Unitary Representations

- G: locally cpt. group
- u-rep. of G is $\pi : G \longrightarrow U(H_\pi)$

$$\pi(xy) = \pi(x)\pi(y)$$

$$\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$$

$$x \xrightarrow[\text{continuous}]{} \pi(x)u, \quad \forall u \in H_\pi,\ x \in G$$

- H_π: the representation space of π
- dim of H_π is called the dim. of π.

Remark: the weak and strong topology coincide on $U(H_\pi)$

Indeed, $T_\alpha \xrightarrow{w} T$ in $U(H_\pi)$

$$\|T_\alpha u - Tu\|^2 = \|T_\alpha u\|^2 - 2\mathrm{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2$$

$$= 2\|u\|^2 - 2R\langle T_\alpha u, Tu\rangle$$

$$\|Tu\|^2 = \|u\|^2$$

$$\longrightarrow 0$$

Example. π of G on locally cpt. Hdf S by

$$[\pi(x)f](s) = f(x^{-1}s)$$

In particular, π_L of G on $L^2(G)$ is

$$[\pi_L(x)f](s) = L_x f(s) = f(x^{-1}s)$$

left regular rep.

π_R of G on $L^2(G,\rho)$, ρ: right Haar measure on G.

$$[\pi_R(x)f](y) = R_x f(y) = f(yx) \quad \text{--- right regular rep.}$$

$\tilde{\pi}_R$ - - - - $L^2(G)$ with left - - - - - - on G

$$[\tilde{\pi}_R(x)f](y) = \Delta(x)^{\frac{1}{2}} R_x f(y) = \Delta(x)^{\frac{1}{2}} f(yx)$$

- π_1, π_2 are u-rep. of G, an <u>intertwining operator</u> for π_1, π_2 is a bounded linear $T: H_{\pi_1} \longrightarrow H_{\pi_2}$ s.t. $T\pi_1(x) = \pi_2(x)T$ denoted by $C(\pi_1, \pi_2)$
- π_1 and π_2 are <u>unitary equivalent</u> if $\exists$ u-operator $U \in C(\pi_1, \pi_2)$ s.t. $\pi_2(x) = U\pi_1(x)U^{-1}$
- $C(\pi, \pi) = C(\pi)$ called <u>commutant or centralizer</u>.

$$T^*\pi(x) = [\pi(x^{-1})T]^* = [T\pi(x^{-1})]^* = \pi(x)T^*$$

- $M \subseteq H_\pi$, M is called <u>invariant subspace</u> for $\pi(x)$ if $\pi(x)M \subset M \quad \forall x \in G$
- If $M \neq \{0\}$ and invariant, then the restriction of π to M is $\pi^M(x) = \pi(x)|M$. it is called <u>subrepresentation</u> of π
- If π admits an invariant subspace that is nontrivial then π is called <u>reducible</u>, otherwise π is irreducible
- <u>direct sum</u>. $\oplus\pi_i$ is the rep. π on $H_\pi = \oplus H_{\pi_i}$ s.t. $\pi(x)\sum v_i = \sum \pi_i(x) v_i$, $v_i \in H_{\pi_i}$

Unitary Representions.

Pro. 3.1 If $\mathcal{M}$ is invariant under π, so is $\mathcal{M}^\perp$.

Proof: $u \in \mathcal{M}$, $v \in \mathcal{M}^\perp$

$$\langle \pi(x)u, v\rangle = \langle u, \pi(x^{-1})v\rangle = 0$$

So $\pi(x)v \in \mathcal{M}^\perp$ □

Cor. 3.2 If π has a nontrivial invariant subspace $\mathcal{M}$, then $\pi = \pi^{\mathcal{M}} \oplus \pi^{\mathcal{M}^\perp}$

def.
- If π is u-rep. of G and $u \in H_\pi$, the closed linear span $\mathcal{M}_u$ of $\{\pi(x)u,\ x \in G\}$ is called the <u>cyclic subspace</u> generated by u.
- If $\mathcal{M}_u = H_\pi$, u is called <u>cyclic vector</u> for π.
- π is called <u>cyclic rep.</u> if it has a cyclic vector.

Pro. 3.3 Every u-rep. is a direct sum of cyclic rep.

Proof: Let π be a rep. of H_π. By Zorn's Lemma $\exists$ maximal collection $\{\mathcal{M}_\alpha\}_{\alpha \in A}$ of mutually orthogonal cyclic subspaces of H_π.

