The material in this lecture is based on

M. Messerschmidt, *Geometric duality theory of cones in dual pairs of vector spaces*, J. Funct. Anal. **269** (2015), no. 7, 2018–2044.

Geometric duality of cones

Let X be a real Banach space pre-ordered by a closed cone C in X.

Definition Let $\alpha > 0$.

X is said to α-max normal if, for any y, z, x, $z \le x \le y$ implies $\|x\| \le \alpha \max\{\|z\|, \|y\|\}$

X is said to be α-sum conormal if, for any $x \in X$, there exists $x_1, x_2 \in C$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| \le \alpha \|x\|$

Theorem

For $\alpha > 0$.

X is α-max normal iff X' is α-sum conormal

Note 1. X is α-max normal iff for $Y = X \oplus_\infty X$, the subsets

$$A = C \oplus (-C)$$

$$D = \{(x,y) \in Y : x = y\}$$

$$B_1 = \{(x,y) \in D : \|x\| \le \alpha\}$$

$$B_2 = \{(x,y) \in Y : \max\{\|x\|, \|y\|\} \le 1\}$$ satisfy

$$(B_2 + A) \cap D \subseteq B_1$$

Note 2

X is α-sum conormal iff for $Y = X \oplus_1 X$ the subsets

$$A = C \oplus (-C)$$

$$D = \{(x,y) \in Y : x + y = 0\}$$

$$B_1 = \{(x,y) \in Y : \|x + y\| \le 1\}$$

$$B_2 = \{(x,y) \in Y : \|x\| + \|y\| \le \alpha\}$$

Satisfy $B_1 \subseteq (B_2 \cap A) + D$.

Definition (Dual pair)

A dual (Y,Z) is a pair of vector spaces Y and Z together with a bilinear map $\langle \cdot,\cdot \rangle\ Y \times Z \to \mathbb{R}$ ($\langle \cdot,\cdot \rangle$ is linear in each variable separately) such that

- $\langle x,y \rangle = 0\ \forall y \in Y$, then $z = 0$
- $\langle x,y \rangle = 0\ \forall z \in Z$, then $y = 0$.

$\langle \cdot,\cdot \rangle$ is called the duality of (Y,Z)

Example

(X,τ) locally convex space. Then (X,X') is a dual pair under $\langle x,x' \rangle = x'(x)$

$y \in Y \quad \langle \cdot, \rangle\ Z \to \mathbb{R}$

$z \mapsto \langle y,z \rangle$

$Y \subseteq Z^* \subseteq \mathbb{R}^Z$

$y \to \langle \cdot, z \rangle$

Consider $(Y, \sigma(Y,Z))$

$(Z, \sigma(Z,Y))$

Thm

$(X, \sigma(Y,Z))' = Z$

$(Z, \sigma(Z,Y))' = Y$

(One-sided polar calculus)

Consider a dual (Y,Z) and $A \subseteq Y$ and $B \subseteq Z$. The one-sided polar of A, denoted by $A^{\odot}$, is defined by

$A^{\odot} = \{ z \in Z : \langle a,z \rangle \leq 1\ \forall a \in A \}$

Simirlary, the one-sided polar of B, denoted by $B^{\odot}$, is defined by

$B^{\odot} = \{ y \in Y : \langle y,b \rangle \leq 1\ \forall b \in B \}$

Lemma

Let (Y,Z) be a dual pair, A a non-empty subset of Y, C a cone in A, $\{A_i\}_{i \in M}$ a collection of subsets of A. Then

1) $A^{\odot}$ contains zero, is convex and $\sigma(Z,Y)$-closed
2) If $A \subseteq B$, then $B^{\odot} \subseteq A^{\odot}$
3) If $\lambda > 0$, then $(\lambda A)^{\odot} = \frac{1}{\lambda} A^{\odot}$
4) $\left(\bigcup_{i \in I} A_i \right)^{\odot} = \bigcap_{i \in I} A_i^{\odot}$
5) $C^{\odot}$ closed cone $C^{\odot} = -C'$

ts

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$\leq q\}$

$-D.$

(6) If A is a $\sigma(Y,Z)$-closed convex set which contains zero, then

$$(A \cap C)^{\circ} = \overline{(A^{\circ} + C^{\circ})}$$

(7) If A is convex and contains zero, then

$$(A + C)^{\circ} = A^{\circ} \cap C^{\circ}$$

(Section 4 in Miek's paper for $\Omega = \mathbb{N}$)

Assumptions: For a real Banach space, the map $\langle \ , \ \rangle : X \times X' \to \mathbb{R}$ will mean $\langle x, x' \rangle = x'(x)$

$$B_X = \{x \in X : \|x\| \leq 1\}$$

For subspaces $Y \subseteq X^{\mathbb{N}}$ and $Z \subseteq X'^{\mathbb{N}}$, if

$Y \times Z \to \mathbb{R}$,

$$\left((y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}\right) \mapsto \sum_{n=1}^{\infty} \langle y_n, z_n \rangle$$

defines a duality, then we will denote it by $\langle\langle \cdot , \cdot \rangle\rangle$.

