



2 Locally Compact Groups

2.1 Topological Groups

Definition

A topological group G is a group equipped with a topology with respect to which the group operations

$G \times G \rightarrow G$ and $G \rightarrow G$
 $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are
cf. Denote the unit element by 1

Remark

- Clearly the maps $x \mapsto xy$,
 $x \mapsto yx$ and $x \mapsto x^{-1}$ are homeomorphisms.
- We shall call a $A \subseteq G$ symmetric,
if $A = \bar{A}^{-1} = \{x^{-1} \mid x \in A\}$

2.1 Proposition

- a. The topology is invariant under translation and inversion, i.e. if U is open in G then xU, Ux, U^{-1}, AU, UA for every $x \in G, A \subseteq G$.
- b. For every neighborhood U of 1 there is a symmetric neighborhood V of 1 st. $VV \subseteq U$.
- c. $H \leq G$ (H subgroup of G) then $\overline{H} \leq G$, and if H is normal in G then also $\overline{H}$.
- d. Every open subgroup of G is closed.
- e. If $A, B \subseteq G$ compact then AB is also cpt.

Suppose $H \leq G$. Let $G/H = \{xH \mid x \in G\}$ and $q: G \rightarrow G/H$ is the canonical quotient map. We define a topology on G/H by $U \subseteq G/H$ is open iff $q^{-1}(U)$ is open in G . q is an open map.
Let $V \subseteq G$ open then $q^{-1}(q(V)) = VH$ which is open by Prop 2.1. b. so $q(V)$ is open.

2.2 Proposition

- Let G be a topol. group and $H \leq G$.
- a. If H closed, then G/H is Hausdorff.
 - b. If G is loc. cpt, so G/H .
 - c. If H is normal, G/H is a topological group.

Proof:

a) From topology we know that if the map $q: G \rightarrow G/H$ is continuous, open and surj. then G/H is Hausdorff iff $\ker(q) = \{(x, y) \in G \times G \mid q(x) = q(y)\}$ is closed.

$$\begin{aligned}\ker(q) &= \{(x, y) \in G \times G \mid xH = yH\} \\ &= \{(x, y) \in G \times G \mid y^{-1}x \in H\}\end{aligned}$$

So $\ker(q)$ is the preimage of H of the map $(x, y) \mapsto xy^{-1}$ hence closed.

b) If U is a compact of G then $q(U)$ is a compact neighborhood of $q(x)$.

c) Clear.

2.3 Corollary

If G is T_1 then G is Hausdorff. If G is not T_1 then $\overline{\{1\}}$ is closed normal subgroup and $G/\overline{\{1\}}$ is Hausdorff top group.

From now on we can assume that every topological group is Hausdorff by this corollary.

2.4 Proposition

Every loc cpt group G has a subgroup G_0 that is open, closed and σ -compact.

Remark

Every loc cpt group G is the disjoint union of the cosets G_0 each of which is also open, closed and homeomorphic to G_0 . Hence, from a topol. view G is just a disjoint union of a σ -cpt. space in part. if connected then G is σ -compact.

Examples

- $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{T}^n = \{z \in \mathbb{C} \mid |z|=1\}^n$ $K \in \mathbb{R}, \mathbb{C}$
- Every lie group
- Groups of matrices, e.g. $GL(n, K) \subset K^{n^2}$ open
- $SL(n, K) \subset K^{n^2}$ closed, $O(n, \mathbb{R}) \subset K^{n^2}$, $U(n) \subset \mathbb{C}^{n^2}$ closed

Definition

- Let $f: G \rightarrow \mathbb{R}$ and $y \in G$ then we define the left and right translates of f by $L_y f(x) = f(y^{-1}x)$

$$R_y f(x) = f(xy)$$

Note that $y \mapsto L_y, y \mapsto R_y$ are group homomorphisms



f is left (resp. right) uniformly continuous if for every $\varepsilon > 0$ there exists a neighborhood V of 1 s.t.
 $\|L_y f - f\|_\infty < \varepsilon$ (resp. $\|R_y f - f\|_\infty < \varepsilon$)
 whenever $y \in V$.

2.6 Proposition

If $f \in C_c(G)$ then f is left and right uniformly ct.

Proof.

