $\underline{n \geq N}$: $x(n+\frac{3}{4}) = 0$

so $\sum_{\ell,k} \lambda_{\ell,k} \underbrace{u_{\ell,k}(n+\tfrac{3}{4})}_{=0} + \sum_m \mu_m \underbrace{e_m(n+\tfrac{3}{4})}_{=0}$

so $\mu_n = 0$

so $\sum_{\ell,m} \lambda_{\ell,m} u_{\ell,m}(n+\frac{1}{k+1}) = 0$ for $k \geq 1$

so $\lambda_{n,k} = 0$.

$\varphi(x) = x'(0)$. $x \in E$.

(a) X is directed and Archimedean.

Arch since $X \subseteq \mathbb{R}^{[0,\infty)}$

Let $x, y \in X$. Then x and y both bounded with bounded support, hence $\exists n \in \mathbb{N}$

$$x, y \leq \underbrace{n(e_1 + \cdots + e_n)}_{\in X}.$$

(b) φ is order bounded:

Sufficient: $\forall a \in X^+$, φ is bounded on $[0, a]$.

Let $a \in X^+$. Then $\exists N \in \mathbb{N}^*$, $C \in (0, \infty)$ st

$\mathrm{supp}(a) \subseteq [0, N]$

$a \leq C\mathbb{1}$

Let $x \in [0, a]$. Then $0 \leq x \leq a$

So $\mathrm{supp}(x) \subseteq [0, N]$

$x \leq C\mathbb{1}$.

(c) There is no $\psi : X \to \mathbb{R}$ linear s.t. $\psi \geq 0$ and $\psi \geq \varphi$.

Suppose such a ψ exists. For $n \in \mathbb{N}, n \geq 1$, take $k_n \in \mathbb{N}$. $k_n > \psi(e_n)$.

Then $\frac{1}{k_n}\psi(e_n) < 1$ and

$u_{n,k_n} \leq e_0 + \frac{1}{k_n} e_n$

So

$$\psi(e_0) = \psi(e_0 + \tfrac{1}{k_n} e_n) - \tfrac{1}{k_n}\psi(e_n)$$
$$\geq \psi(u_{n,k_n}) - \tfrac{1}{k_n}\psi(e_n) \geq \psi(u_{n,k_n}) - 1$$
$$\geq \varphi(u_{n,k_n}) - 1 = n - 1.$$

Contradiction

(d) φ is not regular:

Suppose it is: $\exists\, \varphi_1, \varphi_2 : X \to \mathbb{R}$ positive linear: $\varphi = \varphi_1 - \varphi_2$.

Then $\varphi = \varphi_1 - \varphi_2 \leq \varphi_1$

and φ_1 is positive

Contradicts (c).