Definition

Let X be a Banach space

We recall the l_1-direct sum of countably many copies of X

- $\ell^1(\mathbb{N},X) = \left\{(x_n)_{n\in\mathbb{N}} : x_n \in X, \sum_{n=1}^{\infty} \|x_n\| < \infty\right\}$

The ℓ^∞-direct sum of countably many copies of

- $\ell^\infty(\mathbb{N},X) = \left\{(x_n)_{n\in\mathbb{N}} : x_n \in X, \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$
- $c(\mathbb{N},X)$ closed subspace of $\ell^\infty(\mathbb{N},X)$, where

$$c(\mathbb{N},X) = \left\{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N},X) : \exists x \in X \text{ s.t. } \|x_n - x\| \to 0\right\}.$$

Remark

The duals of $c(\mathbb{N},X)$, $\ell^1(\mathbb{N},X)$ and $\ell^p(\mathbb{N},X)^{(p>1)}$ can be identified by

$\ell^1(\mathbb{N},X')$, $\ell^\infty(\mathbb{N},X')$ and $\ell^q(\mathbb{N},X')$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

Definition

Let X be a Banach space. We define the canonical summation of operator

$$\Sigma : X^{\mathbb{N}} \longrightarrow X \cup \{\infty\}$$

as follows

$$\Sigma (x_n)_{n \in \mathbb{N}} = \begin{cases} \sum_{i=1}^{\infty} x_i & \text{if } \sum_{n=1}^{\infty} x_i \text{ converges in } X \\ \infty & \text{otherwise} \end{cases}$$

We call $D(\Sigma) = \Sigma^{-1}(X)$ the domain of Σ

We define the constant part operator

$$\text{const} : X^{\mathbb{N}} \longrightarrow X \cup \{\infty\}$$

as follows

$$\text{const}(x_n)_{n \in \mathbb{N}} = \begin{cases} x & \text{if } \lim_{n \to \infty} \|x_n - x\| = 0, \\ \infty & \text{otherwise} \end{cases}$$

We call $D(\text{const}) = \text{const}^{-1}(X)$ the domain of const.

Definition

Let X be a Banach space and

- $\{C_n\}_{n \in \mathbb{N}}$ be a collection of cones in X. For $Y \subseteq X^{\mathbb{N}}$, we define
- $(\oplus C)(Y) = \{(x_n)_{n \in \mathbb{N}} \in Y : \forall n\ x_n \in C_n\}$
- $\Sigma_0(Y) = \{(x_n)_{n \in \mathbb{N}} \in Y \cap D(\Sigma) : \sum_{i=1}^{\infty} x_i = 0\}$
- $\Sigma_1(Y) = \{(x_n)_{n \in \mathbb{N}} \in Y \cap D(\Sigma) : \|\sum_{i=1}^{\infty} x_i\| \leq 1\}$
- $\Xi_\infty(Y) = \{(x_n)_{n \in \mathbb{N}} \in Y \cap D(\text{const}) : x_n = \lim_{n \to \infty} x_n = x\}$
- $\Xi_1(Y) = \Xi_\infty(Y) \cap \{(x_n)_{n \in \mathbb{N}} \in Y \cap D(\text{const}) : \|\text{const}(x_n)\| \leq 1\}$

$$= \{(x)_{n \in \mathbb{N}} : \|x\|_X \leq 1\},$$

sets

$\}$

$\leq 1\}$

$\|y\| \leq a\}$

$) + D.$

<u>Lemma</u>

Let X be a Banach space and $\{C_n\}_{n\in\mathbb{N}}$ be a collection of closed cones in X. In the dual pair $(c(\mathbb{N},X), \ell^1(\mathbb{N},X'))$ we have

(i) $\left(\oplus C(c(\mathbb{N},X))\right)^{\odot} = \oplus C^{\odot}(\ell^1(\mathbb{N},X'))$

(ii) $\left(\Xi_\infty(c(\mathbb{N},X))\right)^{\odot} = \Sigma_0(\ell^1(\mathbb{N},X'))$

