

Abstract Harmonic Analysis

Folland: analysis where a locally
cpt. group plays an essential
group.

More specifically, unitary
representations of these groups
+ analysis of space of functions
on these groups, and their
homogeneous spaces G/H (Hecht)

Examples.

$$1^{\circ} G = \mathbb{R}, H = L^2(\mathbb{R})$$

$$\text{Define } [\pi(x)f](y) = f(y-x) \quad \begin{matrix} f \in H \\ x \in G \\ y \in \mathbb{R} \end{matrix}$$

$$\text{Then } \pi: G \rightarrow U(H)$$

is a homomorphism.

Want to understand structure of H
as a G -space

Answer: Fourier analysis
(notably, Plancherel theorem)

Special
loc. cpt.

fract.
intgr.

Four

$\rightarrow C$

2^o $G =$

wh

Special case of Fourier analysis on
loc. cpt. abelian groups. A general
framework that unifies Fourier
integrals ($G = \mathbb{R}^n$) and
Fourier series ($G = \mathbb{T}^n$)

→ Chapter 4.

2° $G = SO(3, \mathbb{R})$. 3x3 real matrices, orthogonal
rotations in $\mathbb{R}^3$ of determinant 1

$d\sigma$ = rotation inv. measure on S^2
 $H = L^2(S^2, d\sigma)$.

Define $[\pi(g)f](\omega) = f(g^{-1}\omega)$ $\begin{matrix} f \in H \\ g \in G \\ \omega \in S^2 \end{matrix}$

Then $\pi: G \rightarrow U(H)$ is a homomorphism.
Want to understand this representation
and other unitary reps. of cpt groups

→ Chapter 5.



Unitary reps of arbitrary G
much more difficult $\searrow GL(n, \mathbb{R})$

Sketched in Ch 7.

Non operator algebras

Layout of the book:

Ch 1. $\rightarrow$ Banach algs, comm. ones
we $\rightarrow$ + Gelfand theory 1.1-1.3
Spectral theory for com.
 C^* -alg 1.5-1.6
 $\vee$ Neumann algs 1.7

$\rightarrow$ Ch 2: setting the scene
 $\rightarrow$ Ch 3: general rep theory
 $\rightarrow$ Ch 4: Fourier analysis on abelian groups
 $\rightarrow$ Ch 5: ————— on cpt groups

Ch 6: How to build reps of G
from reps of $H \subset G$:
"induction"

Ch 7: Dec theory for general loc
cpt. groups. (overview)

Dixmier's books, in
particular C^* -algebras

Recall Banach alg. complex Banach space
that is associative algebra such that
 $\|xy\| \leq \|x\| \|y\| \quad (x, y \in A)$

Involution $*$ $A \rightarrow A$
 $x \mapsto x^*$

s.t.

- 1) $(x^*)^* = x$
- 2) $(xy)^* = y^* x^*$
- 3) $(x+y)^* = x^* + y^*$
- 4) $(\lambda x)^* = \overline{\lambda} x^*$

C^* -algebra: un. B.A. s.t.
 $\|x^* x\| = \|x\|^2$

Examples:

1. X cpt Hld. Then $(C(X) \| \cdot \|_\infty)$ is
a comm. C^* -algebra with $f^* = \overline{f}$.
unital
2. X loc. cpt. Hld. Then $(C_0(X) \| \cdot \|_\infty)$
is a non-unital C^* -alg.
non-unital

3° H Hilbert. Then $L^2(H)$
 is a unital C^* -alg.
 $K(H)$: cpt. operators also C^* -alg.
 (non-unital if $\dim H = \infty$)
 4° $l^1(\mathbb{Z})$: all $a: \mathbb{Z} \rightarrow \mathbb{C}$
 such that $a \in l^1(\mathbb{Z}, \text{counting}$
 measure). $\sum_{n=-\infty}^{\infty} |a_n| < \infty$

Put $(a \times b)_n = \sum_{k \in \mathbb{Z}} a_k b_{n-k}$
 $(= \sum_{k \in G} a_k b_{k^{-1}n})$
 $(a^*)_n = \overline{a_{-n}}$

Then $l^1(\mathbb{Z})$ is an ad. unital BA
 with isometric multiplication
 (after $l^1(G)$)
 id elt. $a_n = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

Then 1.