We only shall proof right uniformly ct
 Given $f \in C_c(G)$ and $\varepsilon > 0$, let $K = \text{supp } f$

Since f is ct there is for every $x \in K$ a neighborhood U_x of 1 s.t. $|f(xy) - f(x)| < \frac{1}{2}\varepsilon$ for every $y \in U_x$ and there symm neighb V_x of 1 s.t. $V_x V_x \subset U_x$. The sets $x V_x$ ($x \in K$) cover K , so there exist $x_1, \dots, x_n \in K$ s.t.
 $K \subset \bigcup_{j=1}^n x_j V_{x_j}$. We show that $\|R_y f - f\|_\infty < \varepsilon$ for

$$y \in V = \bigcap_{j=1}^n V_{x_j}.$$

Case 1. $x \in K$, then there exists some j for which $x_j^{-1} x \in V_{x_j}$ and $xy = x_j (x_j^{-1} xy) \in x_j U_{x_j}$. But then

$$\begin{aligned} |f(xy) - f(x)| &\leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| \\ &= |f(x_j (x_j^{-1} xy)) - f(x_j)| + |f(x_j (x_j^{-1} x)) - f(x_j)| \end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Case 2: $xy \in K$ then there is some $j \in \mathbb{Z}$ such that $x_j^{-1}xy \in V_{x_j}$ and $x_j^{-1}x = x_j^{-1}xyy^{-1} \in U_{x_j}$

So

$$|f(xy) - f(x)| \leq |f(x_j(x_j^{-1}xy)) - f(x_j)| + |f(x_j(x_j^{-1}x)) - f(x_j)|$$

$$< \varepsilon$$

Case 3: $x \notin K, xy \notin K$ Obvious $\square$

2.2 Haar Measure

Definition

A left (resp right) Haar measure on a loc cpt

group G is a nonzero Radon measure μ on G that satisfies $\mu(xE) = \mu(E)$ (resp $\mu(Ex) = \mu(E)$) for every Borel set E and $x \in G$.

2.3 Proposition

Let μ be Radon measure on loc cpt. group G and

$$\tilde{\mu}(E) := \mu(E^{-1})$$

a. μ is a left Haar measure iff $\tilde{\mu}$ is a right Haar measure

b. μ is a left Haar measure iff $\int f_y d\mu = \int f d\mu$ for every $f \in C_c^+(G) = \{f \in C_c(G) \mid f \geq 0 \text{ and } f \neq 0\}$



Proof

a. clear

b. First we assume μ is a left Haar measure. Note that $L_y 1_E = 1_{yE}$. Hence for every simple function $s = \sum_{i=1}^n c_i 1_{E_i}$, $c_i \geq 0$, $\bigcup_{i=1}^n E_i = E$ (disjoint union) we have

$$\begin{aligned} \int s d\mu &= \sum_{i=1}^n c_i \mu(E_i) = \sum_{i=1}^n c_i \mu(yE_i) \\ &= \int L_y s d\mu \end{aligned}$$

Since $\int f d\mu = \sup \{ \int s d\mu \mid s \text{ simple, } 0 \leq s \leq f \}$

for all $f \in C_c(G)$, it follows $\int f d\mu = \int L_y f d\mu$

On the other hand, if $\int f d\mu = \int L_y f d\mu$ for all $f \in C_c^+(G)$ and $y \in G$, then the equation also holds for every $f \in C_c(G) = \text{span } C_c^+(G)$. Since the integral is a positive linear functional and by applying the Riesz-Representation theorem we deduce

$$\begin{aligned} \mu(E) &= \sup \{ \int f d\mu \mid f \in C_c(G), 0 \leq f \leq 1, \text{supp } f \subseteq E \} \\ &= \sup \{ \int L_y f d\mu \mid \text{---} \text{---} \text{---} \} \\ &= \sup \{ \int f d\mu \mid \text{---} \text{---} \text{---}, \text{supp } f \subseteq yE \} \\ &= \mu(yE) \end{aligned}$$

by using $\text{supp } f \subseteq E \iff \text{supp } L_y f \subseteq yE$. Hence μ is left Haar measure. $\square$

2.10 Theorem

Every loc. cpt. group G possesses a left Haar measure

2.10 Proposition

If λ is a left Haar measure on G then
 $\lambda(U) > 0$ for every non-empty open set U
and $\int f d\lambda > 0$ for every $f \in C_c^+(G)$

Proof:

Suppose U is open, nonempty and $\lambda(U) = 0$. Then
 $\lambda(xU) = 0$ for all $x \in G$ and since any cpt set
 K can be covered by finitely many translations

of U , we have $\lambda(K) = 0$ for every compact set K . By inner regularity we get $\lambda(G) = 0$
Contradiction to λ nonzero

Next, given $f \in C_c^+(G)$, let $U := \{x \in G \mid f(x) > \frac{1}{2} \|f\|_\infty\}$
It is clearly open and nonempty. Then
 $\int f d\lambda > \frac{1}{2} \|f\|_\infty \lambda(U) > 0$ □

2.20 Theorem

If λ and μ are left Haar measures on G , there exists
 $c \in (0, \infty)$ st. $\mu = c \cdot \lambda$

Proof

Let $f, g \in C_c^+(G)$ be given. First we prove that $\frac{\int f d\lambda}{\int g d\lambda} = \frac{\int f d\mu}{\int g d\mu}$.