(iii) $\left(\Xi_1(c(\mathbb{N},X))\right)^{\odot} = \Sigma_1(\ell^1(\mathbb{N},X'))$

(iv) $B^{\odot}_{c(\mathbb{N},X)} = B_{\ell^1(\mathbb{N},X')}$

(ii) $\left(\Xi_\infty(c(\mathbb{N},X))\right)^{\odot}$

$$= \left\{ (x)_{n\in\mathbb{N}} \quad x \in X \right\}^{\odot}$$

$$= \left\{ (x_n') \in \ell^1(\mathbb{N},X') : \langle\langle (x)_{n\in\mathbb{N}}, (x_n')\rangle\rangle \leq 1 \text{ where } x \in X \right\}$$

$$= \left\{ (x_n')_{n\in\mathbb{N}} \in \ell^1(\mathbb{N},X') : \sum_{n=1}^{\infty} x_n'(x) \leq 1, \text{ where } x \in X \right\}$$

$$\Sigma_0(\ell^1(\mathbb{N},X')) = \left\{ (x_n')_{n\in\mathbb{N}} \in \ell^1(\mathbb{N},X') \; \sum_{n=1}^{\infty} x_n' = 0 \right\}$$

Clearly $\Sigma_0(\ell^1$

$)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}, X') :$

$$\sum_{n=1}^{\infty} x_n'(x) \leq 1,$$

where $x \in X \}$

Clearly $\Sigma_0(\ell^1(\mathbb{N}, X')) \subseteq \Xi_\infty(c(\mathbb{N}$

Let $(x_n')_{n\in\mathbb{N}} \in \Xi$

and suppose

$(x_n')_{n\in\mathbb{N}} \notin \Sigma$

$\overset{\infty}{\Sigma}$

$\neq$

osed cones

$\left(c(\mathbb{N},X), \ell^1(\mathbb{N},X')\right)$

$\left(\ell^1(\mathbb{N},X')\right)$

$\left(\ell^1(\mathbb{N},X')\right)$

$\ell^1(\mathbb{N},X')\big)\big)$

Definition (General notions of normality and conormality)

Let X be a vector space

Let $C, D \subseteq X$ be cones and

$B_1, B_2 \subseteq X$ convex sets containing zero.

X is said to be **normal** w.r.t (C, D, B_1, B_2)

if $(B_2 + C) \cap D \subseteq B_1$

X is said to be conormal w.r.t (C, D, B_1, B_2)

if

$B_1 \subseteq (B_2 \cap C) + D$

Theorem

Let $q \geq 1$.

X real Banach space and

$\{C_n\}_{n\in\mathbb{N}}$ be a collection of closed cones in X

The space $c(\mathbb{N}, X)$ is normal w.r.t $\left(\oplus C(c(\mathbb{N},X)), \Xi_\infty(c(\mathbb{N},X)), q\,\Xi(c(\mathbb{N},X)), B_{c(\mathbb{N},X)}\right)$

iff

the space $\ell^1(\mathbb{N},X')$ is conormal w.r.t $\left(\oplus C^\omega(\ell^1(\mathbb{N},X')), \Sigma_0(\ell^1(\mathbb{N},X')), \Sigma(\ell^1(\mathbb{N},X')), q\,B_{\ell^1(\mathbb{N},X')}\right)$

Proof

Suppose X is a-max normal.

Let $Y := X \oplus_\infty X$ and $Z = X' \oplus_1 X'$

and the duality $\langle \, , \, \rangle : Y \times Z \to \mathbb{R}$ be given

by $\langle (a,b), (\phi,\psi) \rangle = \phi(a) + \psi(b)$ $\forall \, a,b \in X, \ \phi,\psi \in X'$

Let $D = C \oplus (-C)$

$\Xi_\infty = \{(a,b) \in Y : a = b\}$

$\Xi_1 = \{(a,b) \in \Xi_\infty : \|a\| \le 1\}$,

$B_Y = \{(a,b) \in Y : \max\{\|a\|, \|b\|\} \le 1\}$.

Also, $E = -C' \oplus C'$

$\Sigma_0 = \{(\phi,\psi) \in Z : \phi + \psi = 0\}$

$\Sigma_1 = \{(\phi,\psi) \in Z : \|\phi + \psi\| \le 1\}$, $B_Z = \{(\phi,\psi) \in Z : \|\phi\| + \|\psi\| \le 1\}$.