Thm 14 Let $e \in A$ ('A unital')

• If $x \in A$ and $|\lambda| > \|x\|$ then
 $\lambda e - x$ is invertible in A , and
 $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n$

• If x inv, and $\|y - x\| < \frac{1}{\|x^{-1}\|}$
 then y is invertible

The set of inv. elts is open, and $x \mapsto x^{-1}$
 is ct on this set

Proof. $\frac{1}{\lambda e - x} = \frac{1}{\lambda} \frac{1}{1 - \frac{x}{\lambda}}$
 $\stackrel{\text{geom}}{=} \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{x}{\lambda}\right)^n$
 Wab. $\left\|\left(\frac{x}{\lambda}\right)^n\right\| \leq \left\|\frac{x}{\lambda}\right\|^n$
 • Use first part for the rest



$e \in A, x \in A$, then
 $\sigma(x) = \{\lambda \mid \lambda e - x \text{ not invertible}\}$
(matrix $A = M_n(\mathbb{C})$, $\sigma(x) =$
eigenvalues)
(clear from 14.)

$\sigma(x)$ is compact, and $\sigma(x) \subseteq \{ \lambda \mid |\lambda| \leq \|x\| \}$.
Lemma 15 Let $R(\lambda) = (\lambda e - x)^{-1}$ for
 $\lambda \in \mathbb{C} \setminus \sigma(x)$ resolvent

Then $R: \mathbb{C} \setminus \sigma(x) \rightarrow A$
is analytic.

Proof. Have resolvent equation
 $\mu \neq \lambda \quad \frac{R(\mu) - R(\lambda)}{\mu - \lambda} = -R(\lambda)R(\mu)$

So. $R'(\lambda) = -R(\lambda)^2$ (mult. by $(\lambda e - x)(\mu e - x)$)
 $\square$

Prop 16

Prop 16. Let $e \in A$. If $A \neq \{0\}$, and $x \in A$, then $\sigma(x) \neq \emptyset$

Proof Suppose $\sigma(x) = \emptyset$. Fix $\varphi \in A^*$, then $\lambda \mapsto \varphi(R(\lambda))$ is entire. Note for $|\lambda| \geq \|x\|$, we have $\|R(\lambda)\| \leq \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|x\|}{|\lambda|}} \rightarrow 0$ as $|\lambda| \rightarrow \infty$.
So, (Liouville): $\lambda \mapsto \varphi(R(\lambda))$ is identically zero.

This $\forall \varphi$. $R(\lambda) = 0 \quad \forall \lambda$

Pick a λ . $e = (\lambda e - x) / R(\lambda) = 0$.
Then $A = \{0\}$ $\square$



Thm 17 (Gelfand - Naitan)

If $e \in A \neq \{0\}$ and every non-zero elt is invertible, then $A \cong \mathbb{C}$.

Proof. Let $x \in A$. Know: $\sigma(x) \neq \emptyset$
So $\exists \lambda \in \mathbb{C}$ st. $\lambda e - x$ is not invertible.
Then $\lambda e - x = 0 \Rightarrow x = \lambda e$
Done (it is unique because $e \neq 0$)

Def $e \in A, x \in A$.

$$\rho(x) = \max \{ |\lambda| : \lambda \in \sigma(x) \}$$

spectral radius

Thm 18 (spectral radius formula)
$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}$$

Proof. see book (uses complex analysis)

Now

Mult

Now A commutative, $e \in A$

Multiplicative functional: $h: A \rightarrow \mathbb{C}$

Set of all such: $\sigma(A)$ ^{non-zero homomorphisms} spectrum of A
(sometimes $\Delta(A)$)

Prop $e \in A$ (possibly un-c.) ^{maximal ideal space of A}
 $h: A \rightarrow \mathbb{C}$ non-zero hom.

(a) $h(e) = 1$

(b) $x \text{ inv} \Rightarrow h(x) \neq 0$

(c) $|h(x)| \leq \|x\|$ (so h is bounded) and $\|h\| \leq 1$

Proof (a) Pick $x \in A$ s.t. $h(x) \neq 0$. Then
 $h(e) \cdot h(x) = h(ex) = h(x)$
 $\Rightarrow h(e) = 1$

(b) If $h(x) = 0$, $h(e) = h(x) \cdot h(x^{-1}) = 0$

(c) If $|h| > \|x\|$, the $h \cdot e - x$ is inv.
 $\Rightarrow h(h \cdot e - x) \neq 0 \stackrel{(a)}{\Rightarrow} h(x) \neq 0$ $\square$

2°. Note that A/J is not just a Banach space (J is closed) and an algebra, but actually a Banach alg.
 Since $A/J \neq \{0\}$ Gelf Horm $\exists \gamma: A/J \rightarrow \mathbb{C}$
 because J is maximal ($\rightarrow A/J$ is a field, A comm.)