Let V_0 be a fixed symmetric cpt. nbh of 1 and set

$$A = (\text{supp } f) V_0 \cup V_0 \text{supp } f$$

$$B = (\text{supp } g) V_0 \cup V_0 \text{supp } g$$

Then A, B are compact and for $y \in V_0$ $f(xy) - f(yx)$ and $g(xy) - g(yx)$ are supported in A, B respectively as functions of x . By Prop 26 there exists symmetric nbh $V \subset V_0$ of 1 st $|f(xy) - f(yx)| < \epsilon$ and $|g(xy) - g(yx)| < \epsilon$ for all x when $y \in V$. Now pick $h \in C_c^+(G)$ with $\text{supp}(h) \subset V$ and

let $h(x) = h_1(x) + h_1(x^{-1})$ for all $x \in G$ then $h \in C_c^+(G)$ and $h(x) = h(x^{-1})$ and since V is symmetric $\text{supp } h \subset V$. Since f, g, h are compactly supported, hence supported on sets of finite measure, Tonelli's theorem:

$$\int h d\mu \int f d\lambda = \int \int h(y) f(x) d\lambda(x) d\mu(y) \\ \stackrel{(*)}{=} \int \int h(y) f(yx) d\lambda(x) d\mu(y)$$

And since $h(x) = h(x^{-1})$.

$$\int h d\lambda \int f d\mu = \int \int h(x) f(y) d\lambda(x) d\mu(y) \\ \stackrel{(*)}{=} \int \int h(y^{-1}x) f(y) d\lambda(x) d\mu(y) \\ = \int \int h(x^{-1}y) f(y) d\mu(y) d\lambda(x)$$

$$\begin{aligned} &= \iint h(y) f(xy) d\mu(y) d\lambda(x) \\ &= \iint h(y) f(xy) d\lambda(x) d\mu(y). \end{aligned}$$

$$\begin{aligned} \Rightarrow & \left| \int h d\mu \int f d\lambda - \int f h d\mu \right| \\ &= \left| \iint h(y) (f(xy) - f(yx)) d\lambda(x) d\mu(y) \right| \\ &\leq \varepsilon \lambda(A) \int h d\mu \end{aligned}$$

upon replacing f by g and doing the same calculations.

$$\left| \int h d\mu \int g d\lambda - \int h g d\lambda \right| \leq \varepsilon \lambda(B) \int h d\mu$$

Dividing those inequalities by $\int h d\mu \int f d\mu$, $\int h d\mu \int g d\mu$ respectively and adding them we obtain

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| \leq \varepsilon \left(\frac{\lambda(A)}{\int f d\mu} + \frac{\lambda(B)}{\int g d\mu} \right)$$

Since ε was arbitrary: $\frac{\int f d\lambda}{\int f d\mu} = \frac{\int g d\lambda}{\int g d\mu}$.

Since this holds for every $f, g \in C_c^+(\mathbb{G})$ we conclude that $\int f d\lambda = c \int f d\mu$ ($c = \frac{\int g d\lambda}{\int g d\mu} > 0$) for any $f \in C_c^+(\mathbb{G})$ and hence for any $f \in C_c(\mathbb{G})$. By using uniqueness part of R-R-theorem we get $\lambda = c \cdot \mu$. $\square$



2.4 The Modular function

Let G be a loc. cpt group.

If λ is left Haar measure on G then for every $x \in G$ the measure $\lambda_x(E) = \lambda(Ex)$ for Borel set E is clearly a left Haar measure. By theorem 2.20 we get $\lambda_x = \Delta(x) \lambda$, $\Delta(x) > 0$ and $\Delta(x)$ is independent of choice of λ . Indeed, if μ is another left Haar measure then $\mu = c\lambda$

and $\mu(E) = \mu(Ex) = c\lambda(Ex) = c\Delta(x)\lambda(E) = \Delta(x)\mu(E)$.
The function $\Delta: G \rightarrow (0, \infty)$ is called the modular function.

2.24 Proposition

Δ is a ct homomorphism from G to $\mathbb{R}_+^* = (\mathbb{R}_{>0}, \cdot)$.
Moreover for any $f \in C(\lambda)$.