We showed that the one-sided polars of D, Ξ_∞, Ξ_1 and B_Y in Y is given by

$$E, \Sigma_0, \Sigma_1 \text{ and } B_Z \text{ in } Z$$

By Note 1 $(B_Y + D) \cap \Xi_\infty \subseteq q\,\Xi_1$

Hence Y is normal w.r.t $(D, \Xi_\infty, q\Xi_1, B_Y)$

By Thm we have that Z is conormal w.r.t $(E, \Sigma_0, \Sigma_1, qB_Z)$ in Z. By note 2 X' is a sum conormal.

The converse follow similarly. ∎

Proof

Suppose X

Let $Y :=$

and the

by

Let $D =$

$\Xi_\infty =$

$\Xi_1 =$

$B_Y =$

Also, E

Σ_0

Σ_1

Geometric duality of cones

Let X be a real Banach space pre-ordered by a closed cone C in X

Definition Let $q > 0$.

X is said to q-max normal if, for any y, z, x, $z \le x \le y$ implies $\|x\| \le q \max\{\|z\|, \|y\|\}$

X is said to be q-sum conormal if, for any $x \in X$, there exists $x_1, x_2 \in C$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| \le q\,\|x\|$

Lemma

Let (Y, Z) be a dual pair. Assume that $C, D \subseteq Y$ are $\sigma(Y,Z)$-closed cones and B_1, B_2 be $\sigma(Y,Z)$-closed convex sets containing zero. Then

(1) If Y is normal w.r.t (C, D, B_1, B_2), then $B_1^{\odot} \subseteq \overline{(B_2^{\odot} \cap C^{\odot}) + D^{\odot}}$

(2) If Y is conormal w.r.t (C, D, B_1, B_2), then $(B_2^{\odot} + C^{\odot}) \cap D^{\odot} \subseteq B_1^{\odot}$

(3) If Z is normal w.r.t $(C^{\odot}, D^{\odot}, B_1^{\odot}, B_2^{\odot})$, then $B_1 \subseteq \overline{(B_2 \cap C) + D}$

(4) If Z is conormal w.r.t $(C^{\odot}, D^{\odot}, B_1^{\odot}, B_2^{\odot})$, then $(B_2 + C) \cap D \subseteq B_1$.

Proof

Suppose Y is normal w.r.t (C, D, B_1, B_2). Then $(B_2 + C) \cap D \subseteq B_1$, and hence $B_1^{\odot} \subseteq \left((B_2 + C) \cap D\right.$

$$= \overline{(B_2+C)^{\odot} + D^{\odot}}$$

$$= \overline{(B_2^{\odot} \cap C^{\odot}) + D^{\odot}}$$

(2) Suppose that Y is conormal w.r.t (C, D, B_1, B_2). Then

$$B_1 \subseteq \overline{(B_2 \cap C) + D}. \text{ Then}$$

$$(B_2^{\odot}+C^{\odot}) \cap D^{\odot} \subseteq \overline{B_2^{\odot}+C^{\odot}} \cap D^{\odot} = (B_2 \cap C)^{\odot} \cap D^{\odot} = ((B_2 \cap C) + D)^{\odot} \subseteq B_1^{\odot} \quad \blacksquare$$

Thm (General duality between normality and conormality)

* (i) Z conormal w.r.t $(C^{\odot}, D^{\odot}, B_1^{\odot}, B_2^{\odot}) \Rightarrow$ Y normal w.r.t (C, D, B_1, B_2)

(ii) Y normal w.r.t (C, D, B_1, B_2) and $(B_2^{\odot} \cap C^{\odot}) + D^{\odot}$ is $\sigma(Z, Y)$-closed implies Z conormal w.r.t $(C^{\odot}, D^{\odot}, B_1^{\odot}, B_2^{\odot})$ $\blacksquare$

Theorem

Let $q \geq 1$.
X real Banach space and $\{C_n\}_{n \in \mathbb{N}}$ be a collection of closed cones in X.

The space $c(\mathbb{N}, X)$ is normal w.r.t $\left(\oplus C(c(\mathbb{N}, X))_1, \Xi_{\infty}(c(\mathbb{N}, X))_1, q\,\Xi(c(\mathbb{N}, X)), B_{c(\mathbb{N}, X)}\right)$

iff

the space $\ell^1(\mathbb{N}, X')$ is conormal w.r.t $\left(\oplus C^{\odot}(\ell(\mathbb{N}, X')), \Sigma_{c_0}(\ell(\mathbb{N}, X')), \Sigma_1(\ell(\mathbb{N}, X')), q\, B_{\ell^1(\mathbb{N}, X')}\right)$