Let $\gamma: A \rightarrow A/J$
 Put $h = \gamma \circ \gamma$
 The $\ker h = J$.

3°. Clear

Thm 112 $e \in A$ comm, $A \neq \{0\}$ Then
 $h \rightarrow kh$ is a bijection
 between $\sigma(A)$ and the max ideal.
 In particular $\sigma(A) \neq \emptyset$

Proof:

- surjectivity: previous result
- injectivity: if $kh_1 = kh_2$
 $A = \mathbb{C}e \oplus \ker h_1 = \mathbb{C}e \oplus \ker h_2$
 $x = \lambda + y \Rightarrow \lambda(x) = x = h_2(x)$

$\sigma(A) \neq \emptyset$. $\{0\}$ is a proper ideal in $A \ni e \implies \{0\}$ is contained in a max ideal (algebra use for ω)

Suppose $e \in A$ comm, $A \neq \{0\}$. let $x \in A$. Then $(\lambda e - x)$ not inv.

$\exists$ max ideal $\mathcal{J}$ s.t. $\lambda e - x \in \mathcal{J}$
 $\iff \exists h \in \sigma(A)$ s.t. $h(\lambda e - x) = 0$
 $\iff \exists h \in \sigma(A)$ s.t. $h(x) = \lambda$

Def $\hat{x} : \sigma(A) \rightarrow \mathbb{C}$ cf
 $\bigcap_{B \in A^*} B(A^*)$
 by $\hat{x}(h) = h(x)$
 (restriction of $A \hookrightarrow A^{**}$)
 let $\Gamma(x) = \hat{x}$ Gelfand transform

Thm 1/3 $e \in A$, A comm., $A \neq \{0\}$

Then

1: $\hat{\pi}: A \rightarrow C(\sigma(A))$ is dy. hahn.

2: $\sigma(x) = \text{Range of } \hat{x}$

3: $x \text{ inv} \iff \hat{x} \text{ has no zeros on } \sigma(A)$

4: $\|\hat{x}\|_\infty = p(x)$

Proof: 1: $(xy)^\wedge(h) = h(xy) = h(x)h(y)$
 $= \hat{x}(h)\hat{y}(h)$
 $= (\hat{x}\hat{y})(h)$

2: Clear

3: $x \text{ inv} \iff 0 \notin \sigma(x) \iff \hat{x} \text{ has no zeros on } \sigma(A)$

4: Clear

□

! Then 1/3 $e \in A$, A comm., $A \neq \{0\}$

Then

1: $\hat{\pi}: A \rightarrow \mathcal{C}(\sigma(A))$ - is dy. lin.

2: $\sigma(x) = \text{Range of } \hat{x}$

3: $x \text{ inv.} \iff \hat{x} \text{ has no zeroes on } \sigma(A)$

4: $\|\hat{x}\|_\infty = p(x)$

Proof: 1: $(xy)^\wedge(h) = h(xy) = h(x)h(y)$
 $= \hat{x}(h)\hat{y}(h)$
 $= (\hat{x}\hat{y})(h)$

2: Clear

3: $x \text{ inv.} \iff 0 \notin \sigma(x) \iff \hat{x} \text{ has no zeroes on } \sigma(A)$

4: Clear

□



Cor 1.18 (Wiener) Let $f \in C(\mathbb{T})$

Suppose that

$$1^\circ f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}$$

$$\text{with } \sum_{n \in \mathbb{Z}} |a_n| < \infty$$

2° f has no zeros on $\mathbb{T}$.

Then $\frac{1}{f}$ also has an abs. conv. F. series

Proof. Consider $A = L^1(\mathbb{Z})$. It can be shown (see Thm 1.17 or better - Thm 4.6.c)

that $\sigma(L^1(\mathbb{Z})) \cong \mathbb{T}$ via

$$h \mapsto \hat{h}(a) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}.$$

Now interpret this.