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda \quad (*)$$

Proof:

$$\begin{aligned} \text{For any } x, y \in G, E \subset G \text{ Borel set: } \Delta(xy) \lambda(E) &= \lambda(Exy) \\ &= \Delta(y) \lambda(Ex) = \Delta(y) \Delta(x) \lambda(E) \Rightarrow \Delta(xy) = \Delta(x) \Delta(y) \end{aligned}$$

Moreover since $1_E(x) = 1_{Eg^{-1}}(x)$:

$$\int 1_E(x) d\lambda(x) = \lambda(Eg^{-1}) = \Delta(g^{-1}) \cdot \int 1_E(x) d\lambda(x)$$

The general case follows by approximation with simple functions

Finally, by 2.6 is $y \mapsto R_y f$ ct from G to $C_c(G)$. Now prove that $y \mapsto \int R_y f d\lambda$ is ct. Let $K := \text{supp } f$ and V a compact nbh of 1 then VK is cpt and $\lambda(VK) < \infty$. $(x_\alpha)_{\alpha \in A} \rightarrow x$

First: $x=1$. Then let $\varepsilon > 0$ given, by 2.6 there exists nbh U of 1 of $\|R_{y^{-1}} f - f\| < \frac{\varepsilon}{\lambda(VK)}$ for every $y \in U$

By replacing U by $U \cap V$ we can assume that $U \subset V$. Then clearly $\text{supp}(R_y f - f) \subset VK \forall y \in U$. This ensures that there exists $\delta > 0$ st. for $x \geq \delta$ then $x_\alpha \in U$ and

$|\int R_{x_\alpha} f d\lambda - \int f d\lambda| \leq \|R_{x_\alpha} f - f\|_\infty \lambda(VK) < \varepsilon$.
In general if $\lim_{\alpha} x_\alpha = x \in G$ then $\lim_{\alpha} x_\alpha x^{-1} = 1$ and since $R_{x^{-1}} f \in C_c(G)$:

$$\lim_{\alpha} \int R_{x_\alpha} f d\lambda = \lim_{\alpha} \int R_{x_\alpha x^{-1}} R_x f d\lambda = \int R_x f d\lambda.$$

And since $\int f d\lambda$ is independent of y and $y \mapsto y^{-1}$ is ct, Δ is ct, by (*). $\square$



If we set $y_0 = y^{-1}$ in (*) and make substitution $x \rightarrow xy_0$ we obtain

$$\Delta(y_0) \int f(x) d\lambda(x) = \int f(xy_0^{-1}) d\lambda(x) \\ = \int f(x) d\lambda(xy_0)$$

$$\Rightarrow d\lambda(xy_0) = \Delta(y_0) d\lambda(x)$$

Definition

G is called unimodular, if $\Delta = 1$, i.e. the left Haar measure is right Haar measure.

Note: Every abelian group and discrete is unimodular.

2.2.2 Proposition

If K is a compact subgroup of G then $\Delta_K \equiv 1$.

Proof:

$\Delta(K)$ is compact subgroup $\mathbb{R}_+$, hence equal to $\{1\}$. $\square$

2.2.8 Corollary

If G compact, then G is unimodular.

2.3.1 Proposition

Let λ be the left Haar measure on G and $\rho(E) = \lambda(E^{-1})$. Then λ and ρ are strongly equivalent, i.e. there exists a $f: G \rightarrow (0, \infty)$ st $\int f d\rho = \int f d\lambda$ for all $f \in C_c(G)$.

when $\Delta_x \equiv 1$

equal to δx ?

on G and
roughly equivalent

$$dS = \int \rho dx$$

In particular $f(x) = \Delta(x^{-1})$ so we can write
 $dS(x) = \Delta(x^{-1}) d\lambda(x)$

This formula can be restated in following ways:
 $d\lambda(x^{-1}) = \Delta(x^{-1}) d\lambda(x)$, $dS(x^{-1}) = \Delta(x) dS(x)$.

When G is not unimodular, $\mathcal{P}(\lambda)$ and $\mathcal{P}(S)$ are not the same. There are two ways of passing from one to the other.

$\tilde{f}(x) = f(x^{-1})$, $M_P f(x) = \Delta(x)^{1/p} f(x)$
Then $f \mapsto \tilde{f}$ and $f \mapsto M_P f$ are isometric isomorphisms.

from $\mathcal{P}(\lambda)$ to $\mathcal{P}(S)$, because $d\lambda(x) = dS(x^{-1}) = \Delta(x) dS(x)$
(composition gives an isometric isomorphism of $\mathcal{P}(\lambda)$)
 $M_P^{-1} \tilde{f}(x) = (M_P f) = \Delta(x)^{-1/p} f(x^{-1})$