So $H_\pi = \oplus \mathcal{M}_\alpha$. So $\forall u \in H_\pi$ $\exists u_\alpha \in \mathcal{M}_\alpha$ s.t. $u = \sum_\alpha u_\alpha$

$$\pi(x)u = \pi(x)\left(\sum_\alpha u_\alpha\right) = \sum_\alpha \pi_\alpha(x) u_\alpha = \sum_\alpha \pi(x)|_{\mathcal{M}_\alpha}$$

$$= \oplus \pi^{\mathcal{M}_\alpha}$$

□

Unitary Representions

(3.5) Schur's Lemma

(a) A u-rep. π of G is irreducible iff $C(\pi)$ contains only scalar multiples of the identity

(b) Suppose π_1, π_2 are irreducible u-rep of G

If π_1, π_2 are equivalent, then $\dim C(\pi_1, \pi_2) = 1$

otherwise $C(\pi_1, \pi_2) = \{0\}$

Proof (a) If π is reducible, $\exists M \neq \{0\}, H_\pi$

s.t. $\pi M \subset M$. By Pro 3.4

$\exists$ nontrivial projection $P \in C(\pi)$ $\}$

If $T \in C(\pi)$, $T \neq cI$, then

Let $A = \frac{1}{2}(T + T^*)$, $B = \frac{1}{2i}(T - T^*)$ $\in C(\pi)$

$T \in C(\pi)$, $T^* \in C(\pi)$

Assume $A \neq cI$. $A \in C(\pi)$ $A\pi(x) = \pi(x)A$

As A is self-adjoint, there exists a projection $P(A)$, s.t. $P(A)\pi(x) = \pi(x)P(A)$

So by Pro. 3.4, $\pi(x)$ is reducible $\}$

(b) If $T \in C(\pi_1, \pi_2)$, then

$$T^*\pi_2(x) = [\pi_2(x^{-1})T]^* = [T\pi_1(x^{-1})]^* = \pi_1(x) \cdot T^*$$

So $T^* \in C(\pi_2, \pi_1)$

Hence $TT^* \in C(\pi_2)$, $T^*T \in C(\pi_1)$

By (a) $TT^* = CI$, $T^*T = CI$.

So either $T = 0$ or $C^{-\frac{1}{2}}T$ is unitary

· $T = 0$ precisely when π_1 and π_2 are equivalent.

· $C^{-\frac{1}{2}}T$ is unitary, i.e. $C(\pi_1, \pi_2)$ contains of scalar multiples of unitary operator.

If $T_1, T_2 \in C(\pi_1, \pi_2)$, then

$T_2^{-1}T_1 = T_2^*T_1 \in C(\pi_1)$, So $T_2^{-1}T_1 = cI$

So $T_1 = cT_2$. So $\dim C(\pi_1, \pi_2) = 1$ □

Cor. 3.6 If G is Abelian, then every irreducible rep. of G is one-dimensional.

Proof: $xy = yx$ in G. So $\pi(x)\pi(y) = \pi(y)\pi(x)$

So $\pi(x) \in C(\pi)$. If π is irreducible, by (3.5)(a)

$\pi(x) = c_x I$, $\forall x \in G$

Then every one-dimensional subspace of H_π is invariant.

So $\dim H_\pi = 1$ □

Def: A *-rep of $L^1(G)$ on Hilbert space H, is a *-homomorphism π from $L^1(G)$ to $\mathcal{L}(H)$

· π is nondegenerate if there is no nonzero $v \in H$ s.t. $\pi(f)v = 0$, $\forall f \in L^1(G)$

Unitary Representations

- $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$, $\quad f \in L^1(G)$
- u-rep. π of G $\exists$ rep of $L^1(G)$

  s.t. $\pi(f) = \int f(x)\pi(x)\,dx$, $\quad f \in L^1(G)$

(3.7) $\langle \pi(f)u, v\rangle = \int f(x)\langle \pi(x)u, v\rangle\,dx$, $\quad u, v \in H_\pi$