$f = \hat{a}$ for some $a \in A$. Know $\hat{a}$ has no zeros. So a is invertible in A .

$$\text{Then } e = a \times a^{-1} \Rightarrow 1 = \hat{a} \cdot \hat{a}^{-1} = f \cdot \hat{a}^{-1}$$



Prop. If $x \in A$ comm C^* alg
If $h \in \sigma(A)$ and $x \in X$
then $h(x^*) = \overline{h(x)}$

Proof. First let $x = x^*$. Write
 $h(x) = \alpha + ip$ ($\alpha, p \in \mathbb{R}$)
Then $\forall t \in \mathbb{R}$, $h(x+ite) = \alpha + i(p+it)$
and $(x+ite)^*(x+ite) =$
 $= (x+ite)(x+ite) = x^2 + t^2 e$

So

$$\begin{aligned} \alpha^2 + (p+t)^2 &= \|h(x+ite)\|^2 \\ &\leq \|x+ite\|^2 \\ &\stackrel{C^*}{=} \|(x+ite)^*(x+ite)\| \\ &= \|x^2 + t^2 e\| \\ &\leq \|x\|^2 + t^2 \\ \Rightarrow \alpha^2 + p^2 + 2pt &\leq \|x\|^2 \quad \forall t \in \mathbb{R} \\ \Rightarrow p &= 0. \end{aligned}$$

Arbitrary $x \in A$: $x = \operatorname{Re} x + i \operatorname{Im} x$

Then $h(x^*) = h\left(\frac{x+x^*}{2} + i \frac{x-x^*}{2i}\right)$

$$= h(\operatorname{Re} x - i \operatorname{Im} x)$$

$$= h(\operatorname{Re} x) - i h(\operatorname{Im} x)$$

$$= \overline{h(\operatorname{Re} x) + i h(\operatorname{Im} x)}$$

$$= \overline{h(x)}.$$

Continue, $\|x\| = \|x^*\|$

Proof. Let $y = x^* x$. Then $y^* = y$, so

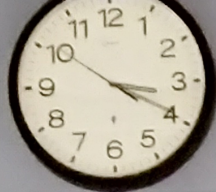
$$\|y^{2^k}\| = \|(y^{2^{k-1}})^* y^{2^{k-1}}\| \leq \|y^{2^{k-1}}\|^2$$

$$= \dots = \|y\|^{2^k} \quad \forall k = 1, 2, \dots$$

So, $\|y\|_\infty = \rho(y) = \lim_{k \rightarrow \infty} \left(\|y^{2^k}\|^{1/2^k} \right) = \|y\|$

Then $\|x\|^2 = \|x^* x\| = \|y\| = \|y\|_\infty$

$$= \left\| \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \right\|_1 = \|x\|_1$$



Thm 1.20 Comm. Gelfand-Naimark
 If A is a comm. C^* -alg, then
 $\Gamma: A \rightarrow C(\sigma(A))$
 is an isometric $*$ -isomorphism
 (onto)

Proof. $\Gamma(A) \subset C(\sigma(A))$ is
 an algebra, self-adjoint, contains
 the constants and separates the
 points. St. W. $\Gamma(A)$ is dense in

$C(\sigma(A))$

Know from isometric property,
 $\Gamma(A)$ is complete, so closed in $C(\sigma(A))$

Now. A comm. BA, $e \notin A$
 Example: $L^1(\mathbb{R})$ or generally
 $L^1(G)$ if G is not discrete

Trick $\tilde{A} = A \oplus \mathbb{C}$ unitisation of A
 $(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$
 $x, y \in A, \lambda, \mu \in \mathbb{C}$

$$\|(x, \lambda)\| := \|x\| + |\lambda|$$

Then $\tilde{A}$ is a comm B.A. with identity
elt $(0, 1)$

$A \hookrightarrow \tilde{A}$
 $x \mapsto (x, 0)$ is an isometric embedding
of A as an ideal

If $h \in \sigma(A)$, we can (try to) extend h to
non-zero complex

$\tilde{h} \in \sigma(\tilde{A})$

$$\tilde{h}(x, \lambda) = \tilde{h}((x, 1) + \lambda(0, 1)) = h(x) + \lambda \cdot 1$$

So, only one way to do it.

Check this actually works

Conversely, if $\tilde{h} \in \sigma(\tilde{A})$, then $\tilde{h}$ extends
to A Catch: this could be zero.

