Ideals In pre-Riesz Space

*. Suppose that I is a directed subspace of a POVS $\mathbb{X}$ TFAE:

1) I is full

2) I is solid.

3) I is solvex

Def 1 A set A of a POVS $\mathbb{X}$ is called <u>full</u> if for $\forall y, z \in A$ we have the interval $[y, z] := \{x \in \mathbb{X},\ y \leq x \leq z\} \subset A$.

Lemma 2 I is a linear subspace of a POVS $\mathbb{X}$. TFAE.

1) I is full.

2) For $\forall\ y \in I$ and $x \in \mathbb{X}$ with $-y \leq x \leq y$, one has $x \in I$

3) For $\forall\ y \in I$ and $x \in \mathbb{X}$ with $0 \leq x \leq y$, then $x \in I$

Proof: 1) $\Rightarrow$ 2) $\Rightarrow$ 3)

3) $\Rightarrow$ 1) Let $y, z \in I$ and $x \in \mathbb{X}$.

s.t. $y \leq x \leq z$

$0 \leq x - y \leq z - y \in I$

$\Rightarrow \quad x - y \in I$

$x = (x-y) + y \in I$ $\square$

Def: 3 Let V be a subspace of $\mathbb{X}$. Define the <u>directed part</u> of V to be the subspace $(V \cap K) - (V \cap K)$

Lemma 4. I is a full subspace iff the directed part of I is a full subspace.

Proof. I and the directed part of I have the same elements and 3) of Lemma 2.

Def 5. A subset A of a POVS $\mathcal{E}$ is called solid if for $x \in \mathcal{E}$ and there exists $a \in A$ with $\{-x, x\}^u \supseteq \{-a, a\}^u$, then $x \in A$.

6. Theorem 6. If I is a solid subspace of $\mathcal{E}$, then I is also full.

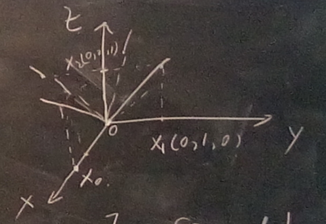
Proof: If $y \in I$ with I is solid and $x \in \mathcal{E}$ are such that $0 \leq x \leq y$

$$\{-x, x\}^u \supseteq \{-y, y\}^u$$

As I is solid $x \in I$.

By 3) of Lemma 2. we know I is full □

7. Example. Let K be the cone of R^3 consisting of positive-linear combinations of $(1,0,1)$, $(0,1,1)$ $(-1,0,1)$ and $(0,-1,1)$



The space I_0 spanned by $x_0 = (1,0,0)$

$$\{-x_0, x_0\}^u = K + (0,0,1)$$

$$\subset \{-x_i, x_i\} \quad i = 1, 2$$

But the directed part of $I_0 = (0,0,0)$

? The directed part of a full subspace is solid

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2) $\Leftrightarrow$ 1)

Theorem 8. (1) If I is full and directed then I is solid.

(2) The directed part of a full (solid) subspace I is solid.

Proof: (1). Assume that I is full and directed. let $y \in I$ and $x \in X$ be s.t.

$$\{-y, y\}^u \subseteq \{-x, x\}^u.$$

As I is directed, there is a $v \in \{-y, y\}^u \cap I$. Then $v \geq -x$ and $v \geq x$. thus.

$$-v \leq x \leq v$$

As I is full. $x \in I$ $\square$

2). combine Lemma 4 and 1) $\square$

Def. 9 A set A in a POVS X is **solvex** if $x \in X$. $x_1, \ldots x_n \in A$ and $\lambda_1, \ldots \lambda_n \in (0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ s.t.:

$$\left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_k \in \{-1, 1\} \right\}^u \subseteq \{-x, x\}^u$$

then $x \in A$.