Because $|\langle \pi(f)u, v\rangle| \le \|f\|_1 \|u\| \|v\|$

So $\|\pi(f)\| \le \|f\|_1$

Theorem 3.9. Let π be a u-rep. of G. The map $f \to \pi(f)$ is a nondegenerate *-rep of $L^1(G)$ on H_π. Moreover, for $x \in G$, $f \in L^1(G)$

$$\pi(x)\pi(f) = \pi(L_x f), \quad \pi(f)\pi(x) = \Delta(x^{-1})\pi(R_x f)$$

Proof: $f \longrightarrow \pi(f)$ Linear

$$\pi(f * g) = \iint f(y)g(y^{-1}x)\pi(x)\,dy\,dx$$
$$= \iint f(y)g(x)\pi(xy)\,dx\,dy$$
$$= \iint f(y)g(x)\pi(x)\pi(y)\,dx\,dy = \pi(f)\pi(g)$$

$$\pi(f^*) = \int \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)\,dx$$
$$= \int \overline{f(x^{-1})}\pi(x)\,d(x^{-1})$$
$$= \int \overline{f(x)}\pi(x^{-1})\,dx = \int [f(x)\pi(x)]^*\,dx = \pi(f)^*$$

$$\pi(x)\pi(f) = \int \pi(x) f(x) \pi(y) dy$$
$$= \int f(x) \pi(xy) dy$$
$$= \int f(x^{-1}y) \pi(y) dy = \pi(L_x f)$$

$$\pi(f)\pi(x) = \int f(y) \pi(y) \pi(x) dy$$
$$= \int f(y) \pi(yx) dy$$
$$= \int f(yx^{-1}) \pi(y) \Delta(x^{-1}) dy = \Delta(x^{-1}) \pi(R_{x^{-1}} f)$$

show π nondegenerate. Let $u \neq 0 \in H_\pi$.
Take neighborhoof V of 1 in G
s.t. $\|[\pi(x) - \pi(1)] u\| < \|u\| \quad \forall x \in V$

Let $f = |V|^{-1} \chi_V$, then
$$\|\pi(f)u - u\| = \frac{1}{|V|} \left\| \int_V (\pi(x) u - u) dx \right\|$$
$$< \frac{1}{|V|} \cdot |V| \cdot \|u\| = \|u\|$$
In particular $\pi(f) u \neq 0$, so it is nondegenerate $\square$

Unitary Representions

The 3.11 Suppose π is a nondegenerate $*$-rep. of $L^1(G)$ on H, then π arises from a unique u-rep. of G on H according

$$\langle \pi(f)u, v\rangle = \int f(x)\langle \pi(x)u, v\rangle\, dx.$$

Proof: Let ψ_u be:

- $\operatorname{supp}\psi_u$ compact, $\operatorname{supp}\psi_u \subset U$
- $\psi_u \geq 0$, $\psi_u(x^{-1}) = \psi_u(x)$
- $\int \psi_u\, dx = 1$

Then for $f \in L^1$, $\psi_u * f \longrightarrow f$.

So $(L_x\psi_u)*f = L_x(\psi_u * f) \longrightarrow L_x f$

So $\pi(L_x\psi_u)\pi(f)v \longrightarrow \pi(L_x f)v$, $x \in G$, $v \in H$

Let $D = \operatorname{span}\{\pi(f)v : f \in L^1, v \in H\}$

Then D dense in H, because of $u \perp D$, then

$0 = \langle u, \pi(f)v\rangle = \langle \pi(f^*)u, v\rangle$, $\forall f, v$

As π is nondegenerate, so $u = 0$

Define $\tilde{\pi}(x) : D \longrightarrow D$, by $\tilde{\pi}(x)\pi(f)v = \pi(L_x f)v$.

So $\pi(L_x\psi_u)$ converges strongly to $\tilde{\pi}(x)$ on D

Since $\sum_j \pi(f_j)v_j = 0 \Rightarrow \sum \pi(L_x f_j)v_j = \lim_u \sum_j \pi(L_x\psi_u)\pi(f_j)v_j = 0$

So $\tilde{\pi}(x) : D \to D$ is well defined.

Moreover: $\|\pi(L_x\psi_u)\| \leq \|L_x\psi_u\|_1 = 1$

Therefore $\tilde{\pi}(x)$ extends unique to H in

$$\|\tilde{\pi}(x)\| \le 1, \qquad \tilde{\pi}(x)\pi(f) = \pi(L_x f)$$

- Claim $\tilde{\pi}$ is a u-rep. of G.
  - $\tilde{\pi}(xy)\pi(f) = \pi(L_{xy} f) = \pi(L_x L_y f) = \tilde{\pi}(x)\pi(L_y f) = \tilde{\pi}(x)\tilde{\pi}(y)\pi(f)$