At any rate, have bijection

$$\begin{aligned} \sigma(A) \cup \{0\} &\xleftrightarrow{\sim} \sigma(\tilde{A}) \\ \text{Seq: if } h \in \sigma(A), \text{ then } \|h(x)\| &= \|\tilde{h}(x, 0)\| \\ &\leq \|(x, 0)\| = \|x\| \end{aligned}$$



So again $\sigma(A) \subset \mathbb{D}_1(A^*)$

Supply $\sigma(A)$ with weak* - top

Note $h_\alpha \xrightarrow{w^*} h$ in $\sigma(A)$

$$\Leftrightarrow h_\alpha(x) \rightarrow h(x) \quad \forall x \in A$$

$$\Leftrightarrow h_\alpha(x) + \lambda \rightarrow h(x) + \lambda \quad \forall x \in A$$

$$\Leftrightarrow \tilde{h}_\alpha(x, \lambda) \rightarrow \tilde{h}(x, \lambda) \quad \forall (x, \lambda) \in \tilde{A}$$

$$\Leftrightarrow \tilde{h}_\alpha \xrightarrow{w^*} \tilde{h} \text{ in } \sigma(\tilde{A})$$

$$\text{So } j: \sigma(A) \hookrightarrow \sigma(\tilde{A})$$
$$h \mapsto \tilde{h}$$

is a homeomorphism between $\sigma(A)$ and its image.

$$\delta_{\sigma(A)}$$

$$\sigma(\tilde{A}) = \sigma(A) \cup \{\delta\}$$

So $\sigma(A)$ is locally cpt. Ho

Note.

$$\hat{x}(h) = h(x) = \tilde{h}(x, 0)$$

$$\downarrow \epsilon \in \mathbb{R} \quad (x, 0)^\wedge(\tilde{h})$$

$$= (x, 0)^\wedge(j(h))$$

$$(x, 0)^\wedge(\tilde{0}) = 0.$$

So, after identifying $\mathbb{R}/A$ with $j(\mathbb{R}/A)$, $\hat{x}$ coincides with the restriction of $(x, 0)^\wedge$ to $\mathbb{R}/A$ and $(x, 0)^\wedge$ (missing part) =

So $\hat{x}$ vanishes at infinity on $\mathbb{R}/A$.



$$f: X \rightarrow \mathbb{C} \text{ is cl}$$

$$f|_U =$$

$$X \setminus \{0\} \in C_0(X \setminus \{0\})$$

All is all.

Then (1.30) A comm, $A \neq \{0\}$, $e \notin A$
 Then $\sigma(A)$ is loc. cpt. HD
 and $\Gamma: A \rightarrow C(\sigma(A))$
 is an alg hom $\uparrow$
 Furthermore.

$$\begin{aligned}
 \|x\| &\hookrightarrow (= \|x_0\|) = L \|x_0\|^{1/n} \\
 &= L \|x\|^{1/n} \xrightarrow{h \rightarrow \infty}
 \end{aligned}$$

Will see. fact that $\Gamma(A) \subset C_0(\sigma(A))$
 yields the Riemann-Lebesgue Lemma
 • if $f \in L^1(\mathbb{R})$ then $\hat{f} \in C_0(\mathbb{R})$
 • if $f \in L^1[0, 2\pi]$, then

$$\lim_{|n| \rightarrow \infty} \hat{f}(n) = 0$$

$\subset C_0(\sigma(A))$
 the lemma
 $C_0(\mathbb{R})$

$$\| = \|x\| + \|x\|$$

Thm (p. 16) X loc. cpl. Hld. Then
 $\sigma(C_0(X)) \cong X$
 and after identification Γ is
 the identity map

Thm 131 $A \neq \{0\}$ comm C^* -alg, $e \notin A$.
 Then $\Gamma: A \rightarrow C_0(\sigma(A))$ is an
 isomorphism

Proof. Prop 127. $\hat{A}$ can be used into a C^* -alg.
 under an equivalent norm to the general one

So $\hat{X}$ vanishes at infinity on $\sigma(A)$



$f: X \rightarrow \mathbb{C}$ is cl

$f(0) =$

$\forall x \in \sigma(A) \in C_0(X)$

All is all

and still $A \hookrightarrow \hat{A}$ isom