Ideals in pre-Riesz Space

*. Suppose that I is a directed subspace of a POVS X. TFAE:

1) I is full $\quad$ 2) $\Leftrightarrow$ 1)
2) I is solid $\quad$ 3) $\Rightarrow$ 2) $\Leftrightarrow$ 1)

$\quad$ 6) $\Rightarrow$ 5) $\Rightarrow$ 4)

3) I is solvex
4) There exists a POVS (Y, L) and a positive linear map $T: X \to Y$ s.t. $I = T^{-1}(\{0\})$
5) There exists a POVS (Y, L) and a Riesz* hom $T: X \to Y$ s.t. $I = T^{-1}(\{0\})$
6) There exists a POVS (Y, L) and a Riesz hom. $T: X \to Y$ s.t. $I = T^{-1}(\{0\})$

Def 10. Let (Y, L) be a POVS and $T: X \to Y$ be a liner map. T is called Riesz hom. if for $\forall x, y \in X$ one has

$$T(\{x, y\}^u)^l = \{T(x), T(y)\}^{ul}$$

Def. 11 Let (Y, L) be a POVS and $T: X \to Y$ be a linear map. T is called Riesz* hom. if for any finite set $F \subset X$ one has

$$T[F^{ul}] \subseteq (T[F])^{ul}.$$

Proposition 12. Let (Y, L) be a POVS and $T: X \to Y$ be a Positive linear operator s.t. for $\forall w \in L$, there is $x \in K$ with $T(x) = w$. If the kernel $T^{-1}(\{0\})$ is directed, then T is a Riesz hom.

iff

a pos

$T^{-1}(\{0\})$

Proof. Let $x, y \in E$. As T is positive, then

$$\{T(x), T(y)\}^{ul} \subseteq T(\{x, y\}^u)^l.$$

It remains to show. $T(\{x, y\}^u)^l \subseteq \{T(x), T(y)\}^{ul}$

We claim that. $\{T(x), T(y)\}^u \subseteq T(\{x, y\}^u)$

Let $\underline{w \in \{T(x), T(y)\}^u}$, $w \geq T(x)$,

$w \geq T(y)$. Then $\underline{w - T(x) \in L}$. there is

a $x_1 \in K$ with. $T(x_1) = \underline{w - T(x)}$

Put $x_2 = x + x_1$, then $x_2 \geq x$ and

$$T(x_2) = T(x) + T(x_1) = w.$$

In a similar way, there is $y_2 \geq y$ and $T(y_2) = w$

Clearly, $y_2 - x_2 \in T^{-1}(\{0\})$. Consider that $T^{-1}(\{0\})$ is

dirated. there is a $z \in T^{-1}(\{0\})$ with $z \geq 0$

and $z \geq y_2 - x_2$. put $v = x_2 + z$ and also

$$T(v) = T(x_2) + T(z) = w. \quad v \geq \underline{x_2 \geq x}$$

and $v \geq x_2 + (y_2 - x_2) = \underline{y_2 \geq y}$

Hence, $v \in \{x, y\}^u$ and $T(v) = w$

So $w \in T(\{x, y\}^u)$ $\square$

Corollary 13. If I is a directed full subspace then the quotient map

$$q : E \to E/I$$

is a Riesz hom.

$$I = q^{-1}(\{0\})$$

Corollary 14. Let $f : E \to \mathbb{R}$ be a positive functional s.t there exists $x \in K$ with $f(x) > 0$. If $f^{-1}(\{0\})$ is directed then f is Riesz hom.

Ideals In pre-Riesz Space

X. Suppose that I is a directed subspace of a POVS X TFAE:

1) I is full $2) \Leftrightarrow 1)$
2) I is solid $3) \Rightarrow 2) \Rightarrow 1)$
  $6) \Rightarrow 5) \Rightarrow 4)$
3) I is solvex
4) There exists a POVS (Y, L) and a positive linear map $T: X \to Y$ s.t $I = T^{-1}(\{0\})$
5) There exists a POVS (Y, L) and a Riesz* hom. $T: X \to Y$ s.t $I = T^{-1}(\{0\})$
6) There exists a POVS (Y, L) and a Riesz hom. $T: X \to Y$ s.t $I = T^{-1}(\{0\})$

Corollary 13: $1) \Rightarrow 6)$

Proposition 14: $1) \Leftrightarrow 4)$

$$3) \Rightarrow 2) \Rightarrow 1)$$
$$\Updownarrow$$
$$6) \Rightarrow 5) \Rightarrow 4)$$

Proposition 14. The subspace I is full iff there exists a POVS (Y, L) and a pos. linear map $T: X \to Y$. s.t $I = T^{-1}(\{0\})$

Proof: If I is a full subspace, then the quotient space $Y = X/I$ is POVS. then quotient map $T := q$ is pos. and $I = T^{-1}(\{0\})$.