    So $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ on D, so on H
- $\tilde{\pi}(1) = I$. so $\tilde{\pi}$ is a homomorphism.
- As $\|u\| = \|\tilde{\pi}(x^{-1})\tilde{\pi}(x)u\| \le \|\tilde{\pi}(x)u\| \le \|u\|$

  So $\tilde{\pi}(x)$ is isometry, hence unitary.
- If $x_\alpha \to x$ in G. then $L_{x_\alpha} f \to L_x f$ in L^1

  So $\tilde{\pi}(x_\alpha)\pi(f) = \pi(L_{x_\alpha} f) \xrightarrow{\text{strong}} \pi(L_x f) = \tilde{\pi}(x)\pi(f)$

So $\tilde{\pi}(x_\alpha) \to \tilde{\pi}(x)$ strongly on D

As D is dense in H, by a $\varepsilon/3$ argument

$\tilde{\pi}(x_\alpha) \to \tilde{\pi}(x)$ on H. so $\tilde{\pi}$ is continuous

To show $\tilde{\pi}(f) = \pi(f)$ $\quad f \in L^1$

$$\begin{aligned}\pi(f)\pi(g) = \pi(f * g) &= \int f(y)\,\pi(L_y g)\,dy \\ &= \int f(y)\,\tilde{\pi}(y)\pi(g)\,dy \\ &= \left[\int f(y)\,\tilde{\pi}(y)\,dy\right]\pi(g) = \tilde{\pi}(f)\pi(g)\end{aligned}$$

Thus $\tilde{\pi}(f) = \pi(f)$ on D hence on H

To show $\tilde{\pi}$ is unique, Suppose $\exists\ \hat{\pi}(f) = \pi(f)$. $f \in L^1(G)$

$\langle \hat{\pi}(x)u, v\rangle = \langle \tilde{\pi}(x)u, v\rangle$. $x \in G$. $u, v \in H$

Hence, $\hat{\pi}(x) = \tilde{\pi}(x)$, $x \in G$ $\square$

Unitary Representions.

The. 3.12 Let π be a u-rep. of G

i> The C^*-algebra generated by $\pi(G)$ and $\pi(L^1(G))$ have same closure in the strong and weak operator topologies

ii> $T \in \mathcal{L}(H_\pi)$ belong to $C(\pi)$ iff. $T\pi(f) = \pi(f)T$ $f \in L^1(G)$

iii> A closed subspace $M \subset H_\pi$, is invariant under π iff $\pi(f) M \subset M$, $f \in L^1(G)$

Proof <i> Claim. If $g \in C_c(G)$, then $\pi(g)$ is strong limit of $\Sigma_E = \sum g(x_j)\pi(x_j)|E_j|$

$E = \{E_j\}$ is the finite partition of $supp(g)$, $x_j \in E_j$

Indeed, $\varepsilon > 0$ and, $u_1 \cdots u_n \in H_\pi$

Since $x \longrightarrow g(x)\pi(x)u_m$ uniformly continuous.

So. $\| g(x)\pi(x)u_m - g(y)\pi(y)u_m \| < \varepsilon$, $m = 1, \ldots, n$ $x, y \in E_j$

So. $\| \Sigma_E u_m - \pi(g) u_m \|$

$$= \| \sum g(x_j)\pi(x_j)|E_j| u_m - \sum \int_{E_j} g(x)\pi(x)dx\, u_m \|$$

$$= \| \sum g(x_j)\pi(x_j)|E_j| u_m - \sum g(y_j)\pi(y_j)|E_j| u_m \|$$

$$< \varepsilon \sum |E_j| = \varepsilon\, supp(g)$$

Thus $\pi(g)$ is the strong limit of Σ_E

If $f \in L^1(G)$, $\exists$ some $g \in C_c(G)$, s.t. $g \longrightarrow f$ in L^1

$x, y \in E$;

$U_n\|$

$)U_n\|$

So $\pi(g) \to \pi'(f)$

As claimed, $\Sigma_E \xrightarrow{\text{strong}} \pi(g)$, and Σ_E are in the algebra generated by $\pi(G)$.

On the other hand, by The. 3.11,

$$\pi(L_x \psi_u) \to \pi(x) \quad \text{as } U \to \{1\}$$

So algebras generated by $\pi(G)$ and $\pi(L^1(G))$ have same strong closure and also weak closure.

ii) If $T \in C(\pi)$, then T commutes with every element of the weak closure of the algebra generated by $\pi(G)$, in particular $\pi(f)$. Vice versa

iii) By <ii> and Pro. 3.4 $\square$