If $T: X \to Y$ is positive, $x \in X$ and $y, t \in I := T^{-1}(\{0\})$, with $y \le x \le t$.

$$0 = T(y) \le T(x) \le T(t) = 0$$

$x \in I$ $\quad \overset{\shortparallel}{0}$

$\square$

l iff

d a pos.

$T^{-1}(\{0\})$

then the

POVS.

s pos.

$x \in X$

$x \le z$.

$\square$

Theorem 15. Let (Y, L) be a POVS. If $T: X \to Y$ is a Riesz* hom, then the kernel $T^{-1}(\{0\})$ is a solvex subspace of X.

Proof: Let $x \in X$, $x_1, \dots, x_n \in I$ and $\lambda_1, \dots, \lambda_n \in (0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ be s.t.

$$\left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k,\ \varepsilon_k \in \{-1, 1\} \right\}^{u} \subseteq \{-x, x\}^{u}$$

In other words.

$$\{-x, x\}^{ul} \subseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k,\ \varepsilon_k \in \{-1, 1\} \right\}^{ul}$$

we only need to show $x \in I$

By applying T to this inclusion and due to the reformulation of the Riesz* hom for finitely elements in [15, Theorem 5.3]

$$\begin{aligned} T(\{-x, x\}^{ul}) &\subseteq T\left(\left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k,\ \varepsilon_k \in \{-1, 1\} \right\}^{ul} \right) \\ &\subseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k T(x_k),\ \varepsilon_k \in \{-1, 1\} \right\}^{ul} \\ &= \{0\}^{ul} = -L \end{aligned}$$

As $x \in \{-x, x\}^{ul}$ and $-x \in \{-x, x\}^{ul}$.

So $T(x) \in -L$, $T(x) \le 0$, $T(-x) \le 0$

Then $T(x) = 0$, $x \in I$

$\square$

$Y, L)$ and
$= T^{-1}(\{0\})$
nd a
$T^{-1}(\{0\})$

Def 16. A linear subspace D of a POVS Y is called order dense if $\forall\, y \in Y$, one has

$$y = \inf\{d \in D,\ d \geq y\}$$

Proposition 17. Let Y be a directed POVS and X an order dense subspace of Y

i) If J is a solvex subspace of Y then $J \cap X$ is solvex in X

Proof: i) Let $x \in X$, $x_1, \dots, x_n \in J \cap X$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ with $\sum_{k=1}^{n} \lambda_k = 1$ be s.t.

$$\{-x, x\}^u \cap X \supseteq \left\{\sum_{k=1}^{n} \varepsilon_k \lambda_k x_k,\ \varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}\right\}^u \cap X$$

If $v \in Y$ is an upper bound of the set $\left\{\sum_{k=1}^{n} \varepsilon_k \lambda_k x_k,\ \varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}\right\}$, then

$$\{u \in X,\ u \geq v\} \subseteq \left\{\sum_{k=1}^{n} \varepsilon_k \lambda_k x_k;\ \varepsilon_1 \dots \varepsilon_n \in \{-1,1\}\right\}^u \cap X \subseteq \{x, -x\}^u \cap X$$

which implies $\underline{v = \inf\{u \in X : u \geq v\}} \geq x$.

In a similar way $v \geq -x$. So

$$\{x, -x\}^u \supseteq \left\{\sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}\right\}^u$$

Proof: i) Let $x \in Z$, $x_1, \dots, x_n \in J \cap Z$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ with $\sum_{k=1}^{n} \lambda_k = 1$ be s.t.

$$\{-x, x\}^u \cap Z \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\} \right\}^u \cap Z$$

If $v \in Y$ is an upper bound of the set $\left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\} \right\}$, then

$$\{u \in Z, u \geq v\} \subseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_1 \dots \varepsilon_n \in \{-1, 1\} \right\}^u \cap Z \subseteq \{x, -x\}^u \cap Z$$

which implies $\underline{v = \inf\{u \in Z : u \geq v\}} \geq x$.

In a similar way $v \geq -x$. So

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k : \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\} \right\}^u$$

As J is solvex in Y

So $x \in J$. So $x \in J \cap Z$

Hence, $J \cap Z$ is also solvex.

Proposition 18 Let Y be a vector lattice and let Z be an order dense subspace of Y. For an ideal $\underline{I}$ in Z, TFAE

1) There is an ideal J in Y s.t.

$$I = J \cap Z$$

2) I is $\underline{\text{solvex}}$ in Z